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# A CAHN–HILLIARD MODEL FOR CELL MOTILITY*

ALESSANDRO CUCCHI†, ANTOINE MELLET‡, AND NICOLAS MEUNIER§

**Abstract.** We introduce and study a diffuse interface model describing cell motility. We provide a detailed rigorous analysis of the model in dimension 1 and formally derive the sharp interface limit in any dimension. The model integrates the most important physical processes involved in cell motility, such as incompressibility, internal stresses exerted by the cytoskeleton seen as an active gel, and dynamic contact lines. The resulting nonlinear system couples a degenerate fourth order parabolic equation of Cahn–Hilliard type for the phase variable with a convection-reaction-diffusion equation for the active potential. The sharp interface limit leads to a Hele-Shaw type free boundary problem which includes the effects of surface tension and an additional destabilizing term at the free boundary. This additional term can be seen as a nonlinear Robin type boundary condition with the "wrong" sign. Such a boundary condition reflects the active nature of the cell, e.g., protrusion formation. We rigorously investigate the properties of this model in one dimension and prove the appearance of nontrivial traveling wave solutions for the limit problem when the key physical parameter exceeds a certain critical value. Although minimal, this new Hele-Shaw model, with Robin's unconventional boundary condition, is rich enough to describe the universal property of migrating cells that has been recently described by various theoretical biophysical models.

**Key words.** Cahn–Hilliard equations, nonlinear fourth order parabolic equations, Hele-Shaw free boundary problems, sharp interface limit, hysteresis phenomena

**AMS subject classifications.** 35Q92, 35K55, 35K65, 35R35

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## 1. Introduction, model, and results.

This paper is devoted to the analysis of the following system of equations, which we introduce in this paper as a simple model for cell motility (see section 2):

$$\begin{cases} \partial_t \rho = \operatorname{div}\left(\rho \nabla \left[\gamma\left(-\varepsilon\Delta\rho + \frac{1}{\varepsilon}W'(\rho)\right) + \phi\right]\right) , \\ \partial_t \phi - \varepsilon \Delta \phi = \frac{1}{\varepsilon}\left(\beta\rho - \phi\right) , \end{cases} \tag{1.1}$$

for $x \in \Omega \subset \mathbb{R}^n$, $t > 0$, with $\varepsilon > 0$, $\gamma > 0$, $\beta \geq 0$, and $W$ a double-well potential satisfying

$$W(0) = W(1) = 0, \quad W(\rho) > 0 \text{ if } \rho \neq 0, 1 \tag{1.2}$$

(for instance, $W(\rho) = \rho^2(1-\rho)^2$). This system will be supplemented by appropriate boundary conditions on $\partial\Omega$ and initial conditions.

System (1.1) involves a fourth order degenerate Cahn–Hilliard equation, coupled to a second order diffusion equation. We make the following simple observations:

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†Map5, UMR 8145 CNRS, Université Paris Descartes, 75006 Paris, France (alessandro.cucchi@parisdescartes.fr).
‡University of Maryland, College Park, MD 20742 (mellet@math.umd.edu).
§LaMME, UMR 8071 CNRS, Université Évry Val d'Essonne, 91025, Évry, France (nicolas.meunier@univ-evry.fr).

- When the potential $\phi$ is zero (for instance, if $\beta = 0$ and $\phi(t = 0) = 0$), then (1.1) is a classical Cahn–Hilliard equation with degenerate mobility, whose sharp interface limit ($\varepsilon$ goes to zero) is the Hele-Shaw free boundary model with surface tension (or one-phase Mullins–Sekerka free boundary problem; see [22]). For this model it is well known that the ball is a stable stationary solution.
- When the surface tension parameter $\gamma$ is zero, system (1.1) is a repulsive Keller–Segel type system (although without diffusion in the $\rho$ equation): the potential $\phi$ describes a chemorepulsion type of phenomenon (see [15]). When $\varepsilon \ll 1$, the dynamic is close to that of the porous medium equation $\partial_t \rho = \frac{1}{2}\beta\Delta\rho^2$ which does not have stationary solutions in $\mathbb{R}^n$ since the support of the solution will spread for all $t > 0$ (except possibly for an initial waiting time).

As we will explain below, these two competing mechanisms are what makes this model interesting in the context of cell motility. In particular it is well suited to describing the active character of the membrane of the cell and the formation of protrusions.

Our goal in this paper is threefolds. First, we will prove the existence of nonnegative solutions for the coupled system (1.1) in the one-dimensional case. This is nontrivial since it involves the usual difficulties in dealing with a fourth order degenerate equation (similar to the thin film equation) together with the coupling with the evolution of the potential $\phi$. Then we will consider the sharp interface limit $\varepsilon \to 0$ and formally derive a free boundary problem in which the stabilizing effect of surface tension is competing with the destabilizing effect of the chemorepulsion mechanism. More precisely, we will see that when $\varepsilon \ll 1$, the dynamic of the cell (represented here by the support $\Sigma(t)$ of $\rho$) is described by the following Hele-Shaw free boundary problem:

$$\begin{cases} -\Delta q = 0 & \text{in } \Sigma(t), \\ q = \bar{\gamma}\kappa(t) + \beta F(V) & \text{on } \partial\Sigma(t), \\ V = -\nabla q \cdot n & \text{on } \partial\Sigma(t), \end{cases} \tag{1.3}$$

where $\kappa(t)$ denotes the mean-curvature of the boundary $\partial\Sigma(t)$ (with the convention that the curvature of a sphere is positive) and $V$ denotes its normal outward velocity. Importantly, the function $F : \mathbb{R} \to \mathbb{R}$, whose definition is given in (1.15), will be proved to be a decreasing function of $V$. As a consequence, the active term $\beta F(V)$ at the boundary has a destabilizing effect which leads to hysteresis phenomena. Indeed, we can rewrite the boundary condition in (1.3) as

$$q - \beta F(-\nabla q \cdot n) = \bar{\gamma}\kappa(t),$$

which is a (nonlinear) Robin type condition with the "wrong" sign (which might lead to multiple solutions). Note in particular that this condition has the opposite effect of the (linear) Robin condition used in the so-called undercooling Hele-Shaw problem [17, 18, 20, 33, 30].

As mentioned above, system (1.1) is a simple model for cell motility. One of the most remarkable characteristics of eukaryotic cells is their ability to reach and maintain an asymmetric shape spontaneously or in response to external signals. This cellular property, called front-rear polarization, results from symmetry breaking in its internal organization and is necessary for efficient cell migration. In the last part of this paper, we will rigorously establish these properties for the free boundary problem (1.3) in one dimension, proving in particular the existence of multiple traveling wave solutions (thus including nonstationary ones) when the parameter $\beta$ is large enough.

From a modeling point of view, our system is very simple. We use only two quantities to describe the cell: the phase field (or order parameter) $\rho$, describing everything that lies inside the cell (cytoskeleton, solvent, molecular motors, etc.), and myosin II, a molecular motor that assembles in minifilaments, interacts with actin, behaves as active crosslinkers, and generates contractile or dilative stresses in the cytoskeleton network, whose concentration is denoted by $\phi$. The main assumptions that lead to (1.1) are the following: (i) the cell velocity, $v$, is given by the local actin flow, (ii) myosin II in the bulk is slowly diffusing, (iii) actin filaments undergo uniform bulk polymerization and depolymerization, (iv) the osmotic pressure involved in the network stress acts to saturate the linear instability causing gel phase separation and to smooth the interface between cytosol-rich and cytosol-poor regions. The underlying processes are friction of the cytosol on the substrate together with the active character of the myosin II. We refer to section 2 for a detailed presentation of the model with biological motivations.

Before stating our results, let us briefly comment on the existing literature. Phase-field models have been widely used in the biophysical community to describe cell motility. These models, reviewed in [23, 36], are mostly computational. In [5] a phase-field model, first introduced in [37], was mathematically studied. It consists of a second order parabolic equation for a scalar phase-field function coupled with a vectorial parabolic equation for the actin filament network polarity. The derivation of the sharp interface limit leads to a volume preserving curvature driven motion with an additional nonlinear term due to adhesion to the substrate and protrusion by the cytoskeleton. This sharp interface limit and the limiting model are rigorously studied in dimension 1 in [4, 5, 6]. Numerical simulations allow one to observe discontinuity of interface velocities and hysteresis phenomena. In [4, 6], nonstationary traveling wave solutions are rigorously obtained for the limit problem and the phase-field model. These aforementioned mathematical works deal with second order Allen–Cahn models that lead to mean-curvature flow type of free boundary problems. By comparison, we consider here a fourth order Cahn–Hilliard model and derive a Hele-Shaw type free boundary model with surface tension, which is well suited to describing cell motility (see, e.g., [36] and references therein) and has the advantage of being volume preserving without the addition of a Lagrange multiplier. However, the analysis of this fourth order degenerate equation is more delicate, even in one dimension. Sharp interface limits for such Cahn–Hilliard equations are formally studied in [32, 22]. The main novelty of our derivation is the role played by the potential $\phi$, which leads to the non-linear and destabilizing term $\beta F(V)$ in (1.3). The resulting hysteresis phenomena that we rigorously establish in dimension 1 is in good agreement with recent models used in the biophysical community [29, 9].

**1.1. Weak existence in dimension 1.** Our first result is concerned with the existence of a weak solution for the coupled system (1.1) in dimension 1. To simplify the notation, we take all parameters to be 1, so the system becomes

$$\begin{cases} \partial_t \rho = \partial_x \left( \rho \, \partial_x \left( -\partial_{xx} \rho + W'(\rho) + \phi \right) \right), & x \in \Omega, \ t > 0, \\ \partial_t \phi = \partial_{xx} \phi - \phi + \rho, & x \in \Omega, \ t > 0, \end{cases} \tag{1.4}$$

where $\Omega$ is a fixed domain in $\mathbb{R}$, i.e., an open interval of the form $\Omega = (a,b)$ with initial conditions

$$\rho(x,0) = \rho_{in}(x), \qquad \phi(x,0) = \phi_{in}(x) \quad \text{for } x \in \Omega \tag{1.5}$$

and boundary conditions

$$\partial_x \phi_{|\partial\Omega} = 0, \tag{1.6}$$

$$(\rho\,\partial_x\,(-\partial_{xx}\rho + W'(\rho) + \phi]))_{|\partial\Omega} = (\rho\,\partial_x\,(-\partial_{xx}\rho + W'(\rho)))_{|\partial\Omega} = 0, \tag{1.7}$$

$$\partial_x \rho_{|\partial\Omega} = 0, \tag{1.8}$$

Note that (1.7) is a no-flux boundary condition for $\rho$, which guarantees that the mass $\int_\Omega \rho\,dx$ is preserved.

We will prove the following theorem.

THEOREM 1.1. *Assume that the potential $W$ is a nonnegative function in $C^2(\mathbb{R})$. Then for all $T > 0$ and all nonnegative initial data $(\rho_{in}(x), \phi_{in}(x))$ satisfying*

$$\rho_{in} \in H^1(\Omega), \quad W(\rho_{in}) \in L^1(\Omega), \quad \phi_{in} \in L^2(\Omega), \tag{1.9}$$

*the system of equations (1.4)–(1.8) has a weak solution $(\rho(x,t), \phi(x,t))$ satisfying*

$$\rho \geq 0, \quad \rho \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \quad W(\rho) \in L^\infty(0,T;L^1(\Omega)),$$

$$\rho\,\partial_{xxx}\rho \in L^2(\{\rho > 0\}),$$

*and*

$$\phi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

*In particular, the first equation in (1.4) is satisfied in the following sense: for all $\varphi \in \mathcal{D}(\Omega_T)$,*

$$\iint_{\Omega_T} \rho\,\partial_t\varphi\,dx\,dt - \iint_{\{\rho>0\}} \rho\,\partial_x(-\partial_{xx}\rho + W'(\rho) + \phi)\partial_x\varphi\,dx\,dt = -\int_\Omega \rho_{in}(x)\,\varphi(x,0)\,dx,$$

*where $\Omega_T = \Omega \times [0,T)$ and $\rho$ satisfies the boundary condition $\partial_x\rho = 0$ if $\rho > 0$ on $\partial\Omega$ and the following mass conservation property:*

$$\int_\Omega \rho(x,t)\,dx = \int_\Omega \rho_{in}(x)\,dx \qquad \forall\ t > 0.$$

*The diffusion equation with Neumann boundary conditions for $\phi$ is satisfied in the usual weak formulation.*

Note that this type of weak formulation is classical for degenerate fourth order equations such as the thin film equation (see [7, 8]). Since the equation for $\rho$ is degenerate when $\rho = 0$, the first step in the proof of Theorem 1.1 is to perform a regularization procedure by introducing a uniformly parabolic equation of order 4 whose solution is $\rho^\delta$ where $\delta > 0$ is the regularization parameter. Existence of smooth solutions is well known for this kind of uniformly parabolic equation. However, because the equation is of order 4, it is a classical fact that the solution might change its sign, even though the initial condition is nonnegative. The existence of a solution $(\rho^\delta, \phi^\delta)$ to the coupled system of regularized equations is proved via a fixed point argument on the potential $\phi^\delta$. The second step consists in passing to the limit $\delta \to 0$. This requires techniques that are classical in the study of the thin film equation, as done in [7]. In particular, an entropy type inequality allows us to show that the limit $\rho$ is a nonnegative function.

Note that the uniqueness of the solution is a classical open problem for the thin film equation (i.e., when $\phi = 0$ and $W = 0$).

**1.2. Sharp interface limit in any dimension.** Next, we will formally derive the sharp interface limit $\varepsilon \ll 1$ for the system (1.1) in any dimension. Note that if we take $\beta = 0$, the system decouples and we are led to consider the following degenerate Cahn–Hilliard equation (with a given potential):

$$\partial_t \rho^\varepsilon = \operatorname{div}\left(\rho^\varepsilon \nabla \left[\gamma\left(-\varepsilon\Delta\rho^\varepsilon + \frac{1}{\varepsilon}W'(\rho^\varepsilon) + \phi\right)\right]\right). \tag{1.10}$$

We recall that for the classical Cahn–Hilliard equation with constant mobility the formal asymptotic limit $\varepsilon \to 0$ was derived by Pego in [32]. The limit leads to phase separations and the free boundary separating the two phases evolves according to a two-phase Mullins–Sekerka type free boundary problem. In the degenerate case that we consider here, a similar formal analysis was performed by Glasner in [22] for nonnegative solutions of (1.10) when $\phi = 0$. Importantly, the fact that such a fourth order parabolic equation admits nonnegative solutions is due to the degeneracy of the mobility coefficient when $\rho = 0$. In this case, the limit is described by a one-phase Mullins–Sekerka type free boundary problem, also known in dimension 2 as Hele-Shaw flow with surface tension.

In this section, we consider the slightly more general system (supplemented with the boundary conditions (2.7), (2.8), and (2.9)):

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon v^\varepsilon) = 0\,, \\ v^\varepsilon = -\nabla\left[\gamma\left(-\varepsilon\Delta\rho^\varepsilon + \frac{1}{\varepsilon}W'(\rho^\varepsilon)\right) + \phi^\varepsilon\right], \\ \partial_t\phi^\varepsilon + \alpha\operatorname{div}(\phi^\varepsilon v^\varepsilon) = \frac{1}{\varepsilon}\left(\eta^2\Delta\phi^\varepsilon + \beta\rho^\varepsilon - \phi^\varepsilon\right), \end{cases} \tag{1.11}$$

where the term $\alpha \operatorname{div}(\phi v)$ (with $\alpha \in [0,1]$) in the last equation accounts for the fact that the myosin II can be actively transported by the local actin flow. The parameter $\eta \geq 0$ allows us to characterize the role played by the diffusivity of $\phi$ in the asymptotic behavior of $\rho$ in Theorem 1.2 below. Note that the system (1.1) is a particular case of (1.11) corresponding to $\alpha = 0$ and $\eta = \varepsilon$ and its limit is the object of part (ii) in the following theorem.

