

Finite time horizon mixed control of vibrational systems*

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We consider a vibrational system control problem over a finite time horizon. The performance measure of the system is taken to be p -mixed H_2 norm which generalizes the standard H_2 norm. We present an algorithm for efficient calculation of this norm in the case when the system is parameter dependent and the number of inputs or outputs of the system is significantly smaller than the order of the system. Our approach is based on a novel procedure which is not based on solving Lyapunov equations and which takes into account the structure of the system. We use a characterization of the H_2 norm given in terms of integrals which we solve using adaptive quadrature rules. This enables us to use recycling strategies as well as parallelization. The efficiency of the new algorithm allows for an analysis of the influence of various system parameters and different finite time horizons on the value of the p -mixed H_2 norm. We illustrate our approach by numerical examples concerning an n -mass oscillator with one damper.

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1. Introduction

The topic of this paper are vibrational systems, a class of systems which models oscillating physical systems. We are interested in those systems where the vibrations are unwanted and where one wants to design a system which reduces or minimizes the effects of a particular type of vibratory disturbance.

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More precisely, we deal with a linear vibrational system given by the following matrix algebraic-differential equation:

$$G = \begin{cases} M\ddot{x} + C\dot{x} + Kx = B_2u, \\ x(0) = x_0, \dot{x}(0) = \dot{x}_0, \\ w = \begin{bmatrix} E_1x \\ E_2\dot{x} \end{bmatrix}. \end{cases} \quad (1)$$

Here the mass matrix M and the stiffness matrix K are real, symmetric positive definite matrices of order $n \in \mathbb{N}$. The damping matrix C is real positive definite matrix and it is given as a sum of the internal damping matrix C_{int} (which is predefined) and the external damping matrix C_{ext} (which is subject to a design choice), that is $C = C_{\text{int}} + C_{\text{ext}}$. The internal damping C_{int} is usually taken to be a small multiple of the so called critical damping, that is, $C_{\text{int}} = \alpha C_{\text{crit}}$, for some $\alpha > 0$, where the critical damping C_{crit} is (see, e.g., [41, 32, 44]) given by

$$C_{\text{crit}} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}. \quad (2)$$

The external damping matrix describes the (passive) dampers of the system. In our case it will depend on positive real parameters $v_i, i = 1, \dots, q, q \ll n$, (so called viscosities) and matrices corresponding to the positions of the dampers.

The vector function $x: [0, \infty) \rightarrow \mathbb{R}^n$ contains the state variables and x_0, \dot{x}_0 are the initial data. System disturbances are denoted by the vector function $u: [0, \infty) \rightarrow \mathbb{R}^m$ and the matrix $B_2 \in \mathbb{R}^{n \times m}$. The output or the measurement vector function w is determined by the output matrices $E_1, E_2 \in \mathbb{R}^{k \times n}$.

The problem of determining the optimal damping matrix C_{ext} which will ensure optimal evanescence of the state x from (1) is well studied. There is a vast literature in this field of research and this optimization problem has been intensively considered in the last two decades, see, e.g., [8, 12, 30, 37, 44, 32, 41, 14]. The minimization of vibrations was also intensively studied in engineering and applied mathematics. Here, we list only a couple of references: [4, 33, 35, 17, 26, 22].

For damping optimization there exist several optimization criteria depending on different application areas. An overview of such criteria can be found, e.g., in [44] or [29]. From the control theory perspective for the optimization criteria the H_2 or H_∞ norms can be used. Within this setting, several authors considered model order reduction approaches in order to determine the optimal damping parameters efficiently; for more details see [36, 7, 1, 10]. Some criteria are based on eigenvalues, such as spectral abscissa criterion (for more details, see, e.g., [20, 31, 45]), while other criteria are based on the total energy of the system, such as the total average energy. Total average energy was considered widely in the last two decades, see, e.g., [42, 43, 40, 38, 14, 30]. In [8, 9] the authors considered dimension reduction techniques that allowed efficient calculation of the total average energy.

In all the aforementioned papers, the time horizon of the system (1) was taken to be infinite. We are interested in the case of the finite time horizon, i.e. we study the system (1) in the time interval $[0, T], T < \infty$. The infinite time horizon case is a natural choice in the cases when the vibration occurs over a longer period of time or the system is perpetually disturbed. But in the case of short duration phenomena such as earthquakes, finite time horizon is a much more suitable choice. From the mathematical point of view, the infinite time horizon leads to a computationally simpler optimization criterion, which is a very important aspect when

designing the optimal damping structure. Hence if the finite time T is large enough, it makes sense to choose $T = \infty$ even though this might lead to a (usually slightly) suboptimal design. But if the finite time is sufficiently small (but not too small as to render the design problem infeasible) then the choice $T = \infty$ is not viable and so the finite time horizon problem is the one that merits a closer investigation.

Our choice of the optimization criterion will be based on the p -mixed H_2 norm, which was investigated in [32] where the authors considered performance measure that takes into account the total average energy, and also the H_2 norm of the system. This criterion contains both the H_2 norm and the total average energy criteria as the special cases and takes into account both the initial data as well as external disturbances. This criterion will be also used in this paper as well, but instead of the infinite time horizon, we will consider the finite one. In particular, the p -mixed H_2 norm of a system for the finite time horizon can be calculated by

$$\text{Trace } E^\top E \left(\int_0^T e^{At} Z e^{A^\top t} dt \right),$$

where the matrix A comes from the linearization of the system (1), the matrix Z encodes the information about dangerous external forces and initial conditions, and the matrix E is given in terms of output matrices E_1, E_2 . We will introduce this criterion in more details in the next section.

This objective function can be calculated directly using the following formula:

$$\text{Trace}(X - e^{AT} X e^{A^\top T}), \text{ where } X \text{ is such that } AX + XA^\top = -Z. \quad (3)$$

We would like to emphasize that the use of formula (3) requires, besides solving Lyapunov equation, also calculating the matrix exponential, for more details see, e.g. [15, 3, 28, 23, 25].

The technique we develop can also be used in a more general setting of the finite horizon H_2 control in the case when the number of sensors and actuators is small, the system is parameter-dependent and one needs to calculate the appropriate H_2 norm for a large number of different parameters.

We do not pursue this line of research here to make the exposition less technical. Moreover our approach consists of two main parts; the offline part where we calculate matrices that do not depend on viscosities and the online part in which we organize calculations in such a way so that recycling of computationally demanding parts can be achieved efficiently.

One can also consider a model order reduction oriented approach for the finite time horizon problem which will result in a model of reduced order. This can be applied even for a large-scale systems, see, e.g. [24, 34, 27] where authors considered model order reduction for a finite time horizon. However, when the order of a system is reduced, then our new approach can again be employed for an efficient calculation of the p -mixed H_2 system norm.

