

THE SECOND MOMENT PHENOMENON FOR MONOCHROMATIC SUBGRAPHS

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ABSTRACT. What is the chance that among a group of n friends, there are s friends all of whom have the same birthday? This is the celebrated birthday problem which can be formulated as the existence of a monochromatic s -clique K_s (s -matching birthdays) in the complete graph K_n , where every vertex of K_n is uniformly colored with 365 colors (corresponding to birthdays). More generally, for a general connected graph H , let $T(H, G_n)$ be the number of monochromatic copies of H in a uniformly random coloring of the vertices of the graph G_n with c_n colors. In this paper we show that $T(H, G_n)$ converges to $\text{Pois}(\lambda)$ whenever $\mathbb{E}T(H, G_n) \rightarrow \lambda$ and $\text{Var} T(H, G_n) \rightarrow \lambda$, that is, the asymptotic Poisson distribution of $T(H, G_n)$ is determined just by the convergence of its mean and variance. Moreover, this condition is necessary if and only if H is a star-graph. In fact, the second-moment phenomenon is a consequence of a more general theorem about the convergence of $T(H, G_n)$ to a finite linear combination of independent Poisson random variables. As an application, we derive the limiting distribution of $T(H, G_n)$, when $G_n \sim G(n, p)$ is the Erdős-Rényi random graph. Multiple phase-transitions emerge as p varies from 0 to 1, depending on whether the graph H is balanced or unbalanced.

1. INTRODUCTION

Let G_n be a simple labeled undirected graph with vertex set $V(G_n) := \{1, 2, \dots, |V(G_n)|\}$, edge set $E(G_n)$, and adjacency matrix $A(G_n) = \{a_{ij}(G_n), i, j \in V(G_n)\}$. In a *uniformly random c_n -coloring* of G_n , the vertices of G_n are colored with c_n colors as follows:

$$\mathbb{P}(v \in V(G_n) \text{ is colored with color } a \in \{1, 2, \dots, c_n\}) = \frac{1}{c_n}, \quad (1.1)$$

independent from the other vertices. Let X_v denote the color of the vertex $v \in V(G_n)$ in a uniformly random c_n -coloring of G_n . A subgraph F of G_n with vertex set $V(F) = \{v_1, \dots, v_{|V(F)|}\}$ is said to be *monochromatic* if $X_{v_1} = \dots = X_{v_{|V(F)|}}$.

In this paper we consider the problem of determining the limiting distribution of the number of monochromatic copies of a general connected simple graph H , in a uniformly random c_n -coloring of a graph sequence G_n . Formally, this is defined as

$$T(H, G_n) := \frac{1}{|Aut(H)|} \sum_{\mathbf{s} \in V(G_n)^{|V(H)|}} \prod_{(a,b) \in E(H)} a_{s_a s_b}(G_n) \mathbf{1}\{X_{\mathbf{s}}\}, \quad (1.2)$$

where:

- For a finite set S and a positive integer N , S_N denotes the set of all N -tuples $\mathbf{s} = (s_1, \dots, s_N) \in S^N$ with distinct entries.¹ Thus, the cardinality of S_N is $\frac{|S|!}{(|S|-N)!}$.

2010 *Mathematics Subject Classification.* 05C15, 60C05, 60F05, 05D99.

Key words and phrases. Birthday paradox, Combinatorial probability, Graph coloring, Poisson approximation.

*Research partially supported by NSF grant DMS-1712037.

¹For a set S , the set S^N denotes the N -fold cartesian product $S \times S \times \dots \times S$.

– For any $\mathbf{s} = (s_1, \dots, s_{|V(H)|}) \in V(G_n)^{|V(H)|}$,

$$\mathbf{1}\{X_{=\mathbf{s}}\} := \mathbf{1}\{X_{s_1} = \dots = X_{s_{|V(H)|}}\}.$$

– $\text{Aut}(H)$ is the *automorphism group* of H , that is, the group of permutations σ of the vertex set $V(H)$ such that $(x, y) \in E(H)$ if and only if $(\sigma(x), \sigma(y)) \in E(H)$.

Hereafter, we assume that H is simple and connected, with $|V(H)| \geq 2$, and $V(H) = \{1, 2, \dots, |V(H)|\}$. Note that for the case $H = K_2$ is an edge, the statistic (1.2) counts the number of monochromatic edges. This statistic arises in several contexts, for example, as the Hamiltonian of the Ising/Potts models on G_n [5], in non-parametric two-sample tests [14], and as a generalization of the birthday paradox [4, 10, 11, 12]: If G_n is a friendship-network graph colored uniformly with $c_n = 365$ colors (corresponding to birthdays and assuming the birthdays are uniformly distributed across the year), then two friends will have the same birthday whenever the corresponding edge in the graph G_n is monochromatic.² Therefore, $\mathbb{P}(T(K_2, G_n) > 0)$ is the probability that there are two friends with the same birthday. Note that $\mathbb{P}(T(K_2, G_n) > 0) = 1 - \mathbb{P}(T(K_2, G_n) = 0) = 1 - \chi_{G_n}(c_n)/c_n^{|V(G_n)|}$, where $\chi_{G_n}(c_n)$ counts the number of proper colorings of G_n using c_n colors. The function χ_{G_n} is known as the *chromatic polynomial* of G_n , and is a central object in graph theory [13, 17, 18]. More generally, s -matching birthdays in a friendship network G_n corresponds to the case $H = K_s$ (the complete graph on s vertices) in (1.2). The asymptotics of birthday collisions have found many applications, for example, in the study of coincidences [12, Problem 3], hash-function attacks in cryptology [21], and the discrete logarithm problem [15, 19].

In this paper, we study the asymptotic distribution of $T(H, G_n)$, in the regime where $\mathbb{E}(T(H, G_n)) = O(1)$. It is well-known that the limiting distribution of $T(K_2, G_n)$, exhibits a *first-moment phenomenon*, that is, $T(K_2, G_n) \xrightarrow{D} \text{Pois}(\lambda)$, for any graph sequence G_n such that $\mathbb{E}(T(K_2, G_n)) = \frac{1}{c_n}|E(G_n)| \rightarrow \lambda$. This was shown by Barbour et al. [4, Theorem 5.G], using the Stein's method for Poisson approximation. Recently, Bhattacharya et al. [7, Theorem 1.1] gave a new proof of this result based on the method of moments, which illustrates interesting connections to extremal combinatorics. The first-moment phenomenon is not true for general graphs H : it is easy to construct examples where $\mathbb{E}(T(H, G_n)) \rightarrow \lambda$, but $T(H, G_n) \not\xrightarrow{D} \text{Pois}(\lambda)$ [7, Section 8], if $H \neq K_2$. In this paper, we show that the limiting distribution of $T(H, G_n)$, for a general connected graph H , exhibits a *second-moment phenomenon*: $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$ whenever $\mathbb{E}T(H, G_n) \rightarrow \lambda$ and $\text{Var}T(H, G_n) \rightarrow \lambda$, that is, the limiting Poisson distribution of $T(H, G_n)$ is determined by the convergence of its mean and variance. This complements and generalizes the result for $T(K_2, G_n)$, since, in this case, the variance condition $\text{Var}T(K_2, G_n) \rightarrow \lambda$ is automatically implied by the mean condition $\mathbb{E}T(K_2, G_n) \rightarrow \lambda$. Using this result, the limiting distribution of $T(H, G_n)$ in the Erdős-Rényi random graph is derived, where interesting phase-transitions emerge.

1.1. The Second Moment Phenomenon. Throughout the paper, we will assume that H is a finite, simple, and connected graph with no isolated vertices, and G_n a sequence of growing simple graphs, with the vertices colored uniformly with c_n colors. We will also assume that $c_n \rightarrow \infty$ as $n \rightarrow \infty$, unless specified otherwise.

Theorem 1.1. *Let $H \neq K_2$ be as above, and $\{G_n\}_{n \geq 1}$ a sequence of graphs colored uniformly with c_n colors, such that*

$$\lim_{n \rightarrow \infty} \mathbb{E}T(H, G_n) = \lambda \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{Var}T(H, G_n) = \lambda. \quad (1.3)$$

²When the underlying graph $G_n = K_n$ is the complete graph K_n on n vertices, this reduces to the classical birthday problem.

Then $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$.

Note that Theorem 1.1 assumes that $H \neq K_2$, which corresponds to monochromatic edges. In this case, it is easy to check that

$$\mathbb{E}(T(K_2, G_n)) = \frac{|E(G_n)|}{c_n} \quad \text{and} \quad \text{Var}(T(K_2, G_n)) = \frac{|E(G_n)|}{c_n} \left(1 - \frac{1}{c_n}\right).$$

Therefore, the assumption $\mathbb{E}(T(K_2, G_n)) \rightarrow \lambda$ automatically ensures that $\text{Var}(T(K_2, G_n)) \rightarrow \lambda$. As a consequence, the variance condition (1.3) cannot be leveraged, when $H = K_2$, and the proof presented in this paper breaks down. However, as mentioned earlier, the conclusion in Theorem 1.1 still holds when $H = K_2$, that is, $T(K_2, G_n) \xrightarrow{D} \text{Pois}(\lambda)$, whenever $\mathbb{E}(T(K_2, G_n)) \rightarrow \lambda$ (refer to [4, Theorem 5.G] and [7, Theorem 1.1] for two different proofs of this result). Therefore, *the second-moment phenomenon* holds for all connected graph H , that is, the limiting Poisson distribution of the $T(H, G_n)$ is determined by the convergence of its first two moments.

The proof of Theorem 1.1 is described in Section 2. In fact, this theorem is a consequence of a more general result (Theorem 2.1) where we derive a general sufficient condition under which $T(H, G_n)$ is a finite linear combination of independent Poisson random variables. The proof is based on a truncated moment-comparison technique, and has two main steps:

- We begin with a truncation step: This involves defining a *remainder term*, which (informally) counts the number of tuples $\mathbf{s} \in V(G_n)_{|V(H)|}$ such that the number of copies of H passing through a subset of indices in \mathbf{s} is ‘large’. The first step is to show that the remainder term converges to zero in L_1 , because of the variance assumption in (1.3) (Lemma 2.1).
- To analyze the *main term*, which is $T(H, G_n)$ minus the remainder term, we use the ‘independent approximation’, which shows that the moments of the random variable obtained by replacing the indicators $\mathbf{1}\{X=\mathbf{s}\}$ by independent $\text{Ber}(\frac{1}{c_n^{|V(H)|-1}})$ variables, for every subset of vertices in G_n of size $|V(H)|$, are asymptotically close (Lemma 2.2). The result then follows by deriving the asymptotic distribution of the approximating variable, which is a finite linear combination of independent Bernoulli random variables, each of which converges to a Poisson distribution (Lemma 2.4).

The truncation step is necessary because, $T(H, G_n)$, for a general graph H , does not converge in moments (see Theorem 1.2 below), and hence, its limiting distribution, cannot be captured by a direct moment-based argument.

Remark 1.1. Another natural approach to proving a limiting Poisson distribution is through the Stein’s method for Poisson approximation [2, 4, 8, 9]. In fact, the well-known Stein’s method based on dependency graphs [9, Theorem 15], bounds the convergence rate in terms of covariances (but, not in terms of the mean and the variance). Arratia et al. [3] used this to obtain rates of convergence for the number of monochromatic cliques in a uniform coloring of a complete graph (see also Chatterjee et al. [9]). However, this cannot be used to prove Theorem 1.1 for a general graph H , as the condition imposed by the convergence of the mean and the variance is, in general, weaker than what is required by a generic dependency graph construction (refer to Remark 4.1 for a specific example). Moreover, our general result (Theorem 2.1) goes beyond the Poisson regime, and captures the asymptotic regime where $T(H, G_n)$ is a finite linear combination of Poisson variables.

Next, we consider the converse to Theorem 1.1, that is, whether the Poisson convergence of $T(H, G_n)$ implies the convergence of the first two moments. The following theorem shows that this is true if and only if H is a *star-graph*, that is, $H = K_{1,r}$ for some integer $r \geq 1$.

Theorem 1.2. Fix an integer $r \geq 1$, a real number $\lambda > 0$, and a sequence of graphs $\{G_n\}_{n \geq 1}$ colored uniformly with c_n colors. Then $T(K_{1,r}, G_n) \xrightarrow{D} \text{Pois}(\lambda)$ if and only if

$$\lim_{n \rightarrow \infty} \mathbb{E}T(K_{1,r}, G_n) = \lambda \quad \text{and} \quad \lim_{n \rightarrow \infty} \text{Var} T(K_{1,r}, G_n) = \lambda. \quad (1.4)$$

Moreover, if H is connected and is not a star-graph, then there exists a sequence of graphs $\{G_n(H)\}_{n \geq 1}$ such that $T(H, G_n(H)) \xrightarrow{D} \text{Pois}(\lambda)$, but (1.4) does not hold.

The proof of the theorem is given in Section 3. In fact, the proof shows that when H is a star-graph, we have convergence in all moments, that is, (1.4) implies that $T(K_{1,r}, G_n) \rightarrow \text{Pois}(\lambda)$ in distribution and in all moments, and conversely, $T(K_{1,r}, G_n)$ converges in distribution to $\text{Pois}(\lambda)$ implies the convergence of all moments of $T(K_{1,r}, G_n)$ to the corresponding moments of $\text{Pois}(\lambda)$.

Remark 1.2. The second moment phenomenon for the Poisson distribution complements the well-known *fourth-moment phenomenon*, which asserts that the limiting normal distribution of certain homogeneous forms is implied by the convergence of the corresponding sequence of fourth moments (refer to Nourdin et al. [22] and the references therein, for general fourth-moment theorems and invariance principles, and Bhattacharya et al. [7, Theorem 1.3] for an example of this phenomenon in random graph coloring). In this regard, it would be interesting to see if the Poisson second-moment phenomenon extends beyond monochromatic subgraphs to general integer-valued homogeneous forms.

1.2. Application to Erdős-Rényi Random Graphs. Theorem 1.1 can be easily extended to random graphs, when the limits in Theorem 2.1 hold in probability, under the assumption that the graph and its coloring are jointly independent (see Lemma 4.1 for details). Using this we can derive the limiting distribution of $T(H, G_n)$, where $G_n \sim G(n, p)$ is the Erdős-Rényi random graph, colored uniformly with c_n colors (independently of the graph), such that

$$\mathbb{E}T(H, G_n) = \frac{|V(H)|! \binom{n}{|V(H)|} p^{|E(H)|}}{|Aut(H)| c_n^{|V(H)|-1}} \rightarrow \lambda. \quad (1.5)$$

This implies $c_n = \Theta(n^{\frac{|V(H)|}{|V(H)|-1}} p^{\frac{|E(H)|}{|V(H)|-1}})$. Also the condition $c_n \rightarrow \infty$ implies $n^{\frac{|V(H)|}{|E(H)|}} p \rightarrow \infty$.

Under the above scaling, Theorem 2.1 can be used to characterize the limiting distribution of $T(H, G_n)$ for all connected graphs H , where $G_n \sim G(n, p)$ and $p = p(n) \in (0, 1)$. Here multiple interesting phase transitions occur depending on whether the graph H is balanced or unbalanced. We begin by recalling the notion of balancedness of a graph.

Definition 1.1. [16, Chapter 3] For a finite connected graph H , define

$$m(H) = \max_{H_1 \subseteq H} \frac{|E(H_1)|}{|V(H_1)|}, \quad (1.6)$$

where the maximum is over all non-empty subgraphs H_1 of H . The graph H is said to be *balanced*, if $m(H) = \frac{|E(H)|}{|V(H)|}$, and *unbalanced* otherwise. Moreover, the graph H is said to be *strictly balanced* if $\frac{|E(H')|}{|V(H')|} < \frac{|E(H)|}{|V(H)|} = m(H)$, for all proper subgraphs H' of H .

In the balanced case, where the asymptotic distribution of $T(H, G_n)$ undergoes a phase transition from $\text{Pois}(\lambda)$ to a linear combination of independent Poissons, depending on whether $p(n) \rightarrow 0$ or $p(n) := p$ is fixed, respectively.