THEOREM 1.2. *Assume that the potential $W$ is in $C^2(\mathbb{R})$ and satisfies* (1.2)*. Formally, the solution $\rho^\varepsilon(x,t)$ of the system* (1.11) *converges as $\varepsilon \to 0$ to $\rho^0(t) = \chi_{\Sigma(t)}$, where the evolution of $\Sigma(t)$ is described by the following Hele-Shaw type free boundary problems:*

(i) *If $\eta > 0$ is fixed, then the normal velocity $V$ of $\partial\Sigma(t)$ is determined by*

$$\begin{cases} -\Delta q = 0 & \text{in } \Sigma(t), \\ q = \bar{\gamma}\kappa(x,t) + \phi^0(x,t) & \text{on } \partial\Sigma(t), \\ V = -\nabla q \cdot n & \text{on } \partial\Sigma(t), \end{cases} \tag{1.12}$$

*where $\kappa(x,t)$ denotes the mean-curvature of the boundary $\partial\Sigma(t)$ and for all $t > 0$ the function $\phi^0(\cdot,t)$ is the solution of*

$$\phi^0 - \eta^2\Delta\phi^0 = \beta\chi_{\Sigma(t)} \qquad \text{in } \Omega, \tag{1.13}$$

*with Neumann boundary conditions on* $\partial\Omega$. *The constant* $\bar\gamma$ *is given by*

$$\bar\gamma = \gamma\sqrt{2}\int_0^1 \sqrt{W(x)}\,dx.$$

(ii) *If* $\eta = \tau\varepsilon$ *for some fixed* $\tau$, *then the normal velocity* $V$ *of* $\partial\Sigma(t)$ *is determined by*

$$\begin{cases} -\Delta q = 0 & \text{in } \Sigma(t),\\ q = \bar\gamma\kappa(t) + \beta F_\tau((1-\alpha)V) & \text{on } \partial\Sigma(t),\\ V = -\nabla q\cdot n & \text{on } \partial\Sigma(t), \end{cases} \tag{1.14}$$

*where the function* $F_\tau : \mathbb{R}\to\mathbb{R}$ *is defined below (see* (1.15)*).*

The function $F_\tau(V)$ appearing in the limiting equation (1.14) models the effects of the active potential $\phi^\varepsilon$ in the sharp interface limit when $\eta\sim\varepsilon$. Note that in that case, the function $\phi$ becomes discontinuous across the interface $\partial\Sigma(t)$ and a blow-up analysis in the neighborhood of the interface will be necessary. This function is defined as follows. First, we denote by $\psi(z)$ the unique solution of

$$\psi'(z) = \sqrt{2\,W(\,\psi(z)\,)},\qquad \lim_{z\to-\infty}\psi(z) = 0,\qquad \lim_{z\to+\infty}\psi(z) = 1.$$

This function $\psi$ describes the blow-up transition profile for the function $\rho^\varepsilon$; see section 4. Then, for any $V\in\mathbb{R}$, we set

$$F_\tau(V\,) := \int_{-\infty}^{\infty}\Phi_\tau(V,z)\,\psi'(z)\,dz, \tag{1.15}$$

where $\Phi_\tau(V,z)$ is the unique bounded solution (see Proposition 4.2) of

$$\tau^2\Phi'' - V\Phi' - \Phi + \psi = 0.$$

*Remark* 1.3. The difference between the asymptotic equations (1.12) and (1.14) shows that the limit $\varepsilon\to 0$ is very sensitive to the particular choice we make for the evolution of the potential $\phi^\varepsilon$. In both cases, the free boundary condition is of the form $q(x,t) = \bar\gamma\kappa(x,t) + h(x,t)$, where $h$ is related to $\phi^\varepsilon$ and $\rho^\varepsilon$ in the following way: formally at least (see Remark 4.3), $\phi^\varepsilon\nabla\rho^\varepsilon$ converges as $\varepsilon\to 0$ and in the sense of measure to the vector valued measure

$$-h(x,t)\,n(x,t)\mathcal{H}^{n-1}|_{\partial\Sigma(t)}.$$

In particular, models other than the ones considered here are possible. For instance, if we assume that $\phi^\varepsilon$ is given for all $t>0$ by

$$\phi^\varepsilon = \beta\rho^\varepsilon(x,t),$$

then the Dirichlet condition in (1.12) is replaced with the simpler condition:

$$q = \gamma\kappa(t) + \frac{1}{2}\beta,$$

which can be an interesting problem if $\beta$ is taken to be a function of $x$ and $t$ rather than a constant.

Note that when $\alpha = 1$ or $\beta = 0$, the model (1.14) reduces to the classical Hele-Shaw flow with surface tension. In dimension 2, the existence of solutions for this problem was proved by Chen [12] (weak solutions) and by Constantin and Pugh [16] (analytic solutions). The existence of classical solutions is proved by Escher and Simonett [19] for a large class of initial data and in any dimension. Hele-Shaw models with surface tension, coupled to diffusion equations, have also been studied in the context of tumor growth models [14, 3]. Finally, we point out that the rigorous derivation of Hele-Shaw free boundary problem from the Cahn–Hilliard equation is a notoriously difficult problem. For the uncoupled and nondegenerate problem, the convergence was first proved by Alikakos, Bates, and Chen [1] under the assumption that the limiting problem has a smooth solution. In [13], Chen proved the rigorous convergence of the Cahn–Hilliard equation to the varifold solutions of the Hele-Shaw model with weak convergence methods. Finally, Le [28] established new convergence results using the gamma convergence of gradient flows approach introduced by Sandier and Serfaty in [35]. No rigorous results are known in the degenerate case that we are considering here. Note also that the coupling with the function $\phi$ adds an additional difficulty since the gradient flow structure seems lost in that case.

**1.3. Properties of the asymptotic models (1.12) and (1.14).** The first system (1.12)–(1.13), which we derive when $\eta = \mathcal{O}(1)$, is closely related to a model for viscous ferrofluids studied by Otto in [31]. It can be written as a gradient flow for an appropriate energy function with respect to the Wasserstein distance over the manifold of characteristic functions with fixed mass. Approximated solutions can thus be constructed via a JKO type time discretization and a conditional existence result is proved in [31]. In this model, the potential $\phi$ has a destabilizing effect on the Hele-Shaw flow and the dynamics is the result of the competition between the regularizing effect of surface tension and the destabilizing effect of the potential. For small surface tension, fingering instabilities appear and Otto in [31] investigates the asymptotic behavior of approximated solutions of (1.12)–(1.13) when $\bar{\gamma} \ll \eta \ll 1$: The perimeter of $\Sigma(t)$ goes to infinity and the characteristic function $\chi_{\Sigma(t)}$ converges weakly to a function which takes value in $[0, 1]$ and whose dynamics is described by a porous media equation. In that limit, the integrity of the cell is lost, so this is not an appropriate regime in the context of cell motility. However, these fingering instabilities suggest that when $\bar{\gamma} \sim 1$ and $\beta$ is large enough, the model is well suited to describe the formation of protrusions on the cell's membrane. Further investigation of this model in dimension 2, and in particular the existence of a traveling waves solution, will be the object of some future work.

We focus now on the second model (1.14), which appears to be new, though it bears some similarities with the model introduced and studied in [4, 5, 6, 34] for cell motility. In those papers, a second order mean-curvature flow model is derived, with velocity law given by

$$V = \kappa + G_\beta(V) - \lambda(t) \tag{1.16}$$

for a function $G_\beta$ which is defined in a similar manner as our $F$. The model (1.16) is derived as the sharp interface limit of a second order Allen–Cahn equation coupled to a diffusion equation similar to our equation for $\phi$. Because such a second order equation does not preserve the volume, the model has to include a Lagrange multiplier $\lambda(t)$. By contrast, the fourth order Cahn–Hilliard equation that constitutes the starting point of our paper (as well as the limiting Hele-Shaw flow that we derive) naturally preserves the volume of the cell and does not require the introduction of a Lagrange multiplier.

Another significant difference is that the solution of our model (1.4) satisfies $\rho^\varepsilon \geq 0$—provided the initial data is nonnegative—while the solution of the Allen–Cahn equation considered in [4, 5, 6, 34] can change sign.

In the asymptotic model (1.16), the velocity $V(x,t)$ must be found at each point $x \in \partial\Sigma(t)$ by solving the algebraic equation (1.16). For some choices of potential $W$ and large enough $\beta$, it can be proved that this equation has more than one solution. This leads to hysteresis phenomena (the velocity of the cell at a given time is not uniquely determined by its asymptotic shape) and to the existence of nonstationary traveling wave like solutions, the existence of which is investigated in [5].

The asymptotic dynamics of our model (1.14) is different and more delicate to characterize than that of the mean-curvature flow (1.16) since the velocity $V$ is determined by a nonlocal (Dirichlet to Neumann type) equation set on $\partial\Sigma(t)$. Indeed, we can understand the model (1.14) by rewriting the equation for $q(x,t)$ as a Robin boundary problem (we take $\alpha = 0$ for simplicity):

$$\begin{cases} -\Delta q = 0 & \text{in } \Sigma(t), \\ q - \beta F_\tau(-\nabla q \cdot n) = \bar{\gamma}\kappa(t) & \text{on } \partial\Sigma(t), \end{cases} \tag{1.17}$$

with $V = -\nabla q \cdot n$. The analysis of this boundary value problem depends heavily on the behavior (and monotonicity) of the function $F_\tau$. We will prove in particular the following (see section 5).

PROPOSITION 1.4. *The function $V \mapsto F_\tau(V)$ is differentiable monotone decreasing and satisfies*

$$\lim_{V\to+\infty} F_\tau(V) = 0, \quad \lim_{V\to-\infty} F_\tau(V) = 1.$$

Note that a similar Robin boundary condition, but with $F_\tau(V) = V$ (or more generally with $F$ any *increasing* function), is sometimes used as a stabilizing/regularizing term in Hele-Show flow (the corresponding model is known as the Hele-Shaw model with kinetic undercooling—see [17, 18] and the references therein). The effect of $F_\tau$ in our case is opposite and thus destabilizing. This "wrong" monotonicity of $F_\tau$ leads to some interesting behaviors. In particular, it is easy to check that for a given set $\Sigma(t)$, the function $q(\cdot,t)$, solution of (1.17), is a critical point of the functional

$$\mathcal{F}(q) := \frac{1}{2}\int_{\Sigma(t)} |\nabla q|^2\,dx - \int_{\partial\Sigma(t)} \beta G\left(\beta^{-1}\left(q - \bar{\gamma}\kappa(x,t)\right)\right)\,dS(x), \tag{1.18}$$

where the function $G$ is a convex function satisfying $G' = -F_\tau^{-1}$. The fact that the functional (1.18) is the difference of two convex functionals suggests that, at least for some values of $\beta$, (1.17) might have more than one solution, thus leading, as in [5], to some interesting hysteresis phenomena.

Intuitively, this is in good agreement with the universal law for cell migration that was highlighted in [29]: for some parameters values, the effect of the potential is high enough to counterbalance the smoothing character of the curvature term and will lead to the polarization of the cell and the existence of persistent trajectories.

In this work we are interested in making this informal discussion rigorous in the one-dimensional case. Since there is no mechanism that could split a cell in one dimension, we consider solutions for which $\Sigma(t)$ is an interval $(a(t), b(t))$. Furthermore, it is easy to check that the measure of $\Sigma(t)$ is preserved by (1.14) (this is a consequence of the conservation of mass $\frac{d}{dt}\int \rho^\varepsilon\,dx = 0$). Thus, if we denote $\ell = |\Sigma(t)|$, we get

$$\Sigma(t) = (a(t), b(t)), \qquad b(t) = a(t) + \ell$$

and the normal velocity is given by $-a'(t)$ at the left end boundary point and by $a'(t)$ at the right end boundary point. We will then prove the following.

THEOREM 1.5. *There exists a critical value* $\gamma_c := \frac{-1}{2F'_\tau(0)} > 0$ *such that if* $\frac{\beta(1-\alpha)}{\ell} \leq \gamma_c$*, then the unique solution of* (1.14) *in dimension* 1 *is the stationary solution*

$$\Sigma(t) = (a(0), b(0)).$$

*If* $\frac{\beta(1-\alpha)}{\ell} > \gamma_c$*, then* (1.14) *has at least two solutions besides the stationary solution (still in dimension* 1*), which moves with speed* $\pm\frac{1}{1-\alpha}c$ *for some speed* $c > 0$ *which depends on the double-well potential* $W$ *and on the parameter* $\frac{\beta(1-\alpha)}{\ell}$*.*

This theorem proves that, at least in one dimension, our asymptotic model has a nontrivial dynamics and exhibits hysteresis phenomena when $\frac{\beta(1-\alpha)}{\ell} > \gamma_c$. Indeed, (1.14) does not provide any mechanisms to pick one solution rather than another. Presumably this means that small variations in the function $\rho^\varepsilon_{in}(x)$ (which converges to $\chi_{\Sigma(0)}$) could lead to radically different behavior of the cell (stationary solution versus moving traveling wave). Note also that there is nothing that would prevent a solution from changing velocity in a discontinuous way (for example, a solution that moves with positive speed could suddenly stop). This indicate an unstable process in which a small variation in the media can cause a stationary cell to suddenly start moving, or a moving cell to change direction. Such behaviors are precisely what is observed experimentally.

Finally, we point out that numerical simulations show that nonstationary traveling like solutions exists also for the $\varepsilon$ model (1.1) when $\beta$ is large enough. A rigorous proof of this fact as well as a detailed analysis of the model in two dimensions will be the object of future work.

### 1.4. Outline of the work.

The rest of the paper is organized as follows. Section 2 describes in further details the biological hypothesis that lead to our model. Section 3 is devoted to the proof of the existence of solutions in the one-dimensional case (Theorem 1.1). In section 4, the sharp interface limit is formally derived (Theorem 1.2). The properties of the function $F_\tau$, and in particular Proposition 1.4, are established in section 5, and a rigorous analysis of the asymptotic problem in dimension 1 and the proof of Theorem 1.5 is given in section 6.

## 2. Model description.

Cell motility is involved in key physiological processes such as wound healing, morphogenesis, and immunological response. In recent decades, research in cell biology has made spectacular progress, which has identified many of the molecular protagonists involved. In particular, the actin cytoskeleton, composed of actin filaments organized into bundles and networks, has been shown to be an essential element of the motility machinery. Actin filaments continuously polymerize at their "plus" end near the cell membrane and depolymerize at their "minus" end within the cell. This polar behavior can give rise to spontaneous flows. In addition, molecular motors, such as myosin II, assemble into minifilaments, interact with actin, behave as active crosslinking agents, and generate contractile or dilative stresses in the network. Finally, this activity in the cytoskeleton occurs continuously thanks to a constant source of energy input, via the hydrolysis of ATP, and it leads to nonequilibrium behavior likely to generate instabilities. The resulting system is intrinsically out of equilibrium, designated as the active system. The description of this active system has attracted much attention in the physics community. The desire to construct a minimal model of cellular motility justifies a macroscopic description of the actin cytoskeleton. It has been the focus of active gel theory [25, 24, 26], a hydrodynamic approach providing a framework for the quantitative understanding of cellular motility [9].

The motility of eukaryotic cells is closely related to the maintenance of functional asymmetry. This depends on the polymerization and depolymerization of the actin filaments and the active stresses in the actin network. It has been shown that in the presence of significant friction with the solid substrate, the dynamics of the actin gel can be approximated by a two-dimensional flow [26, 9]. Here, to describe the motility of actin-based crawling cells, we consider a two-component two-dimensional fluid bounded by a membrane of arbitrary shape. We focus on the description of cytosol, actin, and myosin II.

We use the term cytosol to describe both fluid and gel fractions of the cytoplasm. Indeed, in recent years, it has become clear that the coupling between the cytoskeleton and cytosol plays an important role in many cellular mechanical phenomena [11, 9, 27]. The result is an interdependent dynamics that we describe now.

