ON LIPSCHITZ-LIKE PROPERTY FOR POLYHEDRAL MOVING SETS

EWA M. BEDNARCZUK¹ AND KRZYSZTOF E. RUTKOWSKI²

ABSTRACT. We give sufficient conditions for Lipschitz-likeness of a class of polyhedral set-valued mappings in Hilbert spaces based on Relaxed Constant Rank Constraint Qualification (RCRCQ) proposed recently by Minchenko and Stakhovsky. To this aim we prove the R-regularity of the considered set-valued mapping and correct the respective proof given by these authors.

1. INTRODUCTION

Let \mathcal{H}, \mathcal{G} be a Hilbert space and $D \subset \mathcal{G}$ be a nonempty set. Let $\mathbb{C} : \mathcal{D} \rightrightarrows \mathcal{H}$ be a multifunction defined as $\mathbb{C}(p) := C(p)$, where

$$C(p) = \left\{ x \in \mathcal{H} \mid \langle x \mid g_i(p) \rangle = f_i(p), \quad i \in I_1, \\ \langle x \mid g_i(p) \rangle \leq f_i(p), \quad i \in I_2 \end{array} \right\},$$
(1.1)

and $f_i: \mathcal{D} \to \mathbb{R}$, $g_i: \mathcal{D} \to \mathcal{H}$, $i \in I_1 \cup I_2$, $I_1 = \{1, \ldots, m\}$, $I_2 = \{m+1, \ldots, n\}$ are Lipschitz on \mathcal{D} with Lipschitz constants ℓ_{f_i}, ℓ_{g_i} , respectively.

In finite dimensional case $(\mathcal{H} = \mathbb{R}^{n_1}, \mathcal{G} = \mathbb{R}^{n_2})$ the sufficient conditions for R-regularity of multifunction \mathbb{C} and more general set-valued mappings have been proposed in [8, Theorem 4]. R-regularity of the multifunction \mathbb{C} at $(\bar{p}, \bar{x}) \in \text{gph} \mathbb{C}$ is defined as follows.

Definition 1. Multifunction \mathbb{C} : $\mathcal{D} \rightrightarrows \mathcal{H}$ given by (1.1) is said to be *R*-regular at a point (\bar{p}, \bar{x}) , if for all (p, x) in a neighbourhood of (\bar{p}, \bar{x}) ,

$$dist \ (x, \mathbb{C}(p)) \le \alpha \max\{0, \ |\langle x \mid g_i(p) \rangle - f_i(p)|, \ i \in I_1, \ \langle x \mid g_i(p) \rangle - f_i(p), \ i \in I_2\}$$

for some $\alpha > 0$.

The aim of the paper is to investigate the Lipschitz-like property of the multifunction \mathbb{C} at $(\bar{p}, \bar{x}) \in \operatorname{gph} \mathbb{C}$ defined as follows.

Definition 2. Multifunction \mathbb{C} is Lipschitz-like at a point (\bar{p}, \bar{x}) , if there exist a constant $\ell > 0$, a neighbourhood $U(\bar{p})$ and a neighbourhood $V(\bar{x})$ such that for all $p_1, p_2 \in U(\bar{p})$

$$\mathbb{C}(p_1) \cap V(\bar{x}) \subset \mathbb{C}(p_2) + \ell \| p_1 - p_2 \| \mathbb{B},$$

where B denotes the open unit ball in the space \mathcal{H} .

²⁰¹⁰ Mathematics Subject Classification. 41A50, 46C05, 49K27, 52A07, 90C31.

Key words and phrases. metric regularity, moving polyhedral sets, relaxed constant rank constraint qualification.

¹ Systems Research Institute of the Polish Academy of Sciences, Warsaw University of Technology, e.bednarczuk@mini.pw.edu.pl .

² Warsaw University of Technology, k.rutkowski@mini.pw.edu.pl .

To this aim we provide Proposition 1 which is the infinite-dimensional version of Lemma 3 of [8] applied to our set-valued mapping (1.1). However, the proof of [8, Lemma 3] which is important for the proof of [8, Theorem 4] is incorrect. It is also our aim to provide the correct proof of [8, Lemma 3] in our case.

2. Preliminaries

Let $p \in \mathcal{D}$, $w \in \mathcal{H}$, $w \notin C(p)$. Projection of w onto C(p) is defined as

$$P_{C(p)}(w) = \arg\min_{x \in C(p)} \|w - x\|,$$
(2.1)

or equivalently

$$P_{C(p)}(w) = \arg\min_{x \in C(p)} \frac{1}{2} \|w - x\|^2.$$
(2.2)

Put $f_w(x) = ||x - w||$ and

$$f_{P_{C(p)}(w)}^{*}(x) = ||x - w|| + \frac{\langle x - w \mid x - P_{C(p)}(w) \rangle}{||P_{C(p)}(w) - w||}$$

Denote $G_i(x,p) = \langle x \mid g_i(p) \rangle - f_i(p)$, $i \in I_1 \cup I_2$ and $\overline{G}_i(x,p) = G_i(x,p)$ for $g_i(p) = a_i$, $a_i \in \mathcal{H}, i \in I_1 \cup I_2$, i.e., $g_i, i \in I_1 \cup I_2$ does not depend on p. Let G(x,p) and $\overline{G}(x,p)$ be defined as

$$G(x,p) = [G_i(x,p)]_{i=1,...,n}, \quad \bar{G}(x,p) = [\bar{G}_i(x,p)]_{i=1,...,n}$$

Let $\lambda \in \mathbb{R}^n$ and

$$L_w(p, x, \lambda) := f_w(x) + \langle \lambda \mid G(x, p) \rangle,$$

$$L_w^*(p, x, \lambda) := f_{P_{G(p)}(w)}^*(x) + \langle \lambda \mid G(x, p) \rangle.$$

The sets of Lagrange multipliers corresponding to (2.1) are defined as

$$\begin{split} \Lambda_w(p,x) &:= \{\lambda \in \mathbb{R}^n \mid \nabla_x L_w(p,x,\lambda) = 0, \ \lambda_i \geq 0, \ \text{and} \ \lambda_i G_i(x,p) = 0, \ i \in I_2\}, \\ \Lambda_w^*(p,x) &:= \{\lambda \in \mathbb{R}^n \mid \nabla_x L_w^*(p,x,\lambda) = 0, \ \lambda_i \geq 0, \ \text{and} \ \lambda_i G_i(x,p) = 0, \ i \in I_2\}. \end{split}$$

