INVERSE RANDOM SOURCE SCATTERING FOR THE HELMHOLTZ EQUATION WITH ATTENUATION*

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Abstract. In this paper, a new model is proposed for the inverse random source scattering problem of the Helmholtz equation with attenuation. The source is assumed to be driven by a fractional Gaussian field whose covariance is represented by a classical pseudo-differential operator. The work contains three contributions. First, the connection is established between fractional Gaussian fields and rough sources characterized by their principal symbols. Second, the direct source scattering problem is shown to be well-posed in the distribution sense. Third, we demonstrate that the micro-correlation strength of the random source can be uniquely determined by the passive measurements of the wave field in a set which is disjoint with the support of the strength function. The analysis relies on careful studies on the Green function and Fourier integrals for the Helmholtz equation.

Key words. Inverse scattering problem, the Helmholtz equation, random source, fractional Gaussian field, pseudo-differential operator, principal symbol

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1. Introduction. The inverse source scattering in waves is an important and active research subject in inverse scattering theory. It is an important mathematical tool for the solution of many medical imaging modalities [3, 14]. The inverse source scattering problems are to determine the unknown sources that generate prescribed wave patterns. These problems have attracted much research. The mathematical and numerical results can be found in [7, 8, 18] and the references cited therein.

Stochastic modeling is widely introduced to mathematical systems due to unpredictability of the environments, incomplete knowledge of the systems and measurements, and fine-scale fluctuations in simulation. In many situations, the source, hence the wave field, may not be deterministic but are rather modeled by random processes [13]. Due to the extra challenge of randomness and uncertainties, little is known for the inverse random source scattering problems.

In this paper, we consider the Helmholtz equation with a random source

$$\Delta u + (k^2 + {\rm i}k\sigma)u = f, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $d = 2$ or 3, $k > 0$ is the wavenumber, the attenuation coefficient $\sigma \geq 0$ describes the electrical conductivity of the medium, u denotes the wave field, and f is a random function representing the electric current density.

In [4], the white noise model was studied for the inverse random source problem of the stochastic Helmholtz equation without attenuation

$$\Delta u + k^2 u = g + h\dot{W}, \quad x \in \mathbb{R}^d,$$

where g and h are deterministic and compactly supported functions, and $\dot{W}$ is the spatial white noise. It was shown that g and h can be determined by statistics of the wave fields at multiple frequencies. The white noise model can also be found in [6] and [5] for the one-dimensional problem and the stochastic elastic wave equation,

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respectively. Recently, the model of a generalized Gaussian field was developed to handle random processes [9, 15]. The random function is said to be microlocally isotropic of order $2s$ if the covariance operator is a pseudo-differential operator with principal symbol given by $\mu(x)|\xi|^{-2s}$, where $\mu \geq 0$ is a smooth and compactly support function and is called the micro-correlation strength of the random function. It was shown that μ can be uniquely determined by the wave field averaged over the frequency band at a single realization of the random function. This model was also investigated in [20, 21] for the inverse random source problems of the elastic wave equation and the Helmholtz equation without attenuation. In these work, the parameter $s \in [\frac{d}{2}, \frac{d}{2}+1)$ and the random functions are smoother than the white noise (cf. Lemma 2.6): it can be interpreted as a distribution in $W^{-\epsilon,p}(\mathbb{R}^d)$ for any $\epsilon > 0$ and $p \in (1, \infty)$ if $s = \frac{d}{2}$; it is a function in $C^{0,\alpha}(\mathbb{R}^d)$ for any $\alpha \in (0, s - \frac{d}{2})$ if $s \in (\frac{d}{2}, \frac{d}{2}+1)$.

In this work, we consider a new model for the Helmholtz equation (1.1), where the random source f is driven by a fractional Gaussian field with $s \in [0, \frac{d}{2}+1)$. There are three contributions. First, we demonstrate that the fractional Gaussian fields include the classical fractional Brownian fields. Moreover, we establish the connection between the fractional Gaussian fields and rough sources characterized by their principal symbols. Second, we examine the regularity of the random source and show that the direct scattering problem is well-posed in the distribution sense. Third, for the inverse problem, we prove that the strength of the random source μ can be uniquely determined by the high frequency limit of the second moment of the wave field. In particular, if $\sigma = 0$, the strength μ can also be determined uniquely by the amplitude of the wave field averaged over the frequency band at a single realization of the random source. It is worthy to be pointed out that (1) if $s \in [0, \frac{d}{2}]$, the random function is a distribution in $f \in W^{s-\frac{d}{2}-\epsilon,p}$ for any $\epsilon > 0$ and $p \in (1, \infty)$ (cf. Lemma 2.6), which is rougher than those considered in [9, 15, 20, 21]; (2) if $\sigma = 0$ and $s \in [\frac{d}{2}, \frac{d}{2}+1)$, the results obtained in this paper coincides with the ones given in [20].

The paper is organized as follows. In Section 2, the random source model is introduced. The relationship is established between the fractional Gaussian field and the classical fractional Brownian motion; the regularity is studied for the random source. Section 3 addresses the well-posedness and regularity of the solution for the direct problem. The inverse problem is discussed in Section 4, where the two- and three-dimensional problems are considered separately. The paper is concluded with some general remarks and directions for future work in Section 5.

2. Random source. In this section, we give a general description of the random source on $\mathbb{R}^d$. Let f be a real-valued centered random field defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Introduce the following Sobolev spaces. The details can be found in [2].

- $W^{s,p} := W^{s,p}(\mathbb{R}^d)$ for $s \in \mathbb{R}$ and $p \in (1, \infty)$. In particular, if $p = 2$, denote $H^s := W^{s,2}$.
- Denote by $W^{s,p}_{\rm loc}$ the space of functions which are locally in $W^{s,p}$. More precisely, for any precompact subset $\mathcal{O} \subset \mathbb{R}^d$, $u|_{\mathcal{O}} \in W^{s,p}(\mathcal{O})$.
- Denote by $W^{s,p}_{\rm comp}$ the space of functions in $W^{s,p}$ with compact support.
- Denote by $W^{s,p}_0(\mathcal{O})$ the closure of $C_0^\infty(\mathcal{O})$ in $W^{s,p}(\mathcal{O})$ with $\mathcal{O} \subset \mathbb{R}^d$. In particular, if $\mathcal{O} = \mathbb{R}^d$, $W^{s,p}_0 = W^{s,p}$.

Let $f : \Omega \to \mathcal{S}'$ be measurable such that the mapping $\omega \mapsto \langle f(\omega), \phi \rangle$ defines a Gaussian random variable for any $\phi \in C_0^\infty$. Here, $\mathcal{S}'$ is the space of distributions on $\mathbb{R}^d$, which is the dual space of the Schwartz space $\mathcal{S}$. The covariance operator

$Q_f : C_0^\infty \to \mathcal{S}'$ is given by

$$\langle \varphi, Q_f \psi \rangle = \mathbb{E}[\langle f, \varphi\rangle \langle f, \psi\rangle] \quad \forall\, \varphi, \psi \in C_0^\infty,$$

where $\langle\cdot,\cdot\rangle$ denotes the dual product. Denote by $K_f(x,y)$ the Schwartz kernel of Q_f, which satisfies

$$\langle \varphi, Q_f \psi \rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_f(x,y)\varphi(x)\psi(y)dxdy.$$

Hence we have the following formal expression of the Schwartz kernel:

$$K_f(x,y) = \mathbb{E}[f(x)f(y)].$$

Assumption 2.1. *The source f is assumed to have a compact support contained in $\mathcal{D} \subset \mathbb{R}^d$. The covariance operator Q_f of f is a classical pseudo-differential operator with the principal symbol $\mu(x)|\xi|^{-2s}$, where $s \in \left[0, \frac{d}{2}+1\right)$ and $0 \le \mu \in C_0^\infty(\mathcal{D})$.*

The positive function μ stands for the micro-correlation strength of the random field f. The assumption implies that the covariance operator Q_f satisfies

$$(Q_f\psi)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot\xi} c(x,\xi)\hat{\psi}(\xi)d\xi \quad \forall\, \psi \in C_0^\infty,$$

where the symbol $c(x,\xi)$ has the leading term $\mu(x)|\xi|^{-2s}$ and

$$\hat{\psi}(\xi) = \mathcal{F}[\psi](\xi) = \int_{\mathbb{R}^d} e^{-\mathrm{i}x\cdot\xi}\psi(x)dx$$

is the Fourier transform of ψ [16, 17]. By the expression of $Q_f\psi$, we can deduce the relationship between the kernel K_f and the symbol $c(x,\xi)$. In fact, noting that

$$\begin{aligned}
\langle \varphi, Q_f\psi\rangle &= \int_{\mathbb{R}^d}\varphi(x)\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot\xi}c(x,\xi)\hat{\psi}(\xi)d\xi\right]dx\\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\varphi(x)\int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot\xi}c(x,\xi)\left[\int_{\mathbb{R}^d} e^{-\mathrm{i}y\cdot\xi}\psi(y)dy\right]d\xi dx\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left[\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\mathrm{i}(x-y)\cdot\xi}c(x,\xi)d\xi\right]\varphi(x)\psi(y)dxdy,
\end{aligned}$$

we get that the kernel K_f is an oscillatory integral of the form

$$K_f(x,y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\mathrm{i}(x-y)\cdot\xi}c(x,\xi)d\xi. \tag{2.1}$$

2.1. Fractional Gaussian fields. We introduce the fractional Gaussian fields, which can be used to generate random fields satisfying Assumption 2.1.

Definition 2.2. *The fractional Gaussian field h^s on $\mathbb{R}^d$ with parameter $s \in \mathbb{R}$ is given by*

$$h^s := (-\Delta)^{-\frac{s}{2}}\dot{W},$$

where $(-\Delta)^{-\frac{s}{2}}$ is the fractional Laplacian on $\mathbb{R}^d$ defined by

$$(-\Delta)^{\alpha}u = \mathcal{F}^{-1}\left[|\xi|^{2\alpha}\mathcal{F}[u](\xi)\right], \quad \alpha \in \mathbb{R}, \tag{2.2}$$

and $\dot W \in \mathcal{S}'$ *is the white noise on* $\mathbb{R}^d$ *determined by the covariance operator* $Q_{\dot W}$: $L^2 \to L^2$ *as follows:*

$$\langle \varphi, Q_{\dot W}\psi\rangle := \mathbb{E}[\langle \dot W, \varphi\rangle\langle \dot W, \psi\rangle] = (\varphi,\psi)_{L^2} \quad \forall\, \varphi,\psi \in L^2.$$

We denote by $\mathbb{G}_s(\mathbb{R}^d)$ the space of fractional Gaussian fields with parameter s. Let $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$ if h^s is a fractional Gaussian field on $\mathbb{R}^d$ with parameter s. If $d=1$ and $s=1$, h^1 turns to be the classical one-dimensional Brownian motion. If $s=0$, $h^0=\dot W$ is the white noise on $\mathbb{R}^d$. If $s<0$, h^s is even rougher than the white noise. We refer to [24] and references therein for more details about the fractional Gaussian fields and the fractional Laplacian.