Throughout the paper we will use the following notation. The symbol $\|\cdot\|$ denotes the standard vector norm or matrix 2-norm, depending on the context. If p and q are vectors, like in many programming languages including Julia or Matlab, the notation $A(p, q)$ will denote the submatrix of A obtained by intersection of rows determined with elements of vector p and columns determined with elements of vector q . Similarly, $i : k : j$ denotes the vector of integers from i to j with increments of k . For integers j and k , we denote by $\delta_{j,k}$ the Kronecker delta symbol, i.e. $\delta_{j,k} = 1$ if $j = k$, otherwise $\delta_{j,k} = 0$. Also, for a matrix A and a scalar s , $sI - A$ denotes the matrix $sI - A$.

The paper is structured as follows. The finite time horizon p -mixed H_2 norm is introduced in section 2. In section 3 we first derive formulae for auxiliary vectors x_j . Then, in subsection 3.2, we investigate the case of one-dimensional damping and derive final formula for the finite time horizon p -mixed H_2 norm. Using the derived formulae we present an approach for an estimation of the finite time horizon p -mixed H_2 norm in section 4. In particular, in subsection 4.1 we analyse estimation of the integration interval which is used in our algorithm for the calculation of the finite time horizon p -mixed H_2 norm presented in subsection 4.2. In section 5 we illustrate the efficiency of the new approach through numerical experiments.

2. Finite time horizon p -mixed H_2 norm

2.1. Preliminaries

Differential equation in (1) can be transformed to the first order system in the phase space. For that purpose let Φ be a matrix which simultaneously diagonalizes M and K , that is

$$\Phi^\top M \Phi = I, \Phi^\top K \Phi = \Omega^2 = \text{diag}(\omega_1^2, \dots, \omega_n^2), \quad (4)$$

where positive numbers $\omega_1, \omega_2, \dots, \omega_n$ are undamped eigenfrequencies of the system, i.e. square roots of the eigenvalues corresponding to the system with $C = 0$ (that is, the eigenvalues of $Q(\lambda) = \lambda^2 M - K$). In this case the matrix Φ diagonalizes the internal damping matrix defined by (2), that is $\Phi^\top C_{\text{int}} \Phi = \nu \Omega$, for $\nu > 0$, with $\nu = 2\alpha$. Now, using the substitutions $y_1 = \Omega \Phi^{-1} x$ and $y_2 = \Phi^{-1} \dot{x}$, the differential equation in (1) can be written as

$$\dot{y} = Ay + Bu, \quad y(0) = y_0, \quad (5)$$

where

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\nu\Omega - D \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \Phi^\top B_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (6)$$

with $D = \Phi^\top C_{\text{ext}} \Phi$. The output is determined by

$$w = Ey \quad \text{with} \quad E = \begin{bmatrix} E_1 \Phi \Omega^{-1} & 0 \\ 0 & E_2 \Phi \end{bmatrix}.$$

Here y_0 contains the corresponding transformation of the initial data, see, e.g. [12, 44, 41]. This is the so-called *modal representation* of the system (1).

The solution of (1) hence can be written as

$$w(t) = E e^{At} y_0 + E \int_0^t e^{A(t-\tau)} B u(\tau) d\tau. \quad (7)$$

Let the parameter p satisfy $0 \leq p \leq 1$. The p -mixed H_2 norm of a system G , denoted by $\|G\|_{2,p}$, is defined as

$$\|G\|_{2,p}^2 = (1-p)\|G\|_2^2 + p\|G\|_{2,\text{hom}}^2. \quad (8)$$

In (8) $\|\cdot\|_2$ denotes the standard H_2 norm given by

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(\hat{G}(i\omega))^* \hat{G}(i\omega) d\omega,$$

where \hat{G} denotes the transfer function of the system G , i.e. the Laplace transform of the mapping $u \mapsto w$. It can be shown, see e.g. [18], that the formula for $\|G\|_2$ in the time domain is given by

$$\|G\|_2^2 = \text{Trace} \left(\int_0^\infty E^\top e^{At} B B^\top e^{A^\top t} E dt \right).$$

With $\|\cdot\|_{2,\text{hom}}$, in (8), we denote the H_2 norm of the corresponding homogeneous ($u = 0$) system given by

$$\|G\|_{2,\text{hom}}^2 = \int_{\|y_0\|^2=1} \int_0^\infty \|w(t; y_0)\|^2 dt d\sigma \stackrel{(7)}{=} \int_{\|y_0\|^2=1} y_0^\top \left(\int_0^\infty e^{A^\top t} E^\top E e^{At} dt \right) y_0 d\sigma,$$

where σ is an averaging (surface) measure on the unit sphere \mathbb{R}^{2n} . More precisely, for a given measure in \mathbb{R}^{2n} , the corresponding surface measure σ is given by the Minkowski–Steiner formula [19]. Both $\|\cdot\|_2$ and $\|\cdot\|_{2,\text{hom}}$ can be expressed in terms of solutions of Lyapunov equations; for details see [32]. The p -mixed H_2 norm can be calculated as (see [32, eq. (14)]):

$$\text{Trace}(E^\top E X), \text{ where } X \text{ is such that } AX + XA^\top = -pZ_\sigma - (1-p)BB^\top, \quad (10)$$

where the matrix Z_σ depends on the choice of averaging measure σ on the set of unit initial data.

2.2. Definition of the finite time horizon p -mixed H_2 norm

The Lyapunov equation in (10) occurs, in both the H_2 term and the homogeneous H_2 term, from the integrals in the time domain of the form $\int_0^\infty e^{At} \cdot e^{A^\top t} dt$. It is easy to see that in the case of the finite time horizon the formula corresponding to (10) is

$$\text{Trace} \left(E^\top E \int_0^T e^{At} (pZ_\sigma + (1-p)BB^\top) e^{A^\top t} dt \right). \quad (11)$$

It is well known and used frequently in computations that such an expression again can be written in terms of a Lyapunov equation, and so the last expression can be written as (see, e.g., [3, 15, 32])

$$\text{Trace} \left(E^\top E \tilde{X} \right), \quad (12)$$

where X is such that $A\tilde{X} + \tilde{X}A^\top = e^{AT}(pZ_\sigma + (1-p)BB^\top)e^{A^\top T} - pZ_\sigma - (1-p)BB^\top$.