### 2.1. Cytosol description.

*Cytosol mass balance.* To develop a phase-field model, we begin by introducing a so-called phase-field variable $\rho$ that will describe the cytosol. The phase-field variable acts as a marker that will be almost constant (in our case 0 or 1) in the bulk regions and will smoothly transition between these values in an interfacial region of small thickness.

Recall that here cytosol designates everything that makes up a cell: solvent, actin polymers forming the filaments, actin monomers, nucleus, etc. The conservation of mass is represented by the continuity equation

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad \text{in } \Omega\,, \tag{2.1}$$

where $\Omega$ is a two-dimensional bounded domain, a drop, representing the environment in which the cell evolves (a laboratory, e.g.).

We assume that the velocity of the drop is given by the actin flow velocity. Indeed it is well known that the result of the depolymerization of the actin filaments which is isotropically distributed in the bulk and of the polymerization which occurs at the boundary gives rise to a flow of actin that is directed toward the center in the cell. Hence, in (2.1), the phase field can be thought of as a scalar that is convected by the actin flow $u$.

*Forces on the gel describing the cytosol.* Neglecting the dynamics of the actin polarization field [11], the cytosol can be described as an incompressible isotropic viscous fluid. Since the flows involved in cell motility occur at low Reynolds numbers, we neglect inertia and assume that the cytosol is at mechanical equilibrium. The forces acting on the cytosol are as follows: a force due to stresses in the actin network, the actin filaments-solvent friction, and the actin filaments-substrate friction. Neglecting actin filaments-solvent friction force, which is much smaller than the polymer-substrate force (see [11]), and also the exchange of momentum between the actin filaments and the solvent, the stress balance on the cytosol reads

$$-\operatorname{div}\,(\sigma - \Pi \ \mathrm{Id}) = -\xi u \qquad \text{in } \Omega\,, \tag{2.2}$$

where $\xi$ is the friction coefficient on the substrate, $\sigma$ is the Cauchy stress tensor describing the cytosol stress, and $\Pi$ is the osmotic pressure of the cytosol.

*Remark* 2.1. We note that in (2.2) the actin filaments-substrate friction force is written as $-\xi u$, where $u$ is the actin velocity. A more realistic form would be to account for the dependence of the friction force on actin filaments volume fraction. Our choice for such a force in (2.2) corresponds to a linearization and an approximation of the actin filaments volume fraction by a constant value.

The cytosol stress $\sigma$ contains passive and active contributions. Indeed, molecular motors are able to transmit stresses. Here, we consider the liquid limit of the gel, valid on large timescales. Within this limit, the passive part of the stress, resulting from the convection of the actin filaments and the remodeling by crosslinking, has the viscous form $\eta\left(\nabla u + \nabla u^T\right)/2$, where $\eta$ is the viscosity. However, this contribution is known to be very weak [11]. It does not qualitatively affect the flow and we omit it. Here, in the spirit of [9], we consider the limit when the coefficient of friction $\xi$ is strong and we neglect the viscosity $\eta$. The active part, on the other hand, is essential for motility. Since the active stress resulting from the motor activity on the filaments increases with the presence of myosin motors, a simple choice for the network stress is

$$\sigma = -\phi \mathbb{I}\,, \tag{2.3}$$

where $\phi$ is the myosin concentration and $\mathbb{I}$ is the identity tensor. We consider negative values of the activity coefficient as it corresponds to extensile behavior of myosin.

Let us now focus on the description of the osmotic pressure which acts to saturate the linear instability causing gel phase separation and to smooth the interface between cytosol-rich and cytosol-poor regions. A simple, phenomenological form for $\Pi$ is

$$\Pi = \gamma\left(-\varepsilon\Delta\rho + \frac{1}{\varepsilon}W'(\rho)\right)\,, \tag{2.4}$$

where $\gamma$ is a positive coefficient and $W$ is a double-well potential with minima at 1 and 0. Finally, combining (2.3) and (2.4) we obtain the equation for $\rho$

$$\partial_t \rho = \frac{1}{\xi}\mathrm{div}\left(\rho\nabla\left(\gamma\left(-\varepsilon\Delta\rho + \frac{1}{\varepsilon}W'(\rho)\right) + \phi\right)\right)\,. \tag{2.5}$$

**2.2. Myosin description.** The second module in our model is a convection-reaction-diffusion equation for the myosin concentration $\phi$, which relies on several assumptions that we describe now. Myosin motors are known to interact with actin filaments, hence we assume rapid adsorption of myosin on the adhered actin cytoskeleton. Therefore, on one hand, the effective myosin velocity is given by $v = \alpha u$ where $\alpha > 0$ is the quasi-static fraction of adsorbed molecules convected by the local actin flow $u$. On the other hand, let us now turn to actin description. Assuming that the diffusion of the free monomers is sufficiently rapid so that we may consider their concentration to be fixed at the cytosol value, we consider that the actin filaments undergo a uniform bulk polymerization with the rate $k_p\rho$. In addition, we assume bulk depolymerization with the rate $k_d$. Since myosin motors interact with actin filaments, the myosin creation and death rates are related to $k_p\rho$ and $k_d$.

To describe the random events in the myosin dynamics, we include a diffusion coefficient that is very small. Finally we assume that the actin (and hence also myosin) creation and death rates are very fast in comparison to both the convection and the diffusion rates, meaning that the equation satisfied by the myosin concentration is

$$\partial_t \phi + \alpha\,\mathrm{div}\,(\phi u) = \varepsilon\Delta\phi + \frac{1}{\varepsilon}\left(\beta\rho - \phi\right) \qquad \text{in } \Omega\,, \tag{2.6}$$

where $\varepsilon > 0$ is a small parameter whose inverse is related to a relaxation time and $\beta > 0$ corresponds to a polymerization rate and where $u$ is given by (2.2). It is to be noticed that for simplicity we only consider one parameter $\beta$, the other being fixed to 1, for the myosin creation and death terms.

**2.3. Boundary conditions.** Equations (2.5)–(2.6) form our model for cell motility. They are set in the domain $\Omega \subset \mathbb{R}^2$ and must thus be supplemented with boundary conditions. Because of the divergence form, (2.5) preserves the mass $\int_\Omega \rho\, dx$ provided it is supplemented with no-flux boundary conditions

$$\rho\, \partial_n \left( \gamma \left( -\varepsilon \Delta \rho + \frac{1}{\varepsilon} W'(\rho) \right) + \phi \right) = 0 \qquad \text{on } \partial\Omega, \tag{2.7}$$

where $n$ is the unit normal outward vector. Morover, since it is a parabolic equation of order 4, we need one more boundary condition, and we impose the following Neumann boundary conditions:

$$\partial_n \rho = 0 \qquad \text{on } \partial\Omega. \tag{2.8}$$

The equation (2.6) for $\phi$ is supplemented with Neumann boundary conditions:

$$\partial_n \phi = 0 \qquad \text{on } \partial\Omega. \tag{2.9}$$

**3. Existence of solution for (1.4): Proof of Theorem 1.1.** In this section, we drop the parameters $\varepsilon$, $\beta$, and $\gamma$ from our equations to simplify the notation throughout the proof. The system (1.4) becomes

$$\begin{cases} \partial_t \rho = \partial_x \left( \rho \partial_x \left[ -\partial_{xx} \rho + W'(\rho) + \phi \right] \right), & x \in \Omega,\ t > 0, \\ \partial_t \phi = \partial_{xx} \phi - \phi + \rho, & x \in \Omega,\ t > 0, \end{cases} \tag{3.1}$$

where $\Omega$ is a fixed domain in $\mathbb{R}$ (i.e., open interval of the form $\Omega = (a, b)$) and we recall that $W$ is a nonnegative function in $W^{2,\infty}_{loc}$. The system is supplemented with initial and boundary conditions (1.5), (1.6)–(1.8).

**3.1. Sketch of the proof of Theorem 1.1.** Since the equation for $\rho$ is degenerate when $\rho = 0$, we perform a classical regularization procedure by introducing the positive mobility coefficient

$$f_{\delta,M}(\rho^\delta) = \min\{M, \delta + |\rho^\delta|\}.$$

Our first task will then be to show that for all $0 < \delta \le M < \infty$, there exist $\rho^\delta$ and $\phi^\delta$ solutions of

$$\begin{cases} \partial_t \rho^\delta = \partial_x \left( f_{\delta,M}(\rho^\delta) \partial_x q^\delta \right) & \text{in } \Omega \times (0,T), \\ q^\delta = -\partial_{xx} \rho^\delta + W'(\rho^\delta) + \phi^\delta & \text{in } \Omega \times (0,T), \\ f_{\delta,M}(\rho^\delta) \partial_x q^\delta = 0 & \text{on } \partial\Omega \times (0,T), \\ \partial_x \rho^\delta(x) = 0 & \text{on } \partial\Omega \times (0,T), \\ \rho(x,0) = \rho_{in}(x) & \text{in } \Omega \end{cases} \tag{3.2}$$

and

$$\begin{cases} \partial_t \phi^\delta = \partial_{xx} \phi^\delta - \phi^\delta + \rho^\delta & \text{in } \Omega \times (0,T), \\ \partial_x \phi^\delta = 0 & \text{on } \partial\Omega \times (0,T), \\ \phi^\delta(x,0) = \phi^\delta_{in}(x) & \text{in } \Omega. \end{cases} \tag{3.3}$$

Note that we have regularized the initial data for the potential $\phi$. We define $\phi_{in}^{\delta} = j_\delta \star \bar{\phi}_{in}$, where $j_\delta$ is the usual sequence of mollifiers and $\bar{\phi}_{in}$ is the extension of $\phi_{in}$ to $\mathbb{R}$ by zero. We then have $\phi_{in}^{\delta} \in H^1(\Omega)$ for all $\delta > 0$, $\|\phi_{in}^{\delta}\|_{L^2(\Omega)} \leq \|\phi_{in}\|_{L^2(\Omega)}$, and

$$\phi_{in}^{\delta} \to \phi_{in} \text{ in } L^2(\Omega) \text{ as } \delta \to 0.$$

We are going to prove the existence of a solution to the coupled system of equations (3.2)–(3.3) by a fixed point argument on the potential $\phi^\delta$. The proof of Theorem 1.1 then consists in passing to the limit $\delta \to 0$. We will prove in particular that $\rho^\delta(x,t)$ converges uniformly to a function $\rho(x,t)$ which satisfies

$$0 \leq \rho(x,t) \leq C_0$$

for some constant $C_0$ depending on $\Omega$, $\rho_{in}$, $\phi_{in}$, and $T$, but independent of $M$. By choosing $M \geq C_0$, we will deduce that $f_{\delta,M}(\rho^\delta) \to \rho$ and that $\rho$ satisfies (3.1).

The fixed point argument is relatively classical and uses appropriate energy estimates for (3.2) and (3.3) which are detailed below. We note that the regularization of the mobility coefficient in (3.2) makes the equation uniformly parabolic and provides the existence of smooth solutions. However, because the equation is of order 4, it is a classical fact that the solution might take negative values even though we can then prove that the limit $\rho$ is a nonnegative function.

**3.2. A priori estimates.** The null flux boundary condition ensures that smooth solutions to (3.2) satisfy

$$\int_\Omega \rho^\delta(x,t)\,dx = \int_\Omega \rho_{in}(x)\,dx. \tag{3.4}$$

Furthermore, if we define the usual Cahn–Hilliard energy

$$E\left[\rho^\delta\right] := \int_\Omega \left(\frac{1}{2}|\partial_x \rho^\delta|^2 + W\left(\rho^\delta\right)\right) dx, \tag{3.5}$$

we get (still for smooth solutions)

$$\frac{dE\left[\rho^\delta(\cdot,t)\right]}{dt} + \frac{1}{2}\int_\Omega f_{\delta,M}(\rho^\delta)\left|\partial_x[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]\right|^2 dx \leq \frac{1}{2}\int_\Omega f_{\delta,M}(\rho^\delta)|\partial_x\phi^\delta|^2\,dx. \tag{3.6}$$

Indeed, multiplying the first equation of (3.2) by $[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]$, we get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \left(\frac{1}{2}|\partial_x\rho^\delta|^2 + W(\rho^\delta)\right) dx &= \int_\Omega \left[-\partial_{xx}\rho^\delta + W'(\rho^\delta)\right]\partial_t\rho^\delta\,dx \\
&= -\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]\,|^2\,dx \\
&\quad - \int_\Omega f_{\delta,M}(\rho^\delta)\,\partial_x[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]\,\partial_x\phi^\delta\,dx \\
&\leq -\frac{1}{2}\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]\,|^2\,dx \\
&\quad + \frac{1}{2}\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x\phi^\delta|^2\,dx.
\end{aligned}$$

The mass conservation (3.4) and the energy inequality (3.6), together with classical estimates for the parabolic equation (3.3), play a crucial role in what follows.

**3.3. Solution for $\delta > 0$: Fixed point argument.** As explained above, the first step is to prove the existence of a solution for the regularized system (3.2)–(3.3). More precisely, we will prove the following.

Proposition 3.1. *Assume that $\rho_{in}$ satisfies* (1.9) *and that $\phi_{in}^{\delta} \in H^1(\Omega)$. Then, for all $\delta \in (0, M)$, there exists a solution $(\rho^\delta, \phi^\delta) \in (L^\infty(0,T;H^1(\Omega)))^2$ to the coupled system of equations* (3.2)–(3.3) *which satisfies the a priori estimates* (3.4) *and* (3.6)*.*

The proof of this result relies on a fixed point argument: Given $\phi \in L^2(0,T;H^1(\Omega))$, we consider the function $\rho \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega))$ a weak solution to

$$
(3.7) \qquad \begin{cases} \partial_t \rho = \partial_x \left( f_{\delta,M}(\rho) \partial_x \left[ -\partial_{xx}\rho + W'(\rho) + \phi \right] \right) & \text{in } \Omega \times (0,T), \\ f_{\delta,M}(\rho) \partial_x \left[ -\partial_{xx}\rho + W'(\rho) + \phi \right] = 0 & \text{on } \partial\Omega \times (0,T), \\ \partial_x \rho = 0 & \text{on } \partial\Omega \times (0,T), \\ \rho(x,0) = \rho_{in}(x) & \text{in } \Omega. \end{cases}
$$

We then define $\widetilde{\phi}$ the solution of

$$
(3.8) \qquad \begin{cases} \partial_t \widetilde{\phi} = \partial_{xx}\widetilde{\phi} - \widetilde{\phi} + \rho & \text{in } \Omega \times (0,T), \\ \partial_x \widetilde{\phi} = 0 & \text{on } \partial\Omega \times (0,T), \\ \widetilde{\phi}(x,0) = \phi_{in}^{\delta}(x) & \text{in } \Omega. \end{cases}
$$

Introducing the operator

$$
(3.9) \qquad \begin{aligned} \mathcal{T} : L^2(0,T;H^1(\Omega)) &\to L^2(0,T;H^1(\Omega)) \\ \phi &\to \widetilde{\phi}, \end{aligned}
$$

we see that any fixed point $\mathcal{T}(\phi) = \phi$ will provide a solution to (3.2)–(3.3).

For a given function $\rho \in L^\infty(0,T;H^1(\Omega))$, the existence of a unique solution $\widetilde{\phi}$ to (3.8) is classical. Moreover $\widetilde{\phi}$ satisfies

$$
(3.10) \qquad \frac{d}{dt} \int_\Omega \frac{1}{2} |\widetilde{\phi}|^2 \, dx + \int_\Omega |\widetilde{\phi}|^2 + |\partial_x \widetilde{\phi}|^2 \, dx = \int_\Omega \rho \widetilde{\phi} \, dx
$$

and

$$
(3.11) \qquad \frac{d}{dt} \int_\Omega \frac{1}{2} |\partial_x \widetilde{\phi}|^2 \, dx + \int_\Omega |\partial_x \widetilde{\phi}|^2 + |\partial_{xx} \widetilde{\phi}|^2 \, dx = - \int_\Omega \rho \partial_{xx} \widetilde{\phi} \, dx.
$$

The following result is classical and it justifies the existence of $\rho$ (given the function $\phi$) and hence the construction of the operator $\mathcal{T}$.