To make sense of the expression $h^s = (-\Delta)^{-\frac{s}{2}}\dot W$, we define

$$\mathcal{S}_r := \begin{cases} \{\varphi \in \mathcal{S} : \int_{\mathbb{R}^d} x^\alpha \varphi(x)dx = 0 \quad \forall\, |\alpha| \le r\} & \text{if } r \ge 0\\ \mathcal{S} & \text{if } r<0\end{cases}$$

Denote by T_s the closure of $\mathcal{S}_{s-\frac{d}{2}}$ in H^{-s}. Then the expression $h^s = (-\Delta)^{-\frac{s}{2}}\dot W$ in Definition 2.2 is interpreted by

$$\langle h^s, \varphi\rangle := \langle \dot W, (-\Delta)^{-\frac{s}{2}}\varphi\rangle = \int_{\mathbb{R}^d} (-\Delta)^{-\frac{s}{2}}\varphi(x)dW(x) \quad \forall\, \varphi \in T_s.$$

The kernel K_{h^s} for the covariance operator Q_{h^s} of h^s satisfies

$$\langle \varphi, Q_{h^s}\psi\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_{h^s}(x,y)\varphi(x)\psi(y)dxdy \quad \forall\, \varphi,\psi \in C_0^\infty \cap T_s. \tag{2.3}$$

Moreover, the kernel has the following expression. The proof can be found in [24].

Lemma 2.3. *Let* $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$ *with parameter* $s \in [0,\infty)$. *Denote* $H := s - \frac{d}{2}$.

(i) If $s \in (0,\infty)$ *and* H *is not a nonnegative integer, then*

$$K_{h^s}(x,y) = C_1(s,d)|x-y|^{2H},$$

where $C_1(s,d) = 2^{-2s}\pi^{-\frac{d}{2}}\Gamma(\frac{d}{2}-s)/\Gamma(s)$ *with* $\Gamma(\cdot)$ *being the Gamma function.*

(ii) If $s \in (0,\infty)$ *and* H *is a nonnegative integer, then*

$$K_{h^s}(x,y) = C_2(s,d)|x-y|^{2H}\ln|x-y|,$$

where $C_2(s,d) = (-1)^{H+1}2^{-2s+1}\pi^{-\frac{d}{2}}/(H!\Gamma(s))$.

(iii) If $s=0$, *then*

$$K_{h^s}(x,y) = \delta(x-y),$$

where $\delta(\cdot)$ *is the Dirac delta function centered at* 0.

2.2. Relationship with classical fractional Brownian fields. For any $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$, we define its generalized Hurst parameter $H = s - \frac{d}{2}$. If $s \in (\frac{d}{2}, \frac{d}{2}+1)$, h^s coincides with the classical fractional Brownian fields B^H determined by the covariance operator Q_{B^H}:

$$\langle \varphi, Q_{B^H}\psi\rangle = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{2}\left[|x|^{2H} + |y|^{2H} - |x-y|^{2H}\right]\varphi(x)\psi(y)dxdy, \tag{2.4}$$

where the Hurst parameter $H \in (0,1)$.

Lemma 2.4. *Let $s \in (\frac{d}{2}, \frac{d}{2}+1)$ and $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$. Then the stochastic process defined by*

$$\tilde{h}^s(x) = \langle h^s, \delta_x - \delta_0 \rangle$$

has the same distribution as the fractional Brownian field B^H with $H = s - \frac{d}{2} \in (0,1)$ up to a multiplicative constant, where $\delta_x(\cdot) \in H^{-s}$ is the Dirac measure centered at $x \in \mathbb{R}^d$.

Proof. By Theorem 2.3, the kernel of the covariance operator reads

$$\begin{aligned}
\mathbb{E}[\tilde{h}^s(x)\tilde{h}^s(y)] &= \mathbb{E}[\langle h^s, \delta_x - \delta_0\rangle \langle h^s, \delta_y - \delta_0\rangle] \\
&= C_1(s,d) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |r_1 - r_2|^{2H} (\delta_x - \delta_0)(r_1)(\delta_y - \delta_0)(r_2) dr_1 dr_2 \\
&= C_1(s,d)\left(|x-y|^{2H} - |x|^{2H} - |y|^{2H}\right),
\end{aligned}$$

which is a scalar multiple of the kernel of the covariance operator Q_{B^H} defined in (2.4). The result then follows from the fact that the distribution of a centered Gaussian random field is unique determined by its covariance operator. □

Note that $\langle h^s, \delta_x - \delta_0\rangle$ is actually a translation of h^s. It indicates that we can identify $h^s \sim \mathbb{G}_s(\mathbb{R}^d)$ as the fractional Brownian field B^H with $H = s - \frac{d}{2} \in (0,1)$ by fixing its value to be zero at the origin. Define a random function

$$f(x,\omega) := a(x) h^s(x,\omega), \quad x \in \mathbb{R}^d,\ \omega \in \Omega, \tag{2.5}$$

where $s \in (\frac{d}{2}, \frac{d}{2}+1)$ and $a \in C_0^\infty$ with $\mathrm{supp}(a) \subset \mathcal{D}$. We claim that such an f defined above satisfies Assumption 2.1. More precisely, the covariance operator Q_f of f has the principal symbol $a^2(x)|\xi|^{-2s}$ up to a multiplicative constant.

Proposition 2.5. *The random field f defined in* (2.5) *with $s \in [0, \frac{d}{2}+1)$ satisfies Assumption 2.1 with $\mu = a^2$.*

Proof. According to the expression of the kernel (2.1) and Definition 2.2, the covariance operator Q_{h^s} of h^s satisfies

$$\begin{aligned}
\langle \varphi, Q_{h^s}\psi\rangle &= \mathbb{E}\left[\langle h^s, \varphi\rangle \langle h^s, \psi\rangle\right] = \mathbb{E}\left[\int_{\mathbb{R}^d} (-\Delta)^{-\frac{s}{2}}(a\varphi) dW \int_{\mathbb{R}^d} (-\Delta)^{-\frac{s}{2}}(a\psi) dW\right] \\
&= \int_{\mathbb{R}^d} (-\Delta)^{-\frac{s}{2}}(a\varphi)(-\Delta)^{-\frac{s}{2}}(a\psi) dx \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \overline{\mathcal{F}\left[(-\Delta)^{-\frac{s}{2}}(a\varphi)\right](\xi)}\mathcal{F}\left[(-\Delta)^{-\frac{s}{2}}(a\psi)\right](\xi) d\xi,
\end{aligned}$$

where the Plancherel theorem is used in the last step. It follows from the definition of the fractional Laplacian given in (2.2) that we get

$$\begin{aligned}
\langle \varphi, Q_{h^s}\psi\rangle &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{-2s} \overline{\widehat{(a\varphi)}(\xi)}\widehat{(a\psi)}(\xi) d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} |\xi|^{-2s}\left[\int_{\mathbb{R}^d} a(x)\varphi(x) e^{\mathrm{i}x\cdot\xi} dx\right]\left[\int_{\mathbb{R}^d} a(y)\psi(y) e^{-\mathrm{i}y\cdot\xi} dy\right] d\xi \\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\psi(y) e^{\mathrm{i}(x-y)\cdot\xi} a^2(x)|\xi|^{-2s} d\xi dx dy \\
&\quad - \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\psi(y) e^{\mathrm{i}(x-y)\cdot\xi} a(x)(a(x)-a(y))|\xi|^{-2s} d\xi dx dy \\
&:= I_1 + I_2.
\end{aligned}$$

Noting that $a(x)-a(y)=a'(\theta x+(1-\theta)y)(x-y)$ for some $\theta\in(0,1)$ and

$$\int_{\mathbb{R}^d} e^{\mathrm{i}(x-y)\cdot\xi}|\xi|^{-2s}d\xi=(-\Delta)^{-s}\delta(x-y),$$

we obtain that the term I_2 is more regular than the term I_1. The proof is completed by comparing the term I_1 with (2.1). □

2.3. Regularity of random sources. By Proposition 2.5, for any function f satisfying Assumption 2.1 with parameter $s\in[0,\frac{d}{2}+1)$, its principal symbol has the same order as the principal symbol of the random field ah^s. Without loss of generality, we only need to investigate the regularity of random fields given by $f=ah^s$, where $a\in C_0^\infty$ and $\mathrm{supp}(a)\subset\mathcal{D}$. Moreover, we assume that f is a centered random field to avoid using the modification $\langle h^s,\delta_x-\delta_0\rangle$.

Lemma 2.6. *Let $s\in[0,\frac{d}{2}+1)$ and $h\sim\mathbb{G}_s(\mathbb{R}^d)$. Define the random field $f:=ah^s$ with $a\in C_0^\infty$ and $supp(a)\subset\mathcal{D}$.*

(i) If $s\in(\frac{d}{2},\frac{d}{2}+1)$, then $f\in C^{0,\alpha}$ a.s. for all $\alpha\in(0,s-\frac{d}{2})$.

(ii) If $s\in[0,\frac{d}{2}]$, then $f\in W^{s-\frac{d}{2}-\epsilon,p}$ a.s. for any $\epsilon>0$ and $p\in(1,\infty)$.

Proof. (i) If $s\in(\frac{d}{2},\frac{d}{2}+1)$, it follows from Lemma 2.4 that f has the same distribution as aB^H, where B^H is a classical fractional Brownian field on $\mathbb{R}^d$ with the Hurst parameter $H=s-\frac{d}{2}\in(0,1)$. Note that B^H is $(H-\epsilon)$-Hölder continuous for any $\epsilon\in(0,H)$. Hence, $f\in C^{0,\alpha}$ with $\alpha\in(0,H)=(0,s-\frac{d}{2})$.