Then

$$\nabla_{x}L_{w}(p, P_{C(p)}(w), \lambda) = \frac{P_{C(p)}(w) - w}{\|P_{C(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_{i}g_{i}(p),$$

$$\nabla_{x}L_{w}^{*}(p, P_{C(p)}(w), \lambda) = 2\frac{P_{C(p)}(w) - w}{\|P_{C(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_{i}g_{i}(p).$$
(2.3)

Let us note that when $w \notin C(p)$ condition $\nabla_x L_w(p, P_{C(p)}(w), \lambda) = 0$ is equivalent to the following

$$\frac{w - P_{C(p)}(w)}{\|P_{C(p)}(w) - w\|} = \sum_{i=1}^{n} \lambda_i g_i(p) \quad \Leftrightarrow \quad w - P_{C(p)}(w) = \sum_{i=1}^{n} \hat{\lambda}_i g_i(p), \tag{2.4}$$

where $\hat{\lambda}_i = \lambda_i \| P_{C(p)}(w) - w \|$, $i = 1, \dots, n$.

Let us recall that the Kuratowski limit of $\mathbb C$ at $\bar p$ is given as

$$\liminf_{p \to \bar{p}} \mathbb{C}(p) = \{ y \in \mathcal{H} \mid \forall p_k \to \bar{p} \exists y_k \in \mathbb{C}(p_k) \mid y_k \to y \}.$$

Equivalently, $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$ if and only if

$$\forall V(\bar{x}) \exists U(\bar{p}) \text{ s.t. } \mathbb{C}(p) \cap V(\bar{x}) \neq \emptyset \quad \text{for } p \in U(\bar{p}).$$
(2.5)

For any $(p, x) \in \mathcal{D} \times \mathcal{H}$ let $I_p(x) := \{i \in I_1 \cup I_2 \mid \langle x - f_i(p) \mid g_i(p) \rangle = 0\}$ denote the active index set for $p \in \mathcal{D}$ at $x \in \mathcal{H}$.

Definition 3 (Relaxed Constant Rank Constraint Qualification). The relaxed constant rank constraint qualification (RCRCQ) holds for multifunction $\mathbb{C} : \mathcal{D} \rightrightarrows \mathcal{H}$ given by (1.1) at $(\bar{p}, \bar{x}), \bar{x} \in C(\bar{p})$, if there exists a neighbourhood $U(\bar{p})$ of \bar{p} such that, for any index set $J, I_1 \subset J \subset I_{\bar{p}}(\bar{x})$, for every $p \in U(\bar{p})$ the system of vectors $\{g_i(p), i \in J\}$ has constant rank. Precisely, for any $J, I_1 \subset J \subset I_{\bar{p}}(\bar{x})$

$$\mathsf{rank}(q_i(p), i \in J) = \mathsf{rank}(q_i(\bar{p}), i \in J)$$
 for all $p \in U(\bar{p})$.

For more general constraint sets this definition has been introduced in [8, Definition 1]. In [6] several kinds of relations between constraint qualifications (for $C(\bar{p})$) has been established including RCRCQ and the classical Mangasarian Fromovitz Constraint Qualification (MFCQ).

The following diagram provides the summary of the existing results concerning R-regularity, calmness, metrical subregularity, metric regularity of sets and multifunctions $\mathbb{C}(p)$. Let us note however that it also applies to more general forms of sets and multifunctions.



In the diagram multifunction \mathbb{G} is defined as $\mathbb{G} = \overline{G} + K$, where $K = \{0\}^m \times \mathbb{R}^{n-m}_+$. Implication given as dotted line under additional assumption has been proposed in [8, Theorem 4]. However, as mentioned in Introduction the proof of [8, Theorem 4] is incorrect. In the next section we present a counterexample to the proof of [8, Lemma 3] and propose a new proof in our settings.

3. MAIN RESULT

We start with the proposition which relates RCRCQ condition to the boundedness (with respect to p, w) of Lagrange multiplier set

$$\Lambda_w^M(p, P_{C(p)}(w)) := \{ \lambda \in \Lambda_w(p, P_{C(p)}(w)) \mid \sum_{i=1}^n |\lambda_i| \le M \}.$$

Proposition 1. Let multifunction \mathbb{C} given by (1.1) satisfy RCRCQ at $(\bar{x}, \bar{p}) \in \operatorname{gph} \mathbb{C}$. Assume that $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. Then there exist numbers M > 0, $\delta > 0$, $\delta_0 > 0$ such that

$$\Lambda_w^M(p, P_{C(p)}(w)) \neq \emptyset \quad \text{ for } p \in \bar{p} + \delta_0 B, \ w \in \bar{x} + \delta B, \ w \notin C(p).$$

The content of Proposition 1 coincides with the content of [8, Lemma 3]. The proof of Proposition 1 we present below is essentially different from the proof of Lemma 3 of [8]. The proof of [8, Lemma 3] is incorrect which can be shown by the the following example.

Example 1. Let \mathbb{C} : $\mathbb{R}^2 \to \mathbb{R}^2$ be defined as follows

$$C(p) := \left\{ x \in \mathbb{R}^2 \middle| \begin{array}{l} \langle x \mid (1,0) \rangle = 0\\ \langle x \mid (0,1) \rangle = 0\\ \langle x \mid p \rangle \le 0 \end{array} \right\}$$
(3.1)

and $\bar{p} = \bar{x} = (0,0)$. We have $C(p) = \{(0,0)\}$ for all $p = (r_1, r_2) \in \mathbb{R}^2$ and

(1) RCRCQ holds for multifunction C at z₀ = ((0,0), (0,0)) ∈ gph(C),
 (2) (0,0) ∈ lim inf C(p).

$$(0,0) \subset \liminf_{p \to (0,0)} \mathbb{C}($$

We have $g_1(p) = (1,0)$, $g_2(p) = (0,1)$, $g_3(p) = p$ for all $p \in \mathbb{R}^2$ and $G_1(p,x) = \langle x \mid (1,0) \rangle$, $G_2(p,x) = \langle x \mid (0,1) \rangle$, $G_3(p,x) = \langle x \mid p \rangle$ and the assumptions of [8, Lemma 3] are satisfied.

The proof of [8, Lemma 3] relies on showing that for any sequences $p_k \to \bar{p}, w_k \to \bar{x}, w_k \notin C(p_k)$ there exist

$$\lambda_k \in \Lambda_{w_k}^M(p_k, P_{C(p_k)}(w_k))$$
 for some $M \ge 0$ and all $k \in \mathbb{N}$.