Proposition 3.2. *Under the conditions of Proposition* 3.1*, and for all function $\phi \in L^2(0,T;H^1(\Omega))$, there exists a unique solution*

$$
\rho \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^3(\Omega))
$$

*to* (3.7) *and it satisfies* (3.4) *and* (3.6)*.*

Note that the first boundary condition in (3.7) is satisfied in a weak sense, while the second condition holds in the classical sense.

Proposition 3.1 is now a consequence of the following two lemmas.

Lemma 3.3. *The operator $\mathcal{T}$ defined by* (3.9) *is compact in $L^2(0,T,H^1(\Omega))$.*

LEMMA 3.4. *There exists a constant $C$ depending on $\Omega$, $\rho_{in}$, $\phi_{in}$, $T$, and $M$ (but not on $\delta$) such that for all $\phi \in L^2(0,T;H^1(\Omega))$ satisfying $\phi = \sigma\mathcal{T}(\phi)$ for some $\sigma \in [0,1]$, we have*

$$\|\phi\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{3.12}$$

Before proving these two lemmas, we point out that they imply Proposition 3.1.

*Proof of Proposition* 3.1. In view of Lemmas 3.3 and 3.4 we can apply the Leray–Schauder fixed point theorem (see [21]). We deduce that $\mathcal{T}$ has a fixed point in $L^2(0,T,H^1(\Omega))$. This fixed point is a solution to (3.2)–(3.3). ◻

*Remark* 3.5. Note that this fixed point satisfies the bound (3.12) where the constant $C$ does not depend on $\delta$ but only on $\|\rho_{in}\|_{L^1(\Omega)}$, $E[\rho_{in}]$. This bound will thus be useful in the next part of the proof when passing to the limit $\delta \to 0$.

*Proof of Lemma* 3.3. Inequality (3.6) and the definition of $f_{\delta,M}$ imply in particular that

$$\begin{aligned}\sup_{t\in[0,T]} E[\rho(\cdot,t)] &\leq E[\rho_{in}] + \frac{1}{2}\int_0^T\int_\Omega f_{\delta,M}(\rho)|\partial_x\phi|^2\,dx\,dt\\ &\leq E[\rho_{in}] + \frac{M}{2}\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt\end{aligned}$$

and a classical Poincaré type inequality together with (3.4) implies

$$\begin{aligned}\|\rho(\cdot,t)\|_{L^2(\Omega)} &\leq C(\Omega)\left(\int_\Omega \rho(x,t)\,dx + \left(\int_\Omega |\partial_x\rho(x,t)|^2\,dx\right)^{1/2}\right)\\ &\leq C(\Omega)\left(\int_\Omega \rho_{in}(x)\,dx + E[\rho(\cdot,t)]^{1/2}\right).\end{aligned}$$

We deduce

$$\|\rho\|^2_{L^\infty(0,T;H^1(\Omega))} \leq C(\Omega)\left(\|\rho_{in}\|_{L^1(\Omega)} + E[\rho_{in}] + M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt\right).$$

This bound, together with the inequality (3.10), yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{2}|\widetilde{\phi}|^2\,dx + \frac{1}{2}\int_\Omega |\widetilde{\phi}|^2 + |\partial_x\widetilde{\phi}|^2\,dx &\leq \int_\Omega |\rho|^2\,dx\\ &\leq C(\Omega,\rho_{in}) + C(\Omega)M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt,\end{aligned}$$

and so

$$\begin{aligned}\|\widetilde{\phi}\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\widetilde{\phi}\|^2_{L^2(0,T;H^1(\Omega))} &\leq \|\phi^\delta_{in}\|^2_{L^2(\Omega)}\\ &\quad + \left(C(\Omega,\rho_{in}) + C(\Omega)M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt\right)T.\end{aligned}$$

Similarly, (3.11) yields

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\partial_x\widetilde{\phi}|^2\,dx + \frac{1}{2}\int_\Omega |\partial_x\widetilde{\phi}|^2 + |\partial_{xx}\widetilde{\phi}|^2\,dx \le \int_\Omega |\rho|^2\,dx$$
$$\le C(\Omega,\rho_{in}) + C(\Omega)M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt,$$

and so

$$\sup_{t\in(0,T)}\int_\Omega \frac{1}{2}|\partial_x\widetilde{\phi}|^2\,dx + \frac{1}{2}\int_0^T\int_\Omega |\partial_x\widetilde{\phi}|^2 + |\partial_{xx}\widetilde{\phi}|^2\,dx\,dt$$
$$\le \|\phi_{in}^\delta\|^2_{H^1(\Omega)} + \left(C(\Omega,\rho_{in}) + C(\Omega)M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt\right)T.$$

We have thus proved that

$$\|\widetilde{\phi}\|^2_{L^2(0,T;H^2(\Omega))} \le \|\phi_{in}^\delta\|^2_{H^1(\Omega)} + \left(C(\Omega,\rho_{in}) + C(\Omega)M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt\right)T. \tag{3.13}$$

Using now (3.8), we also deduce

$$\|\partial_t\widetilde{\phi}\|^2_{L^2(0,T;L^2(\Omega))} \le \|\widetilde{\phi}\|^2_{L^2(0,T,H^2(\Omega))} + \|\rho\|^2_{L^2(0,T;L^2(\Omega))}$$
$$\le C(\phi_{in}^\delta) + \left(C(\Omega,\rho_{in}) + C(\Omega)M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt\right)T. \tag{3.14}$$

In particular, (3.13) and (3.14) imply that if $\phi$ belongs to a bounded subset of $L^2(0,T,H^1(\Omega))$, then $\widetilde{\phi}$ is such that

$$\|\widetilde{\phi}\|_{L^2(0,T;H^2(\Omega)} \le C \qquad \text{and} \qquad \|\partial_t\widetilde{\phi}\|_{L^2(0,T;L^2(\Omega))} \le C.$$

By the Aubin–Lions lemma, [2], it follows that $\mathcal{T}(\phi) = \widetilde{\phi}$ belongs to a compact subset of $L^2(0,T,H^1(\Omega))$, which completes the proof of Lemma 3.3. ∎

*Proof of Lemma* 3.4. Let $\phi \in L^2(0,T;H^1(\Omega))$ satisfy $\phi = \sigma\mathcal{T}(\phi)$ for some $\sigma \in [0,1]$ and let $\rho$ be the corresponding solution to (3.7). Then the function $\phi$ satisfies

$$\partial_t\phi = \partial_{xx}\phi - \phi + \sigma\rho \qquad \text{in } \Omega\times(0,T).$$

In particular we have

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\phi|^2\,dx + \int_\Omega |\phi|^2 + |\partial_x\phi|^2\,dx = \sigma\int_\Omega \rho\phi\,dx,$$

and since $\sigma\in[0,1]$ it follows that

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\phi|^2\,dx + \frac{1}{2}\int_\Omega |\phi|^2 + |\partial_x\phi|^2\,dx \le \int_\Omega \rho^2\,dx.$$

Recalling (3.6) we see that

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\partial_x\rho|^2 + W(\rho)\,dx \le M\int_\Omega |\partial_x\phi|^2\,dx.$$

Combining the two last inequalities (after multiplying the first one by $M$), we deduce

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\partial_x\rho|^2 + W(\rho) + 2M|\phi|^2\,dx + M\int_\Omega |\phi|^2 + |\partial_x\phi|^2\,dx$$
$$\leq 4M\int_\Omega \rho^2\,dx$$
$$\leq C(M,\rho_{in})\left(1+\int_\Omega |\partial_x\rho|^2\,dx\right),$$

and Gronwall's lemma yields that

$$\sup_{t\in(0,T)}\int_\Omega \frac{1}{2}|\partial_x\rho|^2 + W(\rho) + 2M|\phi|^2\,dx \leq Ce^{CT}$$

for some constant $C$ depending on $\Omega$, $M$, $\rho_{in}$, and $\|\phi^\delta_{in}\|_{L^2(\Omega)}$. Since the $H^1$ regularization of $\phi_{in}$ was chosen in order to satisfy $\|\phi^\delta_{in}\|_{L^2(\Omega)} \leq \|\phi_{in}\|_{L^2(\Omega)}$, the constant $C$ only depends on $\Omega$, $M$, $\rho_{in}$, and $\phi_{in}$. In particular it is independent of $\delta$.

In turn this gives

$$\int_0^T\int_\Omega |\phi|^2 + |\partial_x\phi|^2\,dx\,dt \leq CTe^{CT}$$

for some (other) constant $C$ also depending on $\Omega$, $M$, $\rho_{in}$, and $\phi_{in}$. ◻

**3.4. Limit $\delta\to 0$.** We now want to pass to the limit $\delta \to 0$. We proceed as follows:

1. First, using an energy type inequality, we will prove that $\rho^\delta$ is bounded in $C^{1/2,1/8}(\Omega\times(0,T))$ and thus converges uniformly (up to a subsequence) to a function $\rho(x,t)$ (and $\phi^\delta$ converges toward $\phi$ weakly in $L^2(0,T;H^1(\Omega))$.
2. Using an entropy type inequality, we will show that the limit function $\rho$ is nonnegative. As a consequence, we will see that it satisfies a $L^\infty$ bound independent of $M$.
3. Finally, we will pass to the limit in (3.2) and prove that $\rho$ satisfies the limiting equation in an appropriate weak form (the fact that $\phi$ satisfies the limiting equation is obvious).

*A priori bounds and convergence of* $(\rho^\delta,\phi^\delta)$. Recalling Remark 3.5 we see that the fixed point that we have constructed is such that

$$\|\phi^\delta\|_{L^2(0,T;H^1(\Omega))} \leq C(\Omega,\rho_{in},\phi_{in},M,T) \tag{3.15}$$

with the constant independent on $\delta$. Furthermore, integrating (3.6), we find

$$\sup_{t\in(0,T]} E[\rho^\delta(\cdot,t)] + \frac{1}{2}\int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)\left|\partial_x[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]\right|^2\,dx\,dt$$
$$\leq E[\rho_{in}] + \frac{1}{2}\int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)|\partial_x\phi^\delta|^2\,dx\,dt$$
$$\leq E[\rho_{in}] + \frac{M}{2}\int_0^T\int_\Omega |\partial_x\phi^\delta|^2\,dx\,dt.$$

Hence using (3.15), we obtain

$$\sup_{t\in(0,T]} E[\rho^\delta(\cdot,t)] + \frac{1}{2}\int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)\left|\partial_x[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]\right|^2\,dx\,dt \tag{3.16}$$
$$\leq C(\Omega,\rho_{in},\phi_{in},M,T)$$

with the constant independent on $\delta$. Finally, using (3.4) and the definition of $E$ together with the Poincaré inequality, inequality (3.16) implies

$$\|\rho^\delta\|^2_{L^\infty(0,T,H^1(\Omega))} \le C\|\rho_{in}\|^2_{L^1(\Omega)} + \sup_{t\in(0,T]} E[\rho^\delta(\cdot,t)] \le C(\Omega,\rho_{in},\phi_{in},M,T). \tag{3.17}$$

Classical Sobolev embeddings then yield that $\rho^\delta$ is Hölder continuous:

$$|\rho^\delta(x,t)-\rho^\delta(y,t)| \le C\,|x-y|^{1/2} \qquad \forall (x,y)\in\Omega^2, \quad \text{a.e. } t\in(0,T), \tag{3.18}$$

where $C$ is a constant independent of $\delta$.

We can also check that the flux $f_{\delta,M}(\rho^\delta)\,\partial_x q^\delta$ in (3.2) is bounded in $L^2((0,T)\times\Omega)$. Indeed, we have

$$\begin{aligned}
\int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x q^\delta|^2\,dx\,dt &\le \int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x[\partial_{xx}\rho^\delta - W'(\rho^\delta)]\,|^2\,dx\,dt \\
&\quad + \int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x\phi|^2\,dx\,dt \\
&\le \int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x[\partial_{xx}\rho^\delta - W'(\rho^\delta)]\,|^2\,dx\,dt \\
&\quad + M\int_0^T\int_\Omega |\partial_x\phi|^2\,dx\,dt \\
&\le C(\Omega,\rho_{in},\phi_{in},M,T),
\end{aligned}$$

where we have used (3.15) and (3.16). Consequently, one has

$$\int_0^T\!\!\int_\Omega f_{\delta,M}(\rho^\delta)^2|\partial_x q^\delta|^2\,dx\,dt \le M\int_0^T\!\!\int_\Omega f_{\delta,M}(\rho^\delta)\,|\partial_x q^\delta|^2\,dx\,dt \le C(\Omega,\rho_{in},\phi_{in},M,T). \tag{3.19}$$

Classically, the Hölder regularity of $\rho^\delta$ with respect to $x$ and the fact that the flux is bounded in $L^2$ yield some Hölder regularity with respect to $t$ as stated in the following lemma; see [7].

Lemma 3.6. *There exists a constant $C$ independent by $\delta$ such that*

$$|\rho^\delta(x,t)-\rho^\delta(y,s)| \le C\left(|x-y|^{1/2} + |t-s|^{1/8}\right)$$

*for all $(x,y)\in\Omega^2$ and $(t,s)\in[0,T)^2$.*

In particular, the sequence $\{\rho^\delta\}_{\delta>0}$ is uniformly bounded and equicontinuous in $\Omega\times[0,T)$. By the Ascoli–Arzela theorem, up to a subsequence, there exists a function $\rho$ such that

$$\rho^\delta \to \rho \quad \text{uniformly in } \Omega\times[0,T) \quad \text{as } \delta\to 0. \tag{3.20}$$

In view of (3.15), we can also choose the subsequence so that

$$\phi^\delta \rightharpoonup \phi \qquad \text{weakly in } L^2(0,T;H^1(\Omega)).$$

Furthermore, passing to the limit in (3.4), (3.15), and (3.16), we get

$$\int_\Omega \rho(x,t)\,dx = \int_\Omega \rho_{in}(x)\,dx, \tag{3.21}$$

$$\|\phi\|_{L^2(0,T;H^1(\Omega))} \le C(\Omega, \rho_{in}, \phi_{in}, M, T), \tag{3.22}$$

and

$$\sup_{t\in(0,T]} E[\rho(\cdot,t)] + \frac{1}{2}\int_0^T \int_\Omega |k(x,t)|^2\, dx\, dt \le C(\Omega, \rho_{in}, \phi_{in}, M, T), \tag{3.23}$$

where $k(x,t)$ denotes the weak limit in $L^2$ of $\sqrt{f_{\delta,M}(\rho^\delta)}\,|\partial_x q^\delta|$ (we have used that the $L^2$ norm is lower semicontinuous with respect to the weak convergence).

*Nonnegativity property of the limiting density $\rho$.* We now prove that if the initial condition $\rho_{in}$ is nonnegative, then the limit function $\rho$ is also nonnegative.

PROPOSITION 3.7 (nonnegativity). *Let $\rho^\delta$ be a solution of the regularized equation (3.2) and assume that $\rho^\delta$ converges uniformly in $x$ and $t$ to a function $\rho$. If $\rho_{in} \ge 0$ in $\Omega$, then $\rho(x,t) \ge 0$ for all $(x,t) \in \Omega \times [0,T)$.*

We use here a classical argument first introduced in [7].