(ii) We first consider the case $s=\frac{d}{2}$ and hence $H=s-\frac{d}{2}=0$. By Lemma 2.3, the covariance operator Q_f satisfies

$$\begin{aligned}\langle\varphi,Q_f\psi\rangle&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}K_f(x,y)\varphi(x)\psi(y)dxdy\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}a(x)a(y)K_{h^s}(x,y)\varphi(x)\psi(y)dxdy\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}C_2(s,d)a(x)a(y)\ln|x-y|\varphi(x)\psi(y)dxdy\end{aligned}$$

for all $\varphi,\psi\in C_0^\infty\cap T_s$. We may choose $K_f(x,y)=C_2(s,d)a(x)a(y)\ln|x-y|$ in this case. Following a similar proof to that of Theorem 2 in [19], we consider the Bessel potential operator $\mathcal{J}_\epsilon:=(I-\Delta)^{-\frac{\epsilon}{2}}$ with $\epsilon>0$, where I is the identify operator. It can be expressed through the kernel in the form $G_\epsilon(x,y)=C(\epsilon,d)|x-y|^{-d+\epsilon}+S(x,y)$ such that

$$\mathcal{J}_\epsilon u=\int_{\mathbb{R}^d}G_\epsilon(x,y)u(y)dy,$$

where $C(\epsilon,d)$ is a constant depending on ϵ and d, and $S(x,y)$ is the more regular residual. Note that $\mathcal{J}_\epsilon:W^{t,p}\to W^{t+\epsilon,p}$ is an isomorphism for $t\in\mathbb{R}$ and $p\in(1,\infty)$ (see e.g. [25, Section 3.3]). It then suffices to show that $\mathcal{J}_\epsilon f\in L^p$ a.s. for any $\epsilon>0$ and $p\in(1,\infty)$. In fact, this result is obvious since the kernel of $\mathcal{J}_\epsilon f$ is uniformly bounded:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|a(x)a(y)\frac{\ln|x-y|}{|x-y|^{d-\epsilon}}\right|dxdy\lesssim\int_0^R\left|\frac{\ln r}{r^{d-\epsilon}}\right|r^{d-1}dr<\infty$$

for some positive number R such that $\mathcal{D}\subset B(0,R)$.

If $s\in[0,\frac{d}{2})$, the result can be obtained directly by noticing $f=a(-\Delta)^{-\frac{s}{2}+\frac{d}{4}}\tilde{f}$ with $\tilde{f}:=(-\Delta)^{-\frac{d}{4}}\dot{W}\in W^{-\epsilon,p}$ and the result obtained above for $s=\frac{d}{2}$. □

3. Direct scattering problem. This section is to investigate the well-posedness and study the regularity of the solution for the direct scattering problem.

3.1. Fundamental solution. Let $\kappa^2 = k^2 + {\rm i}k\sigma$. A simple calculation yields that

$$\Re[\kappa] = \kappa_{\rm r} = \left(\frac{\sqrt{k^4 + k^2\sigma^2} + k^2}{2}\right)^{\frac{1}{2}}, \quad \Im[\kappa] = \kappa_{\rm i} = \left(\frac{\sqrt{k^4 + k^2\sigma^2} - k^2}{2}\right)^{\frac{1}{2}},$$

and

$$\lim_{k\to\infty} \frac{\kappa_{\rm r}}{k} = 1, \quad \lim_{k\to\infty} \kappa_{\rm i} = \frac{\sigma}{2}. \tag{3.1}$$

Then the Helmholtz equation (1.1) can be written as

$$\Delta u + \kappa^2 u = f \quad \text{in } \mathbb{R}^d. \tag{3.2}$$

The Helmholtz equation (3.2) with a complex-valued wavenumber has the fundamental solution

$$\Phi_\kappa(x,y) = \begin{cases} \frac{\rm i}{4} H_0^{(1)}(\kappa|x-y|), & d=2, \\ \frac{1}{4\pi}\frac{e^{{\rm i}\kappa|x-y|}}{|x-y|}, & d=3, \end{cases}$$

where $H_0^{(1)}$ is the Hankel function of the first kind with order 0.

Lemma 3.1. *For any given $x \in \mathbb{R}^d$, it holds that $\Phi_\kappa(x,\cdot) \in W^{1,p}_{\rm loc}$, where $p \in (1,2)$ if $d = 2$ and $p \in (1, \frac{3}{2})$ if $d = 3$.*

Proof. Let $D \subset \mathbb{R}^d$ be any bounded domain. Denote $r^* := \sup_{y\in D}|x-y|$, then $D \subset B_{r^*}(x)$.

For $d = 2$, we have $\Phi_\kappa(x,y) = \frac{\rm i}{4}H_0^{(1)}(\kappa|x-y|)$. It suffices to show that $H_0^{(1)}(\kappa|x-\cdot|) \in L^p(D)$ and $D_y^\alpha H_0^{(1)}(\kappa|x-\cdot|) \in L^p(D)$ with $0 < |\alpha| \le 1$. Note that

$$\left|H_\nu^{(1)}(z)\right| \le e^{-\Im[z]\left(1-\frac{\Theta^2}{|z|^2}\right)^{\frac{1}{2}}} \left|H_\nu^{(1)}(\Theta)\right|$$

for any $\nu \in \mathbb{R}$ and any real number Θ satisfying $0 < \Theta \le |z|$ (cf. [11, Lemma 2.2]). By choosing $z = \kappa|x-y|$ and $\Theta = \Re(z) = \kappa_{\rm r}|x-y|$, on one hand, we have

$$\begin{aligned}\int_D \left|H_0^{(1)}(\kappa|x-y|)\right|^p dy &\le \int_D e^{-p\frac{\kappa_{\rm i}^2}{|\kappa|}|x-y|}\left|H_0^{(1)}(\kappa_{\rm r}|x-y|)\right|^p dy \\ &\lesssim \int_0^{r^*} e^{-p\frac{\kappa_{\rm i}^2}{|\kappa|}r}\left|H_0^{(1)}(\kappa_{\rm r}r)\right|^p rdr.\end{aligned}$$

For the above integral, since $H_0^{(1)}(\kappa_{\rm r}r) \sim \frac{2{\rm i}}{\pi}\ln(\kappa_{\rm r}r)$ as $r \to 0$ (cf. [1, eq. (9.1.8)]), we only need to consider the integral

$$\int_0^{r^*} e^{-p\frac{\kappa_{\rm i}^2}{|\kappa|}r}|\ln(\kappa_{\rm r}r)|^p rdr < \infty,$$

which leads to $H_0^{(1)}(\kappa|x-\cdot|) \in L^p(D)$. On the other hand, we have

$$\partial_{y_i}H_0^{(1)}(\kappa|x-y|) = \kappa H_0^{(1)'}(\kappa|x-y|)\frac{y_i-x_i}{|x-y|} = -\kappa H_1^{(1)}(\kappa|x-y|)\frac{y_i-x_i}{|x-y|}, \quad i=1,2.$$

Hence

$$\begin{aligned}\int_D \left|\partial_{y_i} H_0^{(1)}(\kappa|x-y|)\right|^p dy =&|\kappa|^p \int_D \left|H_1^{(1)}(\kappa|x-y|)\right|^p \left|\frac{y_i-x_i}{|x-y|}\right|^p dy\\ &\lesssim \int_D e^{-p\frac{\kappa_{\rm i}^2}{|\kappa|}|x-y|}\left|H_1^{(1)}(\kappa_{\rm r}|x-y|)\right|^p dy\\ &\lesssim \int_0^{r^*} e^{-p\frac{\kappa_{\rm i}^2}{|\kappa|}r}\left|H_1^{(1)}(\kappa_{\rm r}r)\right|^p rdr,\end{aligned}$$

where $H_1^{(1)}$ is the Hankel function of the first kind with order 1 and it has the asymptotic expansion $H_1^{(1)}(\kappa_{\rm r}r)\sim\frac{2{\rm i}}{\pi}\frac{1}{\kappa_{\rm r}r}$ as $r\to 0$ (cf. [1, eq. (9.1.9)]). Since

$$\int_0^{r^*} e^{-p\frac{\kappa_{\rm i}^2}{|\kappa|}r}\frac{1}{r^p}rdr<\infty,\quad p\in(1,2),$$

we obtain that $D_y^\alpha H_0^{(1)}(\kappa|x-\cdot|)\in L^p(D)$.

For $d=3$, we have $\Phi_\kappa(x,y)=\frac{1}{4\pi}\frac{e^{{\rm i}\kappa|x-y|}}{|x-y|}$ and $\partial_{y_i}\Phi_\kappa(x,y)=\frac{e^{{\rm i}\kappa|x-y|}(y_i-x_i)}{4\pi|x-y|^3}({\rm i}\kappa|x-y|-1)$, $i=1,2,3$. Noting for $p\in(1,\frac32)$ that

$$\int_D\left|\frac{e^{{\rm i}\kappa|x-y|}}{|x-y|}\right|^p dy\lesssim\int_0^{r^*}\frac{1}{r^p}r^2dr<\infty$$

and

$$\int_D\left|\frac{e^{{\rm i}\kappa|x-y|}(y_i-x_i)}{|x-y|^3}\right|^p dy\lesssim\int_0^{r^*}\frac{1}{r^{2p}}r^2dr<\infty,$$

we complete the proof. □

3.2. Well-posedness and regularity. Using the fundamental solution Φ_κ, we define a volume potential

$$(V_\kappa f)(x):=-\int_{\mathbb{R}^d}\Phi_\kappa(x,y)f(y)dy.$$

The mollifier V_κ has the following property. The proof can be found in [19, 20].

LEMMA 3.2. *Let $\mathcal{O}$ and $\mathcal{U}$ be two bounded domains in $\mathbb{R}^d$. The operator V_κ : $H_0^{-\beta}(\mathcal{O})\to H^\beta(\mathcal{U})$ is bounded for $\beta\in\big(0,2-\frac d2\big]$.*

THEOREM 3.3. *Let $p\in(\frac d2,2]$, $s\in\big(d(\frac1p+\frac12)-2,\frac d2\big]$, and $H=s-\frac d2\in(\frac dp-2,0]$. Assume that $f\in W^{H-\epsilon,p}_{\rm comp}$ for any $\epsilon>0$. Then the scattering problem (1.1) admits a unique solution $u\in W^{-H+\epsilon,q}_{\rm loc}$ a.s. in the distribution sense with q satisfying $\frac1p+\frac1q=1$. Moreover, the solution is given by*

$$u(x;k)=-\int_{\mathbb{R}^d}\Phi_\kappa(x,y)f(y)dy.$$

Proof. We only need to show the existence of the solution since the uniqueness follows directly from the deterministic case. Let $\mathcal{D}$ be a bounded domain such that ${\rm supp}(f)\subset\mathcal{D}$. Then $f\in W^{H-\epsilon,p}(\mathcal{D})$. For any $x\in\mathbb{R}^d$, define the volume potential

$$u_*(x;k):=-\int_{\mathcal{D}}\Phi_\kappa(x,y)f(y)dy=-\int_{\mathbb{R}^d}\Phi_\kappa(x,y)f(y)dy.$$

First we show that u_* is a solution of (1.1) in the distribution sense. In fact, we have for any $v \in C_0^\infty$ that

$$
\begin{aligned}
&\langle \Delta u_* + \kappa^2 u_*, v\rangle = -\langle \nabla u_*, \nabla v\rangle + \kappa^2 \langle u_*, v\rangle \\
&= \int_{\mathbb{R}^d} \nabla_x \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy \Big] \nabla v(x) dx - \kappa^2 \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy \Big] v(x) dx \\
&= -\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta_x \Phi_\kappa(x,y) v(x) f(y) dx dy - \kappa^2 \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy \Big] v(x) dx \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(\kappa^2 \Phi_\kappa(x,y) + \delta(x-y)\big) v(x) f(y) dx dy - \kappa^2 \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy \Big] v(x) dx \\
&= \langle f, v\rangle.
\end{aligned}
$$