Below we show that the way of choosing λ_k which are to satisfy the above property is incorrect in general. More precisely, we show that for \mathbb{C} defined by (3.1) there are sequences $p_k \to \bar{p}, w_k \to \bar{x}$ and $\lambda_k \in \Lambda_{w_k}(p_k, P_{C(p_k)}(w_k))$ chosen as in the proof of [8, Lemma 3] with $\|\lambda_k\| \to +\infty$.

Let $p_k = (\frac{1}{k^2}, \frac{1}{k^2}) \rightarrow (0, 0)$, $w_k = (\frac{1}{k}, \frac{2}{k})$. We have that $x_k = (0, 0) = \prod_{C(p_k)} (w_k)$ and in the notation of the proof of [8, Lemma 3], $z_k = ((\frac{1}{k^2}, \frac{1}{k^2}), (0, 0))$. We have $I_{p_k}(x_k) = I^* = \{1, 2, 3\}$ and

$$0 = \frac{P_{C(p_k)(w_k)} - w_k}{\|P_{C(p_k)(w_k)} - w_k\|} + \sum_{i \in I_{p_k}(x_k)} \lambda_i g_i(p_k)$$

$$\Leftrightarrow \quad \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \lambda_1(1, 0) + \lambda_2(0, 1) + \lambda_3\left(\frac{1}{k^2}, \frac{1}{k^2}\right)$$

There exists a maximal linearly independent subfamily $\{g_i(p_k), i \in \{2,3\}\}$ in the family $\{g_i(p_k), i \in \{1,2,3\}\}$ such that $(0, \frac{1}{\sqrt{5}}, \frac{k^2}{\sqrt{5}}) \in \Lambda_{(\frac{1}{k^2}, \frac{2}{k^2})}((\frac{1}{k^2}, \frac{1}{k^2}), (0,0))$ for all $k \in \mathbb{N}$.

In the notation of the proof of [8, Lemma 3] we have $J(z_k) = J^0 = \{2,3\}$. RCRCQ at the point z_0 implies that

$$2 = \operatorname{rank} \{ g_i(\bar{p}), \ i \in \{1, 2, 3\} \} = \operatorname{rank} \{ g_i(p), \ i \in \{1, 2, 3\} \}$$

for all points $z \in \mathbb{R}^2$. Moreover for all z_k , $k = 1, 2, \ldots$ we have

$$2 = \operatorname{rank} \{ g_i(p_k), \ i \in \{1, 2, 3\} \} = \operatorname{rank} \{ g_i(p), \ i \in \{2, 3\} \}.$$

Observe that rank $\{g_1(p), g_2(p)\} = 2$ for all $p \in U((0,0))$. Hence, in the notation of the proof of [8, Lemma 3], $J^{00} = \{1,2\}$ and the function Φ takes the form

$$\begin{aligned} G_1(p, x) &= \Phi(G_1(p, x), G_2(p, x)), \\ G_2(p, x) &= \Phi(G_1(p, x), G_2(p, x)), \\ G_3(p, x) &= \Phi(G_1(p, x), G_2(p, x)) = r_1 G_1(p, x) + r_2 G_2(p, x) \\ (\text{since } \langle x \mid p \rangle &= \langle x \mid \langle p \mid (1, 0) \rangle \cdot (1, 0) \rangle + \langle x \mid \langle p \mid (0, 1) \rangle \cdot (0, 1) \rangle). \end{aligned}$$

On the other hand,

$$g_3(p_k) = \nabla_x \Phi(G_1(p_k, x_k), G_2(p_k, x_k)) = \frac{1}{k}(1, 0) + \frac{1}{k}(0, 1),$$

$$g_3(\bar{p}) = \nabla_x \Phi(G_1(\bar{p}, \bar{x}), G_2(\bar{p}, \bar{x})) = 0 \cdot (1, 0) + 0 \cdot (0, 1),$$

and vectors $g_2((0,0)) = (1,0)$, $g_3((0,0)) = (0,0)$ are linearly dependent. Moreover, $\|(0,\frac{1}{\sqrt{5}},\frac{k^2}{\sqrt{5}})\| = \sqrt{\frac{1}{5} + \frac{k^4}{5}} \to +\infty$ and

$$(0,0) = \lim_{k \to +\infty} \frac{1}{\sqrt{\frac{1}{5} + \frac{k^4}{5}}} (\frac{1}{k}, \frac{2}{k}) = \lim_{k \to +\infty} \frac{\sqrt{5}}{k^2 \sqrt{\frac{1}{k^4} + 1}} (\frac{1}{k}, \frac{2}{k})$$
$$= \lim_{k \to +\infty} \frac{\sqrt{5}}{k^3 \sqrt{\frac{1}{k^4} + 1}} (0,1) + \frac{\sqrt{5}}{k \sqrt{\frac{1}{k^4} + 1}} (\frac{1}{k^2}, \frac{1}{k^2}) = 0(0,1) + 0(0,0).$$

The example shows that the construction proposed in the proof of [8, Lemma 3] may lead to the contradiction of the conclusion. The reason is that in the proof of [8, Lemma 3] the set J^0 is chosen in an incorrect way and function Φ does not depend on p directly.

Proof of Proposition 1. On the contrary suppose, that there exist sequences $p_k \rightarrow \bar{p}$, $w_k \to \bar{x}$ such that $w_k \notin C(p_k)$ and

dist
$$(0, \Lambda_{w_k}(p_k, P_{C(p_k)}(w_k)) \to +\infty.$$
 (3.2)

Due to the fact that $\bar{x} \in \liminf_{p \to \bar{x}} \mathbb{C}(p)$, we may assume without loss of generality that $C(p_k) \neq \emptyset$ for each p_k , and there exists $\hat{x}_k \in C(p_k)$ such that $\hat{x}_k \to \bar{x}$.