Indeed, the function $X(t) = e^{At}(pZ_\sigma + (1-p)BB^\top)e^{A^\top t}$ is the solution of the Cauchy problem

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) + \tilde{X}(t)A^\top, \quad \tilde{X}(0) = pZ_\sigma + (1-p)BB^\top,$$

and by integrating the differential equation from 0 to T we obtain

$$\tilde{X}(T) - \tilde{X}(0) = A \int_0^T \tilde{X}(t) dt + \int_0^T \tilde{X}(t) dt A^\top.$$

If we denote by X the solution of the Lyapunov equation

$$AX + XA^\top = -pZ_\sigma - (1-p)BB^\top,$$

it is easy to see that $\tilde{X} = X - e^{AT} X e^{A^\top T}$, hence (12) can be written as

$$\text{Trace} \left(E^\top E \left(X - e^{AT} X e^{A^\top T} \right) \right), \quad (13)$$

where X is such that $AX + XA^\top = -pZ_\sigma - (1-p)BB^\top$. By duality, (13) is equivalent to

$$\text{Trace} \left((pZ_\sigma + (1-p)BB^\top) \left(X - e^{AT} X e^{A^\top T} \right) \right), \quad (14)$$

where X is such that $A^\top X + XA = -E^\top E$.

Although (14) is much easier to compute than directly (11) in the case of one system, we will show that in some instances (11) has an advantage. More precisely, if E has a simple structure, Z_σ and B are low rank matrices and the goal is to compute (11) (or (14)) for a large number of external low rank damping matrices C_{ext} , which is a frequent situation when optimizing C_{ext} , the expression 11 can be used to construct an efficient algorithm for such a task.

In the sequel we will consider the typical case when

$$E^\top E = \frac{1}{2}I \quad \text{and} \quad pZ_\sigma + (1-p)BB^\top = Z, \quad (15)$$

$$\text{with} \quad Z = \begin{bmatrix} pZ_1 & 0 \\ 0 & Z_1 \end{bmatrix} \quad \text{and} \quad Z_1 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \quad (16)$$

Indeed, when modeling vibrational systems, the matrix B_2 is usually designed as a band-pass filter where only the dangerous frequencies are passed through. In terms of the modal representation (5)-(6), this would mean that B_2 has the form $B_2 = Z_1$, where the dangerous frequencies are $\omega_1, \dots, \omega_r$. Typically r is a much smaller number than n which will also be beneficial for our approach. For the dangerous frequencies we choose those which have a significant influence on the behavior of the system, e.g., those that may lead to system resonances. In damping applications it is typical to only damp dangerous frequencies, hence our choice of the matrix B_2 . The measure σ typically is chosen in such a way that it attenuates initial data which are not dangerous. In particular, the surface measure can be chosen in such a way that it is generated by the Lebesgue measure on the subspace spanned by the vectors $[x_i, 0]^\top$ and $[0, x_i]^\top$, $i = 1, \dots, r$, where x_i are the eigenvectors of ω_i , and on the rest of \mathbb{R}^{2n} is generated by the Dirac measure concentrated at zero. Then, in the modal representation we obtain $Z_\sigma = c \text{diag}(Z_1, Z_1)$, where the constant $c > 0$ only depends on the dimension. This corresponds to ignoring the initial data spanned by non-dangerous frequencies. Since the squared L_2 norm of the output, $\|w\|_2^2$, typically corresponds to the energy of the whole system, in the modal representation we obtain $E^\top E = \frac{1}{2}I$. See [32, Section 4] for the detailed explanation. Hence, by appropriate scaling, we obtain (15)-(16). Moreover, in the case when $\|w\|_2^2$ equals the energy corresponding to dangerous frequencies, we obtain $E^\top E = \frac{1}{2}Z$, which can be also covered by our procedure; see (1) below.

Therefore, our goal is to efficiently calculate

$$\text{Trace} \left(\int_0^T e^{At} Z e^{A^\top t} dt \right), \quad (17)$$

with A given in (6) and Z given in (16).

Our technique is not limited to this particular choice of matrices B_2 , E_1 , E_2 and the measure σ . As long as the corresponding matrix $pZ_\sigma + (1-p)BB^\top$ is of low rank and the matrix $E^\top E$ has a simple structure, one can construct a similar procedure. We limit our attention to this particular case to not overburden the paper with technicalities.

3. Derivation of the formula for the finite time horizon p -mixed H_2 norm

In this section we use (17) directly to obtain a formula for the finite time horizon p -mixed H_2 norm. For the purpose of the further structured calculations, we will use the fact that A can be written as

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\nu\Omega - D \end{bmatrix} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\nu\Omega \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} = A_0 - A_1. \quad (18)$$

Using the fact that the matrix exponential function is the inverse Laplace transform of the corresponding resolvent, we have

$$\begin{aligned} \text{Trace} \left(\int_0^T e^{At} Z e^{A^\top t} dt \right) &= \int_0^T \text{Trace}((\sqrt{Z} e^{A^\top t})^\top (\sqrt{Z} e^{A^\top t})) dt \\ &= \sum_{j,k=1}^{2n} \int_0^T \left((\sqrt{Z} e^{A^\top t})_{j,k} \right)^2 dt \\ &= \sum_{j,k=1}^{2n} \int_0^T \left(\frac{1}{2\pi i} \left(\sqrt{Z} \int_{-i\infty}^{+i\infty} e^{\lambda t} (\lambda - A^\top)^{-1} d\lambda \right)_{j,k} \right)^2 dt \\ &= \sum_{j,k=1}^{2n} \int_0^T \left(\frac{1}{2\pi} \left(\int_{-\infty}^{+\infty} e^{ist} \sqrt{Z} (is - A^\top)^{-1} ds \right)_{j,k} \right)^2 dt. \end{aligned}$$

Due to the structure of the matrix Z , the summation index of j goes from 1 to r and from $n+1$ to $n+r$, so we obtain

$$\begin{aligned} \text{Trace} \left(\int_0^T e^{At} Z e^{A^\top t} dt \right) &= \frac{p}{4\pi^2} \sum_{j=1}^r \sum_{k=1}^{2n} \int_0^T \left(\int_{-\infty}^{+\infty} h_{jk}(t,s) ds \right)^2 dt \\ &\quad + \frac{1}{4\pi^2} \sum_{j=n+1}^{n+r} \sum_{k=1}^{2n} \int_0^T \left(\int_{-\infty}^{+\infty} h_{jk}(t,s) ds \right)^2 dt, \quad (20) \end{aligned}$$

where

$$h_{jk}(t,s) = (\cos st + i \sin st) e_j^\top (is - A^\top)^{-1} e_k, \quad (21)$$

and e_j denotes the j -th canonical vector in \mathbb{C}^n , \mathbb{C}^{2n} , \mathbb{R}^n or \mathbb{R}^{2n} , depending on the context.

Remark 1. Note that if we would have chosen $E^\top E = \frac{1}{2}Z$, then the sums in (20) would be given by $p \sum_{j,k=1}^r + \sum_{j,k=n+1}^{n+r}$ and hence the Algorithm 1 from subsection (4.2) can be easily modified to cover this case as well.

To construct an efficient algorithm for the calculation of the last sum, we will carefully study the terms $e_j^\top (is - A^\top)^{-1} e_k$ using the structure of the matrix A , distinguishing those terms which do not depend on the C_{ext} .