It then suffices to show that $u_* \in W_{\rm loc}^{-H+\epsilon,q}$, which is equivalent to show that $\phi u_* \in W^{-H+\epsilon,q}$ for any $\phi \in C_0^\infty$ with support $\mathcal{U} \subset \mathbb{R}^d$. Define a weighted potential

$$
(\tilde{V}_\kappa f)(x) := -\phi(x) \int_{\mathbb{R}^d} \Phi_\kappa(x,y) f(y) dy, \quad x \in \mathcal{U}.
$$

By Lemma 3.2, the operator $\tilde{V}_\kappa : H_0^{-\beta}(\mathcal{D}) \to H^\beta(\mathcal{U})$ is bounded for $\beta \in \big(0, 2 - \frac{d}{2}\big]$. Noting the Sobolev embedding theorem with fractional index that $W^{r,p}$ is embedded continuously into $W^{t,q}$ with $r \ge t$ and $\frac{1}{q} = \frac{1}{p} - \frac{r-t}{d}$, we get that $W_0^{H-\epsilon,p}(\mathcal{D}) \hookrightarrow H_0^{-\beta}(\mathcal{D})$ with $-H+\epsilon \le \beta$ and $-H+\epsilon = d(\frac{1}{2} - \frac{1}{p}) + \beta \in (0, 2 - \frac{d}{p}]$, and $H^\beta(\mathcal{U}) \hookrightarrow W^{-H+\epsilon,q}(\mathcal{U})$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consequently, $\tilde{V}_\kappa : W_0^{H-\epsilon,p}(\mathcal{D}) \to W^{-H+\epsilon,q}(\mathcal{U})$ is bounded, which shows that $\phi u_* = \tilde{V}_\kappa f \in W^{-H+\epsilon,q}$ and completes the proof. ◻

REMARK 3.4. *It follows from Lemma 2.6 that the random source is a continuous function for $s \in (\frac{d}{2}, \frac{d}{2}+1)$. The well-posedness of the scattering problem* (1.1) *is well known since the source f is compactly supported and regular enough [10].*

4. Inverse scattering problem. This section addresses the inverse scattering problem. The goal is to determine the strength μ of the random source f. We discuss the two- and three-dimensional cases, separately.

4.1. Two-dimensional case. First we consider $d = 2$ in which $s \in [0, \frac{d}{2}+1) = [0, 2)$. Recall that the Hankel function has the following asymptotic expansion [1]:

$$
H_0^{(1)}(z) \sim \sum_{j=0}^{\infty} a_j z^{-\left(j+\frac{1}{2}\right)} e^{{\rm i}z}, \quad z \in \mathbb{C},\ |z| \to \infty, \tag{4.1}
$$

where $a_0 = \sqrt{\frac{2}{\pi}} e^{-\frac{{\rm i}\pi}{4}}$ and $a_j = \sqrt{\frac{2}{\pi}} \left(\frac{\rm i}{8}\right)^j \Big(\prod_{l=1}^{j} (2l-1)^2 / j!\Big) e^{-\frac{{\rm i}\pi}{4}}, j \ge 1$. Denoting

$$
H_{0,N}^{(1)}(z) := \sum_{j=0}^{N} a_j z^{-\left(j+\frac{1}{2}\right)} e^{{\rm i}z}, \quad \Phi_\kappa^N(x,y) := \frac{\rm i}{4} H_{0,N}^{(1)}(\kappa|x-y|),
$$

we have

$$
\Phi_\kappa(x,y) = \Phi_\kappa^N(x,y) + O\big(|\kappa|x-y||^{-(N+\frac{3}{2})}\big), \quad N \in \mathbb{N},
$$

as $|\kappa|x-y||\to\infty$ due to $\kappa_{\rm i}>0$. Based on the truncated fundamental solution $\Phi_\kappa^2(x,y)$ by choosing $N=2$, we consider the approximate solution

$$u^2(x;k)=-\int_{\mathbb{R}^2}\Phi_\kappa^2(x,y)f(y)dy=-\frac{{\rm i}a_0}{4}\int_{\mathbb{R}^2}(\kappa|x-y|)^{-\frac12}e^{{\rm i}\kappa|x-y|}f(y)dy$$
$$-\frac{{\rm i}a_1}{4}\int_{\mathbb{R}^2}(\kappa|x-y|)^{-\frac32}e^{{\rm i}\kappa|x-y|}f(y)dy-\frac{{\rm i}a_2}{4}\int_{\mathbb{R}^2}(\kappa|x-y|)^{-\frac52}e^{{\rm i}\kappa|x-y|}f(y)dy,\quad x\in\mathbb{R}^2.$$

Let $\mathcal{U}\subset\mathbb{R}^2$ be a bounded domain satisfying ${\rm dist}(\mathcal{U},\mathcal{D})=r_0>0$. First we show that the strength μ of the source f given in Assumption 2.1 can be reconstructed uniquely by the variance of the solution u on $\mathcal{U}$.

PROPOSITION 4.1. *Let $k\geq1$ and the assumptions in Theorem 3.3 hold. Then the following estimate holds:*

$$\mathbb{E}|u^2(x;k)|^2=T_\kappa(x)|\kappa|^{-1}\kappa_{\rm r}^{-2s}+O\left(\kappa_{\rm r}^{-2s-2}\right),\quad x\in\mathcal{U},$$

where

$$T_\kappa(x):=\frac{1}{2^3\pi}\int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\rm i}|x-y|}}{|x-y|}\mu(y)\,dy.$$

Proof. For any $x\in\mathcal{U}$, we have from straightforward calculations that

$$\begin{aligned}\mathbb{E}|u^2(x;k)|^2=&\frac{|a_0|^2}{16|\kappa|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{{\rm i}\kappa|x-y|-{\rm i}\bar\kappa|x-z|}}{|x-y|^{\frac12}|x-z|^{\frac12}}\mathbb{E}[f(y)f(z)]dydz\\&+\Re\left[\frac{a_0\bar a_1}{8|\kappa|\bar\kappa}\right]\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{{\rm i}\kappa|x-y|-{\rm i}\bar\kappa|x-z|}}{|x-y|^{\frac12}|x-z|^{\frac32}}\mathbb{E}[f(y)f(z)]dydz\\&+\frac{|a_1|^2}{16|\kappa|^3}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{{\rm i}\kappa|x-y|-{\rm i}\bar\kappa|x-z|}}{|x-y|^{\frac32}|x-z|^{\frac32}}\mathbb{E}[f(y)f(z)]dydz\\&+\Re\left[\frac{a_0\bar a_2}{8|\kappa|\bar\kappa^2}\right]\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{{\rm i}\kappa|x-y|-{\rm i}\bar\kappa|x-z|}}{|x-y|^{\frac12}|x-z|^{\frac52}}\mathbb{E}[f(y)f(z)]dydz\\&+\Re\left[\frac{a_1\bar a_2}{8|\kappa|^3\bar\kappa}\right]\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{{\rm i}\kappa|x-y|-{\rm i}\bar\kappa|x-z|}}{|x-y|^{\frac32}|x-z|^{\frac52}}\mathbb{E}[f(y)f(z)]dydz\\&+\frac{|a_2|^2}{16|\kappa|^5}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{{\rm i}\kappa|x-y|-{\rm i}\bar\kappa|x-z|}}{|x-y|^{\frac52}|x-z|^{\frac52}}\mathbb{E}[f(y)f(z)]dydz.\end{aligned}\tag{4.2}$$

To estimate all the above terms, it suffices to consider the integral

$$I_{l_1,l_2}(x;k):=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{e^{{\rm i}\kappa|x-y|-{\rm i}\overline{\kappa}|x-z|}}{|x-y|^{\frac12+l_1}|x-z|^{\frac12+l_2}}K_f(y,z)\theta(x)dydz,\quad l_1,l_2\in\{0,1,2\},$$

where $\theta\in C_0^\infty$ such that $\theta|_{\mathcal{U}}\equiv1$ and ${\rm supp}(\theta)\subset\mathbb{R}^2\backslash\overline{\mathcal{D}}$. Define $C_1(y,z,x):=K_f(y,z)\theta(x)$ and $c_1(y,x,\xi):=c(y,\xi)\theta(x)$ with $c(y,\xi)$ being the symbol of the covariance operator Q_f of the random field f. Furthermore, $c_1\in S^{-2s}$ with S^m being the space of symbols of order m, $m\in\mathbb{R}$, has the principal symbol

$$c_1^p(y,x,\xi)=\mu(y)\theta(x)|\xi|^{-2s}.$$

Based on (2.1), we have

$$C_1(y,z,x)=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{\mathrm{i}(y-z)\cdot\xi}c_1(y,x,\xi)d\xi,$$

which is compactly supported in $\mathcal{D}^\theta:=\mathcal{D}\times\mathcal{D}\times\mathrm{supp}(\theta)$. Moreover, C_1 is a conormal distribution in $\mathbb{R}^6$ of Hörmander type having conormal singularity on the surface $S:=\{(y,z,x)\in\mathbb{R}^6:y-z=0\}$ and is invariant under a change of coordinates [17].