RCRCQ at (\bar{p}, \bar{x}) implies that RCRCQ holds also at all the points near the point (\bar{p}, \bar{x}) . Without loss of generality one may assume that RCRCQ holds at all $(p_k, P_{C(p_k)}(w_k))$, $k \in \mathbb{N}$. Consequently, $\Lambda_{w_k}(p_k, P_{C(p_k)}(w_k)) \neq \emptyset$ for all $k = 1, 2, \ldots$

Passing to subsequences, if necessary, we may assume that $(p_k, w_k) \in V(\bar{p}, \bar{w})$, where by RCRCQ, $V(\bar{p}, \bar{w})$ is such that for any J, $I_1 \subset J \subset I_1 \cup I_2$

rank
$$\{g_i(p_k), i \in J\} = \text{rank} \{g_i(\bar{p}), i \in J\}.$$
 (3.3)

By Theorem 2,

$$w_k - P_{C(p_k)}(w_k) = \sum_{i \in I_{p_k}(P_{C(p_k)}(w_k))} \hat{\lambda}_i^k g_i(p_k), \quad k = 1, \dots$$
(3.4)

where $\hat{\lambda}_{i}^{k} \geq 0$, $i \in I_{2} \cap I_{p_{k}}(P_{C(p_{k})}(w_{k}))$. Recall that $I_{p}(P_{C(p)}(w)) := \{i \in I_{1} \cup I_{2} \in I_{2} \}$ $I_2 \mid \langle P_{C(p)}(w) \mid g_i(p) \rangle - f_i(p) = 0 \}$ and $\hat{\lambda}_i^k$, $i \in I_2 \cap I_{p_k}(P_{C(p_k)}(w_k))$ are related to the set $\Lambda_{w_k}(p_k, P_{C(p_k)}(w_k))$ via equivalence (2.4). Then (3.4) takes the form

$$w_{k} - P_{C(p_{k})}(w_{k}) = \sum_{I_{1}} \hat{\lambda}_{i}^{k} g_{i}(p_{k}) + \sum_{I_{p_{k}}(P_{C(p_{k})}(w_{k})\setminus I_{1})} \hat{\lambda}_{i}^{k} g_{i}(p_{k}),$$

$$\hat{\lambda}_{i}^{k} \geq 0, i \in I_{p_{k}}(P_{C(p_{k})}(w_{k})\setminus I_{1} \quad k = 1, \dots$$
(3.5)

By Lemma 3, there exists $I_1^0 \subset I_1$, $I_2^0(w_k, p_k) \subset I_2$, and $\lambda_i(w_k, p_k) \in \mathbb{R}$, $i \in I_1^0$, $\tilde{\lambda}_i(w_k, p_k) > 0$, $i \in I_2^0(w_k, p_k)$ such that

$$w_k - P_{C(p_k)}(w_k) = \sum_{i \in I_1^0} \tilde{\lambda}_i(w_k, p_k) g_i(p) + \sum_{I_2^0(w_k, p_k)} \tilde{\lambda}_i(w_k, p_k) g_i(p_k),$$
(3.6)

where $g_i(p_k)$, $i \in I_1^0 \cup I_2^0(w_k, p_k)$ are linearly independent.

Passing to a subsequence, if necessary, we may assume that for all $k \in \mathbb{N}$, $I_2^0(w_k, p_k)$ is a fixed set, i.e., $I_2^0(w_k, p_k) = I_2^0$.

By RCRCQ, there exists k_0 such that for all $k \ge k_0$

$$\mathsf{rank} \{ g_i(p_k), \ i \in I_1^0 \cup I_2^0 \} = \mathsf{rank} \{ g_i(\bar{p}), \ i \in I_1^0 \cup I_2^0 \}.$$

Put $\lambda_i^k = \frac{\tilde{\lambda}_i^k}{\|w_k - P_{C(p_k)}(w_k)\|}$. For every $k \ge k_0$ we have $\lambda_k(w_k, p_k) \in \Lambda_{w_k}(p_k, P_{C(p_k)}(w_k))$ and by (3.2), $\|\lambda^k(w_k, p_k)\| \to +\infty$. Without loss of generality we may assume that $\lambda^k(w_k, p_k)\|\lambda^k(w_k, p_k)\|^{-1} \to \bar{\lambda}$. Then by (3.6) we obtain

$$0 = \sum_{i \in I_1^0 \cup I_2^0} \bar{\lambda}_i g_i(\bar{p}), \ \bar{\lambda}_i \ge 0, \ i \in I_2^0,$$

where $\|\bar{\lambda}\| = 1$. This contradicts the fact that $g_i(\bar{p})$, $i \in I_1^0 \cup I_2^0$ are linearly independent.

In the next proposition we relate the boundedness of the Lagrange multiplier set $\Lambda_w^M(p, P_{C(p)}(w))$ to the *R*-regularity of \mathbb{C} at (\bar{p}, \bar{x}) . For sets C(p) given as solution sets to parametric systems of nonlinear equations and inequalities in finite dimensional spaces this fact has been already proved in [8, Theorem 2]. The proof we give below is based on the proof of Theorem 2 of [8].

Proposition 2. Let $\bar{p} \in D$, $\bar{x} \in C(\bar{p})$ and $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. Assume that there exist numbers M > 0, $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\Lambda^M_w(p, P_{C(p)}(w)) := \{\lambda \in \Lambda_w(p, P_{C(p)}(v)) \mid \sum_{i=1}^n |\lambda_i| \le M\} \neq \emptyset$$

for all $p \in (\bar{p} + \delta_1 B) \cap S$ and for all $w \in (\bar{x} + \delta_2 B)$, $w \notin C(p)$. Then the multifuction \mathbb{C} is *R*-regular at (\bar{x}, \bar{p}) .

Proof. Since $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$ one can find $\delta_3 > 0$ such that $C(p) \cap \{\bar{x} + 4^{-1}\delta_3B\} \neq \emptyset$ for all $p \in \bar{p} + \delta_3 B$. Let $p \in \bar{p} + 2^{-1}\delta_3 B$, $w \in \bar{x} + 4^{-1}\delta_3 B$. If $w \in C(p)$ then dist (w, C(p)) = 0.

Let $w \notin C(p)$ and $w \in \overline{x} + 4^{-1}\delta_3 B$. Since $C(p) \cap \{\overline{x} + 4^{-1}\delta_3 B\} \neq \emptyset$ there exists $x_1 \in C(p) \cap \{\overline{x} + 4^{-1}\delta_3 B\}$. Then

$$||P_{C(p)}(w) - w|| \le ||w - x_1|| \le ||w - \bar{x}|| + ||x_1 - \bar{x}|| < 2^{-1}\delta_3$$

It follows that $P_{C(p)}(w) \in \bar{x} + \delta_3 B$. Let

$$\lambda \in \bigg\{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n |\lambda_i| \le 2M\bigg\}.$$

Introduce a function

$$h(p,x) = h(p,x,w,\lambda, P_{C(p)}(w)) = \frac{\langle x - w \mid x - P_{C(p)}(w) \rangle}{\|P_{C(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_i G_i(p,x)$$

The function h(p, x) is convex with respect to x on \mathcal{H} .