Theorem 1.3. (*Balanced Graphs*) Let H be a simple connected balanced graph, and $G_n \sim G(n, p)$ be the Erdős-Rényi random graph, with $p := p(n) \in (0, 1)$, colored uniformly with c_n colors such that (1.5) holds. Then the following cases arise:

- (a) If $n^{-\frac{|V(H)|}{|E(H)|}} \ll p(n) \ll 1$, then $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$.
 (b) If $p(n) := p \in (0, 1)$ is fixed,

$$T(H, G_n) \xrightarrow{D} \sum_{F \supseteq H: |V(F)|=|V(H)|} N(H, F) X_F, \quad (1.7)$$

where $X_F \sim \text{Pois}\left(\lambda \cdot \frac{|Aut(H)|}{|Aut(F)|} p^{|E(F)|-|E(H)|} (1-p)^{\binom{|V(H)|}{2}-|E(F)|}\right)$ and the collection $\{X_F : F \supseteq H \text{ and } |V(F)| = |V(H)|\}$ is independent.

Note that the sum in (1.7) above is over the set of non-isomorphic (unlabelled) graphs F , which contain H as a subgraph and has the same number of vertices as H . The proof of Theorem 1.3 is given in Section 4.1.

The situation, however, is more delicate for unbalanced graphs. To explain this, we need the following definition:

Definition 1.2. For an unbalanced graph H , define the exponent

$$\gamma(H) := \min_{H_1 \subset H} \frac{|V(H)| - |V(H_1)|}{|E(H_1)|(|V(H)| - 1) - |E(H)|(|V(H_1)| - 1)}, \quad (1.8)$$

where the minimum is over the set of all proper subgraphs H_1 of H , for which the denominator is positive.

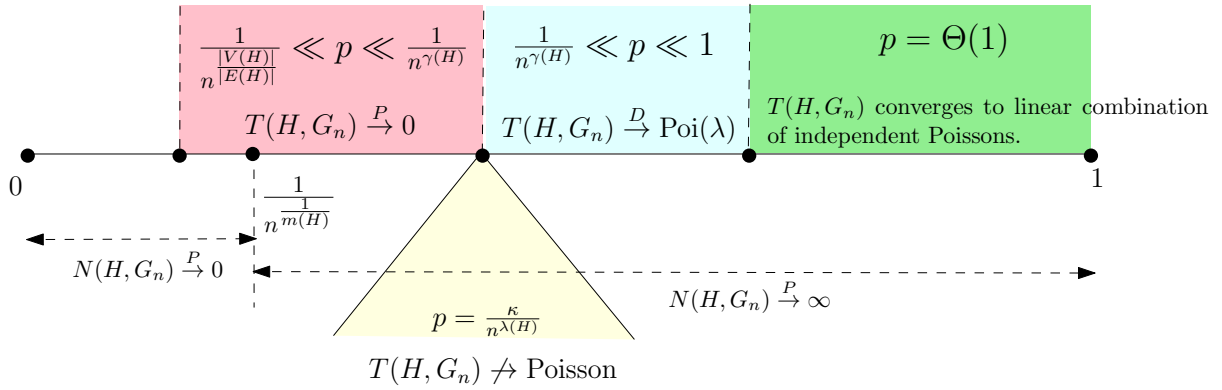


FIGURE 1. Phase transitions of $T(H, G_n)$, for an unbalanced graph H in the Erdős-Rényi random graph $G_n \sim G(n, p)$, as p varies from 0 to 1.

It is easy to verify that $\gamma(H)$ is well-defined and positive, for any unbalanced graph H (see Lemma 4.2).³ When H is unbalanced, the asymptotic distribution of $T(H, G_n)$, where $G_n \sim G(n, p(n))$, undergoes an additional phase-transition, whose location is determined by the exponent $\gamma(H)$.

³Even though, for our results, we only need to define $\gamma(H)$ for unbalanced graphs, it is natural to wonder what happens to the quantity in the RHS of (1.8) for balanced graphs. We show in Lemma 4.2, for H balanced, but not strictly balanced, $\gamma(H)$ as in (1.8), is well-defined and equals to $\frac{1}{m(H)}$. On the other hand, if H is strictly balanced, there are cases where the RHS of (1.8) is finite, and there are cases where it is undefined.

Theorem 1.4. (*Unbalanced Graphs*) Let H be a simple connected unbalanced graph, and $G_n \sim G(n, p)$ be the Erdős-Rényi random graph, with $p := p(n) \in (0, 1)$, colored uniformly with c_n colors, such that (1.5) holds. Then the following cases arise:

- (a) If $n^{-\frac{|V(H)|}{|E(H)|}} \ll p(n) \ll n^{-\gamma(H)}$, then $T(H, G_n) \xrightarrow{P} 0$.
- (b) If $n^{\gamma(H)} p(n) \rightarrow \kappa \in (0, \infty)$ then all moments of $T(H, G_n)$ converges. Moreover, if $T(H, G_n)$ converges in distribution to a random variable W , then W is not Poisson.
- (c) If $n^{-\gamma(H)} \ll p(n) \ll 1$, then $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$.
- (d) If $p(n) := p \in (0, 1)$ is fixed, then $T(H, G_n)$ converges to the RHS of (1.7), that is, a linear combination of independent Poisson random variables.

The proof of Theorem 1.4 is given in Section 4.1. The phase transitions of $T(H, G_n)$, for an unbalanced graph H , are shown in Figure 1.

Remark 1.3. It is well-known that $n^{-\frac{1}{m(H)}}$ is the threshold for the occurrence of H in the random graph $G(n, p)$ [16, Theorem 3.4]. Therefore, for unbalanced graphs, since $\gamma(H) < \frac{1}{m(H)}$ (Lemma 4.2), there exists an interesting regime ($n^{-\frac{1}{m(H)}} \ll p \ll n^{-\gamma(H)}$) where $N(H, G_n)$, the number of copies of H in G_n , goes to infinity, but the number of monochromatic copies $T(H, G_n)$ converges in probability to zero, that is, we do not have convergence of moments. Another surprising feature of unbalanced graphs is that the asymptotic distribution of $T(H, G_n)$ transitions from being degenerate at zero (equivalently, $\text{Pois}(0)$) to $\text{Pois}(\lambda)$, through a non-Poisson limit at the point of criticality ($p = \frac{\kappa}{n^{\gamma(H)}}$). It remains open to show that the limit of $T(H, G_n)$ exists at the critical point, and finding the limiting distribution? Preliminary calculations in a few examples seem to suggest that the limiting moments may not satisfy Stieltjes moment condition [1], and so we cannot conclude existence of limiting distribution from the convergence of moments.

1.3. Organization. The rest of the paper is organized as follows: The general limiting distribution of monochromatic subgraphs and the proof of Theorem 1.1 are given in Section 2. The proof of Theorem 1.2 is given in Section 3. Applications to the Erdős-Rényi random graph (proofs of Theorem 1.3 and Theorem 1.4) and the birthday problem are discussed in Section 4.

2. LIMITING DISTRIBUTION OF MONOCHROMATIC SUBGRAPHS

In this section we derive general sufficient conditions under which the random variable $T(H, G_n)$ converges to a linear combination of independent Poisson random variables. We begin with a few definitions and notations: For a finite simple unlabeled graph F , denote by $\text{hom}_{\text{inj}}(F, G_n)$ the set of injective homomorphisms from F to G_n , that is, the set of injective maps $\phi : V(F) \rightarrow V(G_n)$, such that $(\phi(x), \phi(y)) \in E(G_n)$ whenever $(x, y) \in E(F)$. It is easy to see that

$$|\text{hom}_{\text{inj}}(H, G_n)| = \sum_{s \in V(G_n)^{|V(H)|}} \prod_{(a,b) \in E(H)} a_{s_a s_b}(G_n).$$

Moreover, denote by $N(F, G_n)$ the number of copies of F in G_n , and $N_{\text{ind}}(F, G_n)$ the number of induced copies of F in G_n . Note that

$$N(H, G_n) = \frac{|\text{hom}_{\text{inj}}(H, G_n)|}{|\text{Aut}(H)|} \quad \text{and} \quad \mathbb{E}(T(H, G_n)) = \frac{N(H, G_n)}{c_n^{|V(H)|-1}}. \quad (2.1)$$

Next, we introduce the notion of join of two graphs. These graphs will show up in the analysis of the variance of $T(H, G_n)$.

Definition 2.1. Fix $t \in [1, |V(H)|]$. Let H' be an isomorphic copy of H , with $V(H) = \{1, 2, \dots, |V(H)|\}$ and $V(H') = \{1', 2', \dots, |V(H)|'\}$, where $z' \in V(H')$ is the image of $z \in V(H)$. For two ordered index sets $J_1 = (j_{11}, j_{12}, \dots, j_{1t}) \in [|V(H)|]_t$ and $J_2 = (j_{21}, j_{22}, \dots, j_{2t}) \in [|V(H)|]_t$, denote by $H_t(J_1, J_2)$ the simple graph obtained by the union of H and H' , when the vertex $j_{1a} \in V(H)$ is identified with the vertex $j'_{2a} \in V(H')$, for $a \in [t]$. More precisely,

$$H_t(J_1, J_2) = \left(V(H) \cup \gamma(V(H')), E(H) \cup \gamma(E(H')) \right),$$

where

- $\gamma(V(H')) = \{\gamma(v') : v' \in V(H')\}$, where γ is a relabelling of the vertices of $V(H')$ such that $\gamma(j'_{2a}) = j_{1a}$, for $a \in [t]$, and $\gamma(v') = v'$, for $v' \notin J_2$.
- This induces a relabelling of the edges $\gamma(E(H')) = \{\gamma((u', v')) : (u', v') \in E(H')\}$, where $\gamma((u', v')) = (\gamma(u'), \gamma(v'))$, for $(u', v') \in E(H')$.

The graph $H_t(J_1, J_2)$ will be referred to as the t -join of H with pivots at J_1 and J_2 (see Figure 2). Denote by $\mathcal{J}_t(H) := \{H_t(J_1, J_2) : J_1, J_2 \in [|V(H)|]_t\}$ the collection of all graphs (up to isomorphism) which can be obtained as the t -join of H . Finally, a graph F is said to be a *join* of two isomorphic copies of H , if $F \in \mathcal{J}_t(H)$, for some $t \in [1, |V(H)|]$.

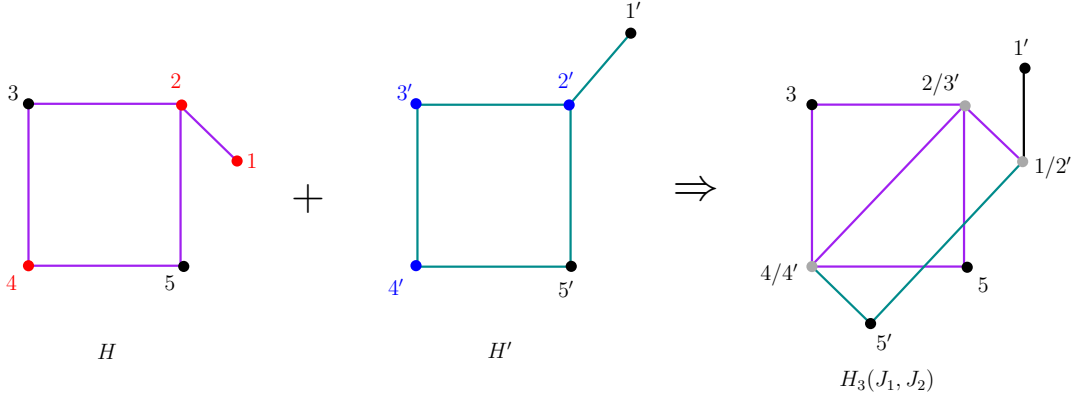


FIGURE 2. 3-join of H (the 4-cycle with an edge hanging from one vertex of the cycle) with pivots $J_1 = (1, 2, 4)$ and $J_2 = (2, 3, 4)$.

Equipped with the above definitions, we can now state our general theorem:

Theorem 2.1. Let H be as in Theorem 1.1, and G_n be a sequence of graphs colored uniformly with c_n colors, such that the following hold:

- For every $k \in [1, N(H, K_{|V(H)|})]$, there exists $\lambda_k \geq 0$ such that

$$\lim_{n \rightarrow \infty} \frac{\sum_{F \in \mathcal{C}_{H,k}} N_{\text{ind}}(F, G_n)}{c_n^{|V(H)|-1}} = \lambda_k, \quad (2.2)$$

where $\mathcal{C}_{H,k} := \{F \supseteq H : |V(F)| = |V(H)| \text{ and } N(H, F) = k\}$.⁴

- For $t \in [2, |V(H)| - 1]$ and every $F \in \mathcal{J}_t(H)$, as $n \rightarrow \infty$, $N(F, G_n) = o(c_n^{2|V(H)|-t-1})$.

⁴Note that the graphs in the set $\mathcal{C}_{H,k}$ are unlabelled. In other words, $\mathcal{C}_{H,k}$ is the collection of non-isomorphic graphs with the same number of vertices as H and containing k copies of H .

Then

$$T(H, G_n) \xrightarrow{D} \sum_{k=1}^{N(H, K_{|V(H)|})} kZ_k, \quad (2.3)$$

where $Z_k \sim \text{Pois}(\lambda_k)$ and the collection $\{Z_k : 1 \leq k \leq N(H, K_{|V(H)|})\}$ is independent.

The second condition ensures that the counts all sub-graphs of G_n which arise as the join of two non-disjoint copies of H on non-identical vertex sets, (that is, $t \neq \{1, |V(H)|\}$) are asymptotically negligible. Moreover, as $\text{Cov}(\mathbf{1}\{X=\mathbf{s}\}, \mathbf{1}\{X=\mathbf{t}\}) = 0$, whenever $\mathbf{s}, \mathbf{t} \in V(G_n)_{|V(H)|}$ have at most 1 index in common, the only terms in $\text{Var} T(H, G_n)$ which contribute are those which arise as a $|V(H)|$ -join of two copies of H . Therefore, Theorem 2.1 captures the asymptotic regime where $T(H, G_n)$ is ‘linear’, and to ensure the existence of the limiting distribution we assume (2.2).

Remark 2.1. An easy sufficient condition for (2.2) is the convergence of $\frac{1}{c_n^{\frac{1}{|V(F)|-1}}} N_{\text{ind}}(F, G_n)$ for every super-graph F of H with $|V(F)| = |V(H)|$. However, condition (2.2) does not require the convergence for every such graph, and is applicable to more general examples, as described below: Define a sequence of graphs G_n as follows:

$$G_n = \begin{cases} \text{disjoint union of } n \text{ isomorphic copies of } C_4 & \text{if } n \text{ is odd} \\ \text{disjoint union of } n \text{ isomorphic copies of } \mathcal{D} & \text{if } n \text{ is even.} \end{cases}$$

where C_4 denotes the 4-cycle and \mathcal{D} is the 4-cycle with one diagonal. Choosing $c_n = \lfloor n^{1/3} \rfloor$, gives $\mathbb{E}(T(C_4, G_n)) \rightarrow 1$. In this case,

$$\frac{\sum_{F \in \mathcal{C}_{H,1}} N_{\text{ind}}(F, G_n)}{c_n^3} = \frac{N_{\text{ind}}(C_4, G_n) + N_{\text{ind}}(\mathcal{D}, G_n)}{c_n^3} \rightarrow 1,$$

and $\frac{1}{c_n^3} \sum_{F \in \mathcal{C}_{H,3}} N_{\text{ind}}(F, G_n) = \frac{1}{c_n^3} N_{\text{ind}}(K_4, G_n) = 0$, and $\frac{1}{c_n^3} \sum_{F \in \mathcal{C}_{H,2}} N_{\text{ind}}(F, G_n) = 0$, since $\mathcal{C}_{H,2}$ is empty. Therefore, Theorem 2.1 implies that $T(C_4, G_n) \xrightarrow{D} \text{Pois}(1)$ (which can also be directly verified, because, in this case, $T(C_4, G_n)$ is a sum of independent $\text{Ber}(\frac{1}{c_n^3})$ variables). However, it is easy to see that individually both $\frac{1}{c_n^3} N_{\text{ind}}(C_4, G_n)$ and $\frac{1}{c_n^3} N_{\text{ind}}(\mathcal{D}, G_n)$ are non-convergent.