To calculate the integral in (4.2), different coordinates systems will be considered. Define an invertible transformation $\tau:\mathbb{R}^6\to\mathbb{R}^6$ by

$$\tau(y,z,x)=(g,h,x),$$

where $g=(g_1,g_2)$ and $h=(h_1,h_2)$ with

$$g_1=\frac12(|x-y|-|x-z|),\quad g_2=\frac12\left[|x-y|\arcsin\Big(\frac{y_1-x_1}{|x-y|}\Big)-|x-z|\arcsin\Big(\frac{z_1-x_1}{|x-z|}\Big)\right],$$
$$h_1=\frac12(|x-y|+|x-z|),\quad h_2=\frac12\left[|x-y|\arcsin\Big(\frac{y_1-x_1}{|x-y|}\Big)+|x-z|\arcsin\Big(\frac{z_1-x_1}{|x-z|}\Big)\right].$$

Under the new coordinates system, we get

$$\begin{aligned}I_{l_1,l_2}(x;k)&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{\mathrm{i}\kappa_{\mathrm{r}}(|x-y|-|x-z|)-\kappa_{\mathrm{i}}(|x-y|+|x-z|)}\frac{C_1(y,z,x)}{|x-y|^{\frac12+l_1}|x-z|^{\frac12+l_2}}dydz\\&=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}e^{\mathrm{i}2\kappa_{\mathrm{r}}(e_1\cdot g)-2\kappa_{\mathrm{i}}(e_1\cdot h)}C_2(g,h,x)dgdh,\end{aligned}\tag{4.3}$$

where $e_1=(1,0)$ and

$$\begin{aligned}C_2(g,h,x)&=C_1(\tau^{-1}(g,h,x))\frac{\det((\tau^{-1})'(g,h,x))}{((g+h)\cdot e_1)^{\frac12+l_1}\,((h-g)\cdot e_1)^{\frac12+l_2}}\\&=:C_1(\tau^{-1}(g,h,x))L^\tau(g,h,x).\end{aligned}\tag{4.4}$$

To get a detailed expression of C_2 as well as its principal symbol, we define another invertible transformation $\eta:\mathbb{R}^6\to\mathbb{R}^6$ by

$$\eta(y,z,x)=(v,w,x),$$

where $v=y-z$ and $w=y+z$. Consider the pull-back $C_3:=C_1\circ\eta^{-1}$ satisfying

$$\begin{aligned}C_3(v,w,x)&=C_1(\eta^{-1}(v,w,x))=C_1\Big(\frac{v+w}{2},\frac{w-v}{2},x\Big)\\&=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{\mathrm{i}v\cdot\xi}c_1\Big(\frac{v+w}{2},x,\xi\Big)d\xi=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{\mathrm{i}v\cdot\xi}c_3\left(w,x,\xi\right)d\xi,\end{aligned}$$

where we have used the properties of symbols (cf. [17, Lemma 18.2.1]) and that c_3 has the following asymptotic expansion:

$$\begin{aligned}c_3(w,x,\xi)&=e^{-\mathrm{i}\langle D_v,D_\xi\rangle}c_1\Big(\frac{v+w}{2},x,\xi\Big)\Big|_{v=0}\\&\sim\sum_{j=0}^\infty\frac{\langle-\mathrm{i}D_v,D_\xi\rangle^j}{j!}c_1\Big(\frac{v+w}{2},x,\xi\Big)\Big|_{v=0}.\end{aligned}$$

Moreover, the principal symbol of c_3 is

$$c_3^p(w,x,\xi) = c_1^p\Big(\frac{w}{2},x,\xi\Big) = \mu\left(\frac{w}{2}\right)\theta(x)|\xi|^{-2s}.$$

Finally, we define a diffeomorphism $\gamma := \eta\circ\tau^{-1} : (g,h,x)\mapsto(v,w,x)$, which preserves the plane $\{(g,h,x)\in\mathbb{R}^6 : g=0\}$, i.e., if $g=0$ then $v=0$. By Theorem 18.2.9 in [17], the pull-back $C_4 := C_3\circ\gamma$ can be calculated by

$$C_4(g,h,x) = C_3(\gamma(g,h,x)) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}c_4\left(h,x,\xi\right)d\xi,$$

where

$$\begin{aligned} c_4(h,x,\xi) &= c_3\big(\gamma_2(0,h,x),(\gamma_{11}'(0,h,x))^{-\top}\xi\big)\left|\det\left(\gamma_{11}'(0,h,x)\right)\right|^{-1} + r_3(h,x,\xi)\\ &= c_3^p\big(\gamma_2(0,h,x),(\gamma_{11}'(0,h,x))^{-\top}\xi\big)\left|\det\left(\gamma_{11}'(0,h,x)\right)\right|^{-1} + r_4(h,x,\xi). \end{aligned}$$

Here the residuals $r_3, r_4\in S^{-2s-1}$, $\gamma=(\gamma_1,\gamma_2)$ with $\gamma_1(g,h,x)=v$ and $\gamma_2(g,h,x)=(w,x)$, and γ_{11}' is determined by the Jacobian matrix

$$\gamma' = \begin{bmatrix}\gamma_{11}' & \gamma_{12}'\\ \gamma_{21}' & \gamma_{22}'\end{bmatrix}.$$

Hence, $c_4\in S^{-2s}$ is still C^∞-smooth and compactly supported in the variables (h,x) with the principal symbol

$$c_4^p(h,x,\xi) = \mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\left|(\gamma_{11}'(0,h,x))^{-\top}\xi\right|^{-2s}\left|\det(\gamma_{11}'(0,h,x))\right|^{-1}. \tag{4.5}$$

Noting that $C_4 = C_3\circ\gamma = C_1\circ\eta^{-1}\circ\eta\circ\tau^{-1} = C_1\circ\tau^{-1}$ and combining with (4.4), we obtain

$$\begin{aligned} &C_2(g,h,x) = C_4(g,h,x)L^\tau(g,h,x)\\ &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}c_4\left(h,x,\xi\right)L^\tau(g,h,x)d\xi = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}c_5(h,x,\xi)d\xi, \end{aligned} \tag{4.6}$$

where we have used Lemma 18.2.1 in [17] again and the fact that the function $L^\tau(g,h,x)$ is smooth in the domain $\tau(\mathcal{D}^\theta)$. Similar to the asymptotic expansion of c_3, we have

$$c_5(h,x,\xi)\sim\sum_{j=0}^{\infty}\frac{\langle -\mathrm{i}D_g, D_\xi\rangle^j}{j!}\left(c_4\left(h,x,\xi\right)L^\tau(g,h,x)\right)\Big|_{g=0}.$$

Using (4.5) and the expression of L^τ defined in (4.4), and residual $r_5 := c_5 - c_5^p\in S^{-2s-1}$, we obtain the principal symbol

$$\begin{aligned} &c_5^p(h,x,\xi) = c_4^p\left(h,x,\xi\right)L^\tau(0,h,x)\\ &= \mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\left|(\gamma_{11}'(0,h,x))^{-\top}\xi\right|^{-2s}\frac{\det((\tau^{-1})'(0,h,x))}{\left|\det(\gamma_{11}'(0,h,x))\right|(h\cdot e_1)^{1+l_1+l_2}}. \end{aligned} \tag{4.7}$$

Let $\alpha=\frac{h_2}{h_1}$. Simple calculations show that

$$\begin{aligned} \gamma_{11}'(0,h,x) &= \frac{\partial v}{\partial g}(0,h,x) = \begin{bmatrix}\frac{\partial v_1}{\partial g_1} & \frac{\partial v_1}{\partial g_2}\\ \frac{\partial v_2}{\partial g_1} & \frac{\partial v_2}{\partial g_2}\end{bmatrix}(0,h,x)\\ &= 2\begin{bmatrix}\sin\alpha-\alpha\cos\alpha & \cos\alpha\\ \cos\alpha+\alpha\sin\alpha & -\sin\alpha\end{bmatrix} \end{aligned}$$

is invertible since $\det(\gamma_{11}'(0,h,x)) = -4$ and $\gamma_2(0,h,x) = (w(0,h,x),x)$ with

$$w(0,h,x) = \Big(2h_1 \sin\Big(\frac{h_2}{h_1}\Big) + 2x_1, 2h_1 \cos\Big(\frac{h_2}{h_1}\Big) + 2x_2\Big).$$

Moreover, a straightforward calculation gives

$$\begin{aligned}
(\tau^{-1})'(0,h,x) &= \frac{\partial(y,z,x)}{\partial(g,h,x)}\bigg|_{g=0} \\
&= \begin{bmatrix}
\sin\alpha - \alpha\cos\alpha & \cos\alpha & \sin\alpha - \alpha\cos\alpha & \cos\alpha & 1 & 0 \\
\cos\alpha + \alpha\sin\alpha & -\sin\alpha & \cos\alpha + \alpha\sin\alpha & -\sin\alpha & 0 & 1 \\
-\sin\alpha + \alpha\cos\alpha & -\cos\alpha & \sin\alpha - \alpha\cos\alpha & \cos\alpha & 1 & 0 \\
-\cos\alpha - \alpha\sin\alpha & \sin\alpha & \cos\alpha + \alpha\sin\alpha & -\sin\alpha & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{aligned}$$

and $\det((\tau^{-1})'(0,h,x)) = 4$.

Combining (4.3) and (4.6)–(4.7), we obtain

$$\begin{aligned}
I_{l_1,l_2}(x;k) &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\mathrm{i}2\kappa_{\mathrm{r}}(e_1\cdot g) - 2\kappa_{\mathrm{i}}(e_1\cdot h)} \\
&\qquad \times \left[\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{\mathrm{i}g\cdot\xi}\Big(c_4^p(h,x,\xi)L^\tau(0,h,x) + r_5(h,x,\xi)\Big)d\xi\right] dg dh \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}\Big[c_4^p(h,x,\xi)L^\tau(0,h,x) + r_5(h,x,\xi)\Big]\delta(2\kappa_{\mathrm{r}}e_1 + \xi)d\xi dh \\
&= \int_{\mathbb{R}^2} e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}\Big[c_4^p\left(h,x,-2\kappa_{\mathrm{r}}e_1\right)L^\tau(0,h,x) + r_5(h,x,-2\kappa_{\mathrm{r}}e_1)\Big]dh \\
&= \bigg[\int_{\mathbb{R}^2} e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\left|(\gamma_{11}'(0,h,x))^{-\top}(-2\kappa_{\mathrm{r}}e_1)\right|^{-2s} \\
&\qquad \times \frac{1}{(e_1\cdot h)^{1+l_1+l_2}}dh + O(\kappa_{\mathrm{r}}^{-2s-1})\bigg] \\
&= \bigg[\int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}}{(e_1\cdot h)^{1+l_1+l_2}}\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)dh\bigg]\kappa_{\mathrm{r}}^{-2s} + O(\kappa_{\mathrm{r}}^{-2s-1}) \\
&=: M_{l_1,l_2}^\kappa(x)\kappa_{\mathrm{r}}^{-2s} + O(\kappa_{\mathrm{r}}^{-2s-1}),
\end{aligned}$$

where we have used the fact that $\delta(\xi) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{\mathrm{i}x\cdot\xi}d\xi$ in the second step and

$$M_{l_1,l_2}^\kappa(x) = \int_{\mathbb{R}^2}\frac{e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}}{(e_1\cdot h)^{1+l_1+l_2}}\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)dh.$$

To simplify the expression of $M_{l_1,l_2}^\kappa(x)$, we consider another coordinate transformation $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\rho(h) = \zeta := \left(h_1\sin\left(\frac{h_2}{h_1}\right), h_1\cos\left(\frac{h_2}{h_1}\right)\right) + x,$$

which has the Jacobian

$$\det(\rho') = \begin{vmatrix}
\sin\left(\frac{h_2}{h_1}\right) - \frac{h_2}{h_1}\cos\left(\frac{h_2}{h_1}\right) & \cos\left(\frac{h_2}{h_1}\right) \\
\cos\left(\frac{h_2}{h_1}\right) + \frac{h_2}{h_1}\sin\left(\frac{h_2}{h_1}\right) & -\sin\left(\frac{h_2}{h_1}\right)
\end{vmatrix} = -1.$$

Noting that $\det((\rho^{-1})') = \frac{1}{\det(\rho')} = -1$, we get

$$M^{\kappa}_{l_1,l_2}(x) = \int_{\mathbb{R}^2} \frac{e^{-2\kappa_{\rm i}|x-\zeta|}}{|x-\zeta|^{1+l_1+l_2}} \mu(\zeta)\, d\zeta, \quad x \in \mathcal{U}.$$