Let $\lambda \in \Lambda_w^M(p, P_{C(p)}(w))$, $p \in \bar{p} + 2^{-1}\delta_3 B$, $w \in \bar{x} + 4^{-1}\delta_3 B$ such that $w \notin C(p)$. Since $\Lambda_w^*(x, P_{C(p)}(w)) = 2\Lambda_w(x, P_{C(p)}w)) \neq \emptyset$ by (2.3) we have $\lambda^* := 2\lambda \in \Lambda_w^*(x, P_{C(p)}(w))$.

The equality $\nabla_x L_x^*(p, P_{C(p)}(w), \lambda^*) = 0$ can be written in the form

$$\frac{w - P_{C(p)}(w)}{\|w - P_{C(p)}(w)\|} = \frac{P_{C(p)}(w) - v}{\|P_{C(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_i^* g_i(p)$$

where the right side coincides with the gradient $\nabla_x h(p,x)$ of the function

$$h(p,x) = h(p,x,w,\lambda^*, P_{C(p)}(w)) = \frac{\langle x - w \mid x - P_{C(p)}(w) \rangle}{\|P_{C(p)}(w) - w\|} + \sum_{i=1}^n \lambda_i^* G_i(p,x)$$

at the point $y = P_{C(p)}(w)$. Since

$$\langle \nabla_x h(p, P_{C(p)}(w)) \mid w - P_{C(p)}(w) \rangle \le h(p, w) - h(p, P_{C(p)}(w))$$

due to convexity of the function h(p, x) with

$$\lambda^* \in \Lambda^*_w(x, P_{C(p)}(w)) \cap \bigg\{ \lambda \in \mathbb{R}^n \mid \sum_{i=1}^n |\lambda_i| \le 2M \bigg\},\$$

from the last inequality it follows that

$$\begin{split} \|w - P_{C(p)}(w)\| &= \frac{\langle w - P_{C(p)}(w) \mid w - P_{C(p)}(w) \rangle}{\|P_{C(p)}(w) - w\|} \\ &= \left\langle \frac{P_{C(p)}(w) - w}{\|P_{C(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_{i}^{*} g_{i}(p) \mid w - P_{C(p)}(w) \right\rangle \\ &\leq \frac{\langle w - w \mid w - P_{C(p)}(w) \rangle}{\|P_{C(p)}(w) - w\|} + \sum_{i=1}^{n} \lambda_{i}^{*} G_{i}(p, w) \\ &- \frac{\langle P_{C(p)}(w) - w \mid P_{C(p)}(w) - P_{C(p)}(w) \rangle}{\|P_{C(p)}(w) - w\|} - \sum_{i=1}^{n} \lambda_{i}^{*} G_{i}(p, P_{C(p)}(w)) \\ &= \sum_{i=1}^{n} \lambda_{i}^{*} (G_{i}(p, w) - G_{i}(p, P_{C(p)}(w))) = \sum_{i=1}^{n} \lambda_{i}^{*} G_{i}(p, w) = 2 \sum_{i=1}^{n} \lambda_{i} G_{i}(p, w). \end{split}$$

This inequality implies

 $\begin{aligned} & \mathsf{dist}\,(w,C(p)) = \|w - P_{C(p)}(w)\| \le 2\|\lambda\|_1 \max\{0,G_i(p,w), \ i \in I_2, \ |G_i(p,w)|, i \in I_1\} \\ & \le 2M \max\{0,G_i(p,w), \ i \in I_2, \ |G_i(p,w)|, i \in I_1\}. \end{aligned}$

Now we show that if the multifunction \mathbb{C} is *R*-regular at (\bar{p}, \bar{x}) then \mathbb{C} is Lipschitz like at (\bar{p}, \bar{x}) .

Proposition 3. Let \mathcal{H} , \mathcal{G} be a Hilbert spaces and $f_i : \mathcal{D} \to \mathbb{R}$, $g_i : \mathcal{D} \to \mathcal{H}$ are Lipschitz on $\mathcal{D} \subset \mathcal{G}$. If the set-valued mapping $\mathbb{C} : \mathcal{D} \rightrightarrows \mathcal{H}$ given by (1.1) is R-regular at $(\bar{p}, \bar{x}), \bar{p} \in \mathcal{D}, \bar{x} \in C(\bar{p})$ then \mathbb{C} is Lipschitz-like at (\bar{p}, \bar{x})

Proof. By the *R*-regularity of $\mathbb C$ there exists a constant α and a neighbourhood $U(\bar p)$ and a neighbourhood $V(\bar x)$ such that

$$\operatorname{dist}(x, C(p)) \le \alpha \max\{0, |\langle x \mid g_i(p) \rangle - f_i(p)|, i \in I_1, \ \langle x \mid g_i(p) \rangle - f_i(p), \ i \in I_2\}$$

for all (p, x) in neighbourhood $U(\bar{p}) \times V(\bar{x})$. Let (p_1, x_1) , $x_1 \in C(p_1)$ in neighbourhood $U(\bar{p}) \times V(\bar{x})$ and $p_2 \in U(\bar{p})$. Since $C(p_2)$ is closed and convex there exists $x_2 \in C(p_2)$ such that $dist(x_1, C(p_2)) = ||x_1 - x_2||$. Then by *R*-regularity