Remark 2.2. Note that a linear combination of Poisson random variables is a special case of the discrete compound Poisson distribution [23]. To this end, denote by Z the random variable in the RHS of (2.3) and let $\kappa_H = N(H, K_{|V(H)|})$. Then define Y to be a discrete random variable with

$$\mathbb{P}(Y = k) = \frac{\lambda_k}{\sum_{k=1}^{\kappa_H} \lambda_k}, \quad \text{for } k \in \{1, \dots, \kappa_H\}.$$

It is then easy to see that Z has the same distribution as the discrete compound Poisson variable $Z' = \sum_{i=1}^N Y_i$, where $\{Y_1, Y_2, \dots\}$ are independent copies of Y , and $N \sim \text{Pois}(\sum_{k=1}^{\kappa_H} \lambda_k)$, which is independent of $\{Y_1, Y_2, \dots\}$.

The rest of this section is organized as follows: The proof of Theorem 2.1 is given below in Section 2.1 and the proof of Theorem 1.1 is described in Section 2.2.

2.1. Proof of Theorem 2.1. We begin with a few notations and definitions. For an ordered tuple \mathbf{t} with distinct entries, denote by $\bar{\mathbf{t}}$ the (unordered) set formed by the entries of \mathbf{t} (for example, if $\mathbf{t} = (4, 2, 5)$, then $\bar{\mathbf{t}} = \{2, 4, 5\}$).

Given $J \subseteq V(H)$, define $H[J]$ to be the induced subgraph of H on the vertices in J , $H \setminus J$ the graph obtained by removing all vertices in J and the associated edges, and $E_H(J, J^c) = \{(x, y) \in$

$E(H) : x \in J$ and $y \in V(H) \setminus J$. Clearly, $E(H) = E(H[J]) \cup E(H \setminus J) \cup E_H(J, J^c)$ is an edge partition of $E(H)$.

Definition 2.2. Fix $t \in [2, |V(H)|]$ and $J = (j_1, j_2, \dots, j_t) \in V(H)_t$ and $\mathbf{r} = (r_1, r_2, \dots, r_t) \in V(G_n)_t$. Denote by $M_J(\mathbf{r}, H, G_n)$ the number of injective homomorphism $\phi : V(H) \rightarrow V(G_n)$ such that $\phi(j_a) = r_a$, for $a \in [t]$. More formally, define $\psi : \bar{J} \rightarrow [t]$ as $\psi(j_b) = b$, for $b \in [t]$, then

$$M_J(\mathbf{r}, H, G_n) = \prod_{(x,y) \in E(H[\bar{J}])} a_{r_{\psi(x)}r_{\psi(y)}}(G_n) \sum_{\mathbf{s}_{J^c}} \prod_{(x,y) \in E_H(\bar{J}, \bar{J}^c)} a_{r_{\psi(x)}s_y}(G_n) \prod_{(x,y) \in E(H \setminus \bar{J})} a_{s_x s_y}(G_n),$$

where the sum is over tuples $\mathbf{s}_{J^c} := (s_x)_{x \in V(H) \setminus \bar{J}} \in (V(G_n) \setminus \bar{\mathbf{r}})_{|V(H) \setminus \bar{J}|}$.

Example 1. To help parse the above definition, we compute $M_J(\cdot, H, G_n)$ in a few examples:

- $H = K_{1,2}$ is the 2-star with the central vertex labeled 1 and $J = (2, 3)$. Then with $\mathbf{r} = (i, j)$,

$$M_{(2,3)}((i, j), K_{1,2}, G_n) = M_{(3,2)}((i, j), K_{1,2}, G_n) = \sum_{\substack{s=1 \\ s \notin \{i, j\}}}^{|V(G_n)|} a_{is}(G_n) a_{js}(G_n) := t_{G_n}(i, j),$$

where $t_{G_n}(i, j)$ is the number of common neighbors of i, j . Similarly,

$$M_{(1,2)}((i, j), K_{1,2}, G_n) = M_{(1,3)}((i, j), K_{1,2}, G_n) = a_{ij}(G_n)(d_{G_n}(i) - a_{ij}(G_n)), \quad (2.4)$$

where $d_{G_n}(i)$ denotes the degree of the vertex i in G_n . Finally, $M_{(2,1)}((i, j), K_{1,2}, G_n) = M_{(3,1)}((i, j), K_{1,2}, G_n) = a_{ij}(G_n)(d_{G_n}(j) - a_{ij}(G_n))$.

- $H = P_4$, the path of length 3, with vertices labeled $\{1, 2, 3, 4\}$ in order and $J = (2, 4)$, then with $\mathbf{r} = (i, j)$,

$$M_{(2,4)}((i, j), P_4, G_n) = \sum_{\substack{s_1=1 \\ s_1 \notin \{i, j\}}}^{|V(G_n)|} \sum_{\substack{s_3=1 \\ s_3 \notin \{s_1, i, j\}}}^{|V(G_n)|} a_{s_1 i}(G_n) a_{i s_3}(G_n) a_{s_3 j}(G_n).$$

The expressions for other ordered tuples J can be obtained similarly.

We now begin the proof of Theorem 2.1. For $\mathbf{s} \in V(G_n)_{|V(H)|}$ and an ordered subset $J \subseteq [|V(H)|]$ denote by $\mathbf{s}_J = (s_j)_{j \in J}$, subset of indices s_j such that $j \in J$. Then, define

$$\mathcal{A}_\varepsilon(H, G_n) = \left\{ \mathbf{s} \in V(G_n)_{|V(H)|} : M_{J_1}(\mathbf{s}_{J_2}, H, G_n) \leq \varepsilon c_n^{|V(H)|-t}, \right. \\ \left. \text{for all } J_1, J_2 \in V(H)_t, \text{ and all } t \in [2, |V(H)| - 1] \right\}. \quad (2.5)$$

Informally, $\mathcal{A}_\varepsilon(H, G_n)$ counts the number of tuples $\mathbf{s} \in V(G_n)_{|V(H)|}$ such that the number of copies of H passing through a subset of indices in \mathbf{s} is ‘small’.

Example 2. (2-star) If $H = K_{1,2}$ is the 2-star (with central vertex labeled 1), then $\mathcal{A}_\varepsilon(K_{1,2}, G_n)$ consists of all 3-tuples $\mathbf{s} = (s_1, s_2, s_3)$ of distinct vertices of G_n , such that,

- (1) $a_{s_i s_j}(G_n)(d_{G_n}(s_i) - a_{s_i s_j}(G_n)) \leq \varepsilon c_n$ and $a_{s_i s_j}(G_n)(d_{G_n}(s_j) - a_{s_i s_j}(G_n)) \leq \varepsilon c_n$ (recall (2.4)), that is, $\max\{d_{G_n}(s_i), d_{G_n}(s_j)\} = \varepsilon c_n + 1$ if there is an edge between (s_i, s_j) ; and
- (2) s_i and s_j has at most εc_n common neighbors in G_n ,

for every $1 \leq i \neq j \leq 3$.

Next, define the *main term*

$$T_\varepsilon^+(H, G_n) = \frac{1}{|Aut(H)|} \sum_{\mathbf{s} \in \mathcal{A}_\varepsilon(H, G_n)} M(\mathbf{s}, H, G_n) \mathbf{1}\{X=\mathbf{s}\}, \quad (2.6)$$

where $M(\mathbf{s}, H, G_n) = \prod_{(a,b) \in E(H)} a_{s_a s_b}(G_n)$, and the *remainder term*

$$T_\varepsilon^-(H, G_n) = T(H, G_n) - T_\varepsilon^+(H, G_n).$$

2.1.1. The Remainder Term. We shall begin by showing that for each fixed $\varepsilon > 0$, the remainder term $T_\varepsilon^-(H, G_n)$ converges in L^1 to 0 as $n \rightarrow \infty$. Note that, $A \lesssim_\square B$ means $A \leq C \cdot B$, where $C := C(\square) > 0$ is a constant that depends only on the subscripted quantities. Similarly, $A \square \gtrsim B$ is $B \lesssim_\square A$.

Lemma 2.1. *For each fixed $\varepsilon > 0$, $T_\varepsilon^-(H, G_n) \xrightarrow{L^1} 0$ as $n \rightarrow \infty$.*

Proof. To begin with, note that

$$\mathbb{E}T_\varepsilon^-(H, G_n) = \frac{1}{c_n^{|V(H)|-1}} \sum_{\mathbf{s} \in V(G_n)_{|V(H)|} \setminus \mathcal{A}_\varepsilon(H, G_n)} \frac{M(\mathbf{s}, H, G_n)}{|Aut(H)|}.$$

Then, recalling the definition of $\mathcal{A}_\varepsilon(H, G_n)$ from (2.5), by an union bound

$$\begin{aligned} & \mathbb{E}T_\varepsilon^-(H, G_n) \\ & \leq \frac{1}{c_n^{|V(H)|-1}} \sum_{t=2}^{|V(H)|-1} \sum_{\substack{J_1 \in V(H)_t \\ J_2 \in V(H)_t}} \sum_{\mathbf{s} \in V(G_n)_{|V(H)|}} \frac{M(\mathbf{s}, H, G_n)}{|Aut(H)|} \mathbf{1}\{M_{J_1}(\mathbf{s}_{J_2}, H, G_n) > \varepsilon c_n^{|V(H)|-t}\} \\ & \lesssim_H \frac{1}{c_n^{|V(H)|-1}} \sum_{t=2}^{|V(H)|-1} \sum_{\substack{J_1 \in V(H)_t \\ J_2 \in V(H)_t}} \sum_{\mathbf{s} \in V(G_n)_{|V(H)|}} M(\mathbf{s}, H, G_n) \frac{M_{J_1}(\mathbf{s}_{J_2}, H, G_n)}{\varepsilon c_n^{|V(H)|-t}}. \end{aligned} \quad (2.7)$$

In order to complete the proof, it thus suffices to show that

$$\sum_{\mathbf{s} \in V(G_n)_{|V(H)|}} M(\mathbf{s}, H, G_n) M_{J_1}(\mathbf{s}_{J_2}, H, G_n) = o(c_n^{2|V(H)|-t-1}), \quad (2.8)$$

for all $t \in [2, |V(H)| - 1]$ and $J_1 = (j_{11}, \dots, j_{1t})$, $J_2 = (j_{21}, \dots, j_{2t}) \in V(H)_t$ (see Example 3 for a special case).

To this end, we have

$$\begin{aligned} & \sum_{\mathbf{s} \in V(G_n)_{|V(H)|}} M(\mathbf{s}, H, G_n) M_{J_1}(\mathbf{s}_{J_2}, H, G_n) \\ & = \sum_{\mathbf{s}_{J_2}} M_{J_2}(\mathbf{s}_{J_2}, H, G_n) M_{J_1}(\mathbf{s}_{J_2}, H, G_n) \quad (\text{summing over indices in } \mathbf{s}_{J_2}) \\ & = \sum_{\mathbf{r} \in V(G_n)_t} M_{J_2}(\mathbf{r}, H, G_n) M_{J_1}(\mathbf{r}, H, G_n) \quad (\text{changing variable } \mathbf{s}_{J_2} \text{ to } \mathbf{r}) \\ & = \sum_{\mathbf{r} \in V(G_n)_t} |\{(\phi, \psi) \in \text{hom}_{\text{inj}}(H, G_n)^2 : \phi(j_{2a}) = r_a = \psi(j_{1a}) \text{ for all } a \in [t]\}| \\ & = |\{(\phi, \psi) \in \text{hom}_{\text{inj}}(H, G_n)^2 : \phi(j_{2,a}) = \psi(j_{1,a}) \text{ for all } a \in [t]\}| \end{aligned}$$

$$\lesssim_H \sum_{t'=t}^{|V(H)|} \sum_{\substack{J'_1 \supseteq J_1 \\ J'_1 \in V(H)_{t'}}} \sum_{\substack{J'_2 \supseteq J_2 \\ J'_2 \in V(H)_{t'}}} N(H_{t'}(J'_1, J'_2), G_n) \quad (2.9)$$

The last step is based on the observation that a $(\phi, \psi) \in \text{hom}_{\text{inj}}(H, G_n)^2$ satisfying $\phi(j_{2a}) = \psi(j_{1a})$ for all $a \in [t]$, gives rise to a t' join of H with pivots J'_1 and J'_2 for some $t' \in [t, |V(H)|]$ and $J_1 \subseteq J'_1 \subseteq V(H)_{t'}$, $J_2 \subseteq J'_2 \subseteq V(H)_{t'}$ in at most finitely (depending only on $|V(H)|$) many ways. The reason we need to introduce J'_1 and J'_2 , is that $\phi(j_2)$ may equal $\psi(j_1)$ for some $j_1 \notin J_1$ and $j_2 \notin J_2$. To elaborate, J'_2 consists of all those elements j_2 of $V(H)$, for which there exist an element j_1 of $V(H)$ such that $\phi(j_2) = \psi(j_1)$, and $J'_1 = (\psi^{-1}(\phi(j_2)))_{j_2 \in J'_2}$.

Now, note that the sum in (2.9) is a finite sum (depending only on H). Further, for each $t' \in [t, |V(H)| - 1]$, $J_1 \subseteq J'_1 \in V(H)_{t'}$ and $J_2 \subseteq J'_2 \in V(H)_{t'}$,

$$N(H_{t'}(J'_1, J'_2), G_n) = o(c_n^{2|V(H)|-t'-1}) = o(c_n^{2|V(H)|-t-1}),$$

by assumption in Theorem 2.1. Lastly, for $J'_1 \in V(H)_{|V(H)|}$ and $J'_2 \in V(H)_{|V(H)|}$,

$$N(H_{|V(H)|}(J'_1, J'_2), G_n) = O(c_n^{|V(H)|-1}) = o(c_n^{2|V(H)|-t-1}),$$

Therefore, $\lim_{n \rightarrow \infty} \mathbb{E}T_\varepsilon^-(H, G_n) \rightarrow 0$, completing the proof of the lemma. \square

Example 3. (2-star continued) To help the reader parse the above proof, we re-do the calculations for the case $H = K_{1,2}$ (with central vertex labeled 1), and $J_1 = (2, 3)$ and $J_2 = (1, 2)$. In this case, the LHS of (2.8) is

$$\begin{aligned} & \sum_{(s_1, s_2, s_3) \in V(G_n)_3} M((s_1, s_2, s_3), K_{1,2}, G_n) M_{(2,3)}((s_1, s_2), K_{1,2}, G_n) \\ &= \sum_{(s_1, s_2, s_3) \in V(G_n)_3} a_{s_1 s_2}(G_n) a_{s_1 s_3}(G_n) M_{(2,3)}((s_1, s_2), K_{1,2}, G_n) \\ &\leq \sum_{s_1 \neq s_2 \in V(G_n)} a_{s_1 s_2}(G_n) d_{G_n}(s_1) t_{G_n}(s_1, s_2) \quad (\text{recall Example 1}) \\ &\lesssim N(K_3, G_n) + N(\Delta_+, G_n), \end{aligned}$$

where Δ_+ is the $(3, 1)$ -tadpole (the graph obtained by joining a triangle and a single vertex with a bridge). Now, $N(K_3, G_n) \lesssim N(K_{1,2}, G_n) = O(c_n^2) = o(c_n^3)$ and $N(\Delta_+, G_n) = o(c_n^3)$, by assumption in Theorem 2.1, which establishes (2.8), for $H = K_{1,2}$, and $J_1 = (2, 3)$, $J_2 = (1, 2)$.