Combining the above estimates, we obtain

$$\begin{aligned}
\mathbb{E}|u^2(x;k)|^2 &= \frac{|a_0|^2}{16|\kappa|} I_{0,0}(x;k) + \Re\left[\frac{a_0\bar{a}_1}{8|\kappa|\kappa} I_{0,1}(x;k)\right] + \frac{|a_1|^2}{16|\kappa|^3} I_{1,1}(x;k) \\
&\quad + \Re\left[\frac{a_0\bar{a}_2}{8|\kappa|\kappa^2} I_{0,2}(x;k)\right] + \Re\left[\frac{a_1\bar{a}_2}{8|\kappa|^3\kappa} I_{1,2}(x;k)\right] + \frac{|a_2|^2}{16|\kappa|^5} I_{2,2}(x;k) \\
&= \frac{|a_0|^2}{16|\kappa|}\left[M^{\kappa}_{0,0}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right] \\
&\quad + \Re\left[\frac{a_0\bar{a}_1}{8|\kappa|\kappa}\left(M^{\kappa}_{0,1}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right)\right] \\
&\quad + \frac{|a_1|^2}{16|\kappa|^3}\left[M^{\kappa}_{1,1}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right] \\
&\quad + \Re\left[\frac{a_0\bar{a}_2}{8|\kappa|\kappa^2}\left(M^{\kappa}_{0,2}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right)\right] \\
&\quad + \Re\left[\frac{a_1\bar{a}_2}{8|\kappa|^3\kappa}\left(M^{\kappa}_{1,2}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right)\right] \\
&\quad + \frac{|a_2|^2}{16|\kappa|^5}\left[M^{\kappa}_{2,2}(x)\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-1})\right] \\
&= \frac{|a_0|^2}{16} M^{\kappa}_{0,0}(x)|\kappa|^{-1}\kappa_{\rm r}^{-2s} + O(\kappa_{\rm r}^{-2s-2}),
\end{aligned}$$

which completes the proof. □

THEOREM 4.2. *Let $f \in L^2(\Omega, W^{H-\epsilon,p})$ with H, ϵ, and p satisfying the conditions given in Theorem 3.3. Then for any $x \in \mathcal{U}$,*

$$\lim_{k\to\infty} k^{2s+1}\mathbb{E}|u(x;k)|^2 = \frac{1}{2^3\pi}\int_{\mathbb{R}^2} \frac{e^{-\sigma|x-y|}}{|x-y|}\mu(y)\, dy =: T(x).$$

Proof. Note that

$$\begin{aligned}
k^{2s+1}\mathbb{E}|u(x;k)|^2 &= k^{2s+1}\mathbb{E}|u^2(x;k)|^2 + 2k^{2s+1}\mathbb{E}\Re\left[\overline{u^2(x;k)}(u(x;k) - u^2(x;k))\right] \\
&\quad + k^{2s+1}\mathbb{E}\left|u(x;k) - u^2(x;k)\right|^2 \\
&=: V_1(k) + V_2(k) + V_3(k).
\end{aligned}$$

Next we calculate the limits of V_1, V_2, and V_3, respectively.

Using the asymptotic expansions of the Hankel function in (4.1), we get

$$\left|H^{(1)}_n(\kappa|x-y|) - H^{(1)}_{n,N}(\kappa|x-y|)\right| = O\big(|\kappa|x-y||^{-(N+\frac{3}{2})}\big), \quad k\to\infty.$$

Noting $H_0^{(1)'}(z) = -H_1^{(1)}(z)$, we have

$$\left|\partial_{y_i} H^{(1)}_0(\kappa|x-y|) - \partial_{y_i} H^{(1)}_{0,N}(\kappa|x-y|)\right| = O\big(|\kappa|^{-(N+\frac{1}{2})}|x-y|^{-(N+\frac{3}{2})}\big), \quad k\to\infty.$$

Hence

$$\mathbb{E}|u(x;k)-u^2(x;k)|^2 = \mathbb{E}\left|\int_{\mathcal{D}}\left(\Phi_\kappa(x,y)-\Phi_\kappa^2(x,y)\right)f(y)dy\right|^2$$
$$\lesssim \|\Phi_\kappa(x,\cdot)-\Phi_\kappa^2(x,\cdot)\|^2_{W^{1,q}(\mathcal{D})}\mathbb{E}\|f\|^2_{W^{-1,p}(\mathcal{D})}$$
$$\lesssim \|\Phi_\kappa(x,\cdot)-\Phi_\kappa^2(x,\cdot)\|^2_{W^{1,q}(\mathcal{D})}\mathbb{E}\|f\|^2_{W^{H-\epsilon,p}(\mathcal{D})} \lesssim |\kappa|^{-5},$$

where $f\in L^2(\Omega, W^{H-\epsilon,p}_{\rm comp})\subset L^2(\Omega,W^{-1,p}_{\rm comp})$ for $H\in(\frac{d}{p}-2,0]$ and $p\in(1,2]$ and $\frac{1}{q}+\frac{1}{p}=1$ according to Theorem 3.3 with $d=2$. It then indicates that

$$V_3(k)\lesssim k^{2s+1}|\kappa|^{-5}=k^{2s+1}(k^4+k^2\sigma^2)^{-\frac{5}{4}}\to 0$$

as $k\to\infty$ since $s<2$ for $d=2$.

For $V_2(k)$, we have

$$V_2(k)\le 2\left(k^{2s+1}\mathbb{E}|u^2(x;k)|^2\right)^{\frac12}\left(k^{2s+1}\mathbb{E}|u(x;k)-u^2(x;k)|^2\right)^{\frac12}=2V_1(k)^{\frac12}V_3(k)^{\frac12},$$

which converges to 0 if the limit of $V_1(k)$ exists.

For $V_1(k)$, by Proposition 4.1,

$$V_1(k)=T_\kappa(x)k^{2s+1}|\kappa|^{-1}\kappa_{\rm r}^{-2s}+O(k^{2s+1}\kappa_{\rm r}^{-2s-2}).$$

We have from (3.1) that

$$\lim_{k\to\infty}V_1(k)=\lim_{k\to\infty}T_\kappa(x)=\frac{|a_0|^2}{16}\int_{\mathbb{R}^2}\frac{e^{-\sigma|x-y|}}{|x-y|}\mu(y)\,dy,$$

which completes the proof. ☐

Remark 4.3. *It can be seen from the above proof that only two terms are needed in the truncation of* (4.1) *if the source is extremely rough with* $s\in[0,\frac{d}{2})$. *More precisely, it suffices to consider the approximate solution*

$$u^1(x;k):=-\int_{\mathbb{R}^d}\Phi^1_\kappa(x,y)f(y)dy$$

instead of u^2, *where* $V_3(k)\lesssim k^{2s+1}|\kappa|^{-3}\to 0$ *as* $k\to\infty$ *since* $s<\frac{d}{2}=1$.

Theorem 4.4. *The strength* μ *is uniquely determined by*

$$T(x)=\frac{1}{2^3\pi}\int_{\mathbb{R}^2}\frac{e^{-\sigma|x-y|}}{|x-y|}\mu(y)\,dy,\quad x\in\mathcal{U}.$$

Proof. We first consider the function $V(x):=e^{-\sigma|x|}/|x|^l$ for some positive number σ and integer $l\ge 1$, which can be regarded as a composition of functions $U(s)=e^{-\sigma s}/s^l$ and $r(x)=|x|$, i.e., $V(x)=U(r(x))$. A simple calculation shows that

$$\Delta V(x)=U''(r(x))\nabla r(x)\cdot\nabla r(x)+U'(r(x))\Delta r(x)$$
$$=\left[\frac{\sigma^2}{|x|^l}+\frac{2l\sigma}{|x|^{l+1}}+\frac{l(l+1)}{|x|^{l+2}}\right]e^{-\sigma|x|}+\left[\frac{-\sigma}{|x|^l}+\frac{-l}{|x|^{l+1}}\right]e^{-\sigma|x|}\frac{1}{|x|}$$
$$=\left[\frac{l^2}{|x|^{l+2}}+\frac{(2l-1)\sigma}{|x|^{l+1}}+\frac{\sigma^2}{|x|^l}\right]e^{-\sigma|x|}.$$

Hence, if $T(x)$ is known in $\mathcal{U}$, then so is $\Delta^n T(x)$ for any $n \in \mathbb{N}$. It implies that the following integral is determined by the measurement $T(x)$:

$$\begin{aligned}
&\int_{\mathcal{D}} P\Big(\frac{1}{|x-y|}\Big)\frac{e^{-\sigma|x-y|}}{|x-y|}\mu(y)dy\\
=&\int_{r_1}^{r_2} P\Big(\frac{1}{r}\Big)\frac{e^{-\sigma r}}{r}\Big[\int_{|x-y|=r}\mu(y)ds(y)\Big]dr\\
=&\int_{r_1^{-1}}^{r_2^{-1}} P(t)\frac{e^{-\sigma t^{-1}}}{t^{-1}}\Big[\int_{|x-y|=t^{-1}}\mu(y)ds(y)\Big]\Big(-\frac{1}{t^2}\Big)dt\\
=&\int_{r_2^{-1}}^{r_1^{-1}} P(t)\frac{e^{-\sigma t^{-1}}}{t}\Big[\int_{|x-y|=t^{-1}}\mu(y)ds(y)\Big]dt,
\end{aligned}$$

where $P(t) = \sum_{j=0}^{J} c_j t^j$ is any polynomial of order $J \in \mathbb{N}$ with real numbers c_j, $j = 0, \cdots, J$, $r_1 = \min_{y\in\mathcal{D}} |x-y| \geq r_0 > 0$ and $r_2 = \max_{y\in\mathcal{D}} |x-y|$.

Denote $S(x,r) = \int_{|x-y|=r}\mu(y)ds(y)$, which is continuous and compactly supported on $[r_1, r_2]$. Since the polynomial space on the interval $[r_2^{-1}, r_1^{-1}]$ is dense in $C([r_2^{-1}, r_1^{-1}])$, the function $\frac{e^{-\sigma t^{-1}}}{t}S(x,t^{-1})$ can be uniquely determined on $[r_2^{-1}, r_1^{-1}]$, and so does $S(x,t^{-1})$. Hence $S(x,r)$ can be uniquely determined on $[r_1, r_2]$.