$$\begin{aligned} \operatorname{dist}(x_{1}, C(p_{2})) &= \|x_{1} - x_{2}\| \\ &\leq \alpha \max\left\{0, \max_{i \in I_{1}} |\langle x_{1} \mid g_{i}(p_{2})\rangle - f_{i}(p_{2})|, \max_{i \in I_{2}} \langle x_{1} \mid g_{i}(p_{2})\rangle - f_{i}(p_{2})\right\} \\ &\leq \alpha \max\left\{0, \max_{i \in I_{1}} |\langle x_{1} \mid g_{i}(p_{2})\rangle - f_{i}(p_{2}) - (|\langle x_{1} \mid g_{i}(p_{1})\rangle - f_{i}(p_{1}))|, \\ \max_{i \in I_{2}} \langle x_{1} \mid g_{i}(p_{2})\rangle - f_{i}(p_{2}) - (|\langle x_{1} \mid g_{i}(p_{1})\rangle - f_{i}(p_{1}))|\right\} \\ &= \alpha \max\left\{0, \max_{i \in I_{1}} |\langle x_{1} \mid g_{i}(p_{2}) - g_{i}(p_{1})\rangle - (f_{i}(p_{2}) - f_{i}(p_{1}))|, \\ \max_{i \in I_{2}} \langle x_{1} \mid g_{i}(p_{2}) - g_{i}(p_{1})\rangle - (f_{i}(p_{2}) - f_{i}(p_{1}))|\right\} \\ &\leq \alpha \max\left\{\max_{i \in I_{1}} \|x_{1}\| \|g_{i}(p_{2}) - g_{i}(p_{1})\| + \|f_{i}(p_{2}) - f_{i}(p_{1})\|\right\} \\ &= \alpha \max_{i \in I_{2}} \|x_{1}\| \|g_{i}(p_{2}) - g_{i}(p_{1})\| + \|f_{i}(p_{2}) - f_{i}(p_{1})\| \\ &\leq \alpha \max_{i \in I_{1} \cup I_{2}} \|x_{1}\| \|g_{i}(p_{2}) - g_{i}(p_{1})\| + \|f_{i}(p_{2}) - f_{i}(p_{1})\| \\ &\leq \alpha \max_{i \in I_{1} \cup I_{2}} (\|x_{1}\| \|e_{i} + \ell_{f_{i}})\|p_{1} - p_{2}\|, \end{aligned}$$

hence \mathbb{C} is Lipschitz-like at (\bar{x}, \bar{p}) .

The following theorem is our main result.

Theorem 1. Let multifunction \mathbb{C} : $\mathcal{D} \Rightarrow \mathcal{H}$ given by (1.1) satisfy RCRCQ at $(\bar{x}, \bar{p}) \in gph\mathbb{C}$. Assume that $\bar{x} \in \liminf_{p \to \bar{p}} \mathbb{C}(p)$. Then \mathbb{C} is Lipschitz-like at (\bar{p}, \bar{x})

Proof. The proof follows immediately from Proposition 1, Proposition 2, Proposition 3. $\hfill \Box$

4. CONCLUSIONS

In this paper we used RCRCQ to investigate Lipschitz-likeness of set valued mapping \mathbb{C} given by (1.1). In many existing papers (e.g. [1, 4, 3, 5]) the continuity properties of set-valued mappings are related to the Mangasarian-Fromovitz constraint qualification MFCQ. In general, there is no direct relationship between RCRCQ and MFCQ (see [6]).

It depends upon the problem considered which of the two constraint qualifications is more useful.

5. Appendix

Lemma 1. Let $J = \{1, ..., k\}$. Let $g_i : \mathcal{G} \to \mathcal{H}$, $i \in J$ be continuous operators and let \bar{p} be such that $g_i(\bar{p})$, $i \in J$ are linearly independent. Then there exists a neighbourhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$, $g_i(p)$, $i \in J$ are linearly independent.

Proof. The fact that $g_i(\bar{p})$, $i \in J$ are linearly independent is equivalent to the fact that the Gram determinant of $g_i(\bar{p})$, $i \in J$ is nonzero (see for example [2, Lemma 7.5]), i.e.

$$\begin{aligned} \mathsf{Gram}(g_1(\bar{p}), \dots, g_k(\bar{p})) &:= \\ & \left| \begin{array}{ccc} \langle g_1(\bar{p}) \mid g_1(\bar{p}) \rangle & \langle g_1(\bar{p}) \mid g_2(\bar{p}) \rangle & \dots & \langle g_1(\bar{p}) \mid g_k(\bar{p}) \rangle \\ \langle g_2(\bar{p}) \mid g_1(\bar{p}) \rangle & \langle g_2(\bar{p}) \mid g_2(\bar{p}) \rangle & \dots & \langle g_2(\bar{p}) \mid g_k(\bar{p}) \rangle \\ & \dots & \\ \langle g_k(\bar{p}) \mid g_1(\bar{p}) \rangle & \langle g_k(\bar{p}) \mid g_2(\bar{p}) \rangle & \dots & \langle g_k(\bar{p}) \mid g_k(\bar{p}) \rangle \\ \end{aligned} \right| \neq 0. \end{aligned}$$

For any p let

$$\mathcal{F}(p) := \mathsf{Gram}(g_1(p), \dots, g_k(p)) :=$$

$$\begin{vmatrix} \langle g_1(p) \mid g_1(p) \rangle & \langle g_1(p) \mid g_2(p) \rangle & \dots & \langle g_1(p) \mid g_k(p) \rangle \\ \langle g_2(p) \mid g_1(p) \rangle & \langle g_2(p) \mid g_2(p) \rangle & \dots & \langle g_2(p) \mid g_k(p) \rangle \\ \dots \\ \langle g_k(p) \mid g_1(p) \rangle & \langle g_k(p) \mid g_2(p) \rangle & \dots & \langle g_k(p) \mid g_k(p) \rangle \end{vmatrix}$$

Since inner product is a continuous function of arguments and $\mathcal{F} : \mathcal{G} \to \mathbb{R}$ is a combination of continuous functions, there exists a neighbourhood $U(\bar{p})$ such that $\mathcal{F}(p) \neq 0$ for all $p \in U(\bar{p})$. Hence, for all $p \in U(\bar{p})$ vectors $g_i(p)$, $i \in J$ are linearly independent.

Proposition 4. Let $\bar{p} \in D$. Assume that RCRCQ holds at \bar{p} for multifunction \mathbb{C} given by (1.1) and $C(p) \neq \emptyset$ for $p \in U_0(\bar{p})$. Then there exists a neighbourhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$

$$\{ x \mid \langle x \mid g_i(p) \rangle = f_i(p), \ i \in I_1, \ \langle x \mid g_i(p) \rangle \leq f_i(p), \ i \in I_2 \}$$

=
$$\{ x \mid \langle x \mid g_i(p) \rangle = f_i(p), \ i \in I'_1, \ \langle x \mid g_i(p) \rangle \leq f_i(p), \ i \in I_2 \},$$

where $I'_1 \subset I_1$ and $g_i(p)$, $i \in I'_1$ are linearly independent.