2.1.2. The Main Term: Moment Comparison. To analyze $T_\varepsilon^+(H, G_n)$ we use the ‘independent approximation’, where the indicators $\mathbf{1}\{X_{=s}\}$ are replaced by independent Bernoulli variables, for every subset of vertices in G_n of size $|V(H)|$. To this end, define

$$J_\varepsilon^+(H, G_n) = \frac{1}{|Aut(H)|} \sum_{\mathbf{s} \in \mathcal{A}_\varepsilon(H, G_n)} M(\mathbf{s}, H, G_n) J_{\bar{\mathbf{s}}}, \quad (2.10)$$

where $\{J_S : S \subseteq V(G_n) \text{ and } |S| = |V(H)|\}$ is a collection of i.i.d. $\text{Bin}(1, \frac{1}{c_n^{|V(H)|-1}})$ random variables.

Lemma 2.2. For every integer $r \geq 1$,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} |\mathbb{E}T_\varepsilon^+(H, G_n)^r - \mathbb{E}J_\varepsilon^+(H, G_n)^r| = 0.$$

Proof. We begin with the following definition:

Definition 2.3. Let $\mathcal{S}_{\varepsilon,r,b}$ be the collection of all order r -tuples $\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_r)$, where $\mathbf{s}_j = (s_{j1}, s_{j2}, \dots, s_{j|V(H)|})$, for $j \in [r]$, such that

- $\mathbf{s}_j \in \mathcal{A}_\varepsilon(H, G_n)$, for all $j \in [r]$,
- $M(\mathbf{s}_j, H, G_n) = 1$, for all $j \in [r]$.
- There are exactly b distinct $|V(H)|$ -element sets in the collection $\{\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2, \dots, \bar{\mathbf{s}}_r\}$.

Finally, for a graph F , define

$$\mathcal{S}_{\varepsilon,r,b}(F) = \{\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_r) \in \mathcal{S}_{\varepsilon,r,b} : \mathcal{P}(\mathbf{S}) \text{ is isomorphic to } F\}, \quad (2.11)$$

where $\mathcal{P}(\mathbf{S}) = (V(\mathcal{P}(\mathbf{S})), E(\mathcal{P}(\mathbf{S})))$, such that

$$V(\mathcal{P}(\mathbf{S})) = \bigcup_{j=1}^r \bar{\mathbf{s}}_j \quad \text{and} \quad E(\mathcal{P}(\mathbf{S})) = \bigcup_{j=1}^r \{(s_{ja}, s_{jb}) : (a, b) \in E(H)\}. \quad (2.12)$$

For $N \geq 1$, denote by \mathcal{G}_N the set of all labelled graphs on at most N vertices. Moreover, let $\nu(F)$ denote the number of connected components of a graph F . Then by the multinomial expansion,

$$\begin{aligned} |\mathbb{E}T_\varepsilon^+(H, G_n)^r - \mathbb{E}J_\varepsilon^+(H, G_n)^r| &\leq \frac{1}{|Aut(H)|^r} \sum_{b=1}^r \sum_{\mathbf{S} \in \mathcal{S}_{\varepsilon,r,b}} \left| \mathbb{E} \prod_{t=1}^r \mathbf{1}\{X_{=\mathbf{s}_t}\} - \mathbb{E} \prod_{t=1}^r J_{\mathbf{s}_t} \right| \\ &= \frac{1}{|Aut(H)|^r} \sum_{b=1}^r \sum_{\mathbf{S} \in \mathcal{S}_{\varepsilon,r,b}} \left| \frac{1}{c_n^{|V(\mathcal{P}(\mathbf{S})| - \nu(\mathcal{P}(\mathbf{S}))|)} - \frac{1}{c_n^{b|V(H)| - b}} \right| \\ &\lesssim_{H,r} \sum_{b=1}^r \sum_{F \in \mathcal{G}_r|V(H)|} \left| \frac{1}{c_n^{|V(F)| - \nu(F)} - \frac{1}{c_n^{b|V(H)| - b}} \right| |\mathcal{S}_{\varepsilon,r,b}(F)|, \quad (2.13) \end{aligned}$$

Note that if the graph F is connected and $\mathcal{S}_{\varepsilon,r,b}(F)$ is non-empty, $|V(F)| - 1 \leq b|V(H)| - b$, and therefore, in general $|V(F)| - \nu(F) \leq b|V(H)| - b$. Moreover, if $|V(F)| - \nu(F) = b|V(H)| - b$, the corresponding term in the sum in (2.13) is zero. This implies,

$$\begin{aligned} |\mathbb{E}T_\varepsilon^+(H, G_n)^r - \mathbb{E}J_\varepsilon^+(H, G_n)^r| \\ \lesssim_{H,r} \sum_{b=1}^r \sum_{F \in \mathcal{G}_r|V(H)|} \frac{|\mathcal{S}_{\varepsilon,r,b}(F)|}{c_n^{|V(F)| - \nu(F)}} \mathbf{1}\{|V(F)| - \nu(F) < b|V(H)| - b\} \quad (2.14) \end{aligned}$$

To begin with assume that F is connected and $|V(F)| - \nu(F) < b|V(H)| - b$. Then by Lemma 2.3, $|\mathcal{S}_{\varepsilon,r,b}(F)| \lesssim_{H,r} \varepsilon c_n^{|V(F)| - 1}$. Next, if F is disconnected with connected components $F_1, F_2, \dots, F_{\nu(F)}$ such that $|V(F)| - \nu(F) < b|V(H)| - b$, then there exist $r_1, r_2, \dots, r_{\nu(F)}$ and $b_1, b_2, \dots, b_{\nu(F)}$, with $\sum_{j=1}^{\nu(F)} r_j = r$ and $\sum_{j=1}^{\nu(F)} b_j = b$, such that $|V(F_i)| - \nu(F_i) \leq b_i|V(H)| - b_i$, for each $i \in [\nu(F)]$, with strict inequality for some $i \in [\nu(F)]$. More precisely, for the i -th connected component, r_i is the number of tuples $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_{r_i}$ forming F_i , and b_i is the number of distinct $|V(H)|$ -element sets in the collection $\{\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2, \dots, \bar{\mathbf{s}}_{r_i}\}$. Then using Lemma 2.3 below on each connected component gives $|\mathcal{S}_{\varepsilon,r,b}(F)| \lesssim_{H,r} \varepsilon c_n^{|V(F)| - \nu(F)}$. Therefore, every term in the sum in the RHS of (2.14) goes to zero as $n \rightarrow \infty$ followed $\varepsilon \rightarrow 0$. This completes the proof of the lemma, because the outside sum is finite (depending only on H and r). \square

Lemma 2.3. *If F is connected and $\mathcal{S}_{\varepsilon,r,b}(F)$ is non-empty, then $|\mathcal{S}_{\varepsilon,r,b}(F)| \lesssim_{H,r} \varepsilon c_n^{|V(F)| - 1}$. Moreover, if $|V(F)| < b|V(H)| - b + 1$, then $|\mathcal{S}_{\varepsilon,r,b}(F)| \lesssim_{H,r} \varepsilon c_n^{|V(F)| - 1}$.*

Proof. To begin with assume that $|V(F)| < b|V(H)| - b + 1$. Then without loss of generality, consider $\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_r) \in \mathcal{S}_{\varepsilon, r, b}(F)$ in the order given by Lemma A.1. For $0 \leq j \leq |V(H)|$, define

$$\beta_j = \left| \left\{ t \in [2, r] : \left| \bar{\mathbf{s}}_t \cap \left(\bigcup_{a=1}^{t-1} \bar{\mathbf{s}}_a \right) \right| = |V(H)| - j \text{ and } \bar{\mathbf{s}}_t \notin \{\bar{\mathbf{s}}_1, \dots, \bar{\mathbf{s}}_{t-1}\} \right\} \right|. \quad (2.15)$$

The connectedness of F and Lemma A.1 implies that $\beta_{|V(H)|} = 0$ and $\beta_1 + \dots + \beta_{|V(H)|-2} \geq 1$. Note that

$$|V(F)| = |V(H)| + \sum_{j=1}^{|V(H)|-1} j\beta_j \quad \text{and} \quad b = 1 + \sum_{j=0}^{|V(H)|-1} \beta_j.$$

Now, define

$$\mathcal{B} = \left\{ \boldsymbol{\beta} = (\beta_j)_{0 \leq j \leq |V(H)|-1} \in \{0, 1, \dots, r-1\}^{|V(H)|} : |V(H)| + \sum_{j=1}^{|V(H)|-1} j\beta_j = |V(F)|, \sum_{j=1}^{|V(H)|-2} \beta_j \geq 1 \right\}.$$

Hence, for every $\varepsilon \in (0, 1)$, using the fact that $\mathbf{s}_j \in \mathcal{A}_\varepsilon(H, G_n)$, for all $j \in [r]$, gives

$$\begin{aligned} |\mathcal{S}_{\varepsilon, r, b}(F)| &\lesssim_{H, r} \sum_{\boldsymbol{\beta} \in \mathcal{B}} N(H, G_n)^{1+\beta_{|V(H)|-1}} \prod_{j=1}^{|V(H)|-2} (\varepsilon c_n^j)^{\beta_j} \\ &= \sum_{\boldsymbol{\beta} \in \mathcal{B}} N(H, G_n)^{1+\beta_{|V(H)|-1}} \varepsilon^{\sum_{j=1}^{|V(H)|-2} \beta_j} c_n^{\sum_{j=1}^{|V(H)|-2} j\beta_j} \\ &\lesssim_{H, r} \varepsilon \sum_{\boldsymbol{\beta} \in \mathcal{B}} c_n^{(1+\beta_{|V(H)|-1})(|V(H)|-1)} c_n^{\sum_{j=1}^{|V(H)|-2} j\beta_j} \\ &\quad \text{(using } \sum_{j=1}^{|V(H)|-2} \beta_j \geq 1 \text{ and } N(H, G_n) = O(c_n^{|V(H)|-1})\text{)} \\ &= \varepsilon c_n^{|V(F)|-1} |\mathcal{B}| \lesssim_{H, r} \varepsilon c_n^{|V(F)|-1}, \end{aligned} \quad (2.16)$$

where the last step uses the crude estimate $|\mathcal{B}| \leq r^{|V(H)|}$. See Example 4 for an illustration of the argument in the above display in a special case.

Finally, suppose that $|V(F)| \leq b|V(H)| - b + 1$. Since F is connected and $\mathcal{S}_{\varepsilon, r, b}(F)$ is non-empty, there exists $\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_r) \in \mathcal{S}_{\varepsilon, r, b}(F)$ such that $\beta_{|V(H)|} = 0$, where $(\beta_0, \dots, \beta_{|V(H)|})$ is defined as in (2.15). Define

$$\mathcal{B}' = \left\{ \boldsymbol{\beta} = (\beta_j)_{0 \leq j \leq |V(H)|-1} \in \{0, 1, \dots, r-1\}^{|V(H)|} : |V(H)| + \sum_{j=1}^{|V(H)|-1} j\beta_j = |V(F)| \right\}.$$

For a $\boldsymbol{\beta} \in \mathcal{B}'$, $\sum_{j=1}^{|V(H)|-2} \beta_j$ can be zero, but using $\varepsilon = 1$ in (2.16) (with \mathcal{B} replaced by \mathcal{B}')

$$|\mathcal{S}_{\varepsilon, r, b}(F)| \lesssim_{H, r} \sum_{\boldsymbol{\beta} \in \mathcal{B}'} N(H, G_n)^{1+\beta_{|V(H)|-1}} c_n^{\sum_{j=1}^{|V(H)|-2} j\beta_j} \lesssim_{H, r} c_n^{|V(F)|-1},$$

completing the proof of the lemma. \square

Example 4. (2-star continued) Suppose $H = K_{1,2}$, and $F = \mathcal{P}(\mathbf{S})$ is connected, where $\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_r)$. If at the j -th step a single new vertex is added, then the number of ways to choose such a triple \mathbf{s}_j from $\mathcal{A}_\varepsilon(K_{1,2}, G_n)$ is at most $O(\varepsilon c_n)$ (recall Example 2). On the other hand, if

two new vertices are added, the number of possible triples is trivially bounded by $O(N(K_{1,2}, G_n))$. This implies the bound in (2.16) because, the number of times 1 or 2 vertices are added in the sequence \mathbf{S} is β_1 and β_2 , respectively (note that 3 vertices are always added at the first step, which contributes the extra factor of $O(N(K_{1,2}, G_n))$).

2.1.3. *Completing the Proof of Theorem 2.1.* Lemma 2.2 shows the moments of $T_\varepsilon^+(H, G_n)$ and $J_\varepsilon^+(H, G_n)$ are asymptotically close. Now, we derive the limiting distribution of $J_\varepsilon^+(H, G_n)$.

Lemma 2.4. *Let $J_\varepsilon^+(H, G_n)$ be as defined in (2.10). Then for every $\varepsilon > 0$, as $n \rightarrow \infty$,*

$$J_\varepsilon^+(H, G_n) \rightarrow \sum_{k=1}^{N(H, K_{|V(H)|})} k Z_k$$

in distribution and in moments, where $Z_k \sim \text{Pois}(\lambda_k)$ and the collection $\{Z_k : 1 \leq k \leq N(H, K_{|V(H)|})\}$ is independent.

Proof. For each $k \in [1, N(H, K_{|V(H)|})]$, define

$$\mathcal{D}_k(H, G_n) = \{S \subseteq V(G_n) : |S| = |V(H)| \text{ and } N(H, G_n[S]) = k\}, \quad (2.17)$$

where $G_n[S]$ is the subgraph of G_n induced on the set S .⁵

For every subset $S := \{s_1, \dots, s_{|V(H)|}\}$ of $V(G_n)$ of size $|V(H)|$, let $\sigma_0(S) = (s_{\sigma_0(1)}, s_{\sigma_0(2)}, \dots, s_{\sigma_0(|V(H)|)}) \in V(G_n)_{|V(H)|}$, be such that $s_{\sigma_0(1)} < s_{\sigma_0(2)} < \dots < s_{\sigma_0(|V(H)|)}$. Now, define

$$\mathcal{B}_\varepsilon(H, G_n) = \left\{ S \subseteq V(G_n) : |S| = |V(H)| \text{ and } \sigma_0(S) \in \mathcal{A}_\varepsilon(H, G_n) \right\}.$$

Then recalling the definition of $J_\varepsilon^+(H, G_n)$ from (2.10), we have

$$\begin{aligned} J_\varepsilon^+(H, G_n) &= \frac{1}{|\text{Aut}(H)|} \sum_{\mathbf{s} \in \mathcal{A}_\varepsilon(H, G_n)} M(\mathbf{s}, H, G_n) J_{\bar{\mathbf{s}}} \\ &= \frac{1}{|\text{Aut}(H)|} \sum_{k=1}^{N(H, K_{|V(H)|})} \sum_{\substack{\mathbf{s} \in \mathcal{A}_\varepsilon(H, G_n) \\ \bar{\mathbf{s}} \in \mathcal{D}_k(H, G_n)}} M(\mathbf{s}, H, G_n) J_{\bar{\mathbf{s}}} \\ &= \frac{1}{|\text{Aut}(H)|} \sum_{k=1}^{N(H, K_{|V(H)|})} \sum_{S \in \mathcal{D}_k(H, G_n) \cap \mathcal{B}_\varepsilon(H, G_n)} \sum_{\substack{\mathbf{s} \in V(G_n)_{|V(H)|} \\ \bar{\mathbf{s}} = S}} M(\mathbf{s}, H, G_n) J_{\bar{\mathbf{s}}} \\ &= \sum_{k=1}^{N(H, K_{|V(H)|})} \sum_{S \in \mathcal{D}_k(H, G_n) \cap \mathcal{B}_\varepsilon(H, G_n)} N(H, G_n[S]) J_S \\ &= \sum_{k=1}^{N(H, K_{|V(H)|})} k \sum_{S \in \mathcal{D}_k(H, G_n) \cap \mathcal{B}_\varepsilon(H, G_n)} J_S. \end{aligned}$$

⁵For example, $H = K_{1,2}$, then $\mathcal{D}_1(K_{1,2}, G_n)$ is the collection of all induces 2-stars in G_n , $\mathcal{D}_2(K_{1,2}, G_n)$ is empty, and $\mathcal{D}_3(K_{1,2}, G_n)$ is the number of induced triangles in G_n .