To recover the strength μ based on $S(x,t)$, the classical deconvolution is used. More precisely, we consider the convolution between μ and $g(x) = e^{-\frac{|x|^2}{2}}$:

$$(g*\mu)(x) = \int_{r_1}^{r_2} e^{-\frac{r^2}{2}} S(x,r)dr,$$

which is known since $S(x,r)$ can be recovered. Then the Fourier transform yields

$$\mathcal{F}[\mu](\xi) = \frac{\mathcal{F}[g*\mu](\xi)}{\mathcal{F}[g](\xi)} = e^{-\frac{|\xi|^2}{2}}\mathcal{F}[g*\mu](\xi),$$

which implies that μ can be uniquely determined. ∎

4.2. Three-dimensional case. Now we consider $d = 3$. By Theorem 3.3, the solution of the direct problem is

$$u(x;k) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}f(y)dy. \tag{4.8}$$

Following the same procedure as that for the two-dimensional case, we first show that the strength μ is uniquely determined by the variance of the solution u.

THEOREM 4.5. *Assume that $f \in L^2(\Omega, W^{H-\epsilon,p})$ with H, ϵ and p satisfying the conditions given in Theorem 3.3. Then for any $x \in \mathcal{U}$,*

$$\lim_{k\to\infty} k^{2s}\mathbb{E}|u(x;k)|^2 = \frac{1}{2^4\pi^2}\int_{\mathbb{R}^3}\frac{e^{-\sigma|x-y|}}{|x-y|^2}\mu(y)\,dy =: \tilde{T}(x).$$

Proof. Using (4.8), we have for any $x \in \mathcal{U}$ that

$$\begin{aligned}\mathbb{E}|u(x;k)|^2 =&\frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y||x-z|}\mathbb{E}[f(y)f(z)]dydz\\ =&\frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa|x-y|-\mathrm{i}\bar{\kappa}|x-z|}}{|x-y||x-z|}K_f(y,z)\theta(x)dydz\\ =&\frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{\mathrm{i}\kappa_{\mathrm{r}}(|x-y|-|x-z|)-\kappa_{\mathrm{i}}(|x-y|+|x-z|)}\frac{C_1(y,z,x)}{|x-y||x-z|}dydz,\end{aligned}$$

where θ_0^∞ such that $\theta|_{\mathcal{U}} \equiv 1$ and $\mathrm{supp}(\theta) \subset \mathbb{R}^3\backslash\overline{\mathcal{D}}$,

$$C_1(y,z,x) := K_f(y,z)\theta(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}(y-z)\cdot\xi}c_1(y,x,\xi)d\xi.$$

Here $c_1(y,x,\xi) := c(y,\xi)\theta(x)$ with the symbol $c(y,\xi)$ satisfying (2.1). Then the principal symbol of c_1 has the form

$$c_1^p(y,x,\xi) = \mu(y)\theta(x)|\xi|^{-2s}.$$

We first define an invertible transformation $\tau : \mathbb{R}^9 \to \mathbb{R}^9$ by $\tau(y,z,x) = (g,h,x)$, where $g = (g_1,g_2,g_3)$ and $h = (h_1,h_2,h_3)$ with

$$\begin{aligned}g_1 &= \frac{1}{2}\left(|x-y|-|x-z|\right), \quad h_1 = \frac{1}{2}\left(|x-y|+|x-z|\right),\\ g_2 &= \frac{1}{2}\left[|x-y|\arccos\left(\frac{y_3-x_3}{|x-y|}\right)-|x-z|\arccos\left(\frac{z_3-x_3}{|x-z|}\right)\right],\\ h_2 &= \frac{1}{2}\left[|x-y|\arccos\left(\frac{y_3-x_3}{|x-y|}\right)+|x-z|\arccos\left(\frac{z_3-x_3}{|x-z|}\right)\right],\\ g_3 &= \frac{1}{2}\left[|x-y|\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right)-|x-z|\arctan\left(\frac{z_2-x_2}{z_1-x_1}\right)\right],\\ h_3 &= \frac{1}{2}\left[|x-y|\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right)+|x-z|\arctan\left(\frac{z_2-x_2}{z_1-x_1}\right)\right].\end{aligned}$$

Then

$$\mathbb{E}|u(x;k)|^2 = \frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{2\mathrm{i}\kappa_{\mathrm{r}}(e_1\cdot g)-2\kappa_{\mathrm{i}}(e_1\cdot h)}C_2(g,h,x)dgdh,$$

where $e_1 = (1,0,0)$ and

$$\begin{aligned}C_2(g,h,x) &= C_1(\tau^{-1}(g,h,x))\frac{\det\left((\tau^{-1})'(g,h,x)\right)}{((g+h)\cdot e_1)((h-g)\cdot e_1)}\\ &=: C_1(\tau^{-1}(g,h,x))L^\tau(g,h,x).\end{aligned}$$

Next is to get an explicit expression of C_2 with respect to (g,h,x). We define another invertible transformation $\eta : \mathbb{R}^9 \to \mathbb{R}^9$ by $\eta(y,z,x) = (v,w,x)$ with $v = y-z$ and $w = y+z$, and define the diffeomorphism $\gamma := \eta\circ\tau^{-1} : (g,h,x) \mapsto (v,w,x)$. Following the same procedure as that used in Proposition 4.1, by defining $C_3 := C_1\circ\eta^{-1}$, we obtain

$$\begin{aligned}C_3(v,w,x) &= C_1(\eta^{-1}(v,w,x)) = C_1\Big(\frac{v+w}{2},\frac{w-v}{2},x\Big)\\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}v\cdot\xi}c_1\Big(\frac{v+w}{2},x,\xi\Big)d\xi = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}v\cdot\xi}c_3\left(w,x,\xi\right)d\xi,\end{aligned}$$

where c_3 has the principal symbol $c_3^p(w,x,\xi) = c_1^p\left(\frac{v+w}{2},x,\xi\right)|_{v=0} = \mu(\frac{w}{2})|\xi|^{-2s}\theta(x)$. By Theorem 18.2.9 in [17],

$$C_4(g,h,x) := C_3 \circ \gamma(g,h,x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi} c_4(h,x,\xi)d\xi,$$

where c_4 has the principal symbol

$$c_4^p(h,x,\xi) = c_3^p\left(\gamma_2(0,h,x), (\gamma_{11}'(0,h,x))^{-\top}\xi\right)|\det(\gamma_{11}'(0,h,x))|^{-1},$$

and $\gamma_2(0,h,x) = (w(0,h,x),x), \gamma_{11}'(0,h,x) = \frac{\partial v}{\partial g}(0,h,x)$. Noting that $C_4 = C_3\circ\gamma = C_1\circ\eta^{-1}\circ\eta\circ\tau^{-1} = C_1\circ\tau^{-1}$, we are able to give the expression of C_2:

$$\begin{aligned} C_2(g,h,x) &= C_1\circ\tau^{-1}(g,h,x)L^\tau(g,h,x) \\ &= \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi}c_4(h,x,\xi)L^\tau(g,h,x)d\xi = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}g\cdot\xi}c_5(h,x,\xi)d\xi, \end{aligned}$$

where the principal symbol of c_5 is

$$\begin{aligned} c_5^p(h,x,\xi) &= c_4^p(h,x,\xi)L^\tau(0,h,x) = \mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x) \\ &\times\Big|\Big(\frac{\partial v}{\partial g}(0,h,x)\Big)^{-\top}\xi\Big|^{-2s}\Big|\det\Big(\frac{\partial v}{\partial g}(0,h,x)\Big)\Big|^{-1}\frac{\det\left((\tau^{-1})'(0,h,x)\right)}{(h\cdot e_1)^2} \end{aligned}$$

and the residual $r_5 := c_5 - c_5^p \in S^{-2s-1}$.

It then suffices to calculate c_5^p. Noting that

$$\begin{aligned} & h_1+g_1 = |x-y|, \quad h_1-g_1 = |x-z|, \\ & \frac{h_2+g_2}{h_1+g_1} = \arccos\Big(\frac{y_3-x_3}{|x-y|}\Big), \quad \frac{h_2-g_2}{h_1-g_1} = \arccos\Big(\frac{z_3-x_3}{|x-z|}\Big), \\ & \frac{h_3+g_3}{h_1+g_1} = \arctan\Big(\frac{y_2-x_2}{y_1-x_1}\Big), \quad \frac{h_3-g_3}{h_1-g_1} = \arctan\Big(\frac{z_2-x_2}{z_1-x_1}\Big), \end{aligned}$$

we get

$$\begin{aligned} y_1 &= x_1 + (h_1+g_1)\sin\left(\frac{h_2+g_2}{h_1+g_1}\right)\cos\left(\frac{h_3+g_3}{h_1+g_1}\right), \\ y_2 &= x_2 + (h_1+g_1)\sin\left(\frac{h_2+g_2}{h_1+g_1}\right)\sin\left(\frac{h_3+g_3}{h_1+g_1}\right), \\ y_3 &= x_3 + (h_1+g_1)\cos\left(\frac{h_2+g_2}{h_1+g_1}\right), \\ z_1 &= x_1 + (h_1-g_1)\sin\left(\frac{h_2-g_2}{h_1-g_1}\right)\cos\left(\frac{h_3-g_3}{h_1-g_1}\right), \\ z_2 &= x_2 + (h_1-g_1)\sin\left(\frac{h_2-g_2}{h_1-g_1}\right)\sin\left(\frac{h_3-g_3}{h_1-g_1}\right), \\ z_3 &= x_3 + (h_1-g_1)\cos\left(\frac{h_2-g_2}{h_1-g_1}\right). \end{aligned}$$

A simple calculation yields that

$$\frac{\partial v}{\partial g}(0,h,x)=2\begin{bmatrix}\sin\alpha\cos\beta-\alpha\cos\alpha\cos\beta+\beta\sin\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha\sin\beta\\ \sin\alpha\sin\beta-\alpha\cos\alpha\sin\beta-\beta\sin\alpha\cos\beta & \cos\alpha\sin\beta & \sin\alpha\cos\beta\\ \cos\alpha+\alpha\sin\alpha & -\sin\alpha & 0\end{bmatrix},$$

where $\alpha:=\frac{h_2}{h_1},\beta:=\frac{h_3}{h_1}$, and

$$(\tau^{-1})'(0,h,x)=\begin{bmatrix}\frac{1}{2}\frac{\partial v}{\partial g} & \frac{1}{2}\frac{\partial v}{\partial g} & I\\ -\frac{1}{2}\frac{\partial v}{\partial g} & \frac{1}{2}\frac{\partial v}{\partial g} & I\\ 0 & 0 & I\end{bmatrix}.$$

Here I is the 3×3 identity matrix. It can be verified that

$$\det\Big(\frac{\partial v}{\partial g}(0,h,x)\Big)=8\sin\alpha,\quad L^{\tau}(0,h,x)=\frac{8\sin^2\alpha}{(h\cdot e_1)^2},$$

and

$$\Big(\frac{\partial v}{\partial g}(0,h,x)\Big)^{-\top}=\frac{1}{2}\begin{bmatrix}\sin\alpha\cos\beta & \cos\alpha\cos\beta+\alpha\sin\alpha\cos\beta & -\frac{\sin\beta}{\sin\alpha}+\beta\sin\alpha\cos\beta\\ \sin\alpha\sin\beta & \cos\alpha\sin\beta+\alpha\sin\alpha\sin\beta & \frac{\cos\beta}{\sin\alpha}+\beta\sin\alpha\sin\beta\\ \cos\alpha & -\frac{\cos\beta}{\sin\alpha}-\beta\sin\alpha\sin\beta & \beta\cos\alpha\end{bmatrix}.$$