We calculate

$$\begin{aligned} (is - A^\top)^{-1} &= (is - A_0^\top + A_1)^\top^{-1} = \left((is - A_0^\top)(I + (is - A_0^\top)^{-1} A_1) \right)^{-1} \\ &= \left(I + (is - A_0^\top)^{-1} A_1 \right)^{-1} (is - A_0^\top)^{-1}, \end{aligned}$$

hence

$$\begin{aligned} e_j^\top (\text{is} - A^\top)^{-1} e_k &= e_j^\top \left(I + (\text{is} - A_0^\top)^{-1} A_1 \right)^{-1} (\text{is} - A_0^\top)^{-1} e_k \\ &= \left((I + (\text{is} - A_0^\top)^{-1} A_1)^{-\top} e_j \right)^\top (\text{is} - A_0^\top)^{-1} e_k \\ &= \left((I + A_1(\text{is} - A_0)^{-1})^{-1} e_j \right)^\top (\text{is} - A_0^\top)^{-1} e_k. \end{aligned}$$

Let $x_j = (I + A_1(\text{is} - A_0)^{-1})^{-1} e_j$ i.e. $(I + A_1(\text{is} - A_0)^{-1})x_j = e_j$. Let $x_j = [x_j^1 \ x_j^2]^\top$. Note that $e_j^\top (\text{is} - A^\top)^{-1} e_k$ is a scalar product of two vectors, vector x_j depends on s , viscosities and damping positions and vector $(\text{is} - A_0^\top)^{-1} e_k$ depends only on s . Moreover, we would like to derive a formulae for x_j which will be considered in the next subsection.

3.1. Calculation of the vectors x_j

From

$$\text{is} - A_0 = \begin{bmatrix} \text{is} & -\Omega \\ \Omega & \text{is} + \nu\Omega \end{bmatrix}$$

we obtain

$$(\text{is} - A_0)^{-1} = \begin{bmatrix} (\nu\Omega + \text{is})L(s) & \Omega L(s) \\ -\Omega L(s) & \text{is}L(s) \end{bmatrix},$$

where

$$L(s) = (\Omega^2 + \text{is}\nu\Omega - s^2)^{-1} = F(s) - iG(s), \quad (24)$$

where matrices $F(s)$ and $-G(s)$ are real and imaginary part of matrix $L(s)$. Since the matrix $\Omega^2 + \text{is}\nu\Omega - s^2$ is a diagonal complex matrix, its inverse can be calculated directly. Now we have

$$\begin{aligned} I + A_1(\text{is} - A_0)^{-1} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} (\nu\Omega + \text{is})L(s) & \Omega L(s) \\ -\Omega L(s) & \text{is}L(s) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -D\Omega L(s) & I + \text{is}DL(s) \end{bmatrix}. \end{aligned}$$

For $1 \leq j \leq r$ we obtain

$$x_j^1 = e_j, \quad (I + \text{is}DL(s))x_j^2 = D\Omega L(s)e_j$$

and for $n+1 \leq j \leq n+r$

$$x_j^1 = 0, \quad (I + \text{is}DL(s))x_j^2 = e_{j-n}.$$

In the following proposition we obtain the formulae for the real and imaginary parts of complex vectors x_j^2 , for $j = 1, \dots, r, n+1, \dots, n+r$, in terms of the solutions of *real* linear systems (26) and (27).

Proposition 1. *Let matrix $L(s)$ be given as in (24) and let $x_j^2 = x_j^{\Re} + ix_j^{\Im}$, with $x_j^{\Re}, x_j^{\Im} \in \mathbb{R}^n$. Then systems $(I + \text{is}DL(s))x_j^2 = D\Omega L(s)e_j$, for $1 \leq j \leq r$, and $(I + \text{is}DL(s))x_j^2 = e_{j-n}$, for $n+1 \leq j \leq n+r$, are equivalent to the following systems*

$$\begin{aligned} \begin{bmatrix} I & -s(I + sDG(s))^{-1}DF(s) \\ 0 & s^2DF(s)(I + sDG(s))^{-1}DF(s) + I + sDG(s) \end{bmatrix} \begin{bmatrix} x_j^{\Re} \\ x_j^{\Im} \end{bmatrix} \\ = \begin{bmatrix} (I + sDG(s))^{-1}D\Omega F(s)e_j \\ -sDF(s)(I + sDG(s))^{-1}D\Omega F(s)e_j - D\Omega G(s)e_j \end{bmatrix}, \quad (26) \end{aligned}$$

$$\begin{bmatrix} I & -s(I + sDG(s))^{-1}DF(s) \\ 0 & s^2DF(s)(I + sDG(s))^{-1}DF(s) + I + sDG(s) \end{bmatrix} \begin{bmatrix} x_j^{\Re} \\ x_j^{\Im} \end{bmatrix} = \begin{bmatrix} (I + sDG(s))^{-1}e_{j-n} \\ -sDF(s)(I + sDG(s))^{-1}e_{j-n} \end{bmatrix}, \quad (27)$$

respectively.

Proof of this proposition is given in Appendix A.

In the next subsection we analyse formulae for the case of one-dimensional damping which will allow us to construct an efficient procedure for the calculation of the finite time horizon p -mixed H_2 norm.

3.2. The case of one-dimensional damping

The systems of linear equations given above can be solved in a more general setting. But in this paper, to reduce the technicalities, we only treat the particular case when there is just one damper of dimension one, and the only parameter is its viscosity. The procedure can be straightforwardly extended to the multi-parameter case in the case of a small number of parameters. Then the damping matrix D has the following form

$$D = \Phi^\top C_{\text{ext}} \Phi = v \Phi^\top e e^\top \Phi = \gamma U U^\top, \quad (28)$$

where the vector e encodes the position of the damper. The parameter $\gamma > 0$ is a product of viscosity parameter v and the 2-norm of the matrix $\Phi^\top e e^\top \Phi$ and vector $U \in \mathbb{R}^{n \times 1}$ determines the geometry of the damping position.

In the following proposition we give explicit solution of the equations (26)-(27) in terms of the parameter γ that determines the viscosity parameter.