Now, note that, by definition, the collection $\{\sum_{S \in \mathcal{D}_k(H, G_n) \cap \mathcal{B}_\varepsilon(H, G_n)} J_S : 1 \leq k \leq N(H, K_{|V(H)|})\}$ is independent, and for every fixed $k \in [1, N(H, K_{|V(H)|})]$,

$$J_{n,\varepsilon}(k) := \sum_{S \in \mathcal{D}_k(H, G_n) \cap \mathcal{B}_\varepsilon(H, G_n)} J_S \sim \text{Bin} \left(\left| \mathcal{D}_k(H, G_n) \cap \mathcal{B}_\varepsilon(H, G_n) \right|, \frac{1}{c_n^{|V(H)|-1}} \right).$$

Therefore, to prove the lemma it suffices to show that $\mathbb{E}J_{n,\varepsilon}(k) \rightarrow \lambda_k$, for every $k \in [1, N(H, K_{|V(H)|})]$. To this end, note that

$$\begin{aligned} & \left| \mathcal{D}_k(H, G_n) \right| - \left| \mathcal{D}_k(H, G_n) \cap \mathcal{B}_\varepsilon(H, G_n) \right| \\ &= \left| \{S \in \mathcal{D}_k(H, G_n) : \sigma_0(S) \notin \mathcal{A}_\varepsilon(H, G_n)\} \right| \\ &\leq \frac{1}{|\text{Aut}(H)|} \left| \{\mathbf{s} \in V(G_n)_{|V(H)|} : M(\mathbf{s}, H, G_n) = 1 \text{ and } \mathbf{s} \notin \mathcal{A}_\varepsilon(H, G_n)\} \right| \\ &= \frac{1}{|\text{Aut}(H)|} \sum_{\mathbf{s} \in V(G_n)_{|V(H)|}} M(\mathbf{s}, H, G_n) \mathbf{1}\{\mathbf{s} \notin \mathcal{A}_\varepsilon(H, G_n)\} \\ &= c_n^{|V(H)|-1} \mathbb{E}T_\varepsilon^-(H, G_n) \\ &= o(c_n^{|V(H)|-1}) \end{aligned}$$

by Lemma 2.1. Thus, $\mathbb{E}J_{n,\varepsilon}(k) = \frac{1}{c_n^{|V(H)|-1}} |\mathcal{D}_k(H, G_n)| + o(1)$. The lemma now follows from assumption 2.2 of Theorem 2.1, and the observation that $|\mathcal{D}_k(H, G_n)| = \sum_{F \in \mathcal{C}_{H,k}} N_{\text{ind}}(F, G_n)$. \square

Example 5. (2-star continued) If $H = K_{1,2}$, then every set $S \in \mathcal{B}_\varepsilon(K_{1,2}, G_n)$ for which the induced graph $G_n[S]$ is a triangle, contributes to $J_\varepsilon^+(K_{1,2}, G_n)$ the same Bernoulli variable three times, since $N(K_{1,2}, K_3) = 3$. On the other hand, if the induced graph $G_n[S]$ is a 2-star, then S contributes a single Bernoulli variable to $J_\varepsilon^+(K_{1,2}, G_n)$. By the joint independence of the collection J_S over all three-element subsets S of $V(G_n)$, it follows that $J_\varepsilon^+(H, G_n) = J'_{n,\varepsilon} + 3J''_{n,\varepsilon}$, where $J'_{n,\varepsilon}$ and $J''_{n,\varepsilon}$ are independent Binomial random variables. The calculation in the above lemma implies that $\mathbb{E}J'_{n,\varepsilon} = \frac{1}{c_n^2} |\mathcal{D}_1(K_{1,2}, G_n)| + o(1) = \frac{1}{c_n^2} N_{\text{ind}}(K_{1,2}, G_n) + o(1) = \lambda_1 + o(1)$, and, similarly, $\mathbb{E}J''_{n,\varepsilon} = \frac{1}{c_n^2} |\mathcal{D}_3(K_{1,2}, G_n)| + o(1) = \frac{1}{c_n^2} N_{\text{ind}}(K_3, G_n) + o(1) = \lambda_3 + o(1)$ (by assumption (2.2)).

To complete the proof of Theorem 2.1, let Z be the random variable on the RHS of (2.3). The above lemma, combined with Lemma 2.2, implies that, for all $r \geq 1$,

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} |\mathbb{E}(T_\varepsilon^+(H, G_n))^r - \mathbb{E}Z^r| = 0. \quad (2.18)$$

Furthermore, the random variable Z has a finite moment generating function, which implies, by Lemma B.1, that $T_\varepsilon^+(H, G_n) \xrightarrow{D} Z$, as $n \rightarrow \infty$ followed by $\varepsilon \rightarrow 0$. Hence, $T(H, G_n) \xrightarrow{D} Z$ (by Lemma 2.1), as $n \rightarrow \infty$, completing the proof. \square

Recently, Bhattacharya and Mukherjee [6] characterized the limiting distribution of $T(H, G_n)$, when G_n is a sequence of dense graphs converging to a graphon W . In the following remark, we discuss how Theorem 2.1 can be used to re-derive [6, Theorem 1.1], which obtains the limiting distribution $T(H, G_n)$ for a converging sequence of dense graphs, in the Poisson regime.

Remark 2.3. (Dense Graphs) Recall that a *graphon* $W : [0, 1]^2 \rightarrow [0, 1]$ is a measurable function satisfying $W(x, y) = W(y, x)$, for all x, y . A finite simple graph $G = (V(G), E(G))$ can also be represented as a graphon in a natural way: Define $f^G(x, y) = \mathbf{1}\{(\lceil |V(G)|x \rceil, \lceil |V(G)|y \rceil) \in$

$E(G)$, that is, partition $[0, 1]^2$ into $|V(G)|^2$ squares of side length $1/|V(G)|$, and let $f^G(x, y) = 1$ in the (i, j) -th square if $(i, j) \in E(G)$, and 0 otherwise. For a simple graph F with $V(F) = \{1, 2, \dots, |V(F)|\}$, define

$$t(F, W) = \int_{[0,1]^{|V(F)|}} \prod_{(i,j) \in E(F)} W(x_i, x_j) dx_1 dx_2 \cdots dx_{|V(F)|}$$

(continuous analogue of the homomorphism density). The basic definition of graph-limit theory is the following: A sequence of graphs $\{G_n\}_{n \geq 1}$ is said to *converge to* W if for every finite simple graph F , $\lim_{n \rightarrow \infty} t(F, G_n) = t(F, W)$ (refer to Lovász [20] for more on graph limit theory).

In [6, Theorem 1.1] the authors showed that $T(H, G_n)$ converges to a linear combination of independent Poisson random variables, whenever $\mathbb{E}(T(H, G_n)) = O(1)$, and G_n converges to a graphon W such that $t(H, W) > 0$. This result can be derived as a consequence of Theorem 2.1 as follows: If G_n is a sequence of dense graphs, as above, colored with c_n colors such that $\mathbb{E}(T(H, G_n)) \rightarrow \lambda$, then

$$c_n = \Theta(|V(G_n)|^{\frac{|V(H)|}{|V(H)|-1}}),$$

since $N(H, G_n) = \Theta(|V(G_n)|^{|V(H)|})$,⁶ by assumption $t(H, W) > 0$. Therefore, for $t \in [2, |V(H)|-1]$, and $F \in \mathcal{J}_t(H)$,

$$\begin{aligned} N(F, G_n) &= O(|V(G_n)|^{|V(F)|}) = O(|V(G_n)|^{2|V(H)|-t}) = O\left(c_n^{\frac{2|V(H)|-t}{2}}\right) \\ &= o(c_n^{2|V(H)|-t-1}), \end{aligned}$$

which establishes the second assumption of Theorem 2.1. Finally, since the convergence of G_n to a graphon W implies the convergence of the proportion of induced subgraphs in G_n , the limits in (2.2) exist, and, hence, [6, Theorem 1.1] follows:

$$T(H, G_n) \xrightarrow{D} \sum_{F \supseteq H: |V(F)|=|V(H)|} N(H, F) X_F,$$

where $X_F \sim \text{Pois}(\lambda_F)$ (where $\lambda_F := \lim_{n \rightarrow \infty} \frac{1}{c_n^{|V(F)|-1}} N_{\text{ind}}(F, G_n)$ exists because of the convergence of G_n) and the collection $\{X_F : F \supseteq H \text{ and } |V(F)| = |V(H)|\}$ is independent. As usual, we consider only non-isomorphic (unlabelled) super-graphs F of H , whenever we write $F \supseteq H$.

2.2. Proof of Theorem 1.1. Note that, for $s_1, s_2 \in V(G_n)_{|V(H)|}$ such that $\bar{s}_1 \cap \bar{s}_2 \neq \phi$,

$$\text{Cov}(\mathbf{1}\{X_{=s_1}\}, \mathbf{1}\{X_{=s_2}\}) = \frac{1}{c_n^{2|V(H)|-|\bar{s}_1 \cap \bar{s}_2|-1}} - \frac{1}{c_n^{2|V(H)|-2}}.$$

The covariance is 0 if $\bar{s}_1 \cap \bar{s}_2$ is empty or singleton. Therefore,

$$\text{Var } T(H, G_n) = R_{1,n} + R_{2,n} \tag{2.19}$$

where

$$R_{1,n} = \frac{1}{c_n^{|V(H)|-1}} \left(1 - \frac{1}{c_n^{|V(H)|-1}} \right) N(H, G_n) \rightarrow \lambda, \tag{2.20}$$

⁶For two non-negative sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, $a_n = \Theta(b_n)$ means that there exist positive constants C_1, C_2 , such that $C_1 b_n \leq a_n \leq C_2 b_n$, for all n large enough.

since $\mathbb{E}T(H, G_n) = \frac{1}{c_n^{|V(H)|-1}} N(H, G_n) \rightarrow \lambda$, and the covariance terms

$$R_{2,n} = \sum_{t=2}^{|V(H)|} \frac{1}{c_n^{2|V(H)|-t-1}} \left(1 - \frac{1}{c_n^{t-1}}\right) |\mathcal{K}(t, H, G_n)|, \quad (2.21)$$

where $\mathcal{K}(t, H, G_n)$ is the set of all ordered pairs (H_1, H_2) such that $H_1 \neq H_2$ are subgraphs of G_n isomorphic to H , sharing exactly t vertices in common. Now, the assumption $\text{Var } T(H, G_n) \rightarrow \lambda$ and (2.20) implies that $R_{2,n} \rightarrow 0$. Therefore,

$$|\mathcal{K}(t, H, G_n)| = o(c_n^{2|V(H)|-t-1}). \quad (2.22)$$

for every $t \in [2, |V(H)|]$. Further, for every $t \in [2, |V(H)| - 1]$,

$$\sum_{F \in \mathcal{J}_t(H)} N(F, G_n) \lesssim_H |\mathcal{K}(t, H, G_n)|. \quad (2.23)$$

Combining (2.22) and (2.23) imply that $N(F, G_n) = o(c_n^{2|V(H)|-t-1})$ for all $t \in [2, |V(H)| - 1]$ and all $F \in \mathcal{J}_t(H)$.

Next consider $t = |V(H)|$ in (2.22) and note that

$$|\mathcal{K}(|V(H)|, H, G_n)| = \sum_{k=2}^{N(H, K_{|V(H)|})} k(k-1) |\mathcal{D}_k(H, G_n)|, \quad (2.24)$$

where $\mathcal{D}_k(H, G_n)$ is as defined in (2.17). This follows by first choosing the common vertex set from exactly one of the collections $\mathcal{D}_k(H, G_n)$ for $k \in [2, N(H, K_{|V(H)|})]$, and then choosing the pair (H_1, H_2) in $k(k-1)$ ways.⁷

Combining (2.22) and (2.24) gives $|\mathcal{D}_k(H, G_n)| = o(c_n^{|V(H)|-1})$ for all $k \in [2, N(H, K_{|V(H)|})]$. Now, using $|\mathcal{D}_k(H, G_n)| = \sum_{F \in \mathcal{C}_{H,k}} N_{\text{ind}}(F, G_n)$ gives

$$\sum_{F \in \mathcal{C}_{H,k}} N_{\text{ind}}(F, G_n) = o(c_n^{|V(H)|-1}), \quad (2.25)$$

that is, $\lambda_k = 0$ for all $k \in [2, N(H, K_{|V(H)|})]$. Lastly, by a counting argument similar to the one used above,

$$N(H, G_n) = \sum_{k=1}^{N(H, K_{|V(H)|})} k |\mathcal{D}_k(H, G_n)|. \quad (2.26)$$

Since $\frac{1}{c_n^{|V(H)|-1}} N(H, G_n) \rightarrow \lambda$, (2.26) now implies that $\frac{1}{c_n^{|V(H)|-1}} |\mathcal{D}_1(H, G_n)| \rightarrow \lambda$, and hence,

$$\frac{\sum_{F \in \mathcal{C}_{H,1}} N_{\text{ind}}(F, G_n)}{c_n^{|V(H)|-1}} \rightarrow \lambda.$$

Condition (2.2) of Theorem 2.1 is thus satisfied with $\lambda_1 = \lambda$ and $\lambda_k = 0$ for all $k \in [2, N(H, K_{|V(H)|})]$. Theorem 2.1 now implies that $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$, completing the proof of the second-moment phenomenon for monochromatic subgraphs.

⁷For example, if $H = C_4$ is the 4-cycle, and $G_n = K_n$ is the complete graph, the LHS in (2.22) is $6 \binom{n}{4}$ (choose H_1 from G_n in $N(C_4, G_n) = 3 \binom{n}{4}$ ways, which leaves 2 choices for H_2) which matches with the RHS, since $|\mathcal{D}_2(C_4, G_n)| = 0$, and $|\mathcal{D}_3(C_4, G_n)| = \binom{n}{4}$ (every 4-tuple in G_n has an induced K_4 and $N(C_4, K_4) = 3$).

3. PROOF OF THEOREM 1.2

The if part follows directly from Theorem 1.1. The proof of the only-if part is given in Section 3.1. The counter-example when H is not a star-graph is explained in Section 3.2.

3.1. $T(K_{1,r}, G_n) \xrightarrow{D} \text{Pois}(\lambda)$ **implies Convergence of Moments.** We begin by showing that $T(K_{1,r}, G_n) \xrightarrow{D} \text{Pois}(\lambda)$ implies $\mathbb{E}T(K_{1,r}, G_n)$ is bounded.

Lemma 3.1. *Let $\{G_n\}_{n \geq 1}$ be a sequence of deterministic graphs colored uniformly with c_n colors. Then*

$$T(K_{1,r}, G_n) \xrightarrow{P} \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \mathbb{E}T(K_{1,r}, G_n) = 0, \\ \infty & \text{if } \lim_{n \rightarrow \infty} \mathbb{E}T(K_{1,r}, G_n) = \infty. \end{cases}$$

Proof. If $\mathbb{E}T(K_{1,r}, G_n) \rightarrow 0$, then $\mathbb{P}(T(K_{1,r}, G_n) > 0) \leq \mathbb{E}T(K_{1,r}, G_n) \rightarrow 0$.

To show that $T(K_{1,r}, G_n)$ diverges, if $\mathbb{E}T(K_{1,r}, G_n) \rightarrow \infty$, it suffices to show that $\text{Var } T(K_{1,r}, G_n) = o((\mathbb{E}T(K_{1,r}, G_n))^2)$, because this implies that $T(K_{1,r}, G_n)/\mathbb{E}(T(K_{1,r}, G_n)) \xrightarrow{P} 1$, which is possible only if $T(K_{1,r}, G_n) \xrightarrow{P} \infty$.