*Proof.* We define the function $g_\delta$ by

$$g_\delta(s) = \int_1^s \left( \int_1^r \frac{1}{f_{M,\delta}(\tau)}\, d\tau \right) dr. \tag{3.24}$$

It satisfies in particular $g_\delta \ge 0$ and $g_\delta''(\rho^\delta) = \frac{1}{f_{\delta,M}(\rho^\delta)}$. Recalling the boundary condition $f_{\delta,M}(\rho^\delta)\partial_x q^\delta = 0$ on $\partial\Omega \times (0,T)$, with $q^\delta = \partial_{xx}\rho^\delta - W'\left(\rho^\delta\right) - \phi$, one has

$$\begin{aligned}
\frac{d}{dt}\int_\Omega g_\delta(\rho^\delta)\, dx = \int_\Omega g_\delta'(\rho^\delta)\partial_t\rho^\delta\, dx &= \int_\Omega g_\delta'(\rho^\delta)\partial_x[f_{\delta,M}(\rho^\delta)\partial_x q^\delta]\, dx \\
&= -\int_\Omega g_\delta''(\rho^\delta) f_{\delta,M}(\rho^\delta)\partial_x\rho^\delta\partial_x q^\delta\, dx \\
&= -\int_\Omega \partial_x\rho^\delta\partial_x q^\delta\, dx.
\end{aligned}$$

Using the definition of $q^\delta$ and some integration by parts together with the Neumann boundary condition $\partial_x\rho^\delta = 0$ on $\partial\Omega \times (0,T)$, we get

$$\frac{d}{dt}\int_\Omega g_\delta(\rho^\delta)\, dx = -\int_\Omega |\partial_{xx}\rho^\delta|^2\, dx - \int_\Omega |\partial_x\rho^\delta|^2\, W''(\rho^\delta)\, dx - \int_\Omega \partial_x\rho^\delta\partial_x\phi\, dx. \tag{3.25}$$

The last two terms can be bounded as follows (recall that $W$ is in $C^2(\mathbb{R})$ and thus $W''$ is locally bounded):

$$\begin{aligned}
\left|\int_\Omega |\partial_x\rho^\delta|^2\, W''\left(\rho^\delta\right)\, dx\right| &\le C(\|\rho^\delta(\cdot,t)\|_{L^\infty(\Omega)}) \int_\Omega |\partial_x\rho^\delta|^2\, dx, \\
\left|\int_\Omega \partial_x\rho^\delta\partial_x\phi\, dx\right| &\le \frac{1}{2}\int_\Omega |\partial_x\rho^\delta|^2\, dx + \frac{1}{2}\int_\Omega |\partial_x\phi|^2\, dx.
\end{aligned}$$

Recalling (3.16), we obtain

$$\frac{d}{dt}\int_\Omega g_\delta(\rho^\delta)\, dx + \int_\Omega |\partial_{xx}\rho^\delta|^2\, dx \le C(\Omega, \rho_{in}, \phi_{in}, M, T) + \int_\Omega |\partial_x\phi|^2\, dx,$$

and so

$$\sup_{[0,T]}\int_\Omega g_\delta\left(\rho^\delta\right)\,dx+\int_0^T\int_\Omega|\partial_{xx}\rho^\delta|^2\,dx\,dt\le C(\Omega,\rho_{in},\phi_{in},M,T)\\ +\int_0^T\int_\Omega|\partial_x\phi|^2\,dx+\int_\Omega g_\delta(\rho_{in})\,dx.$$

Using (3.15) we deduce

$$\sup_{[0,T]}\int_\Omega g_\delta\left(\rho^\delta\right)\,dx\le C(\Omega,\rho_{in},\phi_{in},M,T)+\int_\Omega g_\delta(\rho_{in})\,dx. \tag{3.26}$$

Next, we check that the right-hand side in (3.26) is bounded. The fact that $f_{\delta,M}(z)\ge\min\{|z|,M\}$ implies that

$$g_\delta(s)\le\begin{cases}s\ln s-s+1 & \text{if } 0\le s\le M,\\ C(M)s^2 & \text{if } s\ge M.\end{cases}$$

In particular, since $\rho_{in}\ge 0$ is such that $\rho_{in}\in L^2(\Omega)$, we deduce

$$\sup_{[0,T]}\int_\Omega g_\delta\left(\rho^\delta\right)\,dx\le C(\Omega,\rho_{in},\phi_{in},M,T).$$

We now conclude the proof by a simple contradiction argument. Suppose that there exists $(x_0,t_0)$ such that $\rho(x_0,t_0)=-\eta$ for some $\eta>0$. The uniform Hölder estimate and the uniform convergence imply that there exists $r>0$ and $\delta_0>0$ such that $\rho^\delta(x,t_0)\le-\eta/4$ for all $\delta<\delta_0$ and all $x\in B_r(x_0)$. The function $g_\delta$ is clearly decreasing for $s<1$, so this implies

$$g_\delta(-\eta/4)\le g_\delta(\rho^\delta(x,t_0))\qquad\text{for all } \delta<\delta_0 \text{ and all } x\in B_r(x_0)$$

and thus

$$g_\delta(-\eta/4)\,|B_r(x_0)|\le\int_{B_r(x_0)}g_\delta(\rho^\delta(x,t_0))\,dx\le\int_\Omega g_\delta(\rho^\delta(x,t_0))\,dx\\ \le C(\Omega,\rho_{in},\phi_{in},M,T).$$

However, it is easy to see that $g_\delta(-\eta/4)\to+\infty$ as $\delta\to 0$ (since $\lim_{\delta\to 0}\frac{1}{f_{\delta,M}(\tau)}$ has a nonintegrable singularity at 0), which leads to a contradiction and completes the proof of the proposition. ∎

*Bound on $\rho$ independent of $M$.* In this section, we show that $\rho$ is bounded in $L^\infty$ independently of $M$, so that when $M$ is large enough we have $\min\{\rho,M\}=\rho$.

This bound will follow from the energy inequality (3.6) and uses in a crucial way the fact that $\rho\ge 0$. The idea is as follows. We recall that

$$\frac{d}{dt}\int_\Omega\frac12|\phi|^2\,dx+\int_\Omega|\phi|^2+|\partial_x\phi|^2\,dx=\int_\Omega\rho\phi\,dx.$$

The nonnegativity of $\rho$, together with the mass conservation (3.4) and classical Sobolev embeddings, implies that the right-hand side is bounded by

$$\left|\int_\Omega\rho\phi\,dx\right|\le\|\phi(\cdot,t)\|_{L^\infty(\Omega)}\int_\Omega\rho\,dx=\|\phi(\cdot,t)\|_{L^\infty(\Omega)}\int_\Omega\rho_{in}\,dx\le C(\rho_{in})\|\phi(\cdot,t)\|_{H^1(\Omega)}.$$

Consequently, we can obtain a bound on $\|\phi\|_{L^2(0,T;H^1(\Omega))}$ which only depends on $\rho_{in}$, $\phi_{in}$, and $T$. Using this bound in the energy inequality for $\rho$, we could then deduce a bound on $\rho$ in $L^\infty(0,T;H^1(\Omega))$ which is independent of $M$. However, we need to be careful with this argument because we cannot pass to the limit in the energy inequality (3.6) (since we only have a weak convergence of $\partial_x\phi^\delta$ in $L^2$). It is thus not clear that (3.6) holds in the limit $\delta \to 0$.

We will first prove the following result.

LEMMA 3.8. *Let*

$$Y^\delta(t) = \int_\Omega |\partial_x \phi^\delta|^2\, dx.$$

*The following inequalities hold:*

$$\int_0^T Y^\delta(t)\, dt \le \int_0^T \int_\Omega |\phi_{in}|^2\, dx + C\int_0^T \left(\int_\Omega |\rho^\delta(x,t)|\, dx\right)^2 dt \tag{3.27}$$

*and*

$$\frac{d}{dt}E[\rho^\delta(\cdot,t)] \le C\left(1 + E[\rho^\delta(\cdot,t)]^{1/2}\right) Y^\delta(t) \tag{3.28}$$

*for a constant $C$ depending only on $\Omega$ and $\int_\Omega \rho_{in}\, dx$.*

Using this result, a Gronwall type argument will then lead to an appropriate bound on $\rho^\delta$ thanks to the following proposition.

PROPOSITION 3.9. *There exists a constant $C$ depending only on $\Omega$, $\int_\Omega \rho_{in}\, dx$, $E[\rho_{in}]$, $\|\phi_{in}\|_{L^2}$, and $T$ such that*

$$\sup_{\Omega\times(0,T)} |\rho^\delta(x,t)| \le C + C\int_0^T \left(\int_\Omega |\rho^\delta(x,t)|\, dx\right)^2 dt.$$

Note that for $\delta > 0$, this proposition does not give any information we did not already have. The important fact is that the upper bound depends on $\delta$ only through $\int_\Omega |\rho^\delta|\, dx$. In the limit, using the uniform convergence of $\rho^\delta$, we get a bound on $\rho$ which only depends on $\int_\Omega |\rho|\, dx$. But, as noted above, the nonnegativity of $\rho$ and the conservation of mass (3.4) imply that

$$\int_\Omega |\rho|\, dx = \int_\Omega \rho\, dx = \int_\Omega \rho_{in}\, dx.$$

Proposition (3.9) therefore implies the following.

COROLLARY 3.10. *There exists a constant $C$ depending only on $\Omega$, $\int_\Omega \rho_{in}\, dx$, $E[\rho_{in}]$, $\|\phi_{in}\|_{L^2}$, and $T > 0$ such that*

$$0 \le \rho(x,t) \le C \qquad \text{in } \Omega\times(0,T).$$

*In particular, we can choose $M \ge C$ so that*

$$f_{\delta,M}(\rho^\delta) \to \min\{M, \rho(x,t)\} = \rho \qquad \text{uniformly in } \Omega\times(0,T) \text{ when } \delta\to 0.$$

We now turn to the proof of Lemma 3.8 and Proposition 3.9.

*Proof of Lemma* 3.8. From the energy inequality for $\phi^\delta$ and classical Sobolev embeddings it follows that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{1}{2}|\phi^\delta|^2\,dx + \int_\Omega |\phi^\delta|^2 + |\partial_x\phi^\delta|^2\,dx &= \int_\Omega \rho^\delta\phi^\delta\,dx \\ &\le \sup_{x\in\Omega}|\phi^\delta(x,t)|\int_\Omega|\rho^\delta|\,dx \\ &\le C\left(\int_\Omega |\phi^\delta|^2 + |\partial_x\phi^\delta|^2\,dx\right)^{1/2}\int_\Omega|\rho^\delta|\,dx.\end{aligned}$$

We deduce that

$$\frac{d}{dt}\int_\Omega \frac{1}{2}|\phi^\delta|^2\,dx + \frac{1}{2}\int_\Omega |\phi^\delta|^2 + |\partial_x\phi^\delta|^2\,dx \le C\left(\int_\Omega|\rho^\delta|\,dx\right)^2,$$

hence

$$\int_0^T\int_\Omega |\phi^\delta|^2 + |\partial_x\phi^\delta|^2\,dx\,dt \le \int_\Omega|\phi_{in}|^2\,dx + C\int_0^T\left(\int_\Omega|\rho^\delta|\,dx\right)^2\,dt, \tag{3.29}$$

which implies (3.27).

Next, using (3.6), we get

$$\begin{aligned}\frac{d}{dt}E[\rho^\delta(\cdot,t)] &\le \sup_{x\in\Omega} f_{\delta,M}(\rho^\delta)\int_\Omega|\partial_x\phi^\delta|^2\,dx \\ &\le \sup_{x\in\Omega}\left(\delta + |\rho^\delta(x,t)|\right)Y^\delta(t)\end{aligned}$$

and (3.28) follows the following consequence of Poincaré inequality:

$$\begin{aligned}\sup_{x\in\Omega}|\rho^\delta(x,t)| &\le C\|\rho^\delta(\cdot,t)\|_{H^1(\Omega)} \\ &\le C\left(\int_\Omega\rho^\delta(x,t)\,dx + \left(\int_\Omega|\partial_x\rho^\delta|^2\;dx\right)^{1/2}\right) \\ &\le C\left(\int_\Omega\rho_{in}(x)\,dx + E[\rho^\delta(\cdot,t)]^{1/2}\right). \qquad (3.30)\end{aligned}$$

□

*Proof of Proposition* 3.9. A Gronwall type argument now yields a bound on $E[\rho^\delta]$. More precisely, we see that the inequality (3.28) implies

$$\frac{d}{dt}\left(1+E[\rho^\delta(\cdot,t)]\right)^{1/2} \le C\,Y^\delta(t),$$

and so

$$\int_\Omega|\partial_x\rho^\delta(x,t)|^2\;dx \le E[\rho^\delta(\cdot,t)] \le C\left[\int_0^T Y^\delta(s)\;ds + 1\right]^2 \qquad \forall t\in(0,T), \tag{3.31}$$

where the constant $C$ depends only on $\int_\Omega\rho_{in}\,dx, \Omega$, and $E[\rho_{in}]$. Combining inequalities (3.31) and (3.27), and using (3.30), we deduce

$$\sup_{\Omega\times(0,T)}|\rho^\delta(x,t)| \le C + C\int_0^T\left(\int_\Omega|\rho^\delta|\,dx\right)^2\,dt,$$

where the constant $C$ depends only on $\Omega$, $\int_\Omega\rho_{in}\,dx$, $E[\rho_{in}]$, $\|\phi_{in}\|_{L^2}$, and $T$. □

*Weak formulation.* We now fix $M \geq C$ where $C$ is the constant given by Corollary 3.10. We consider a set of test functions $\varphi \in \mathcal{D}(\bar{\Omega} \times [0,T))$. Multiplying the first equation in (3.2) by $\varphi$ and integrating on $\Omega_T$, we get

$$\int_0^T \int_\Omega \rho^\delta \partial_t \varphi \, dx\, dt - \int_0^T \int_\Omega f_{\delta,M}(\rho^\delta)\, \partial_x q^\delta \, \partial_x \varphi \, dx\, dt = -\int_\Omega \rho_{in}(x)\, \varphi(x,0)\, dx, \tag{3.32}$$

where we recall that $q^\delta = -\partial_{xx}\rho^\delta + W'(\rho^\delta) + \phi$. We now want to pass to the limit in (3.32). Since $\rho^\delta \to \rho$ uniformly, we have that

$$\int_0^T \int_\Omega \rho^\delta \partial_t \varphi \, dx\, dt \longrightarrow \int_0^T \int_\Omega \rho\, \partial_t \varphi \, dx\, dt \qquad \forall \varphi \in \mathcal{D}(\Omega_T) \text{ as } \delta \to 0. \tag{3.33}$$

Next, recalling (3.19), we see that the function $h^\delta = f_{\delta,M}(\rho^\delta)\, \partial_x q^\delta$ is bounded in $L^2(0,T;L^2(\Omega))$ uniformly with respect to $\delta$. Hence, there exists a function $h \in L^2(0,T;L^2(\Omega))$ such that

$$h^\delta \rightharpoonup h, \qquad \text{weakly in } L^2(0,T;L^2(\Omega)), \text{ for } \delta \to 0.$$

This allows us to write the following convergence as $\delta \to 0$:

$$\int_0^T \int_\Omega f_{\delta,M}(\rho^\delta)\, \partial_x q^\delta \, \partial_x \varphi \, dx\, dt \longrightarrow \int_0^T \int_\Omega h\, \partial_x \varphi \, dx\, dt \qquad \forall\, \varphi \in \mathcal{D}(\Omega_T). \tag{3.34}$$

In order to characterize this function $h$, we first write, for any $\varphi \in \mathcal{D}(\Omega_T)$,

$$\begin{aligned}
\left| \iint_{\{\rho=0\}} h^\delta \varphi \; dx\, dt \right| &\leq C \left( \iint_{\{\rho=0\}} f_{\delta,M}(\rho^\delta)\, dx\, dt \right)^{1/2} \left( \iint_{\{\rho=0\}} f_{\delta,M}(\rho^\delta) |\partial_x q^\delta|^2 \, dx\, dt \right)^{1/2} \\
&\leq C \left( \iint_{\{\rho=0\}} f_{\delta,M}(\rho^\delta)\, dx\, dt \right)^{1/2} \\
&\longrightarrow C \left( \iint_{\{\rho=0\}} \min\{M,\rho\} \; dx\, dt \right)^{1/2} = 0.
\end{aligned}$$

We deduce that $h = 0$ a.e. in $\{\rho = 0\}$. Next, for $\eta > 0$, we consider the set $\{\rho > \eta\}$. The uniform convergence implies that $\rho^\delta(x,t) > \eta/2$ and so $f_{\delta,M}(\rho^\delta) > \eta/2$ in that set for $\delta$ small enough. We deduce

$$\begin{aligned}
\iint_{\{\rho>\eta\}} |\partial_{xxx}\rho^\delta|^2 \, dx\, dt &\leq \frac{2}{\eta} \int_0^T \int_\Omega f_{\delta,M}(\rho^\delta) |\partial_{xxx}\rho^\delta|^2 \, dx\, dt \\
&\leq \frac{2}{\eta} \Bigg( \int_0^T \int_\Omega f_{\delta,M}(\rho^\delta) |\partial_x[-\partial_{xx}\rho^\delta + W'(\rho^\delta)]|^2 \, dx\, dt \\
&\qquad + \int_0^T \int_\Omega f_{\delta,M}(\rho^\delta) |\partial_x W'(\rho^\delta)|^2 \, dx\, dt \Bigg)
\end{aligned}$$

$$\leq \frac{2}{\eta}\left(C+\int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)W''(\rho^\delta)^2|\partial_x\rho^\delta|^2\,dx\,dt\right)$$
$$\leq \frac{C}{\eta}$$

for small $\delta$ (where we used (3.16) and Corollary 3.10).