Proposition 2. *Let $q = 1$ and assume that U and γ define damping matrix D as in (28). Furthermore, let $g(s) = U^\top G(s)U = \sum_{j=1}^n u_j^2 g_j(s)$ and $f(s) = U^\top F(s)U = \sum_{j=1}^n u_j^2 f_j(s)$, where $f_j(s)$ and $g_j(s)$, for $j = 1, \dots, n$ are diagonal elements of $F(s)$ and $G(s)$ given by (24), respectively. Then, the solution of (26) is given by ($1 \leq j \leq r$)*

$$x_j^{\Im}(s) = a(s, \gamma)U, \quad (29)$$

$$x_j^{\Re}(s) = \left(\frac{\gamma f_j(s) u_j \omega_j}{1 + s\gamma g(s)} + \frac{s\gamma f(s)}{1 + s\gamma g(s)} a(s, \gamma) \right) U, \quad (30)$$

where

$$a(s, \gamma) = -\gamma u_j \omega_j \times \frac{g_j(s) + s\gamma(f(s)f_j(s) + 2g(s)g_j(s)) + s^2\gamma^2 g(s)(f(s)f_j(s) + g(s)g_j(s))}{1 + 3s\gamma g(s) + s^2\gamma^2(3g(s)^2 + f(s)^2) + s^3\gamma^3 g(s)(g(s)^2 + f(s)^2)}, \quad (31)$$

and the solution of (27) is given by ($n+1 \leq j \leq n+r$)

$$x_j^{\Im}(s) = b(s, \gamma)U, \quad (32)$$

$$x_j^{\Re}(s) = e_{j-n} + \left(-\frac{s\gamma u_{j-n} g_{j-n}(s)}{1 + s\gamma g(s)} + \frac{s\gamma f(s)}{1 + s\gamma g(s)} b(s, \gamma) \right) U, \quad (33)$$

where

$$b(s, \gamma) = -s\gamma u_{j-n} \frac{f_{j-n}(s) + s\gamma(g(s)f_{j-n}(s) - f(s)g_{j-n}(s))}{(1 + s\gamma g(s))^2 + (s\gamma f(s))^2}. \quad (34)$$

Proof of this proposition is given in Appendix A.

Obviously, x_j^{\Re} and x_j^{\Im} depend on s but not on t and sometimes, to emphasise this, we write $x_j^{\Re}(s)$ and $x_j^{\Im}(s)$.

The following proposition gives formulae for $h_{jk}(t, s)$ defined by (21), which are our main target in light of (20).

Proposition 3. *Let all assumptions from (2) hold. Then, for $h_{jk}(t, s)$, defined by (21), we have the following formulae.*

For $1 \leq j \leq r$ and $1 \leq k \leq n$ we have

$$h_{jk}(t, s) = 2 \cos st \cdot \left((\nu \omega_k f_k(s) + s g_k(s)) \delta_{j,k} + \omega_k \left(f_k(s) (x_j^{\Re})_k + g_k(s) (x_j^{\Im})_k \right) \right).$$

For $1 \leq j \leq r$ and $n+1 \leq k \leq 2n$ we have

$$h_{jk}(t, s) = 2 \cos st \cdot \left(-\omega_{k-n} f_{k-n}(s) \delta_{j,k-n} + s g_{k-n}(s) (x_j^{\Re})_{k-n} - s f_{k-n}(s) (x_j^{\Im})_{k-n} \right).$$

For $n+1 \leq j \leq n+r$ and $1 \leq k \leq n$ we have

$$h_{jk}(t, s) = 2 \cos st \cdot \left(\omega_k \left(f_k(s) (x_j^{\Re})_k + g_k(s) (x_j^{\Im})_k \right) \right).$$

For $n+1 \leq j \leq n+r$ and $n+1 \leq k \leq 2n$ we have

$$h_{jk}(t, s) = 2s \cos st \cdot \left(g_{k-n}(s) (x_j^{\Re})_{k-n} - f_{k-n}(s) (x_j^{\Im})_{k-n} \right).$$

Proof of this proposition is given in Appendix A.

Finally, using derived explicit formulae for h_{jk} we are able to write a formula for the finite time horizon p -mixed H_2 norm, that is, we calculated all the ingredients of the formula (20).

In this section we only considered formulae for the one-parameter case, meaning that $U \in \mathbb{R}^{n \times q}$ is a vector. These formulae can be extended straightforwardly to the multi-parameter case, which can still be used for an efficient p -mixed H_2 norm calculation as long as we have a small number of parameters, meaning $q \ll n$. In general, the matrix U would contain $q > 1$ columns, and the Sherman–Morrison–Woodbury formula would include the inverses of $q \times q$ matrices. The obtained formulae would have a similar structure, which would lead to an algorithm of the same structure as the one given in 1. We have considered only the one-parameter case to simplify the exposition, as in the general case the formulae are more complicated.

Now, in the next section we explain how to use the quadrature rule in order to calculate an approximation of the expression above.

4. Estimation of the integrals

Our approach is based on a numerical integration of a very structured and oscillatory function. This is a widely investigated field and overview of some methods can be found in [16, 21, 13]. In order to be able to recycle data and use our formulae efficiently we will use Gauss quadrature rule for the integration with respect to time and adaptive Simpson rule (see, e.g., [21]) for the

highly oscillatory part. In particular, we will use the following estimate for (20)

$$\begin{aligned} \text{Trace} \left(\int_0^T e^{At} Z e^{A^\top t} dt \right) &\approx \frac{p}{4\pi^2} \sum_{j=1}^r \sum_{k=1}^{2n} \sum_{\alpha=1}^{n_t} \eta_\alpha \left(\sum_{\beta=1}^{n_s} \zeta_\beta h_{jk}(t_\alpha, s_\beta) \right)^2 \\ &+ \frac{1}{4\pi^2} \sum_{j=n+1}^{n+r} \sum_{k=1}^{2n} \sum_{\alpha=1}^{n_t} \eta_\alpha \left(\sum_{\beta=1}^{n_s} \zeta_\beta h_{jk}(t_\alpha, s_\beta) \right)^2, \end{aligned} \quad (35)$$

where $\{(t_\alpha, \eta_\alpha)\}_{\alpha=1}^{n_t}$ are the nodes and weights for the integral over the time variable t and $\{(s_\beta, \zeta_\beta)\}_{\beta=1}^{n_s}$ are the nodes and weights for the integral over the frequency variable s .

For the estimation of $\int_{-\infty}^{+\infty} h_{jk}(t, s) ds$ we will use combination of Simpson rule and adaptive Simpson rule for the highly oscillatory part. First we estimate the indefinite integral by its finite approximation

$$\int_{-\infty}^{+\infty} h_{jk}(t, s) ds = 2 \int_{-\infty}^0 h_{jk}(t, s) ds \approx 2 \int_{-S_{\max}}^0 h_{jk}(t, s) ds.$$

In the next subsection we will derive an upper bound which shows how large parameter S_{\max} should be.

4.1. Estimation of the integration interval

Since the leading term in our integral is $e_j^\top (is - A^\top)^{-1} e_k$, for $j = 1, \dots, r, n+1, \dots, n+r$, $k = 1, \dots, 2n$, we will try to determine efficiently how large (and small) values of the parameter s should be considered. Assume that the parameter γ is taken from the range $[0, \gamma_{\max}]$ and let $\omega_{\max} = \max\{\omega_1, \dots, \omega_n\}$.