Write $\text{Var } T(K_{1,r}, G_n) = R_{1,n} + R_{2,n}$, as in (2.19) (with $H = K_{1,r}$). Clearly,

$$R_{1,n} \leq \mathbb{E}T(K_{1,r}(G_n)) = o((\mathbb{E}T(K_{1,r}, G_n))^2). \quad (3.1)$$

Next, observe that for each $t \in [2, r+1]$,

$$\sum_{\substack{\mathbf{u} \neq \mathbf{v} \in V(G_n) \\ |\bar{\mathbf{u}} \cap \bar{\mathbf{v}}| = t}} \frac{M(\mathbf{u}, K_{1,r}, G_n)M(\mathbf{v}, K_{1,r}, G_n)}{|Aut(K_{1,r})|^2} \lesssim_r \sum_{F \in \mathcal{J}_t(K_{1,r})} N(F, G_n). \quad (3.2)$$

For each $F \in \mathcal{J}_t(K_{1,r})$, by Lemma 3.2, $N(F, G_n) \lesssim_r N(K_{1,r}, G_n)^{\frac{2r-t+1}{r}}$. Therefore, by (2.21) and (3.2),

$$R_{2,n} \lesssim_r \sum_{t=2}^{|V(H)|} \frac{N(K_{1,r}, G_n)^{\frac{2r-t+1}{r}}}{c_n^{2r-t+1}} = \sum_{t=2}^{|V(H)|} (\mathbb{E}T(K_{1,r}, G_n))^{2-\frac{t-1}{r}} = o((\mathbb{E}T(K_{1,r}, G_n))^2). \quad (3.3)$$

Now, (3.1) and (3.3) imply that $\text{Var } T(K_{1,r}, G_n) = o((\mathbb{E}T(K_{1,r}, G_n))^2)$, completing the proof of the lemma. \square

By the above proposition, $T(K_{1,r}, G_n) \xrightarrow{D} \text{Pois}(\lambda)$, implies that $\mathbb{E}T(K_{1,r}, G_n) = \frac{N(K_{1,r}, G_n)}{c_n^r} = \Theta(1)$. Therefore, by Lemma 3.2,

$$N(F, G_n) = O(c_n^{|V(F)| - \nu(F)}), \quad (3.4)$$

for any graph F which is the union of r -stars with $\nu(F)$ connected components. Using this we can show that the moments of $T(K_{1,r}, G_n)$ are bounded. To this end, set $r' = r+1$ and fix an integer $m \geq 1$. Let \mathcal{S} be the collection of all ordered m -tuples $(\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_m)$, where $\mathbf{s}_j := (s_{j1}, \dots, s_{jr'}) \in V(G_n)_{r'}$, for $j \in [m]$, and $M(\mathbf{s}_j, K_{1,r}, G_n) = 1$, for every $j \in [m]$. Then by the multinomial expansion,

$$\mathbb{E}T(K_{1,r}, G_n)^m = \frac{1}{|Aut(K_{1,r})|^m} \sum_{\mathcal{S}} \mathbb{E} \prod_{j=1}^m \mathbf{1}\{X = \mathbf{s}_j\} = \frac{1}{|Aut(K_{1,r})|^m} \sum_{\mathcal{S}} \frac{1}{c_n^{|V(F)| - \nu(F)}}, \quad (3.5)$$

where $F = F(\mathbf{s}_1, \dots, \mathbf{s}_m)$ is the graph on vertex set $V(F) = \bigcup_{j=1}^m \bar{\mathbf{s}}_j$ and edge set $\bigcup_{j=1}^m \{(s_{ja}, s_{jb}) : (a, b) \in E(K_{1,r})\}$, and $\nu(F)$ is the number of connected components of F . Denote by $\mathcal{G}_m(K_{1,r})$ the collection of all unlabelled graphs formed by the join of m isomorphic copies of $K_{1,r}$.⁸

Then (3.5) implies

$$\mathbb{E}T(K_{1,r}, G_n)^m \lesssim_{r,m} \sum_{F \in \mathcal{H}_{r,m}} \frac{N(F, G_n)}{c_n^{|V(F)| - \nu(F)}} = O(1),$$

using (3.4), since $\mathcal{H}_{r,m}$ is a finite set (depending only on r and m). This implies, by uniform integrability, $\mathbb{E}T(K_{1,r}, G_n)^m \rightarrow \mathbb{E}(\text{Pois}(\lambda))^m$, for every $m \geq 1$. In particular, $\mathbb{E}T(K_{1,r}, G_n) \rightarrow \lambda$ and $\text{Var} T(K_{1,r}, G_n) \rightarrow \lambda$, as required in (1.4). Therefore, to complete the proof of the only if part it remains to prove the following lemma:

Lemma 3.2. *Let F be a graph formed by the union of r -stars with $\nu(F)$ connected components. Then for any graph G_n*

$$N(F, G_n) \lesssim_{F,r} N(K_{1,r}, G_n)^{\frac{|V(F)| - \nu(F)}{r}}.$$

Proof. Let $F_1, F_2, \dots, F_{\nu(F)}$ denote the connected components of F . Clearly, F_i contains an r -star for each $1 \leq a \leq \nu(F)$. Hence, for every $1 \leq a \leq \nu(F)$,

$$N(F_a, G_n) \lesssim_{F_a,r} N(K_{1,r}, G_n) (\Delta(G_n))^{|V(F_a)| - r - 1}, \quad (3.6)$$

where $\Delta(G_n)$ is the maximum degree in G_n . On the other hand,

$$N(K_{1,r}, G_n) = \sum_{v \in V(G_n)} \binom{d_v}{r} \geq \binom{\Delta(G_n)}{r} \gtrsim \Delta(G_n)^r.$$

This implies that $(\Delta(G_n))^{|V(F_a)| - r - 1} \lesssim_r N(K_{1,r}, G_n)^{\frac{|V(F_a)| - 1}{r} - 1}$, and from (3.6),

$$N(F_a, G_n) \lesssim_{F_a,r} N(K_{1,r}, G_n)^{\frac{|V(F_a)| - 1}{r}}. \quad (3.7)$$

Since (3.7) is true for every $1 \leq a \leq \nu(F)$,

$$N(F, G_n) \leq \prod_{a=1}^{\nu(F)} N(F_a, G_n) \lesssim_{F,r} N(K_{1,r}, G_n)^{\sum_{a=1}^{\nu(F)} \frac{|V(F_a)| - 1}{r}} = N(K_{1,r}, G_n)^{\frac{|V(F)| - \nu(F)}{r}},$$

completing the proof. \square

3.2. Counterexample when H is not a star-graph. In this section, we construct a graph sequence $G_n(H)$ such that $T(H, G_n(H)) \xrightarrow{D} \text{Pois}(\lambda)$, but (1.3) does not hold, whenever H is connected and is not a star-graph.

Definition 3.1. Fix an integer $n \geq 1$. Let H_1, H_2, \dots, H_n be isomorphic copies of H , with $V(H) = \{1, 2, \dots, |V(H)| - 1, |V(H)|\}$ and $H_a = (V(H_a), E(H_a))$, such that $V(H_a) = \{1, 2, \dots, |V(H)| - 1, z_a\}$, where $\phi(v) = v$, for $v \in [|V(H)| - 1]$, and $\phi(|V(H)|) = z_a$, is an isomorphism of H and H_a , for $a \in [n]$. Define the pyramid of H of height n as follows:

$$\mathcal{P}_n(H) = \left(\bigcup_{a=1}^n V(H_a), \bigcup_{a=1}^n E(H_a) \right).$$

⁸For any graph H , $\mathcal{G}_2(H)$ is the collection of all non-isomorphic graphs obtained the join of 2 copies of H , as in Definition 2.1. For $m \geq 3$, define $\mathcal{G}_m(H)$ inductively, as the collection of all non-isomorphic graphs F , that can be obtained by identifying t vertices of H , for some $t \in [1, |V(H)|]$, with t vertices of some graph $F_1 \in \mathcal{G}_{m-1}(H)$.

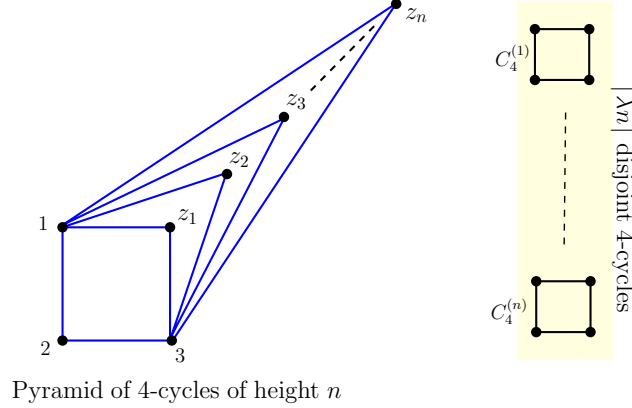


FIGURE 3. Illustration showing Poisson convergence does not imply convergence of moments, when $H = C_4$.

Let $G_n(H)$ be the disjoint union of $\mathcal{P}_n(H)$ and $\lceil \lambda n \rceil$ disjoint copies of H . (Figure 3 illustrates this construction when $H = C_4$ is the 4-cycle).

Lemma 3.3. *Suppose H is connected and is not a star-graph. Let $\mathcal{P}_n(H)$ be a pyramid of H of height n , as defined above. Then every copy of H in $\mathcal{P}_n(H)$ passes through at least two vertices in $\{1, 2, \dots, |V(H)| - 1\}$.*

Proof. Since $\{z_1, z_2, \dots, z_n\}$ is an independent set, by construction, and H is connected, every copy of H in $\mathcal{P}_n(H)$ must pass through at least 1 vertex in $\{1, 2, \dots, |V(H)| - 1\}$. Suppose there exists a copy of H in $\mathcal{P}_n(H)$ which passes through exactly 1 vertex (say k) in $\{1, 2, \dots, |V(H)| - 1\}$. Then every other vertex of H belongs to the set $\{z_1, z_2, \dots, z_n\}$. However, $\{z_1, z_2, \dots, z_n\}$ is an independent set and, therefore, any non-empty connected subgraph of $\mathcal{P}_n(H)$ with vertices in $\{z_1, z_2, \dots, z_n, k\}$ will be a star-graph, which contradicts the assumption of the lemma. \square

Now, choose $c_n = n^{\frac{1}{|V(H)|-1}}$. By the above lemma,

$$\begin{aligned} \mathbb{P}(T(H, \mathcal{P}_n(H)) > 0) &= \mathbb{P}(\text{at least two vertices in } \{1, 2, \dots, |V(H)| - 1\} \text{ have the same color}) \\ &\leq \frac{\binom{|V(H)|-1}{2}}{c_n} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Therefore, $T(H, \mathcal{P}_n(H)) \xrightarrow{P} 0$. However, the number of monochromatic H in $\lceil \lambda n \rceil$ disjoint copies of H follows $\text{Bin}(\lceil \lambda n \rceil, \frac{1}{c_n^{\frac{1}{|V(H)|-1}}}) = \text{Bin}(\lceil \lambda n \rceil, \frac{1}{n})$, which converges to $\text{Pois}(\lambda)$, as $n \rightarrow \infty$. Therefore,

$$T(H, G_n(H)) \xrightarrow{D} \text{Pois}(\lambda).$$

On the other hand, note that $N(H, G_n(H)) = N(H, \mathcal{P}_n(H)) + \lceil \lambda n \rceil$. Then using $N(H, \mathcal{P}_n(H)) \geq n$, gives $\mathbb{E}T(H, G_n(H)) = \frac{1}{c_n^{\frac{1}{|V(H)|-1}}} N(H, G_n) \geq \frac{\lceil \lambda n \rceil + n}{c_n^{\frac{1}{|V(H)|-1}}} \rightarrow \lambda + 1$, that is, (1.4) does not hold.

4. APPLICATIONS OF THEOREM 1.1

In this section we apply Theorem 1.1 in various examples: (1) monochromatic subgraphs in the Erdős-Rényi random graph (Section 4.1), (2) monochromatic cliques in general graphs (Section 4.2), and (3) connections to the birthday paradox (Section 4.3).

4.1. Monochromatic Subgraphs in Erdős-Rényi Random Graphs. Theorem 1.1 can be easily extended to random graphs, when the limits in (1.3) hold in probability, when the graph and its coloring are jointly independent. This is explained in the following lemma, using which we prove Theorem 1.3 and Theorem 1.4, in Section 4.1.1.

Lemma 4.1. *Let $\{G_n\}_{n \geq 1}$ be a sequence of random graphs independent of the coloring distribution $(X_1, \dots, X_{|V(G_n)|})$ such that*

$$\mathbb{E}(T(H, G_n)|G_n) \xrightarrow{P} \lambda, \quad \text{Var}(T(H, G_n)|G_n) \xrightarrow{P} \lambda.$$

Then $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$.

Proof. The given hypothesis implies the existence of positive reals $\varepsilon_n \rightarrow 0$, such that

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = 0, \quad A_n := \{G_n : \max\{|\mathbb{E}(T(H, G_n)|G_n) - \lambda|, |\text{Var}(T(H, G_n)|G_n) - \lambda|\} > \varepsilon_n\}.$$

Thus, given any function $h : \mathbb{Z}_+ \cup \{0\} \mapsto [0, 1]$

$$|\mathbb{E}h(T(H, G_n)) - \mathbb{E}h(\text{Pois}(\lambda))| \leq \mathbb{P}(A_n) + \sup_{G_n \in A_n^c} |\mathbb{E}(h(T(H, G_n))|G_n) - \mathbb{E}h(\text{Pois}(\lambda))|.$$

It thus suffices to prove that the second term in the RHS above converges to 0. If not, there exists a deterministic sequence of graphs $\{G'_n\}_{n \geq 1}$ such that $\mathbb{E}(T(H, G'_n))$ and $\text{Var}(T(H, G'_n))$ both converge to λ , but $T(H, G'_n)$ does not converge to $\text{Pois}(\lambda)$, a contradiction to Theorem 1.1. \square

4.1.1. Proofs of Theorem 1.3 and Theorem 1.4. We begin with some preliminary properties of the exponent $\gamma(H)$ (recall (1.8)).

Lemma 4.2. *Let H be a connected graph. Then the following hold:*

- (a) *If H is unbalanced, then $\gamma(H)$ is well-defined, and $0 < \gamma(H) < \frac{1}{m(H)}$, where $m(H)$ is defined in (1.6). Furthermore, every minimizer of (1.8) is an induced sub-graph H_1 of H .*
- (b) *If H is balanced, but not strictly balanced, then $\gamma(H) = \frac{1}{m(H)}$.*

Proof. Throughout, we assume H is connected. Then we have the following two cases:

- (a) Suppose H is unbalanced. Then there exists $H_1 \subset H$ non-empty such that

$$\frac{|E(H_1)|}{|V(H_1)|} > \frac{|E(H)|}{|V(H)|} \Leftrightarrow |E(H_1)||V(H)| - |E(H)||V(H_1)| > 0.$$

For this H_1 ,

$$\begin{aligned} & |E(H_1)|(|V(H)| - 1) - |E(H)|(|V(H_1)| - 1) \\ &= |E(H_1)||V(H)| - |E(H)||V(H_1)| + |E(H)| - |E(H_1)| > 0. \end{aligned} \quad (4.1)$$

Thus, the minimum in definition of $\gamma(H)$ (recall (1.8)) is not over an empty set, which means $\gamma(H)$ is well-defined. Moreover, as the minimum is taken over finitely many positive items, $\gamma(H) > 0$.

Next, suppose $H_1 \subset H$ such that $m(H) = \frac{|E(H_1)|}{|V(H_1)|}$. To show $\gamma(H) < \frac{1}{m(H)}$ it suffices to show that

$$\frac{|V(H)| - |V(H_1)|}{|E(H_1)||V(H)| - |E(H)||V(H_1)|} < \frac{|V(H_1)|}{|E(H_1)|},$$

which is equivalent to $|E(H_1)||V(H)|(|V(H_1)| - 1) - |V(H_1)||E(H)|(|V(H_1)| - 1) > 0$, that is, $|E(H_1)||V(H)| - |E(H)||V(H_1)| > 0$, which holds since H is unbalanced.

Finally, observe that, for fixed $|V(H_1)|$, the RHS in (1.8) is decreasing in $|E(H_1)|$, which implies that every minimizer of (1.8) is an induced subgraph H_1 of H .