Proof. It is enough to consider the case $g_i(\bar{p})$, $i \in I_1$ are linearly dependent. By RCRCQ there exists a neighbourhood $U_1(\bar{p})$ such that for all $p \in U(\bar{p})$

$$\operatorname{rank} \{g_i(\bar{p}), i \in I_1\} = \operatorname{rank} \{g_i(p), i \in I_1\} = \alpha$$

Let I'_1 be such that $|I'_1| = \alpha$ and $g_i(\bar{p})$, $i \in I'_1$ are linearly independent. By Lemma 1, there exists a neighbourhood $U_2(\bar{p})$ such that for all $p \in U_2(\bar{p})$, $g_i(p)$, $i \in I'_1$ are linearly independent. Let $p \in U_0(\bar{p}) \cap U_1(\bar{p}) \cap U_2(\bar{p})$ x be such that

$$\langle x \mid g_i(p) \rangle = f_i(p), \ i \in I_1, \ \langle x \mid g_i(p) \rangle \le f_i(p), \ i \in I_2.$$

$$(5.1)$$

Since rank $\{g_i(p), i \in I_1\} = |I'_1|, C(p) \neq \emptyset$ and $g_i(p), i \in I'_1$ are linearly independent we have

$$\begin{split} \langle x \mid g_i(p) \rangle &= f_i(p), \ i \in I_1 \\ \iff \langle x \mid g_i(p) \rangle &= f_i(p), \ i \in I'_1 \ \land \ \langle x \mid g_i(p) \rangle = f_i(p), \ i \in I_1 \setminus I'_1 \\ \iff \langle x \mid g_i(p) \rangle &= f_i(p), \ i \in I'_1 \ \land \ \langle x \mid \sum_{j \in I'_1} \alpha^i_j g_j(p) \rangle = f_i(p), \ i \in I_1 \setminus I'_1 \\ \iff \langle x \mid g_i(p) \rangle &= f_i(p), \ i \in I'_1 \ \land \ \sum_{j \in I'_1} \alpha^i_j \langle x \mid g_j(p) \rangle = \sum_{j \in I'_1} \alpha^i_j f_j(p), \ i \in I_1 \setminus I'_1 \\ \iff \langle x \mid g_i(p) \rangle &= f_i(p), \ i \in I'_1, \end{split}$$

where $g_i(p) = \sum_{j \in I'_1} \alpha^i_j g_j(p)$, $f_i(p) = \sum_{j \in I'_1} \alpha^i_j f_j(p)$, $i \in I_1 \setminus I'_1$ and $\alpha^i_j \in \mathbb{R}$, $j \in I'_1$, $i \in I_1 \setminus I'_1$, not all α^i_j , $j \in I'_1$ equal to zero for any $i \in I_1 \setminus I'_1$.

Lemma 2. Let $x = \sum_{i \in J_1} \lambda_i a_i + \sum_{i \in J_2} \lambda_i a_i$, $J_1 \cap J_2 = \emptyset$, J_1, J_2 finite sets, $\lambda_i \in \mathbb{R}$, $i \in J_1$, $\lambda_i \ge 0$, $i \in J_2$ and a_i , $i \in J_1 \cup J_2$ are non-zero vectors. Assume that a_i , $i \in J_1$ are linearly independent. Then there exists $J'_2 \subset J_2$ and λ'_i , $i \in J_1 \cup J'_2$, $\lambda'_i \in \mathbb{R}$, $i \in J_1$, $\lambda'_i > 0$, $i \in J'_2$ such that

$$\sum_{i \in J_1} \lambda_i a_i + \sum_{i \in J_2} \lambda_i a_i = \sum_{i \in J_1} \lambda'_i a_i + \sum_{i \in J'_2} \lambda'_i a_i$$

and a_i , $i \in J_1 \cup J'_2$ are linearly independent.

Proof. Without loss of generality we may assume that $\lambda_i > 0$, $i \in J_2$. If a_i , $i \in J_1 \cup J_2$ are linearly independent, then the assertion is obvious. Suppose that a_i , $i \in J_1 \cup J_2$ are linearly dependent. Then there exists $\hat{J}_1 \subset J_1$ and $\hat{J}_2 \subset J_2$, $\hat{J}_2 \neq \emptyset$ such that

$$\sum_{i \in \hat{J}_1} \beta_i a_i + \sum_{i \in \hat{J}_2} \beta_i a_i = 0, \quad \text{rank} \left\{ a_i, \ i \in \hat{J}_1 \cup \hat{J}_2 \right\} = |\hat{J}_1 \cup \hat{J}_2| - 1$$
(5.2)

for some $\beta_i \neq 0$, $i \in \hat{J}_1 \cup \hat{J}_2$. Then by multiplying both sides of equality equality (5.2) by $\frac{\lambda_k}{\beta_k}$, $k \in \hat{J}_2$ we get

$$\sum_{i \in \hat{J}_1} \frac{\lambda_k}{\beta_k} \beta_i a_i + \sum_{i \in \hat{J}_2} \frac{\lambda_k}{\beta_k} \beta_i a_i = 0.$$

Therefore for any $k \in \hat{J}_2$ we have

$$\begin{aligned} x &= \sum_{i \in J_1} \lambda_i a_i + \sum_{i \in J_2} \lambda_i a_i - \sum_{i \in \hat{J}_1} \frac{\lambda_k}{\beta_k} \beta_i a_i - \sum_{i \in \hat{J}_2} \frac{\lambda_k}{\beta_k} \beta_i a_i \\ &= \sum_{i \in J_1 \setminus J_1'} \lambda_i a_i + \sum_{i \in J_2 \setminus J_2'} \lambda_i a_i + \sum_{i \in \hat{J}_1} (\lambda_i - \frac{\lambda_k}{\beta_k} \beta_i) a_i + \sum_{i \in \hat{J}_2 \setminus \{k\}} (\lambda_i - \frac{\lambda_k}{\beta_k} \beta_i) a_i. \end{aligned}$$

We will show that there exists $k \in \hat{J}_2$ such that for any $i \in \hat{J}_2 \setminus \{k\}$ we have

$$\lambda_i - \frac{\lambda_k}{\beta_k} \beta_i \ge 0.$$

Suppose by contrary that for all $k \in \hat{J}_2$ there exists $i_k \in \hat{J}_2 \setminus \{k\}$ such that

$$\lambda_{i_k} < \frac{\beta_{i_k}}{\beta_k} \lambda_k.$$

Let us note that fact $\lambda_i > 0$ for all $i \in \hat{J}_2$ implies that for all $k \in \hat{J}_2$ we have

$$\frac{\beta_{i_k}}{\beta_k} > \frac{\lambda_{i_k}}{\lambda_k} > 0.$$

Then there exist real numbers $\lambda_{i_1},\ldots,\lambda_{i_q}$, where $i_1,\ldots,i_q\subset \hat{J}_2$, $q\leq |\hat{J}_2|$ and

$$\lambda_{i_1} < \frac{\beta_{i_1}}{\beta_{i_2}} \lambda_{i_2}, \ \dots, \ \lambda_{i_{q-1}} < \frac{\beta_{i_{q-1}}}{\beta_{i_q}} \lambda_{i_q}, \ \lambda_{i_q} < \frac{\beta_{i_q}}{\beta_{i_1}} \lambda_{i_1}.$$