By using the structure of the matrix A given by (18) we have

$$(is - A^\top)^{-1} = \left(I + (is - A_0^\top)^{-1} A_1 \right)^{-1} (is - A_0^\top)^{-1}.$$

When $\|(is - A_0^\top)^{-1} A_1\| < 1$ we have that $I + (is - A_0^\top)^{-1} A_1$ is a non-singular matrix and we have $\|(I + (is - A_0^\top)^{-1} A_1)^{-1}\| \leq (1 - \|(is - A_0^\top)^{-1} A_1\|)^{-1}$. If we use that $\|A_1\| \leq \gamma$, we get

$$\|(is - A^\top)^{-1}\| \leq \frac{\|(is - A_0^\top)^{-1}\|}{1 - \gamma \|(is - A_0^\top)^{-1}\|}. \quad (36)$$

We want to evaluate the norm of the matrix $(is - A_0^\top)^{-1}$, for which it is sufficient to calculate the eigenvalues of the matrix $(is - A_0^\top)^{-1}(-is - A_0)^{-1}$. Since

$$\begin{aligned} (is - A_0^\top)^{-1}(-is - A_0)^{-1} &= (s^2 + is(A_0^\top - A_0) + A_0^\top A_0)^{-1} \\ &= \begin{bmatrix} s^2 + \Omega^2 & -2is\Omega - \nu\Omega^2 \\ 2is\Omega - \nu\Omega^2 & s^2 + (1 + \nu^2)\Omega^2 \end{bmatrix}^{-1}, \end{aligned}$$

it is sufficient to investigate the eigenvalues of B^{-1} , where

$$B = \begin{bmatrix} s^2 + \omega^2 & -2is\omega - \nu\omega^2 \\ 2is\omega - \nu\omega^2 & s^2 + (1 + \nu^2)\omega^2 \end{bmatrix}.$$

We have

$$\det(B - \lambda I) = (s^2 + \omega^2 - \lambda)(s^2 + \omega^2 + \nu^2\omega^2 - \lambda) - (2is\omega - \nu\omega^2)(-2is\omega - \nu\omega^2) = 0.$$

We can write this as

$$\mu(\mu + \nu^2\omega^2) - (\nu^2\omega^4 + 4s^2\omega^2) = 0,$$

for $\mu = s^2 + \omega^2 - \lambda$. This is the quadratic equation in the variable μ and the solutions are given by

$$\mu_{1,2} = \frac{-\nu^2\omega^2 \pm \sqrt{\nu^4\omega^4 + 4(\nu^2\omega^4 + 4s^2\omega^2)}}{2}.$$

Hence the eigenvalues of matrix B are given by

$$\lambda_{1,2} = \frac{\nu^2\omega^2 \mp \sqrt{\nu^4\omega^4 + 4(\nu^2\omega^4 + 4s^2\omega^2)}}{2} + s^2 + \omega^2.$$

If we want $\|(is - A_0^\top)^{-1}\| < \delta$ for some tolerance $\delta > 0$, we must have

$$\frac{1}{\sqrt{\lambda_{1,2}}} < \delta \Rightarrow \lambda_{1,2} \geq \frac{1}{\delta^2},$$

for all $\omega = \omega_1, \dots, \omega_n$. Note that

$$\lambda_{1,2} \geq \frac{\nu^2\omega^2 - \nu^2\omega^2 - 2\nu\omega^2 - 4s\omega}{2} + s^2 + \omega^2 = (s - \omega)^2 - \nu\omega^2.$$

To obtain $\|(is - A^\top)^{-1}\| < \varepsilon$ for a tolerance $\varepsilon > 0$, note that from (36) it follows that it is sufficient to have

$$\|(is - A_0^\top)^{-1}\| \leq \frac{\varepsilon}{1 + \gamma\varepsilon}.$$

Now we can take $\delta = \frac{\varepsilon}{1 + \gamma\varepsilon}$, so it follows

$$(s - \omega)^2 - \nu\omega^2 \geq \frac{(1 + \gamma\varepsilon)^2}{\varepsilon^2}$$

which implies

$$s^2 - 2\omega s + (1 - \nu)\omega^2 - \frac{(1 + \gamma\varepsilon)^2}{\varepsilon^2} \geq 0.$$

This is a quadratic inequality in the variable s which is satisfied for all s for which we have

$$s \geq \omega + \sqrt{\nu\omega^2 + \frac{(1 + \gamma\varepsilon)^2}{\varepsilon^2}}.$$

Hence the inequality $\|(is - A^\top)^{-1}\| < \varepsilon$ will be satisfied if we take

$$S_{\max} = \omega_{\max} + \sqrt{\nu\omega_{\max}^2 + \frac{(1 + \gamma_{\max}\varepsilon)^2}{\varepsilon^2}}. \quad (37)$$

In order to better illustrate the dependence on parameters ω_{\max} and γ_{\max} and the tolerance ε , we can also use the following, slightly worse, bound

$$S_{\max} \geq (1 + \sqrt{\nu})\omega_{\max} + \frac{1}{\varepsilon} + \gamma_{\max}.$$

4.2. Algorithm for the calculation of the finite time horizon p -mixed H_2 norm

To take into account the oscillating nature of the function we integrate, we divide the integration interval in two parts

$$2 \int_{-S_{\max}}^0 h_{jk}(t, s) ds = 2 \left(\int_{-S_{\max}}^{-S_1} h_{jk}(t, s) ds + \int_{-S_1}^0 h_{jk}(t, s) ds \right).$$

In the first integral on the right hand side we will use the standard Simpson rule because this integral does not oscillate as highly as the second one. In the second integral on the right hand side we will use the adaptive Simpson rule since the considered function is highly oscillatory on that segment. Moreover, adaptive Simpson is appropriate for our implementation since we use that the number of nodes is a power of 2 and therefore recycling can be done easily.

Algorithm for the calculation of the finite time horizon p -mixed H_2 norm defined by (35) is given by Algorithm 1.

4.2.1. Algorithm description

The Algorithm 1 consists of two parts; the offline and the online part.

First, we give more details regarding the **offline part**. In the offline part, we use the fact that some intermediate calculations can be effectively stored in matrices that do not depend on viscosity, and therefore, they can be calculated only once. In particular, in the offline part, we prepare the matrix of the type $\mathbb{R}^{n_s \times n_t}$ which entries are $\cos(s_i t_l)$, $i = 1, \dots, n_s$, $l = 1, \dots, n_t$. Also, we prepare matrices of the type $\mathbb{R}^{n_s \times n}$ with rows being the diagonal elements of matrices $F(s_i)$ and $G(s_i)$, $i = 1, \dots, n_s$. As entries of these matrices do not depend on the viscosities they can be calculated in the offline part.