- (b) Now, suppose H balanced, but not strictly balanced. Then there exists a proper subgraph H_1 of H such that $m(H) = \frac{|E(H_1)|}{|V(H_1)|} = \frac{|E(H)|}{|V(H)|}$. Then this H_1 satisfies (4.1), and, therefore $\gamma(H)$ is well defined, positive, and satisfies

$$\gamma(H) \leq \frac{|V(H)| - |V(H_1)|}{|E(H_1)|(|V(H)| - 1) - |E(H)|(|V(H_1)| - 1)} = \frac{|V(H_1)|}{|E(H_1)|} = \frac{1}{m(H)}. \quad (4.2)$$

Next, we are going to show that if H' is a subgraph of H , such that $|E(H')|(|V(H)| - 1) - |E(H)|(|V(H')| - 1) > 0$, then

$$\frac{|V(H)| - |V(H')|}{|E(H')|(|V(H)| - 1) - |E(H)|(|V(H')| - 1)} \geq \frac{|V(H)|}{|E(H)|} = \frac{1}{m(H)}. \quad (4.3)$$

This is equivalent to showing $|V(H')||E(H)|(|V(H)| - 1) - |V(H)||E(H')|(|V(H)| - 1) \geq 0$, which follows by noting that $|V(H')||E(H)| \geq |V(H)||E(H')|$, since H is balanced. Combining (4.2) and (4.3), it follows that $\gamma(H) = \frac{1}{m(H)}$, for H which is balanced, but not strictly balanced. \square

Proof of Theorem 1.4(a): Consider the subgraph H_1 of H such that the minimum in (1.8) is attained, that is,

$$\gamma(H) = \frac{|V(H)| - |V(H_1)|}{|E(H_1)|(|V(H)| - 1) - |E(H)|(|V(H_1)| - 1)}.$$

Note that $\mathbb{P}(T(H, G_n) > 0) \leq \mathbb{P}(T(H_1, G_n) > 0) \leq \mathbb{E}(T(H_1, G_n))$. Therefore,

$$\begin{aligned} \mathbb{P}(T(H, G_n) > 0) &\leq \mathbb{E}(T(H_1, G_n)) \\ &\lesssim_H \frac{n^{|V(H_1)|} p^{|E(H_1)|}}{c_n^{|V(H_1)|-1}} \\ &\lesssim_H \frac{n^{|V(H_1)|} p^{|E(H_1)|}}{n^{\frac{|V(H)|(|V(H_1)|-1)}{|V(H)|-1}} p^{\frac{|E(H)|(|V(H_1)|-1)}{|V(H)|-1}}} \quad (\text{using } c_n = \Theta(n^{\frac{|V(H)|}{|V(H)|-1}} p^{\frac{|E(H)|}{|V(H)|-1}})) \\ &= \left(\frac{n^{|V(H_1)|(|V(H)|-1)} p^{|E(H_1)|(|V(H)|-1)}}{n^{|V(H)|(|V(H_1)|-1)} p^{|E(H)|(|V(H_1)|-1)}} \right)^{\frac{1}{|V(H)|-1}} \\ &= \left(n^{|V(H)|-|V(H_1)|} p^{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)} \right)^{\frac{1}{|V(H)|-1}} \\ &= \left(n^{\gamma(H)} p \right)^{\frac{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)}{|V(H)|-1}}. \end{aligned} \quad (4.4)$$

Since the RHS above goes to 0 by assumption, the proof of Theorem 1.4(a) is complete.

Proof of Theorem 1.4(b): For any integer $r \geq 1$, a direct expansion gives

$$\mathbb{E}T(H, G_n)^r = \sum_{F \in \mathcal{G}_r(H)} c_0(F, H) \frac{\mathbb{E}N(F, G_n)}{c_n^{|V(F)|-\nu(F)}} = \sum_{F \in \mathcal{G}_r(H)} c_1(F, H) \frac{n^{|V(F)|} p^{|E(F)|}}{c_n^{|V(F)|-\nu(F)}},$$

where $c_0(F, H)$, $c_1(F, H)$ are constants free of n , and $\mathcal{G}_r(H)$ is the set of all unlabeled graphs formed by the join of r isomorphic copies of H . The convergence of the moments of $T(H, G_n)$ follows from the lemma below.

Lemma 4.3. *For $F \in \mathcal{G}_r(H)$, define $\eta_n(F) := \frac{1}{c_n^{|V(F)|-\nu(F)}} n^{|V(F)|} p^{|E(F)|}$. Then $\eta(F) := \lim_{n \rightarrow \infty} \eta_n(F)$ exists.*

Proof. First, note that it suffices to prove the lemma for connected F , since in the general case, if F has connected components F'_1, \dots, F'_ν , then $\eta_n(F) = \prod_{i=1}^\nu \eta_n(F'_i)$. If $F \in \mathcal{G}_r(H)$, then each $F'_i \in \mathcal{G}_r(H)$ too, so convergence of each term in the product will show convergence of $\eta_n(F)$. We proceed by induction on r . For $r = 1$, $F = H$, and $\eta_n(F) = \frac{1}{c_n^{|V(H)|-1}} n^{|V(H)|} p^{|E(H)|} \rightarrow \lambda_0 := \lambda |Aut(H)|$, by the assumption (1.5). Now, suppose the result holds for all connected $F \in \mathcal{G}_{r-1}(H)$, and let $F \in \mathcal{G}_r(H)$ be connected. Then, F is the join of F_1 and F_2 , for some connected $F_1 \in \mathcal{G}_{r-1}(H)$ and an isomorphic copy F_2 of H .

Let H_1 be a graph with vertex set $V(F_1) \cap V(F_2)$ and edge set $E(F_1) \cap E(F_2)$. We need to show the convergence of

$$\eta_n(F_r) = \frac{n^{|V(F_r)|} p^{|E(F_r)|}}{c_n^{|V(F_r)|-\nu(F_r)}} = \frac{n^{|V(F_1)|} p^{|E(F_1)|}}{c_n^{|V(F_1)|-1}} \times \frac{n^{|V(H)|-|V(H_1)|} p^{|E(H)|-|E(H_1)|}}{c_n^{|V(H)|-|V(H_1)|}}.$$

The first term in the RHS above converges by induction hypothesis. For the second term, using (1.5) gives

$$\begin{aligned} & \frac{n^{|V(H)|-|V(H_1)|} p^{|E(H)|-|E(H_1)|}}{c_n^{|V(H)|-|V(H_1)|}} \\ &= (1 + o(1)) \lambda_0 \frac{n^{-|V(H_1)|} p^{-|E(H_1)|}}{c_n^{-(|V(H_1)|-1)}} \\ &= (1 + o(1)) \lambda_0 \frac{|V(H)|-|V(H_1)|}{|V(H)|-1} \left(n^{|V(H)|-|V(H_1)|} p^{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)} \right)^{-\frac{1}{|V(H)|-1}}, \end{aligned} \quad (4.5)$$

which converges to $\lambda_0 \frac{|V(H)|-|V(H_1)|}{|V(H)|-1} \kappa^{-\frac{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)}{|V(H)|-1}} \in (0, \infty)$, when H_1 attains the minimum in (1.8), since $n^{\gamma(H)} p \rightarrow \kappa \in (0, \infty)$. Otherwise,

$$n^{\frac{|V(H)|-|V(H_1)|}{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)}} p \gg n^{\gamma(H)} p,$$

and so $\eta_n(F)$ converges to 0. Thus, $\eta_n(F)$ converges for all connected graphs $F \in \mathcal{G}_r(H)$, and the proof of Lemma 4.3 is complete. \square

Now, if $T(H, G_n) \xrightarrow{D} W$ for some random variable W , then $\mathbb{E}T(H, G_n)^r \xrightarrow{D} \mathbb{E}W^r$, for any integer $r \geq 1$. Thus, to show that W is not a Poisson distribution, it suffices to prove that

$$\liminf_{n \rightarrow \infty} \text{Var}(T(H, G_n)) \geq \liminf_{n \rightarrow \infty} \mathbb{E} \text{Var}(T(H, G_n)|G_n) > \lambda. \quad (4.6)$$

Recall from (2.19), $\text{Var}(T(H, G_n)|G_n) = R_{1,n} + R_{2,n}$, where $\mathbb{E}R_{1,n} \rightarrow \lambda$. Therefore, it suffices to show that $\liminf_{n \rightarrow \infty} \mathbb{E}R_{2,n} > 0$. To show this, let H_1 be the subgraph of H for which the minimum in (1.8) is attained, and F_0 be the $|V(H_1)|$ -join (note that $|V(H_1)| < |V(H)|$ by Lemma 4.2) of H and H' , where H' is isomorphic to H , such that $V(H) \cap V(H') = V(H_1)$ and $E(H) \cap E(H') = E(H_1)$. Moreover, let

$$\mathcal{J}_{\geq 2}(H) := \bigcup_{t=2}^{|V(H)|} \mathcal{J}_t(H),$$

where $\mathcal{J}_t(H)$ is as in Definition 2.1. Then, there exist constants $c_2(F, H)$, such that

$$\begin{aligned}
\mathbb{E}R_{2,n} &= (1 + o(1)) \sum_{F \in \mathcal{J}_{\geq 2}(H)} c_2(F, H) \frac{n^{|V(F)|} p^{|E(F)|}}{c_n^{|V(F)|-1}} \\
&\geq (1 + o(1)) c_2(F_0, H) \frac{n^{|V(F_0)|} p^{|E(F_0)|}}{c_n^{|V(F_0)|-1}} \\
&= (1 + o(1)) c_2(F_0, H) \lambda_0 \frac{2^{|V(H)|-|V(H_1)|-1}}{|V(H)|-1} \left(n^{\gamma(H)} p \right)^{-\frac{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)}{|V(H)|-1}} \\
&\hspace{25em} \text{(using (4.5))} \\
&= (1 + o(1)) c_2(F_0, H) \lambda_0 \frac{2^{|V(H)|-|V(H_1)|-1}}{|V(H)|-1} \kappa^{-\frac{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)}{|V(H)|-1}} > 0.
\end{aligned}$$

This implies (4.6), completing the proof of Theorem 1.4(b).

Proofs of Theorem 1.3(a) and Theorem 1.4(c): Note that, in this regime, $p \gg n^{-\frac{1}{m(H)}}$ (by Lemma 4.2), which implies $N(H, G_n) = (1 + o_P(1)) \mathbb{E}(N(H, G_n))$. Therefore,

$$\mathbb{E}(T(H, G_n)|G_n) = \frac{1}{c_n^{|V(H)|-1}} N(H, G_n) = (1 + o_P(1)) \frac{1}{c_n^{|V(H)|-1}} \mathbb{E}N(H, G_n) = (1 + o_P(1)) \lambda,$$

by assumption (1.5). Therefore, by Lemma 4.1, it suffices to check that $\text{Var}(T(H, G_n)|G_n) \xrightarrow{P} \lambda$, which is equivalent to $N(F, G_n) = o_P(c_n^{2|V(H)|-t-1})$, for every $F \in \mathcal{J}_t(H) \setminus \{H\}$ and $t \in [2, |V(H)|]$. Since, $|V(F)| = 2|V(H)| - t$, it suffices to show that

$$\mathbb{E}N(F, G_n) = o(c_n^{|V(F)|-1}), \quad \text{for all connected } F \neq H, \quad (4.7)$$

formed by the join of H and another isomorphic copy H' . To this end, define $H_1 = (V(H) \cap V(H'), E(H) \cap E(H'))$, which is a (possibly disconnected) subgraph of H . Then $|V(F)| = 2|V(H)| - |V(H_1)|$, $|E(F)| = 2|E(H)| - |E(H_1)|$, and

$$\begin{aligned}
\frac{\mathbb{E}(N(F, G_n))}{c_n^{|V(F)|-1}} &\lesssim_F \frac{n^{|V(F)|} p^{|E(F)|}}{c_n^{|V(F)|-1}} \\
&= \frac{n^{2|V(H)|-|V(H_1)|} p^{2|E(H)|-|E(H_1)|}}{c_n^{2|V(H)|-|V(H_1)|-1}} \\
&\lesssim \lambda_0^2 (1 + o(1)) \frac{n^{-|V(H_1)|} p^{-|E(H_1)|}}{c_n^{-|V(H_1)|+1}} \hspace{10em} \text{(using (1.5))} \\
&\lesssim \frac{c_n^{|V(H_1)|-1}}{n^{|V(H_1)|} p^{|E(H_1)|}}. \hspace{10em} (4.8)
\end{aligned}$$

Therefore, to establish (4.7), it suffices to verify that the RHS above goes to zero, as $n \rightarrow \infty$, for every connected $F \neq H$ formed by the join of two isomorphic copies of H .

Now, using $c_n = \Theta(n^{\frac{|V(H)|}{|V(H)|-1}} p^{\frac{|E(H)|}{|V(H)|-1}})$, as in (4.4), the RHS of (4.8) becomes

$$\left(n^{|V(H)|-|V(H_1)|} p^{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)} \right)^{-\frac{1}{|V(H)|-1}}.$$

Therefore, it suffices to show that

$$n^{|V(H)|-|V(H_1)|} p^{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)} \rightarrow \infty. \quad (4.9)$$

Now, depending on whether H is balanced or not, we consider two cases:

- H is balanced: In this case, $n^{-\frac{|V(H)|}{|E(H)|}} \ll p \ll 1$. Using $\frac{|E(H_1)|}{|V(H_1)|} \leq \frac{|E(H)|}{|V(H)|}$, the LHS of (4.9) becomes

$$n^{|V(H)|-|V(H_1)|} p^{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)} \geq \left(np^{\frac{|E(H)|}{|V(H)|}} \right)^{|V(H)|-|V(H_1)|},$$

which implies (4.9), whenever $|V(H_1)| < |V(H)|$, since $np^{\frac{|E(H)|}{|V(H)|}} \rightarrow \infty$ by assumption.

Otherwise, assume $|V(H_1)| = |V(H)|$, in which case the LHS of (4.9) becomes $p^{(|E(H_1)|-|E(H)|)(|V(H)|-1)} \rightarrow \infty$, whenever $|E(H_1)| < |E(H)|$, since $p \rightarrow 0$. Finally, note that $|V(H_1)| = |V(H)|$ and $|E(H_1)| = |E(H)|$, implies $F = H$ which is impossible, by assumption. Therefore, (4.7) holds, and by Lemma 4.1, $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$, completing the proof of Theorem 1.3(a).

- H is unbalanced: In this case, $n^{-\gamma(H)} \ll p \ll 1$.
 - If $|E(H_1)|(|V(H)|-1) - |E(H)|(|V(H_1)|-1) < 0$, then (4.9) is obvious, since $p := p(n) \rightarrow 0$ and $|V(H_1)| \leq |V(H)|$.
 - If $|E(H_1)|(|V(H)|-1) - |E(H)|(|V(H_1)|-1) > 0$ (this implies $|V(H)| > |V(H_1)|$), then by the definition of $\gamma(H)$ (see (1.8)),

$$\gamma(H) \leq \frac{|V(H)| - |V(H_1)|}{|E(H_1)|(|V(H)|-1) - |E(H)|(|V(H_1)|-1)},$$

and so

$$\begin{aligned} n^{|V(H)|-|V(H_1)|} p^{|E(H_1)|(|V(H)|-1)-|E(H)|(|V(H_1)|-1)} \\ \geq n^{|V(H)|-|V(H_1)|} p^{\frac{|V(H)|-|V(H_1)|}{\gamma(H)}} \\ = (n^{\gamma(H)} p)^{\frac{|V(H)|-|V(H_1)|}{\gamma(H)}}, \end{aligned}$$

which implies (4.9), since $n^{\gamma(H)} p \rightarrow \infty$ by assumption.

- If $|E(H_1)|(|V(H)|-1) - |E(H)|(|V(H_1)|-1) = 0$, but $|V(H_1)| < |V(H)|$, then again (4.9) is obvious. Otherwise,

$$|V(H_1)| = |V(H)| \text{ and } |E(H_1)|(|V(H)|-1) - |E(H)|(|V(H_1)|-1) = 0.$$

This implies $|E(H_1)| = |E(H)|$, and hence, $H = F$, which is impossible, by assumption.