However, this implies that

$$\lambda_{i_1} < \frac{\beta_{i_1}}{\beta_{i_2}} \lambda_{i_2} < \frac{\beta_{i_1}}{\beta_{i_2}} \frac{\beta_{i_2}}{\beta_{i_3}} \lambda_{i_3} = \frac{\beta_{i_1}}{\beta_{i_3}} \lambda_{i_3} < \dots < \frac{\beta_{i_1}}{\beta_{i_q}} \lambda_{i_q} < \frac{\beta_{i_1}}{\beta_{i_q}} \frac{\beta_{i_q}}{\beta_{i_1}} \lambda_{i_1} = \lambda_{i_1},$$

which leads to a contradiction. Hence, we can represent x as

$$x = \sum_{i \in J_1} \lambda'_i a_i + \sum_{i \in J'_2} \lambda'_i a_i$$

where $J_2' \subset J_2$, $\lambda_i' > 0$, $i \in J_2'$ and a_i , $i \in J_1 \cup J_2'$ are linearly independent.

The following theorem is particular case of [7, Theorem 11.4] applied to the problem (2.2).

Theorem 2. Let $p \in D$ and $w \notin C(p)$. Then there exist numbers λ_i , $i = I_1 \cup I_2$, not all zero, $\lambda_i \ge 0$, $i \in I_2$, $\lambda_i G_i(P_{C(p)}(w), p) = 0$, $i \in I_2$ such that

$$w - P_{C(p)}(w) = \sum_{I_1 \cup I_2} \lambda_i g_i(p).$$

Lemma 3. Let multifunction \mathbb{C} given by (1.1) satisfy RCRCQ at $\bar{p} \in \mathcal{D}$ and $\bar{x} \in C(\bar{p})$. Assume that $C(p) \neq \emptyset$ for $p \in U_0(\bar{p})$. Then there exists a neighbourhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$, $w \notin C(\bar{p})$ we have

$$w - P_{C(p)}(w) = \sum_{i \in I_1^0} \tilde{\lambda}_i(w, p) g_i(p) + \sum_{I_2^0(w, p)} \tilde{\lambda}_i(w, p) g_i(p),$$

where $\tilde{\lambda}_i(w,p) \in \mathbb{R}$, $i \in I_1^0$, $\tilde{\lambda}_i(w,p) > 0$, $i \in I_2^0(w,p)$, $I_1^0 \subset I_1$, $I_2^0(w,p) \subset I_2$, $g_i(p)$, $i \in I_1^0 \cup I_2^0(w,p)$, $p \in U(\bar{p})$ are linearly independent.

Proof. By Proposition 4, there exists a neighbourhood $U(\bar{p})$ such that for all $p \in U(\bar{p})$ we have

$$\{ x \mid \langle x \mid g_i(p) \rangle = f_i(p), \ i \in I_1, \ \langle x \mid g_i(p) \rangle \leq f_i(p), \ i \in I_2 \}$$

=
$$\{ x \mid \langle x \mid g_i(p) \rangle = f_i(p), \ i \in I_1^0, \ \langle x \mid g_i(p) \rangle \leq f_i(p), \ i \in I_2 \},$$

where $I_1^0 \subset I_1$ and $g_i(p)$, $i \in I_1^0$ are linearly independent. Take $p \in U(\bar{p})$, $w \notin C(p)$. By Theorem 2,

$$w - P_{C(p)}(w) = \sum_{i \in I_1^0} \lambda_i g_i(p) + \sum_{I_2} \lambda_i g_i(p),$$

where $\lambda_i \in \mathbb{R}$, $i \in I_1^0$ and $\lambda_i \ge 0$, $i \in I_2$. By Lemma 2, there exists $I_2^0(w, p) \subset I_2$ and $\tilde{\lambda}_i(w, p) \in \mathbb{R}$, $i \in I_1^0$ and $\tilde{\lambda}_i(w, p) > 0$, $i \in I_2^0(w, p)$ such that

$$w - P_{C(p)}(w) = \sum_{i \in I_1^0} \tilde{\lambda}_i(w, p) g_i(p) + \sum_{I_2^0(w, p)} \tilde{\lambda}_i(w, p) g_i(p) + \sum_{i \in I_1^0} \tilde{\lambda}_i(w, p) + \sum_{i \in I_1^0} \tilde{\lambda}_i$$

and $g_i(p)$, $i \in I_1^0 \cup I_2^0(w, p)$ are linearly independent.

References

- J. Frédéric Bonnans and Alexander Shapiro. Perturbation analysis of optimization problems. Springer Series in Operations Research. Springer-Verlag, New York, 2000.
- [2] Frank Deutsch. Best approximation in inner product spaces, volume 7 of CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer-Verlag, New York, 2001.
- [3] A. L. Dontchev, M. Quincampoix, and N. Zlateva. Aubin criterion for metric regularity. J. Convex Anal., 13(2):281–297, 2006.
- [4] Asen Dontchev and R Rockafellar. Implicit functions and solution mappings. A view from variational analysis. 2nd updated ed. 01 2014.
- [5] R. Henrion, A. Jourani, and J. Outrata. On the calmness of a class of multifunctions. SIAM Journal on Optimization, 13(2):603–618, 2002.
- [6] Alexander Y. Kruger, Leonid Minchenko, and Jiří V. Outrata. On relaxing the mangasarianfromovitz constraint qualification. *Positivity*, 18(1):171–189, Mar 2014.
- [7] D. Louvish and I. V. Girsanov. Lectures on Mathematical Theory of Extremum Problems. Lecture Notes in Economics and Mathematical Systems. Springer Berlin Heidelberg, 2012.
- [8] L. Minchenko and S. Stakhovski. Parametric nonlinear programming problems under the relaxed constant rank condition. SIAM Journal on Optimization, 21(1):314–332, 2011.
- [9] Leonid Minchenko and Sergey Stakhovski. On relaxed constant rank regularity condition in mathematical programming. *Optimization*, 60(4):429–440, 2011.