In particular

$$\partial_{xxx}\rho^\delta \rightharpoonup \partial_{xxx}\rho \qquad \text{weakly in } \ L^2(0,T;L^2(\{\rho>\eta\})), \ \text{ when } \ \delta\to 0.$$

We deduce (using Corollary 3.10) that

$$\begin{aligned} h^\delta &= f_{\delta,M}(\rho^\delta)\left(-\partial_{xxx}\rho^\delta + W''\left(\rho^\delta\right)\partial_x\rho^\delta + \partial_x\phi^\delta\right)\\ &\rightharpoonup \rho\left(-\partial_{xxx}\rho + W''(\rho)\partial_x\rho + \partial_x\phi\right) \qquad \text{weakly in } \ L^2(0,T;L^2(\{\rho>\eta\})). \end{aligned}$$

Since this holds for all $\eta>0$, we deduce that $h=\rho\left(-\partial_{xxx}\rho + W''(\rho)\partial_x\rho + \partial_x\phi\right)$ a.e. in the set $\{\rho>0\}$.

Equation (3.34) thus becomes

$$\int_0^T\int_\Omega f_{\delta,M}(\rho^\delta)\,\partial_x q^\delta\,\partial_x\varphi\,dx\,dt \longrightarrow \iint_{\{\rho>0\}}\rho\left(-\partial_{xxx}\rho + W''(\rho)\partial_x\rho + \partial_x\phi\right)\partial_x\varphi\,dx\,dt$$

for all $\varphi\in\mathcal{D}(\Omega_T)$ and the result follows by passing to the limit in (3.32).

**4. Formal derivation of the asymptotic model: Proof of Theorem 1.2.** In this section, we give a detailed formal derivation of the asymptotic model describing the evolution of $\rho^0=\lim_{\varepsilon\to 0}\rho^\varepsilon$. We rewrite the system (1.11) as follows:

$$\begin{cases} \varepsilon\partial_t\rho^\varepsilon = \operatorname{div}\left(\rho^\varepsilon\nabla q^\varepsilon\right) & \text{in } \Omega\times(0,T),\\ q^\varepsilon = \gamma(-\varepsilon^2\Delta\rho^\varepsilon + W'(\rho^\varepsilon)) + \varepsilon\phi^\varepsilon & \text{in } \Omega\times(0,T),\\ \varepsilon\partial_t\phi^\varepsilon - \alpha\operatorname{div}\left(\phi^\varepsilon\nabla q^\varepsilon\right) = \eta^2\Delta\phi^\varepsilon - \phi^\varepsilon + \beta\rho^\varepsilon & \text{in } \Omega\times(0,T), \end{cases} \tag{4.1}$$

together with appropriate boundary conditions on $\partial\Omega$ when either $\eta$ is a fixed positive parameter or $\eta=\tau\varepsilon$ for some fixed $\tau\geq 0$. Here the domain $\Omega$ is a fixed open subset of $\mathbb{R}^2$.

When $\phi=0$, this is a classical problem (see Pego [32] for the case with constant mobility and Glasner [22] for the degenerate case considered here) and the limit $\varepsilon\to 0$ leads to phase separation. Here, we expect to find, as in [22],

$$\lim_{\varepsilon\to 0}\rho^\varepsilon(x,t) = \rho^0(x,t) = \chi_{\Sigma(t)}(x),$$

where the set $\Sigma(t)$ describes the inside of the cell and $\Sigma(t)^c$ describes the outside of the cell. When $\phi(x,t)$ is a given function independent of $\varepsilon$, the formal derivation of an asymptotic equation for $\partial\Sigma(t)$ is very similar to [22] and it leads to the following Hele-Shaw free boundary problem with surface tension and active potential:

$$\begin{cases} -\Delta q = 0 & \text{in } \Sigma(t),\\ q = \bar\gamma\kappa(t) + \phi(t,x) & \text{on } \partial\Sigma(t),\\ V = -\nabla q\cdot n & \text{on } \partial\Sigma(t), \end{cases} \tag{4.2}$$

where we recall that $\kappa(t)$ denotes the mean-curvature of the boundary $\partial\Sigma(t)$ and $V$ is the normal velocity of $\partial\Sigma(t)$. This derivation relies on asymptotic expansions and matching asymptotic methods. For the sake of completeness, we will provide all the details below, even though our main contribution is the role played by the coupling with the evolution if $\phi^\varepsilon$ (and leads to the definition of the function $F(V)$ in (1.14)).

**4.1. Outer expansions.** We first expand $\rho^\varepsilon$, $q^\varepsilon$, and $\phi^\varepsilon$ as follows:

$$\rho^\varepsilon(x,t) = \rho^0(x,t) + \varepsilon\rho^1(x,t) + \varepsilon^2\rho^2(x,t) + \cdots, \tag{4.3}$$

$$q^\varepsilon(x,t) = q^0(x,t) + \varepsilon q^1(x,t) + \varepsilon^2 q^2(x,t) + \cdots, \tag{4.4}$$

$$\phi^\varepsilon(x,t) = \phi^0(x,t) + \varepsilon\phi^1(x,t) + \varepsilon^2\phi^2(x,t) + \cdots.$$

The function $\rho^0$ (resp., $q^0$ and $\phi^0$) will describe the asymptotic behavior of $\rho^\varepsilon$ (resp., $q^\varepsilon$ and $\phi^\varepsilon$) outside of the small transition layer (of size $\varepsilon$) around the interface $\Gamma(t) = \partial\Sigma(t)$. This expansion is thus usually called the *outer expansion* in the literature.

Plugging these expansions into (4.1), we get in particular

$$\varepsilon\partial_t\rho^0 = \operatorname{div}\left(\rho^0\nabla q^0\right) + \varepsilon\operatorname{div}\left(\rho^1\nabla q^0\right) + \varepsilon\operatorname{div}\left(\rho^0\nabla q^1\right) + \mathcal{O}(\varepsilon^2) \tag{4.5}$$

and

$$\begin{aligned} q^0 + \varepsilon q^1 &= \gamma W'\left(\rho^0 + \varepsilon\rho^1 + \cdots\right) + \varepsilon\phi^0 + \mathcal{O}(\varepsilon^2),\\ &= \gamma W'(\rho^0) + \varepsilon\gamma W''(\rho^0)\rho^1 + \varepsilon\phi^0 + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{4.6}$$

(we do not worry about the equation for $\phi^\varepsilon$ at this point).

Identifying the term of the same order in $\varepsilon$ in (4.5), we get

$$0 = \operatorname{div}(\rho^0\nabla q^0), \tag{4.7}$$

$$\partial_t\rho^0 = \operatorname{div}(\rho^0\nabla q^1) + \operatorname{div}(\rho^1\nabla q^0), \tag{4.8}$$

and doing the same thing with (4.6), we obtain

$$q^0 = \gamma W'(\rho^0), \tag{4.9}$$

$$q^1 = \gamma W''(\rho^0)\rho^1 + \phi^0. \tag{4.10}$$

**4.2. Inner expansions.** Since the functions $\rho^i$ and $q^i$ might be discontinuous across the interface $\Gamma(t)$, these equations hold in each phase $\Sigma(t)$ and $\Sigma^c(t)$ and must be supplemented with boundary conditions along $\Gamma(t)$. In order to derive the appropriate boundary conditions, we need to describe the transition layer and use matching asymptotic methods.

This is done by expanding a rescaled version of the solution near a point on $\Gamma(t)$. This is the so-called *inner expansion* which we describe now: We assume that the width of the transition layer is of order $\varepsilon$, and we approximate the interface separating the inside and outside of the cell by the level set

$$\Gamma^\varepsilon(t) = \{x \,|\, \rho^\varepsilon(x,t) = 1/2\}.$$

We fix $t_0 > 0$ and a point $x_0 \in \Gamma^\varepsilon(t_0)$ and we consider $s \mapsto \zeta(s,t)$ a parametrization of $\Gamma^\varepsilon(t)$ near $x_0$. We denote by $z$ the signed distance from $\Gamma(t) = \partial\Sigma(t)$ ($z > 0$ inside $\Sigma(t)$) and we use $(s,z)$ as an orthogonal local coordinate system in a neighborhood of the interface. For all $(x,t)$ in a small neighborhood of $(x_0,t_0)$, we can write

$$x = X(s,z,t) = \zeta(s,t) + zn(s,t),$$

where $n(s,t)$ is the inward normal unit vector to $\Gamma(t) = \partial\Sigma(t)$ at the point $\zeta(s,t)$ (we recall that the interior of the set $\Sigma(t)$ corresponds to the set $z > 0$). In a small neighborhood of $(x_0,t_0)$, we can invert the change of coordinates $(s,z,t)\mapsto(x,t)$ and we will use the notation

$$s=S(x,t),\quad z=R(x,t)=\pm d(x,\Gamma(t)).$$

We recall that the distance function satisfies in particular $|\nabla R|=1$ in the neighborhood of $\Gamma(t)$ where $z$ is well defined and

$$-\Delta R=\kappa \quad \text{on } \Gamma(t), \tag{4.11}$$

where $\nabla$ and $\Delta$ denote the derivative with respect to the variable $x$ and $\kappa$ is the curvature of $\Gamma$ (with the convention that it is positive if $\Sigma(t)$ is convex).

In order to describe the transition layer, we rescale the normal variable $z$ by defining the functions $\overline{\rho}^\varepsilon,\overline{q}^\varepsilon$, and $\overline{\phi}^\varepsilon$ so that

$$\rho^\varepsilon(x,t)=\overline{\rho}^\varepsilon(z/\varepsilon,s,t),\quad q^\varepsilon(x,t)=\overline{q}^\varepsilon(z/\varepsilon,s,t),\quad \phi^\varepsilon(x,t)=\overline{\phi}^\varepsilon(z/\varepsilon,s,t).$$

A simple computation then shows that

$$\begin{aligned}
\partial_t\rho^\varepsilon(x,t)&=\left[\frac{1}{\varepsilon}R_t\overline{\rho}^\varepsilon_z+S_t\overline{\rho}^\varepsilon_s+\overline{\rho}^\varepsilon_t\right](z/\varepsilon,s,t),\\
\nabla\rho^\varepsilon(x,t)&=\left[\frac{1}{\varepsilon}\nabla R\overline{\rho}^\varepsilon_z+\nabla S\overline{\rho}^\varepsilon_s\right](z/\varepsilon,s,t),\\
\Delta\rho^\varepsilon(x,t)&=\left[\frac{1}{\varepsilon^2}\overline{\rho}^\varepsilon_{zz}+\frac{1}{\varepsilon}\Delta R\overline{\rho}^\varepsilon_z+|\nabla S|^2\overline{\rho}^\varepsilon_{ss}+\Delta S\overline{\rho}^\varepsilon_s\right](z/\varepsilon,s,t),
\end{aligned}$$

where we used the fact that $|\nabla R|=1$ and $\nabla R\cdot\nabla S=0$. In the new coordinates, the system (4.1) leads in particular to

$$\begin{cases}\varepsilon^2\,\overline{\rho}^\varepsilon_zR_t+\varepsilon^3\,(\overline{\rho}^\varepsilon_t+\overline{\rho}^\varepsilon_sS_t)=(\overline{\rho}^\varepsilon\overline{q}^\varepsilon_z)_z+\varepsilon\,\Delta R\,\overline{\rho}^\varepsilon\overline{q}^\varepsilon_z+\varepsilon^2\,[\,\Delta S\overline{\rho}^\varepsilon\overline{q}^\varepsilon_s+|\nabla S|^2\,(\overline{\rho}^\varepsilon\overline{q}^\varepsilon_s)_s\,],\\ \overline{q}^\varepsilon=-\gamma[\,\overline{\rho}^\varepsilon_{zz}+\varepsilon\,\Delta R\overline{\rho}^\varepsilon_z+\varepsilon^2\,(\Delta S\overline{\rho}^\varepsilon_s+|\nabla S|^2\overline{\rho}^\varepsilon_{ss})\,]+\gamma W'(\overline{\rho}^\varepsilon)+\varepsilon\,\overline{\phi}^\varepsilon.\end{cases} \tag{4.12}$$

Proceeding as with the outer expansion, we expand those functions,

$$\begin{cases}\overline{\rho}^\varepsilon(z,s,t)=\overline{\rho}^0(z,s,t)+\varepsilon\overline{\rho}^1(z,s,t)+\varepsilon^2\overline{\rho}^2(z,s,t)+\cdots,\\ \overline{q}^\varepsilon(z,s,t)=\overline{q}^0(z,s,t)+\varepsilon\overline{q}^1(z,s,t)+\varepsilon^2\overline{q}^2(z,s,t)+\cdots,\\ \overline{\phi}^\varepsilon(z,s,t)=\overline{\phi}^0(z,s,t)+\varepsilon\overline{\phi}^1(z,s,t)+\varepsilon^2\overline{\phi}^2(z,s,t)+\cdots,\end{cases} \tag{4.13}$$

and we identify the terms of the same order in $\varepsilon$ after plugging these expansions into (4.12).

*Terms of order zero: The transition profile $\psi(z)$.* After expanding $W'(\overline{\rho}^\varepsilon)=W'(\overline{\rho}^0)+\varepsilon W''(\overline{\rho}^0)\overline{\rho}^1+\dots$, the terms of order zero in (4.12) give

$$\begin{cases}0=(\overline{\rho}^0\overline{q}^0_z)_z,\\ \overline{q}^0=-\gamma\overline{\rho}^0_{zz}+\gamma W'(\overline{\rho}^0).\end{cases} \tag{4.14}$$

Since we are looking for a positive solution $\overline{\rho}^0=\overline{\rho}^0(z,s,t)$ joining two states $\rho_\pm(s,t)$ as $z\to\pm\infty$, the first equation implies that $\overline{q}^0$ does not depend on $z$ (we can use (4.9) and the matching conditions (4.31) for $q$ to make this rigorous), and taking the limit $z \to \pm\infty$ in the second equation leads to

$$\overline{q}^0 = \gamma W'(\rho_+) = \gamma W'(\rho_-). \tag{4.15}$$

Now, by multiplying the second equation of (4.14) by $\rho_z^0$ and by integrating in $z$, we get

$$\frac{\overline{q}^0}{\gamma}\left(\overline{\rho}^0(z) - \rho_-\right) = -\frac{1}{2}(\overline{\rho}_z^0(z))^2 + W(\overline{\rho}^0(z)) - \gamma W(\rho_-). \tag{4.16}$$

When $z \to +\infty$, we find $\frac{\overline{q}^0}{\gamma}(\rho_+ - \rho_-) = W(\rho_+) - \gamma W(\rho_-)$ and so using (4.15), we get that $\rho_-$ and $\rho_+$ are related by the classical relations

$$\begin{cases} \dfrac{W(\rho_+) - W(\rho_-)}{\rho_+ - \rho_-} - W'(\rho_+) = 0, \\ \dfrac{W(\rho_+) - W(\rho_-)}{\rho_+ - \rho_-} - W'(\rho_-) = 0. \end{cases}$$

When $W$ satisfies (1.2), this implies that $\rho_- = 0$ and $\rho_+ = 1$ (and $\overline{q}^0 = 0$ by (4.15)). Equation (4.16) thus becomes

$$(\overline{\rho}_z^0)^2 = 2\,W(\,\overline{\rho}^0\,), \tag{4.17}$$

and so $\overline{\rho}^0(z) = \psi(z)$, where $\psi$ is given by the following lemma.