In the **online part** we organize calculations in such a way that we can recycle computationally demanding parts. First, we define tensors $x_{\Im}, x_{\Re} \in \mathbb{R}^{n_s \times 2r \times n}$, where $x_{\Im}(s_i, j, :) = x_j^{\Im}(s_i)$ and $x_{\Re}(s_i, j, :) = x_j^{\Re}(s_i)$, for $i = 1, \dots, n_s$ and $j = 1, \dots, r, n+1, \dots, n+r$. Here $x_j^{\Im}(s_i)$ and $x_j^{\Re}(s_i)$ are defined in (29) - (33).

Then, in the most computationally demanding part of the algorithm line 11 - line 14, for every $k = 1, \dots, 2n$ and $j = 1, \dots, r, n+1, \dots, n+r$, we calculate the terms $h_{jk}(t_l, s_i)$ (whose formulae are derived in subsection 3.2) and we also use recycling for the parts that do not depend on time variable.

For the computation of the integral in line 12 we use the standard Simpson rule, hence we calculate the terms $h_{jk}(t_l, s_i)$, $i = 1, \dots, n_{s_1}$, $l = 1, \dots, n_t$.

For the integral in line 13 we start from the initial mesh of integration nodes (initially, we have 2^{b_0} nodes). Here, we are using the adaptive Simpson approach, which means that we pick iteratively denser meshes until we reach the prescribed tolerance tol . In this process, we recycle previously calculated function values $h_{jk}(t, s)$ to accelerate computations. Moreover, when the difference between the nodes reaches the maximal number of segment subdivisions (determined by the parameter $2^{b_{\max}}$), the current approximation on the segment is accepted. We emphasize that in the steps 11, 12 and 13 we benefit greatly from recycling. Recycling of the data from the offline part is made easier by the use of the adaptive approach, as we use the equidistant mesh with the number of nodes of the form 2^l .

Remark 2. *The algorithms have been implemented in Julia (see [11]). Julia low level programming enables efficient implementation comparable with standard BLAS routines. In Julia*

Algorithm 1 Algorithm for the calculation of the finite time horizon p -mixed H_2 norm

Require: system matrices: M (mass matrix), K (stiffness matrix), C_{ext} (external damping);

system parameters: ν (determines C_{int});

tolerances tol_s (tolerance for S_{max}), tol (integration tolerance);

the finite time horizon p -mixed H_2 norm parameters: $p \in [0, 1]$ (determine the target norm), r (number of undamped frequencies that need to be damped), T (defines time horizon);

n_t (number of nodes for the integration by t);

v_1, v_2, \dots, v_{n_v} (n_v considered viscosities);

b_{max} and b_0 (maximum and initial number of nodes for adaptive Simpson rule for second integral is $n_2 = 2^{b_{\text{max}}}$ and 2^{b_0});

n_1 (number of nodes for the first integral, $n_s = n_1 + n_2$ is total number of nodes for integration by s);

S_1 (limit for integration by variable s of first integral);

Ensure: Estimation of (17)

Offline part:

1: Determine S_{max} such that equality (37) holds.

2: Determine equidistant nodes $s_1, \dots, s_{n_1} \in [-S_{\text{max}}, -S_1]$, $s_{n_1+1}, \dots, s_{n_s} \in [-S_1, 0]$ for integration by s .

3: Determine nodes $t_1, \dots, t_{n_t} \in [0, T]$ and weights for integration by t .

4: Compute $\cos(s_i t_l)$ for every node $s_i, i = 1, \dots, n_s$ and $t_l, l = 1, \dots, n_t$.

5: Compute matrices $F(s_i)$ and $G(s_i)$ from (24) for all nodes $s_i, i = 1, \dots, n_s$.

6: Compute $f(s_i)$ and $g(s_i)$ defined in Proposition 2 for all nodes $s_i, i = 1, \dots, n_s$.

Online part:

7: **for** considered viscosities v_1, v_2, \dots, v_{n_v} **do**

8: Compute vectors $x_j^{\Im}(s_i)$ and $x_j^{\Re}(s_i)$ from (29)-(33) for all nodes $s_i, i = 1, \dots, n_s$ and all $j = 1, \dots, r, n+1, \dots, n+r$.

9: **for** $j = 1 : r, n+1 : n+r$ **do**

10: **for** $k = 1 : 2n$ **do**

11: Evaluate functions h_{jk} on a given grid (s_i, t_l) using formulae from subsection (3.2) while recycling parts which are time-independent.

12: Use the Simpson rule to compute $\int_{-S_{\text{max}}}^{-S_1} h_{jk}(t, s) ds$.

13: Use the adaptive Simpson rule to compute $\int_{-S_1}^0 h_{jk}(t, s) ds$.

14: Use the quadrature rule to compute $\int_0^T \left(\int_{-S_{\text{max}}}^{S_{\text{max}}} h_{jk}(t, s) ds \right)^2 dt$.

15: **end for**

16: **end for**

17: **end for**

we were able to efficiently implement standard and adaptive Simpson quadratures, including nested loops with simple operations. Of course, efficiency strongly depends on the number of nodes t_j and s_i and on how much recycling is used. Therefore, in the offline phase, we have prepared data needed for the calculation of the target value. Then we have used adaptive Simpson approach for the calculation of the integral over the variable s . The main reason for that is that it allows an implementation of adaptive quadrature that uses equidistant nodes, while on the same time we can recycle data obtained from previous steps as well as the data prepared in the offline part.

4.2.2. Algorithm parallelization

The most demanding parts of our algorithm can be parallelized and therefore our approach can be additionally accelerated. Here we would like to emphasize where we have used Julia's multithreading environment. In particular, we have used the macro `threads`. First, in the offline phase we have used the macro `threads` in line 4 and line 5 since this part includes generation of matrices with rows which are diagonal elements of matrices $F(s_i)$ and $G(s_i)$ for all nodes s_i (that do not depend on the time nodes), while we also form a matrix that stores values $\cos(s_i t_l)$, for all nodes s_i , $i = 1, \dots, n_s$ and all nodes t_l , $l = 1, \dots, n_t$, which depends on the time nodes as well.

Furthermore, the main benefit from parallelization comes from acceleration of the online part. In particular, we have used the macro `threads` in the inner loop for calculating the tensors x_{\Im} and x_{\Re} . Also, the inner loop over k depends on the dimension n , so lines 11 - 14 have been calculated using the macro `threads`.

We would like to emphasize the benefit from using Algorithm 1 compared to the approach that calculates Lyapunov equation given by (3). In the next section, we will calculate the number of floating point operations needed for one evaluation of p -mixed H_2 norm for both approaches.

4.3. Complexity analysis

An alternative approach for the calculation of the finite time horizon p -mixed H_2 norm uses the formula 3 and we will call it a Lyapunov based approach. This approach requires calculation of the Lyapunov equation and calculation of the matrix exponential. This means that Lyapunov based approach needs $n_v \mathcal{O}(n^3)$ floating point operations for calculation of p -mixed H_2 norm for n_v different viscosities.