We then have

$$\begin{aligned}\mathbb{E}|u(x;k)|^2&=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{2\mathrm{i}\kappa_{\mathrm{r}}(e_1\cdot g)-2\kappa_{\mathrm{i}}(e_1\cdot h)}C_2(g,h,x)dgdh\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{2\mathrm{i}\kappa_{\mathrm{r}}(e_1\cdot g)-2\kappa_{\mathrm{i}}(e_1\cdot h)}\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}e^{\mathrm{i}g\cdot\xi}c_5(h,x,\xi)d\xi dgdh\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}c_5(h,x,-2\kappa_{\mathrm{r}}e_1)dh\\ &=\frac{1}{16\pi^2}\int_{\mathbb{R}^3}e^{-2\kappa_{\mathrm{i}}(e_1\cdot h)}\Big[\mu\Big(\frac{w(0,h,x)}{2}\Big)\theta(x)\kappa_{\mathrm{r}}^{-2s}\frac{\sin\alpha}{(h\cdot e_1)^2}+r_5(h,x,-2\kappa_{\mathrm{r}}e_1)\Big]dh,\end{aligned}$$

where

$$\frac{w(0,h,x)}{2}=(h_1\sin\alpha\cos\beta,h_1\sin\alpha\sin\beta,h_1\cos\alpha)+x.$$

Define another coordinate transform $\rho:\mathbb{R}^3\to\mathbb{R}^3$ by

$$\rho(h)=\zeta:=(h_1\sin\alpha\cos\beta,h_1\sin\alpha\sin\beta,h_1\cos\alpha)+x.$$

By noting that $|\zeta-x|=h_1=h\cdot e_1$ and $\det((\rho^{-1})')=\frac{1}{\det(\rho')}$ with

$$\rho'=\begin{bmatrix}\sin\alpha\cos\beta-\alpha\cos\alpha\cos\beta+\beta\sin\alpha\sin\beta & \cos\alpha\cos\beta & -\sin\alpha\sin\beta\\ \sin\alpha\sin\beta-\alpha\cos\alpha\sin\beta-\beta\sin\alpha\cos\beta & \cos\alpha\sin\beta & \sin\alpha\cos\beta\\ \cos\alpha+\alpha\sin\alpha & -\sin\alpha & 0\end{bmatrix},$$

the data $\mathbb{E}|u(x;k)|^2$ turns to be

$$\mathbb{E}|u(x;k)|^2=\left[\frac{1}{2^4\pi^2}\int_{\mathbb{R}^3}\frac{e^{-2\kappa_{\mathrm{i}}|\zeta-x|}}{|\zeta-x|^2}\mu(\zeta)\theta(x)dh\right]\kappa_{\mathrm{r}}^{-2s}+O(\kappa_{\mathrm{r}}^{-2s-1}).$$

Finally, for any $x \in \mathcal{U}$, we have from (3.1) that

$$\lim_{k\to\infty} k^{2s}\mathbb{E}|u(x;k)|^2 = \lim_{k\to\infty} \frac{1}{2^4\pi^2}\int_{\mathbb{R}^3} \frac{e^{-2\kappa_{\rm i}|\zeta-x|}}{|\zeta-x|^2}\mu(\zeta)dh\left(\frac{k}{\kappa_{\rm r}}\right)^{2s} = \tilde{T}(x),$$

which completes the proof. ◻

Repeating basically the same proof as that of Theorem 4.4, we may show the uniqueness of the inverse problem in three dimensions.

THEOREM 4.6. *The strength μ is uniquely determined by*

$$\tilde{T}(x) = \frac{1}{2^4\pi^2}\int_{\mathbb{R}^3} \frac{e^{-\sigma|x-y|}}{|x-y|^2}\mu(y)\,dy, \quad x \in \mathcal{U}.$$

4.3. The case $\sigma = 0$ and ergodicity. If $\sigma = 0$, the model (1.1) reduces to the one considered in [20]. In this case, the ergodicity of the solution can be obtained by following the same way which was investigated in [19,20]. This result makes it possible to uniquely recover the strength μ by a single realization of the measurements.

PROPOSITION 4.7. *Assume that $f \in L^2(\Omega, W^{H-\epsilon,p})$ with H, ϵ and p satisfying the conditions given in Theorem 3.3. Let $s = H + \frac{d}{2}$. Then*

(i) if $d = 2$,

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K k^{2s+1}|u(x;k)|^2 dk = T(x) \quad a.s.,$$

(ii) if $d = 3$,

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K k^{2s}|u(x;k)|^2 dk = \tilde{T}(x) \quad a.s.,$$

where T and $\tilde{T}$ are defined in Theorems 4.2 and 4.5, respectively.

Proof. If $\sigma = 0$, following the same procedure as that of Lemma 3.4 in [20] or Proposition 4.1, we may obtain for any $k_1, k_2 \geq 1$ that

$$\left|\mathbb{E}\left[u^2(x;k_1)\overline{u^2(x;k_2)}\right]\right| \leq C(1+|k_1-k_2|)^{-2s},$$
$$\left|\mathbb{E}\left[u^2(x;k_1)u^2(x;k_2)\right]\right| \leq C(1+|k_1-k_2|)^{-2s}.$$

which, together with the fact that

$$\lim_{K\to\infty} \frac{1}{K-1}\int_1^K X(t)dt = 0, \quad a.s.,$$

if $|\mathbb{E}X(t_1)X(t_2)| \leq C(1+|t_1-t_2|)^{-\varepsilon}$ for a centered real-valued stochastic process X with continuous paths and some $\varepsilon > 0$ (cf. [12,19,20]), one can get the desired results by following the proof in Theorem 3.10 in [20]. The details are omitted for brevity. ◻

5. Conclusion. We have studied the inverse random source scattering problem for the Helmholtz equations with attenuation. The source is assumed to be a fractional Gaussian random field. The relationship is established between the fractional Gaussian fields and the generalized Gaussian random fields. The well-posedness of the direct problem is examined. For the inverse problem, we show that the micro-correlation strength of the random source can be uniquely determined by the passive measurement of the wave fields.

There are some future works which can be considered. For instance, if the medium is inhomogeneous, the solution cannot be expressed explicitly through the fundamental solution. The present method is not applicable, a new approach is needed. Another interesting problem is to consider that both the medium and the source are random functions. Similar problems for the Schrödinger equation were investigated in [22,23]. The Helmholtz equation is more difficult because of the coupling of the medium with the wavenumber. It is an open problem for the Maxwell equations with a random source. The singularity of Green's tensor may limit the roughness of the source. We hope to be able to report the progress on these problems elsewhere in the future.

REFERENCES

[1] M. Abramowitz and I. Stegun, *Tables of Mathematical Functions*, Dover, New York, 1970.

[2] R. Adams and J. Fournier, *Sobolev Spaces,* 2nd ed., Academic Press, Amsterdam, 2003.

[3] H. Ammari, G. Bao, and J. Fleming, *An inverse source problem for Maxwell's equations in magnetoencephalography,* SIAM J. Appl. Math., 62 (2002), pp. 1369–1382.

[4] G. Bao, C. Chen, and P. Li, *Inverse random source scattering problems in several dimensions,* SIAM/ASA J. Uncertain. Quantif., 4 (2016), pp. 1263–1287.

[5] G. Bao, C. Chen, and P. Li, *Inverse random source scattering for elastic waves*, SIAM J. Numer. Anal., 55 (2017), pp. 2616–2643.

[6] G. Bao, S.-N. Chow, P. Li, and H. Zhou, *An inverse random source problem for the Helmholtz equation,* Math. Comp., 83 (2014), pp. 215–233.

[7] G. Bao, P. Li, and Y. Zhao, *Stability for the inverse source problems in elastic and electromagnetic waves*, J. Math. Pures Appl., to appear.

[8] G. Bao, J. Lin, and F. Triki, *A multi-frequency inverse source problem*, J. Differential Equations, 249 (2010), pp. 3443–3465.

[9] P. Caro, T. Helin, and M. Lassas, *Inverse scattering for a random potential,* Anal. Appl. (Singap.), 17 (2019), pp. 513–567.

[10] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory,* 3rd ed., Springer, Berlin, 2013.

[11] Z. Chen and X. Liu, *An adaptive perfectly matched layer technique for time-harmonic scattering problems,* SIAM J. Numer. Anal., 43 (2005), pp. 645–671.

[12] H. Cramer and M. Leadbetter, *Stationary and Related Stochastic Processes,* New York: John Wiley and Sons, 1967.

[13] A. Devaney, *The inverse problem for random sources*, J. Math. Phys., 20 (1979), 1687–1691.

[14] A. S. Fokas, Y. Kurylev, and V. Marinakis, *The unique determination of neuronal currents in the brain via magnetoencephalography,* Inverse Problems, 20 (2004), pp. 1067–1082.

[15] T. Helin, M. Lassas, and L. Oksanen, *Inverse problem for the wave equation with a white noise source,* Comm. Math. Phys., 332 (2014), pp. 933–953.

[16] L. Hörmander, *The analysis of linear partial differential operators I,* Classics in Mathematics, Springer-Verlag, Berlin, 2003.

[17] L. Hörmander, *The analysis of linear partial differential operators III,* Classics in Mathematics, Springer, Berlin, 2007.

[18] V. Isakov and S. Lu, Increasing stability in the inverse source problem with attenuation and many frequencies, SIAM J. Appl. Math., 78 (2018), 1–18.

[19] M. Lassas, L. Päivärinta, and E. Saksman, *Inverse scattering problem for a two dimensional random potential,* Comm. Math. Phys., 279 (2008), pp. 669–703.

[20] J. Li, T. Helin, and P. Li, *Inverse random source problems for time-harmonic acoustic and elastic waves,* arXiv:1811.12478.

[21] J. Li and P. Li, *Inverse elastic scattering for a random source,* SIAM J. Math. Anal., 51 (2019), pp. 4570–4603.

[22] J. Li, H. Liu, and S. Ma, *Determining a random Schrödinger operator: both potential and source are random,* arXiv:1906.01240.

[23] J. Li, H. Liu, and S. Ma, *Determining a random Schrödinger equation with unknown source and potential,* arXiv:1811.00880.

[24] A. Lodhia, S. Sheffield, X. Sun, and S. Watson, *Fractional Gaussian fields: a survey,* Probab. Surv., 13 (2016), pp. 1–56.

[25] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions,* Princeton University Press, Princeton, N.J., 1970.