LEMMA 4.1. *Assume that the double potential* $W$ *satisfies* (1.2). *Then there is a unique profile* $\psi$ *satisfying*

$$\psi'(z) = \sqrt{2\,W(\,\psi(z)\,)}, \qquad \lim_{z\to-\infty} \psi(z) = 0, \qquad \lim_{z\to+\infty} \psi(z) = 1, \tag{4.18}$$

*and such that* $\psi(0) = 1/2$. *When* $W(\rho) = \rho^2(1-\rho)^2$, *we get* $\psi(z) = \frac{1}{1+e^{-\sqrt{2}z}}$.

*Terms of order* 1: *Solvability condition.* Taking the terms of order $\varepsilon^1$ in the system (4.12), we get

$$\begin{cases} 0 = \left(\psi \overline{q}_z^1\right)_z, \\ \overline{q}^1 = -\gamma\overline{\rho}_{zz}^1 + \gamma\kappa^0\,\psi' + \gamma W''(\psi)\,\overline{\rho}^1 + \overline{\phi}^0, \end{cases} \tag{4.19}$$

where we used (4.11) to approximate $-\Delta R$ by the leading order of the curvature of $\Gamma(t)$ denoted here by $\kappa^0$. Assuming that $\overline{q}^1$ is bounded for $z \in \mathbb{R}$ (this will follow from the matching conditions (4.31) once we show that $q^0 = 0$), the first equation in (4.19) implies that $\overline{q}^1$ is independent of $z$.

The second equation in (4.19) then implies that $\rho^1$ solves

$$\gamma L_0[\rho^1] := -\gamma\overline{\rho}_{zz}^1 + \gamma W''(\psi)\,\overline{\rho}^1 = \overline{q}^1 - \gamma\kappa^0\,\psi' - \overline{\phi}^0. \tag{4.20}$$

To derive a solvability condition for this equation, we first differentiate (4.14) with respect to $z$ to find

$$L_0[\psi'] = W''(\psi)\psi' - \psi''' = 0.$$

Multiplying (4.20) by $\psi'(z)$ and integrating, we deduce

$$\int_{-\infty}^{+\infty}\left[\overline{q}^1-\gamma\kappa^0\,\psi'(z)-\overline{\phi}^0(s,t,z)\right]\psi'(z)\,dz=\int_{\mathbb{R}}L_0[\rho^1]\psi'(z)\,dz$$
$$=\int_{\mathbb{R}}\rho^1L_0[\psi'](z)\,dz$$
$$=0,$$

which gives (using (4.18))

$$\overline{q}^1(s,t)=\bar{\gamma}\kappa_0(s,t)+\int_{-\infty}^{\infty}\overline{\phi}^0(s,t,z)\psi'(z)\,dz, \tag{4.21}$$

where

$$\bar{\gamma}=\gamma\int_{-\infty}^{\infty}|\psi'(z)|^2\,dz=\gamma\sqrt{2}\int_0^1\sqrt{W(x)}\,dx.$$

*Terms of order* 2. Finally we consider the terms of order $\varepsilon^2$ in (4.12). Since $\overline{q}^0=0$ and $\overline{q}^1$ does not depend on $z$, we find in particular that

$$\psi'(z)V^0=\left(\psi\,\overline{q}_z^2\right)_z, \tag{4.22}$$

where we approximated $R_t$ by the velocity $V^0$ of the surface $\Gamma$ (note that $R_t>0$ if the $\partial\Sigma$ is moving outward, that is, if $V>0$). Integrating with respect to $z$ and using the matching condition at $z\to-\infty$, we deduce

$$\overline{q}_z^2=V^0. \tag{4.23}$$

*The function* $\overline{\phi}^0$. We finally turn our attention to the function $\overline{\phi}^0$ which appears in the formula (4.21) for $\overline{q}^1$. First, we note that the equation for $\phi^\varepsilon$ in (4.1) yields the following equation for the rescaled function $\overline{\phi}^\varepsilon$:

(4.24)
$$\varepsilon^2\overline{\phi}_z^\varepsilon R_t+\varepsilon^3\left(\overline{\phi}_t^\varepsilon+\overline{\phi}_s^\varepsilon S_t\right)-\alpha\left[\left(\overline{\phi}^\varepsilon\overline{q}_z^\varepsilon\right)_z+\varepsilon^2\left(\overline{\phi}^\varepsilon\overline{q}_s^\varepsilon\right)_s|\nabla S|^2+\varepsilon\overline{\phi}^\varepsilon\overline{q}^\varepsilon\Delta R+\varepsilon^2\overline{\phi}^\varepsilon\overline{q}_s^\varepsilon\Delta S\right]$$
$$=\eta^2\left(\overline{\phi}_{zz}^\varepsilon+\varepsilon\Delta R\overline{\phi}_z^\varepsilon+\varepsilon^2\left[\overline{\phi}_{zz}^\varepsilon|\nabla S|^2+\overline{\phi}_s\nabla S\right]\right)-\varepsilon^2\overline{\phi}^\varepsilon+\varepsilon^2\beta\overline{\rho}^\varepsilon,$$

in which we insert the expansions (4.13).

- *Case* 1*: When* $\eta>0$ *is fixed.* Since $\overline{q}^0=0$ and $\overline{q}^1$ is constant with respect to $z$, the terms of order 0 in (4.24) give
$$\overline{\phi}_{zz}^0=0,$$
which together with the matching boundary conditions (4.31) implies that $\overline{\phi}^0$ is independent of $z$. The term of order 1 then gives
$$\overline{\phi}_{zz}^1=0,$$
which together with the matching boundary conditions (4.31) implies that $\overline{\phi}^1$ is linear.
- *Case* 2*: When* $\eta=\tau\varepsilon\ll 1$. Since $\overline{q}^0=0$ and $\overline{q}^1$ is constant with respect to $z$ the terms of order $\varepsilon^0$ and $\varepsilon^1$ vanish, so we consider the term of order $\varepsilon^2$ in (4.24) and find

$$\overline{\phi}^0_z V^0 - \alpha \left( \overline{\phi}^0 \overline{q}^2_z \right)_z = \tau^2 \overline{\phi}^0_{zz} - \overline{\phi}^0 + \beta \overline{\rho}^0,$$

which is equivalent (using (4.23)) to

$$\tau^2 \overline{\phi}^0_{zz} - (1-\alpha) V^0 \overline{\phi}^0_z + \beta \overline{\rho}^0 - \overline{\phi}^0 = 0,$$

where we recall that $\overline{\rho}^0 = \psi$. We now use the following result, which will be proved later (see Proposition 5.1).

PROPOSITION 4.2. *For any $\tau \geq 0$ and for all $V \in \mathbb{R}$, the equation*

$$\tau^2 \Phi'' - V \Phi' - \Phi + \psi = 0 \tag{4.25}$$

*has a unique (up to translation) bounded solution in $\mathbb{R}$, which we denote $\Phi_\tau(V, z)$.*

We can thus write

$$\overline{\phi}^0(z, s, t) = \beta \, \Phi_\tau((1-\alpha) V^0(s,t), z), \tag{4.26}$$

and (4.21) now gives

$$\overline{q}^1(s,t) = \gamma \, \kappa_0(s,t) + \beta \, F_\tau((1-\alpha) V^0(s,t)\,), \tag{4.27}$$

where the function $F_\tau : \mathbb{R} \to \mathbb{R}$ is defined by

$$F_\tau(V\,) = \int_{-\infty}^{\infty} \Phi_\tau(V, z) \, \psi'(z) \, dz \qquad \forall \; V \in \mathbb{R}. \tag{4.28}$$

*Remark* 4.3. Equation (4.21) shows that the contribution of the potential $\phi^\varepsilon$ to the free boundary condition is given by

$$h(s,t) = \int_{-\infty}^{\infty} \overline{\phi}^0(z, s, t) \psi'(z) \, dz,$$

which using our notation yields

$$\begin{aligned} h(s,t) = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \overline{\phi}^\varepsilon(z,s,t) \psi'(z) \, dz &= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \phi^\varepsilon(\varepsilon z, s, t) \psi'(z) \, dz \\ &= \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \phi^\varepsilon(z, s, t) \varepsilon^{-1} \psi'(z/\varepsilon) \, dz \\ &= \lim_{\varepsilon \to 0} \int_{-\delta}^{\delta} \phi^\varepsilon(z, s, t) \varepsilon^{-1} \psi'(z/\varepsilon) \, dz \\ &= \lim_{\varepsilon \to 0} \int_{-\delta}^{\delta} \phi^\varepsilon(z, s, t) \partial_z \rho^\varepsilon(s, t, z) \, dz \end{aligned}$$

for any $\delta > 0$. We thus have $\lim_{\varepsilon \to 0} \phi^\varepsilon \partial_z \rho^\varepsilon = h(s,t) \delta(z)$.

In particular, if we take $\phi^\varepsilon(x,t) = \beta(x,t) \rho^\varepsilon(x,t)$ for some smooth function $\beta$ (instead of the diffusion equation for $\phi^\varepsilon$), then we get $\overline{\phi}^0(z,s,t) = \beta(\zeta(s,t),t)\psi(z)$ and so

$$\int_{-\infty}^{\infty} \overline{\phi}^0(z) \psi'(z) \, dz = \frac{1}{2} \beta(\zeta(s,t), t),$$

which gives

$$\overline{q}^1(s,t) = \gamma \kappa_0(s,t) + \frac{1}{2} \beta(\zeta(s,t), t). \tag{4.29}$$

**4.3. The matching boundary conditions.** We observe that up to now we have two functions: $\rho^\varepsilon$ defined away from the interface for which we consider the outer expansion, and $\overline{\rho}^\varepsilon$ defined near the interface for which we consider the inner expansion. The behaviors of these functions are related by the so-called matching boundary conditions. These conditions are derived in [10], but we recall the main step of this derivation for the function $\rho^\varepsilon$ for the reader's convenience (see also [32])—the derivation is similar for the functions $\overline{q}$ and $\overline{\phi}$.

We recall that the definition of $\overline{\rho}$ yields

$$\overline{\rho}^\varepsilon(z,s,t) = \rho^\varepsilon(\zeta(s,t) + \varepsilon z\, n(s,t), t),$$

and a Taylor expansion with respect to $\varepsilon$ leads to

$$\sum_{n=0}^{+\infty} \varepsilon^n \overline{\rho}^n(z,s,t) = \sum_{n=0}^{+\infty} \frac{\varepsilon^n}{n!} \left[\frac{d^n}{d\varepsilon^n} \rho^\varepsilon(\zeta(s,t) + \varepsilon z\, n(s,t), t)\right]_{|\,\varepsilon=0}. \tag{4.30}$$

The matching boundary conditions are obtained by taking the limit $z \to \pm\infty$ and $\varepsilon \to 0$ in (4.30) assuming that $\varepsilon z \to 0$ and $\varepsilon z^2 \to 0$. We obtain

(4.31)
$$\begin{cases} \lim_{z\to\pm\infty} \overline{\rho}^0(z,s,t) = \rho^0(\zeta(s,t) \pm 0, t), \\ \overline{\rho}^1(z,s,t) = \rho^1(\zeta(s,t) \pm 0, t) + z\nabla\rho^0(\zeta(s,t) \pm 0, t)\cdot n \qquad \text{as } z \to \pm\infty, \\ \overline{\rho}^2(z,s,t) = \rho^2(\zeta(s,t) \pm 0, t) + z\nabla\rho^1(\zeta(s,t) \pm 0, t)\cdot n \\ \qquad\qquad + \frac{1}{2} z^2 n^T D^2 \rho^0(\zeta(s,t) \pm 0, t)\, n \qquad \text{as } z \to \pm\infty. \end{cases}$$

The same considerations are valid also for the functions $q^\varepsilon$ and $\overline{q}^\varepsilon$ and we impose the same matching conditions on the pressure $\overline{q}^\varepsilon$ for $z \to \pm\infty$ and we get the same formulation of (4.31).

**4.4. Conclusion.** We are now ready to conclude. We recall that $\Gamma(t) = \partial\Sigma$ and we denote by $\rho^0_\pm(x,t)$ the trace of $\rho^0$ on either side of $\Gamma(t)$. The matching condition of order zero in (4.31) and the fact that $\overline{\rho}^0(z,s,t) = \psi(z)$ then lead to

$$\rho^0_+(x,t) = 1 \text{ on } \Gamma(t), \qquad \rho^0_-(x,t) = 0 \text{ on } \Gamma(t).$$

Furthermore, (4.7), (4.9) imply

$$\operatorname{div}(\rho^0 \nabla W'(\rho^0)) = 0,$$

so these boundary conditions lead to

$$\rho^0(x,t) = 1 \text{ in } \Sigma(t), \qquad \rho^0(x,t) = 0 \text{ in } \Sigma(t)^c. \tag{4.32}$$

Since $\overline{q}^0 = 0$, the matching condition of order zero for the pressure leads to $q^0 = 0$. By (4.8), since $\rho^0$ is constant, we then deduce

$$\Delta q^1 = 0 \text{ in } \Sigma(t). \tag{4.33}$$

Since $\overline{q}^1$ is independent by $z$, the second equation in (4.31) for the pressure leads to

$$q^1(x,t) = \overline{q}^1(s,t) \text{ on } \partial\Sigma(t), \tag{4.34}$$

where $\overline{q}^1$ is given by (4.21). Finally, the third equation in (4.31) for the pressure leads to

$$\lim_{z\to\pm\infty} \overline{q}^2_z(z,s,t) = \nabla q^1(\zeta(s,t)\pm 0,t)\cdot n,$$

and using (4.23) we deduce

$$V^0 = \nabla q^1(\zeta(s,t),t)\cdot n, \tag{4.35}$$

where we remember that $n = n(s,t)$ is the inward normal unit vector to $\Gamma(t)$ at the point $\zeta(s,t)$. Equations (4.32), (4.33), (4.34), (4.35), together with (4.27), fully determine the evolution of $\Sigma(t)$ in the case $\eta = \tau\varepsilon \ll 1$. Note that in that case, the terms of order 0 in the equation for $\phi^\varepsilon$ gives (since $q^0 = 0$)

$$\phi^0 = \beta\rho^0 = \beta\chi_{\Sigma(t)}.$$

When $\eta > 0$ is fixed, the term of order 0 in the equation for $\phi^\varepsilon$ gives

$$\phi^0 - \eta^2\Delta\phi^0 = \beta\rho^0 \qquad \text{in } \Omega\setminus\Gamma(t). \tag{4.36}$$

Furthermore, the equations $\overline{\phi}^0_{zz} = 0$ and $\overline{\phi}^1_{zz} = 0$ together with the matching boundary conditions imply that $\phi^0(x,t)$ and $\partial_n\phi^0$ are continuous across the interface $\Gamma(t)$ so that (4.36) holds in the whole set $\Omega$.

**5. Properties of the function $F_\tau$.** In this section, we prove Proposition 4.2—which shows in particular that $F_\tau(V)$ is well defined—as well as Proposition 1.4.

We recall that $\psi(z)$ denotes the solution of (4.18) and we take $\tau = 1$ for simplicity and we write $F$ instead of $F_1$ (the result below, and in particular the bounds (5.3) and the limits (5.4), hold for all $\tau > 0$). Proposition 4.2 follows from the following result.