This implies (4.7), and hence by Lemma 4.1, $T(H, G_n) \xrightarrow{D} \text{Pois}(\lambda)$, completing the proof of Theorem 1.4(c).

Proofs of Theorem 1.3(b) and Theorem 1.4(d): Finally, if $p(n) := p \in (0, 1)$ is fixed, G_n converges to the constant graphon $W^{(p)} = p$, and

$$\frac{N_{\text{ind}}(F, G_n)}{c_n^{|V(F)|-1}} = \frac{\lambda_0(1+o(1))}{|Aut(F)|} p^{|E(F)|-|E(H)|} (1-p)^{(|V(H)|)-|E(F)|},$$

for every super-graph F of H with $|V(F)| = |V(H)|$. The results in Theorem 1.3(b) and Theorem 1.4(d), then follows from Theorem 2.1 and Remark 2.3.

4.2. Monochromatic Cliques. Assumption (1.3) of Theorem 1.1 is equivalent to the conditions:

- (1) $\frac{1}{c_n^{|V(H)|-1}} N(H, G_n) \rightarrow \lambda$, and
- (2) $N(F, G_n) = o(c_n^{2|V(H)|-t-1})$, for every $F \in \mathcal{J}_t(H) \setminus \{H\}$ and $t \in [2, |V(H)|]$.

These conditions simplify considerably when $H = K_s$ is the s -clique. To this end, note that, for every $t \in [2, s-1]$, because of the symmetry of K_s , all t -joins of K_s are isomorphic, that is, $\mathcal{J}_t(K_s) = \{J_t(K_s)\}$, where $J_t(K_s)$ is the graph obtained by the superimposition of two isomorphic copies of K_s , such that the two vertex sets intersect at exactly t vertices. Therefore, for $t \in [2, s-1]$, condition (2) above simplifies to,

$$N(J_t(K_s), G_n) = o(c_n^{2s-t-1}) = o(N(K_s, G_n)^{\frac{2s-t-1}{s-1}}), \quad \text{for every } t \in [2, s-1], \quad (4.10)$$

using $\mathbb{E}(T(K_s, G_n)) = \frac{1}{c_n^{s-1}} N(K_s, G_n) \rightarrow \lambda$. Moreover, the set $\mathcal{J}_s(K_s) \setminus \{K_s\}$ is empty, and condition (2) above, for the case $t = s$, is trivially true. Therefore, we have the following corollary:

Corollary 4.1. $T(K_s, G_n) \xrightarrow{D} \text{Pois}(\lambda)$ whenever $\mathbb{E}(T(K_s, G_n)) \rightarrow \lambda$ and (4.10) holds. \square

In particular, when $H = K_3$ is the triangle, the above corollary implies, $T(K_3, G_n) \xrightarrow{D} \text{Pois}(\lambda)$ whenever $\mathbb{E}(T(K_3, G_n)) \rightarrow \lambda$ and $N(\mathcal{D}, G_n) = o(N(K_3, G_n)^{\frac{3}{2}})$, where \mathcal{D} is the *diamond*: the 4-cycle with a diagonal.

Remark 4.1. As mentioned before, Theorem 1.1, and, in particular, Corollary 4.1, does not follow by applying the Stein's method using a generic dependency graph [2, 9]. For example, let $H = K_3$ be the triangle and denote by \mathcal{X}_3 the set of 3-element subsets of $V(G_n)$ which form a triangle in G_n . Then the graph with vertex set \mathcal{X}_3 which puts an edge between two elements in \mathcal{X}_3 whenever they are non-overlapping, is a valid dependency graph for the collection $(\mathbf{1}\{X=s\})_{s \in \mathcal{X}_3}$. Now, if $\mathbb{E}(T(K_3, G_n)) \rightarrow \lambda$, using this dependency graph in [9, Theorem 15], shows that $T(K_3, G_n) \xrightarrow{D} \text{Pois}(\lambda)$, if

$$N(\mathcal{D}, G_n) = o(N(K_3, G_n)^{\frac{3}{2}}) \quad \text{and} \quad N(\bowtie, G_n) = o(N(K_3, G_n)^{\frac{3}{2}}),$$

where \bowtie denotes two triangles joined at a vertex. This condition is, in general, stronger than Corollary 4.1: For instance, in the wheel graph on W_n on n -vertices,⁹ colored with c_n colors such that $\mathbb{E}(T(K_3, W_n)) = \frac{n}{c_n^3} \rightarrow 1$, it is easy to check that $T(K_3, W_n) \xrightarrow{D} \text{Pois}(1)$, but $N(\bowtie, W_n) = \frac{n(n-1)}{2}$, that is, the above dependency graph construction does not work. This is because, unlike the direct moment-based approach, the generic dependency graph construction is unable to leverage the fact that the $\text{Cov}(\mathbf{1}\{X=s\}, \mathbf{1}\{X=t\}) = 0$, whenever $s, t \in \mathcal{X}_3$ have 1 vertex index in common. It would be interesting to see whether a more sophisticated dependency graph construction or other versions of Stein's method can be used to prove Theorem 1.1, and obtain rates of convergence.

4.3. Birthday Problem. The case $H = K_s$ is the s -clique, is of particular interest, because it generalizes the well-known birthday problem to a general friendship network G_n . In the birthday problem, G_n is a friendship-network graph where the vertices are colored uniformly with $c_n = 365$ colors (corresponding to birthdays). In this case, two friends will have the same birthday whenever the corresponding edge in the graph G_n is monochromatic. Therefore, $\mathbb{P}(T(K_s, G_n) > 0)$ is the probability that there is an s -fold birthday match, that is, there are s friends with the same birthday.

⁹The wheel graph W_n has vertex-set $V(W_n) := \{0, 1, 2, \dots, n\}$, and edge-set $E(W_n) = \{(0, 1), (0, 2), \dots, (0, n), (1, 2), (2, 3), \dots, (n-1, n), (n, 1)\}$.

For example, if the network G_n satisfies (4.10), Corollary (4.1) implies

$$\mathbb{P}(T(K_s, G_n) > 0) \approx 1 - \exp\left(-\frac{N(K_s, G_n)}{c_n^{s-1}}\right) = p, \quad (4.11)$$

from which we can compute the approximate number of people needed to ensure a s -fold birthday match in the network G_n , with probability at least p .

- In the classical birthday problem, the underlying graph $G_n = K_n$ is the complete graph. In this case, $N(K_s, G_n) = \binom{n}{s}$. For example, using $p = \frac{1}{2}$, $s = 4$, and $c_n = 365$ in (4.11), gives that in any group of approximately 167 people, with probability at least 50%, there are four friends all having the same birthday. Diaconis and Mosteller [12] considered the following related example: Suppose a friend reports that she, her husband, and their daughter were all born on the same day of the month (say the 16th). Taking $c_n = 30$ (days in a month), $s = 3$, and $p = \frac{1}{2}$, in (4.11) gives that among birthdays of 16 people, a triple match in day of the month has about 50% chance.
- Another interesting case is birthday coincidences among different types, for example, with two types (boy/girl) one can ask what is the chance there is a boy-girl birthday match among a group of n boys and n girls? More generally, with s -types and n objects in each type, an s -fold birthday coincidence corresponds to an s -clique in the complete s -partite graph with n vertices in each part. For example, using $N(K_3, K_{n,n,n}) = n^3$ and substituting $p = 0.5$, $s = 3$, $c_n = 365$ in the formula gives, in any collection of 3 types (say nationality, for example, American, French, and Indian) of approximately 45 people each, with probability at least 50%, there is a triple birthday match, that is, an American, a French, and an Indian, have the same birthday. Asymptotics of collision times among different objects are useful in developing algorithms for the discrete logarithm problem [15].

Acknowledgements. The authors are grateful to the anonymous referee for the detailed and insightful comments, which greatly improved the quality and the presentation of the paper.

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APPENDIX A. THE ORDERING LEMMA

In this section we prove the ordering lemma used in the proof of Lemma 2.3. To this end, let $\Omega \subset \mathbb{N}$ be finite and $R \geq 3$ a non-negative integer. A collection $\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N)$, where $\mathbf{s}_j \in \Omega_R$, for $j \in [N]$, is said to be *connected* if there exists an *ordering* (permutation) $\sigma : [N] \rightarrow [N]$ such that

$$\mathcal{X}_{\mathbf{S}}(t, \sigma) := \left| \bar{\mathbf{s}}_{\sigma(t)} \cap \left(\bigcup_{a=1}^{t-1} \bar{\mathbf{s}}_{\sigma(a)} \right) \right| \geq 1,$$

for every $t \in [2, N]$.

Lemma A.1. *Suppose $\mathbf{S} = (\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_N) \in \Omega_R^N$ is connected, and $|\bigcup_{j=1}^N \bar{\mathbf{s}}_j| < bR - b + 1$, where b is the number of distinct R -element sets in the collection $\{\bar{\mathbf{s}}_1, \bar{\mathbf{s}}_2, \dots, \bar{\mathbf{s}}_N\}$. Then there exists an ordering $\sigma : [R] \rightarrow [R]$ such that the following hold:*

- $\mathcal{X}_{\mathbf{S}}(t, \sigma) \geq 1$, for every $t \in [2, N]$, and
- $\mathcal{X}_{\mathbf{S}}(t, \sigma) \in [2, R - 1]$, for some $t \in [2, N]$.

Proof. Since \mathbf{S} is connected, there exists an ordering σ , such that $\mathcal{X}_{\mathbf{S}}(t, \sigma) \geq 1$ for every $t \in [2, N]$. Suppose that for every $t \in [2, N]$, $\mathcal{X}_{\mathbf{S}}(t, \sigma) \in \{1, R\}$, and, towards a contradiction, assume that for every $2 \leq t \leq N$, either $\mathcal{X}_{\mathbf{S}}(t, \sigma) = 1$, or $\bar{\mathbf{s}}_{\sigma(t)} \in \{\bar{\mathbf{s}}_{\sigma(1)}, \dots, \bar{\mathbf{s}}_{\sigma(t-1)}\}$. Define

$$k = |\{t \in [2, N] : \mathcal{X}_{\mathbf{S}}(t, \sigma) = 1\}|.$$

Then, $b = 1 + k$ and $|\bigcup_{j=1}^N \bar{\mathbf{s}}_j| = R + k(R - 1)$. This yields a contradiction, because

$$\left| \bigcup_{j=1}^N \bar{\mathbf{s}}_j \right| = R + (b - 1)(R - 1) = bR - b + 1.$$

Hence, there exists $2 \leq t \leq N$ such that $\mathcal{X}_{\mathcal{S}}(t, \sigma) = R$ and $\bar{s}_{\sigma(t)} \notin \{\bar{s}_{\sigma(1)}, \dots, \bar{s}_{\sigma(t-1)}\}$. Define

$$t_0 = \inf \{2 \leq t \leq N : \mathcal{X}_{\mathcal{S}}(t, \sigma) = R \text{ and } \bar{s}_{\sigma(t)} \notin \{\bar{s}_{\sigma(1)}, \dots, \bar{s}_{\sigma(t-1)}\}\} \text{ and}$$

$$t_1 = \inf \{1 \leq t < t_0 : \bar{s}_{\sigma(t_0)} \cap \bar{s}_{\sigma(t_1)} \neq \emptyset\}.$$

Clearly, there exists a permutation $\tau : [R] \rightarrow [R]$ such that $\tau(1) = \sigma(t_0)$, $\tau(2) = \sigma(t_1)$ and $\mathcal{X}_{\mathcal{S}}(t, \tau) \geq 1$ for every $t \in [2, N]$. By the definition of t_0 , it follows that $\bar{s}_{\sigma(t_1)} \neq \bar{s}_{\sigma(t_0)}$, and hence, if $|\bar{s}_{\sigma(t_0)} \cap \bar{s}_{\sigma(t_1)}| \geq 2$, then $\mathcal{X}_{\mathcal{S}}(2, \tau) \in [2, R-1]$, as required.

So, suppose that $|\bar{s}_{\sigma(t_0)} \cap \bar{s}_{\sigma(t_1)}| = 1$, and let $\{s\} = \bar{s}_{\sigma(t_0)} \cap \bar{s}_{\sigma(t_1)}$. Define

$$t_2 = \inf \{t_1 < t < t_0 : (\bar{s}_{\sigma(t_0)} \setminus \{s\}) \cap \bar{s}_{\sigma(t)} \neq \emptyset\}.$$

Once again, there exists a permutation $\kappa : [R] \rightarrow [R]$ such that $\kappa(1) = \sigma(t_0)$, $\kappa(2) = \sigma(t_2)$ and $\mathcal{X}_{\mathcal{S}}(t, \kappa) \geq 1$ for every $t \in [2, N]$. So, if $|\bar{s}_{\sigma(t_0)} \cap \bar{s}_{\sigma(t_2)}| \geq 2$, then $\mathcal{X}_{\mathcal{S}}(2, \kappa) \in [2, R-1]$, as desired.

Hence, assume that $|\bar{s}_{\sigma(t_0)} \cap \bar{s}_{\sigma(t_2)}| = 1$. Now, there exists a permutation $\theta : [R] \rightarrow [R]$ satisfying:

$$\theta(t) = \begin{cases} \sigma(t) & \text{if } 1 \leq t \leq t_2 \\ \sigma(t_0) & \text{if } t = t_2 + 1 \end{cases}$$

and $\mathcal{X}_{\mathcal{S}}(t, \theta) \geq 1$ for every $t \in [2, N]$. Now, it is easy to see that $\mathcal{X}_{\mathcal{S}}(t_2 + 1, \theta) = 2$, completing the proof of lemma A.1. \square

APPENDIX B. CONVERGENCE UNDER DOUBLE LIMIT

Here, we prove the lemma which establishes distributional convergence from moment convergence under the double limit as in (2.18).

Lemma B.1. *Suppose $\{X_{n,\varepsilon}\}_{n \geq 1, \varepsilon > 0}$ be a sequence of real-valued random variables satisfying, for every integer $r \geq 1$,*

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} |\mathbb{E}(X_{n,\varepsilon}^r) - \mathbb{E}(Z^r)| = 0, \quad (\text{B.1})$$

where Z is a random variable with $\mathbb{E}e^{tZ} < \infty$, for any $t \in \mathbb{R}$. Then for any $t \in \mathbb{R}$, we have

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} |\mathbb{E}(e^{itX_{n,\varepsilon}}) - \mathbb{E}(e^{itZ})| = 0,$$

that is, $X_{n,\varepsilon}$ converges in distribution to Z , as $n \rightarrow \infty$ followed by $\varepsilon \rightarrow 0$.

Proof. Fix $K \geq 2$ even. Then for $t \in \mathbb{R}$, by a Taylor's series expansion,

$$\left| e^{it} - \sum_{s=0}^{K-1} \frac{(it)^s}{s!} \right| \leq \frac{|t|^K}{K!}.$$

Using this along with triangle inequality gives,

$$|\mathbb{E}(e^{itX_{n,\varepsilon}}) - \mathbb{E}(e^{itZ})| \leq \sum_{s=0}^{K-1} \frac{|t|^s}{s!} |\mathbb{E}(X_{n,\varepsilon}^s) - \mathbb{E}(Z^s)| + \frac{|t|^K}{K!} \left\{ \mathbb{E}|X_{n,\varepsilon}|^K + \mathbb{E}|Z|^K \right\}.$$

On letting $n \rightarrow \infty$ followed by $\varepsilon \rightarrow 0$ and using (B.1) gives,

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} |\mathbb{E}(e^{itX_{n,\varepsilon}}) - \mathbb{E}(e^{itZ})| \leq \frac{2|t|^K}{K!} \mathbb{E}|Z|^K,$$

From this, the desired conclusion follows by taking limit $K \rightarrow \infty$, along the even integers, on both sides, since $\mathbb{E}(e^{|tZ|}) \leq \mathbb{E}(e^{tZ}) + \mathbb{E}(e^{-tZ}) < \infty$, and recalling that $\mathbb{E}e^{|tZ|} = \sum_{s=0}^{\infty} \frac{|t|^s}{s!} \mathbb{E}|Z|^s$. \square

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