PROPOSITION 5.1. *The unique bounded solution $\Phi$ of*

$$\Phi'' - V\Phi' - \Phi + \psi = 0 \tag{5.1}$$

*is given by*

$$\Phi(V,z) = \int_{\mathbb{R}} G(V,z-s)\psi(s)\,ds, \tag{5.2}$$

*where $G$ is the (Green) function*

$$G(V,z) = \begin{cases} \dfrac{1}{2\nu}e^{(\mu+\nu)z} & \text{for } z<0,\\ \dfrac{1}{2\nu}e^{(\mu-\nu)z} & \text{for } z>0,\end{cases}$$

*with $\mu = \frac{V}{2}$ and $\nu = \sqrt{\left(\frac{V}{2}\right)^2+1}$. It satisfies in particular*

$$0 \le \Phi(V,z) \le 1 \qquad \forall z\in\mathbb{R},\ V\in\mathbb{R}. \tag{5.3}$$

The proof of this proposition is straightforward. We note in particular that $\mu+\nu > 0$ and $\mu-\nu < 0$ are the two roots of the characteristic polynomial $r^2 - Vr - 1 = 0$. The inequalities (5.3) follow from (5.1) and the maximum principle or can be checked directly from the explicit formula (5.2) by using the fact that $0 \le \psi \le 1$ and $G \ge 0$. Indeed,

$$\begin{aligned} 0 \le \Phi(V,z) &\le \frac{1}{2\nu}\int_{-\infty}^{z} e^{-(\nu-\mu)(z-s)}\,ds + \frac{1}{2\nu}\int_{z}^{+\infty} e^{-(\nu+\mu)(s-z)}\,ds \\ &\le \frac{1}{2\nu(\nu-\mu)} + \frac{1}{2\nu(\nu+\mu)} \\ &\le \frac{1}{\nu^2-\mu^2} = 1. \end{aligned}$$

Next, we are interested in the behavior of $\Phi(V,z)$ with respect to $V$. We start with the following.

Proposition 5.2. *The function $\Phi(V,z)$ defined by* (5.2) *satisfies* $0 \le \Phi(V,z) \le 1$ *for all $z \in \mathbb{R}$ and*

$$\lim_{V\to+\infty} \Phi(V,z) = 0, \qquad \lim_{V\to-\infty} \Phi(V,z) = 1 \qquad \forall\ z \in \mathbb{R}. \tag{5.4}$$

*Proof.* We will prove only the second limit (the first one is proved similarly). We note that

$$\nu(V) \sim \frac{1}{2}|V|, \quad \mu(V) = -\frac{1}{2}|V|, \quad \nu+\mu \sim \frac{1}{|V|} \qquad \text{as } V \to -\infty. \tag{5.5}$$

We split the integral in (5.2) into two parts:

$$\Phi(V,z) = \int_{-\infty}^{z} G(V,z-s)\psi(s)\,ds + \int_{z}^{+\infty} G(V,z-s)\psi(s)\,ds.$$

Since $|\psi(s)| \le 1$ for all $s$, we immediately get

$$\begin{aligned} \int_{-\infty}^{z} G(V,z-s)\psi(s)\,ds &\le \int_{-\infty}^{z} G(V,z-s)\,ds \\ &= \int_{0}^{+\infty} G(V,s)\,ds \\ &= \frac{1}{2\nu(\nu-\mu)} \sim \frac{2}{|V|(|V|+|V|)}, \end{aligned}$$

which converges to 0 as $V \to -\infty$. For the other piece of the integral, we write

$$\int_{z}^{+\infty} G(V,z-s)\psi(s)\,ds = \int_{z}^{+\infty} G(V,z-s)\,ds + \int_{z}^{+\infty} G(V,z-s)[\psi(s)-1]\,ds,$$

where (using (5.5))

$$\int_{z}^{+\infty} G(V,z-s)\,ds = \frac{1}{2\nu(\nu+\mu)} \to 1 \text{ as } V \to -\infty,$$

and (using the fact that $|1-\psi(s)| \le e^{-\alpha s}$; see (4.18))

$$\begin{aligned} \left|\int_{z}^{+\infty} G(V,z-s)[\psi(s)-1]\,ds\right| &\le \int_{z}^{+\infty} \frac{1}{2\nu} e^{(\mu+\nu)(z-s)} e^{-\alpha s}\,ds \\ &= \int_{0}^{+\infty} \frac{1}{2\nu} e^{-(\mu+\nu)y} e^{-\alpha(y+z)}\,dy \end{aligned}$$

$$= e^{-\alpha z} \int_0^{+\infty} \frac{1}{2\nu} e^{-(\mu+\nu+\alpha)y}\, dy$$
$$= e^{-\alpha z} \frac{1}{2\nu(\mu+\nu+\alpha)},$$

which converges to zero as $V \to -\infty$ (using (5.5)). Putting the pieces together, we have thus proved that

$$\lim_{V\to-\infty} \Phi(V,z) = 1 \qquad \forall\, z \in \mathbb{R}.$$ ∎

Finally, we recall that $F(V)$ is defined by

$$F(V) = \int_{-\infty}^{+\infty} \Phi(V,z)\psi'(z)\, dz$$

and we turn our attention to the proof of Proposition 1.4, which we split into two lemmas.

LEMMA 5.3. *The function $V \mapsto F(V)$ is differentiable and satisfies $0 < F(V) < 1$ for all $V \in \mathbb{R}$ and*

$$\lim_{V\to+\infty} F(V) = 0, \quad \lim_{V\to-\infty} F(V) = 1.$$

*Proof.* The differentiability with respect to $V$ follows easily from the explicit formula (5.2) for $\Phi$. Also, since $0 < \Phi(V,z) < 1$ and $\psi'(z) > 0$, we clearly have

$$0 < F(V) = \int_{-\infty}^{+\infty} \Phi(V,z)\psi'(z)\, dz < \int_{-\infty}^{+\infty} \psi'(z)\, dz = 1.$$

Next, we note that

$$\lim_{V\to-\infty} \Phi(V,z)\, \psi'(z) = \psi'(z)$$

for all $z \in \mathbb{R}$ and, using (5.3) and the fact that $\psi'(z) > 0$, we have

$$0 \le \Phi(V,z)\, \psi'(z) \le \psi'(z).$$

Using the Lebesgue dominated convergence theorem and the fact that

$$\int_{-\infty}^{\infty} \psi'(z)\, dz = 1,$$

we deduce

$$\lim_{V\to-\infty} F(V) = 1.$$

The other limit is proved similarly. ∎

Finally, we prove the following lemma, which completes the proof of Proposition 1.4.

LEMMA 5.4. *The function $F(V)$ satisfies $F'(V) < 0$ for all $V \in \mathbb{R}$.*

*Proof.* This can be proved directly using the explicit formula (5.2) for $\Phi$, or by use of the maximum principle. For example, by differentiating (5.1) with respect to $z$, we find that the function $\xi(z) = \partial_z \Phi(V,z)$ solves

$$\xi'' - V\xi' - \xi = -\psi'.$$

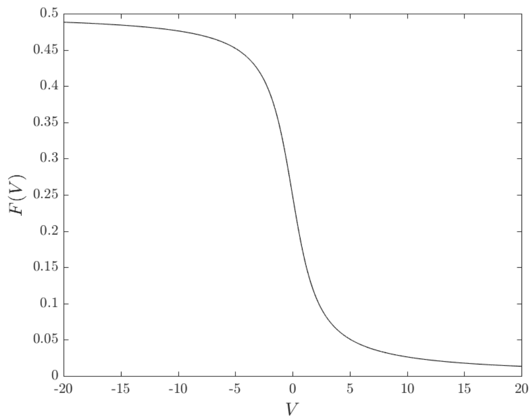


FIG. 1. *Graphical representation of the function* $V \mapsto F(V)$ *for* $V \in [-20, 20]$ *with* $W(\rho) = \rho^2(1-\rho^2)$.

Since $\psi' \geq 0$, the maximum principle (noting that $\lim_{z\to\pm\infty} \xi = 0$) yields $\partial_z \Phi(V, z) \geq 0$ for all $V$ and $z$ in $\mathbb{R}$. By differentiating (5.1) with respect to $V$, we then find that the function $\zeta(z) = \partial_V \Phi(V, z)$ satisfies

$$\zeta'' - V\zeta' - \zeta = \partial_z \Phi(V, z) \geq 0, \tag{5.6}$$

and the maximum principle (noting that $\lim_{z\to\pm\infty} \zeta = 0$) implies that $\partial_V \Phi(V, z) \leq 0$ for all $V$ and $z$ in $\mathbb{R}$. It easily follows that $F'(V) \leq 0$.

Alternatively, we note that (5.6) implies

$$\begin{aligned}\partial_V \Phi(V, z) &= -\int_{-\infty}^{+\infty} G(V, z-s)\partial_z \Phi(V, z)\, ds \\ &= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(V, z-s)G(V, s-t)\psi'(t)\, dt\, ds,\end{aligned}$$

and so

$$F'(V) = -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} G(V, z-s)G(V, s-t)\psi'(t)\psi'(z)\, dt\, ds\, dz,$$

which is negative since every term inside the integral is positive. ∎

## 6. The asymptotic model in dimension 1.

In this section, we investigate the properties of the asymptotic problem (1.14) in dimension $n = 1$. This is considerably simpler than the physical case $n = 2$, but we will see that some interesting behavior can already be observed in that case.

As mentioned in the introduction, in dimension 1, there is no mechanism that could split a cell, so we are interested in solutions for which $\Sigma(t)$ is an interval $(a(t), b(t))$. Furthermore, it is easy to check that the measure of $\Sigma(t)$ is preserved by (1.14) (this is a consequence of the conservation of mass $\frac{d}{dt}\int \rho^\varepsilon\, dx = 0$). Thus, if we denote $\ell = |\Sigma(t)|$, we get

$$\Sigma(t) = (a(t), b(t)), \qquad b(t) = a(t) + \ell$$

and the normal velocity is given by $-a'(t)$ at the left end boundary point and by $a'(t)$ at the right end boundary point.

Since there is no curvature effect in dimension 1, (4.2) for $q(x,t)$ reduces to

$$\begin{cases} -\Delta q = 0 & \text{in } \Sigma(t), \\ q = \beta F(-(1-\alpha)\nabla q \cdot n) & \text{on } \partial\Sigma(t). \end{cases}$$

So $q(x,t)$ is a linear function of the form $q(x,t) = s(t)x + C(t)$ satisfying the boundary conditions

$$q(b(t),t) = \beta F(-(1-\alpha)s(t)), \qquad q(a(t),t) = \beta F((1-\alpha)s(t)).$$

We see that this is possible if and only if $s(t)$ is such that

$$s(t)\ell = \beta\Big[F(-(1-\alpha)s(t)) - F((1-\alpha)s(t))\Big]. \tag{6.1}$$

When $\alpha = 1$, this yields a unique solution $s(t) = 0$. Since

$$V(b(t),t) = -V(a(t),t) = -\partial_x q(b(t),t) = -s(t)$$

this correspond to the stationary solution. But for $\alpha \in [0,1)$, (6.1) is equivalent to

$$(1-\alpha)s(t) \in \mathcal{S}_{\frac{\beta(1-\alpha)}{\ell}}, \tag{6.2}$$

where $\mathcal{S}_\gamma$ denotes the set

$$\mathcal{S}_\gamma = \big\{s \in \mathbb{R} \text{ such that } s = \gamma\big[F(-s) - F(s)\big]\big\}. \tag{6.3}$$

It is clear that we always have $0 \in \mathcal{S}_\gamma$ and so (1.14) has at least one solution (the stationary solution). However, we can prove that when $\gamma$ is large enough, then the set $\mathcal{S}_\gamma$ includes other values.

PROPOSITION 6.1. *There exists a critical $\gamma_c > 0$ such that the following holds:*

- *If $\gamma \le \gamma_c$, then $\mathcal{S}_\gamma = \{0\}$.*
- *If $\gamma > \gamma_c$, then $\mathcal{S}_\gamma \supset \{-s(\gamma), 0, s(\gamma)\}$ for some $s(\gamma) > 0$.*

This proposition proves that a bifurcation phenomenon holds: if $\frac{\beta(1-\alpha)}{\ell} \le \gamma_c$, then (6.2) (and thus the asymptotic problem (4.2)) has only one (stationary) solution, but when $\frac{\beta(1-\alpha)}{\ell} \ge \gamma_c$, there are (at least) two additional solutions, which are the traveling wave like solution moving with constant speed to the left or to the right. We show in Figure 2 the graphical representation of the set $\mathcal{S}_\gamma$ defined in (6.3) for $\gamma \in [0,10]$.

*Proof of Proposition* 6.1. We introduce the function

$$T(s) = F(-s) - F(s).$$

The set $\mathcal{S}_\gamma$ is then the set of solutions $s \in \mathbb{R}$ of the equation

$$\gamma T(s) = s. \tag{6.4}$$

Since $T$ is odd, the nonzero solutions come in pairs so we can focus on the positive solutions. We note that $T(0) = 0$ and Lemma 5.3 implies

$$\lim_{s\to\infty} T(s) = 1. \tag{6.5}$$

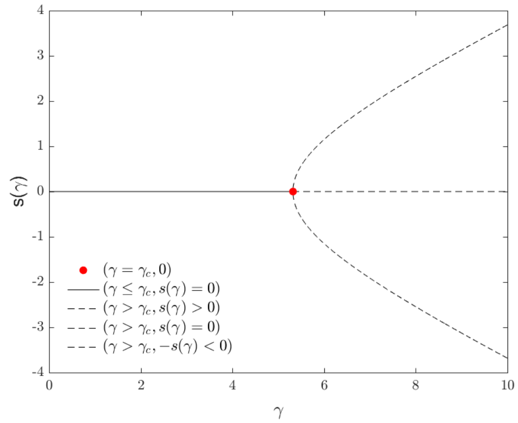


FIG. 2. *Graphical representation of the points $s = s(\gamma)$ belonging to the set $\mathcal{S}_\gamma$ defined in* (6.3) *by varying $\gamma \in [0, 10]$. The function $[F(-s) - F(s)]$ was computed numerically. The red dot in the intersection of the graphs represents the (numerical) bifurcation value $\gamma_c$. For $\gamma \leq \gamma_c$, there is only one point $s(\gamma) = 0$ in $\mathcal{S}_\gamma$, while for $\gamma > \gamma_c$ there are (at least in the interval $(\gamma_c, 10]$) three points $\{-s(\gamma), 0, s(\gamma)\}$ for some $s(\gamma) > 0$ beloging to $\mathcal{S}_\gamma$. In particular, the (numerical) bifurcation value is $\gamma_c \simeq 5.3$.*

To prove Proposition 6.1, we can, for instance, define the map $h(\gamma) = \min_{s>0}(\frac{1}{\gamma}s - T(s))$. It is well defined since $\lim_{s\to\infty} \frac{1}{\gamma}s - T(s) = +\infty$ and the function $h$ is clearly monotone decreasing. Furthermore, if we pick $\bar{s} > 0$ so that $T(\bar{s}) \geq 1/2$ (which we can do thanks to (6.5)), then we have $h(\gamma) \leq 0$ as soon as $\gamma > 2\bar{s}$. We then define

$$\gamma_c = \min\{\gamma \geq 0 \text{ s.t. } h(\gamma) \leq 0\} < \infty,$$

and check that the result holds.

We can in fact be more precise. The monotonicity of $F$ implies that $T(s) > 0$ for $s > 0$ (and $T(s) < 0$ for $s < 0$). The relation (6.4) thus defines a unique $\gamma > 0$ such that $s \in \mathcal{S}_\gamma$ for all $s \neq 0$:

$$\gamma(s) = \frac{s}{T(s)}.$$

This implies in particular that

$$\gamma_c = \lim_{s\to 0} \frac{s}{T(s)} = \frac{1}{T'(0)} = \frac{-1}{2F'(0)} > 0. \qquad \square$$

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