This Lyapunov based approach can be accelerated by using model order reduction techniques as we mentioned in the introduction, but also it can be accelerated by using the low rank structure that appears in our case. In particular, the objective function can be calculated using an approach that is based on function calculation using low-rank updates; see, e.g., [5, 6]. This can be applied by calculating the matrix exponential function and by solving the Lyapunov equation given by 3. Comparison with this approach is not given since we do not have relevant implementation of this approach. Moreover, we would like to emphasize that when the finite horizon T is changed, the Lyapunov equation 3 does not need to be solved again, but the matrix exponential function needs to be calculated repeatedly. On the other hand, our approach includes formulae constructed in such a way so that the most expensive part (from the calculation point of view) can be recycled. Below we give more details on the complexity of our approach.

In the analysis of the complexity of Algorithm 1, we will separately study the online and the offline part. Moreover, there are two levels of offline parts. One includes certain complexity that does not depend even on the external damping and therefore can be done only once for all viscosities, damping positions and different time horizons. Main cost within this part comes from the simultaneous diagonalization defined by (4) that requires $\mathcal{O}(n^3)$ (floating point) operations, but only once. Note that the simultaneous diagonalization is always necessary if the internal damping is defined in terms of the critical damping in (2). Then, once the damping positions are fixed, we can evaluate Algorithm (1) as follows

- Offline
 - preparation of offline data includes line 1 - line 6 that requires: $n_s\mathcal{O}(n)$ operations
- Online, for each viscosity line 8 - line 14 include
 - line 8 that requires: $n_s\mathcal{O}(rn)$ operations
 - for all j and k we have line 11 - line 14 where
 - line 11 requires: $\mathcal{O}(n_s n)$ operations
 - line 12 requires: $\mathcal{O}(n_t n_1)$ operations
 - line 13 requires: $\mathcal{O}(n_t n_2)$ operations
 - line 14 requires: $\mathcal{O}(n_t)$ operations

This means that line 11 - line 14 require $n_s\mathcal{O}(rn) + n_t n_s\mathcal{O}(rn) + n_s\mathcal{O}(rn^2)$ floating point operations.

Taking all into account, we can conclude that p -mixed H_2 norm for n_v different viscosities can be calculated by Algorithm 1 using $n_v n_s\mathcal{O}(rn^2)$ floating point operations excluding the possible computational cost of the simultaneous diagonalization (4), which is independent on the choice of all parameters except matrices M and K . In particular, it does not depend on the parameters T , p , and the choice of the external damping matrix C_{ext} and so the cost can be taken as negligible in the framework of optimization or online simulation. Moreover, our estimation on the required number of floating point operations depends on n_s which does not depend directly on n .

Here we would like to emphasize that in our approach when parameter T , which defines the time horizon, is changed we can use recycling on that level too. In particular, line 11 is the most demanding step in the algorithm but it does not depend on the time nodes. Moreover, when T is slightly changed we need to evaluate Algorithm 1 for very small n_t while all the data that do not depend on time nodes can be recycled. From the complexity perspective this means that the floating point operations from line 8 and line 11 now belong to the offline part. Taking all this into account we obtain that calculation of p -mixed H_2 norm for n_T different time horizons can be calculated by Algorithm 1 using

$$n_s\mathcal{O}(rn) + n_s\mathcal{O}(rn^2) + n_T n_t n_s\mathcal{O}(rn)$$

floating point operations. In contrast, Lyapunov based approach needs $n_T\mathcal{O}(n^3)$ floating point operations for the calculation of p -mixed H_2 norm for n_T different time horizons.

This analysis shows that we can efficiently analyze the influence of different final times T on the p -mixed H_2 norm. Therefore, one approach for practical determination of a good final time T could be based on the determination of T for which the p -mixed H_2 norm stagnates. In the next section, we will illustrate this in an numerical example.

We will consider one damper, but the damping position will be changed, so the external damping is defined by (28) and the internal damping C_{int} is defined as in (2), with $\alpha = 0.005$. In this example for the illustration and comparison purposes we will consider four different damping positions, that is, we will consider that vector e from (28), that encodes damping positions, is equal to e_i where $i = 10, 80, 110, 160$.

We consider damping of 1% smallest eigenfrequencies of the system which means that we have $r = \frac{n}{100} = 2$ and parameter p that defines p -mixed H_2 norm is taken to be 0.5.

In this example we will show the influence of the parameter T in (17), that is the influence of the time horizon in the p -mixed H_2 norm. This influence is shown on (2).

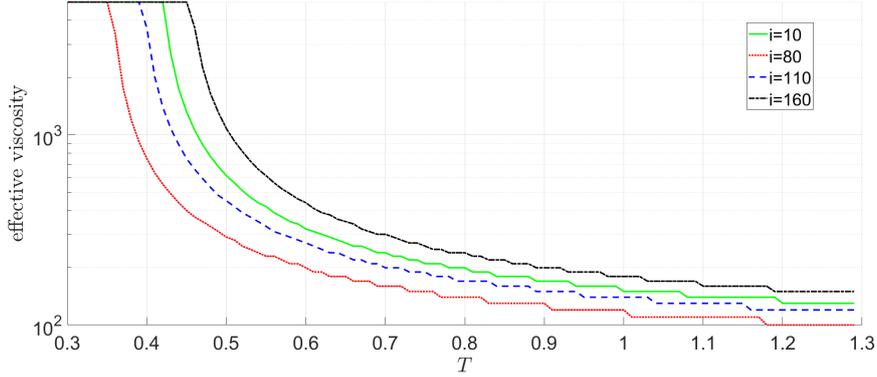


Figure 2: (Example 1) The influence of the integration time on the magnitude of the effective viscosity for dimension $n = 200$ and for four different damping positions. By effective viscosity we mean the threshold value after which the finite time horizon p -mixed H_2 norm drops significantly. The computation is done using the workstation.

As is to be expected, for very small times T it is hard to achieve significant damping effects, therefore a very large viscosity is needed to significantly reduce the finite time horizon p -mixed H_2 norm, which is usually physically infeasible. On the other hand, when T is increased we observe major decay in effective viscosities for all considered damping positions. This means that for moderate times T effective viscosities vary within the appropriate values, and for large times T the curve is close to the case $T = \infty$. Of course, it is hard to state what do we mean by moderate T , but exactly this decay gives us this information. In this example, from our analysis we can observe that relevant time horizon with reasonable effective viscosity starts at around 1.

Example 2. In this example we will have the same configuration as in the previous example, but with the dimension $n = 2000$. We have calculated an approximation of the finite time horizon p -mixed H_2 norm by using Algorithm 1 with the following initial requirements:

$$\begin{aligned}
 \text{tol} &= 10^{-5}, & n_t &= 20, \\
 n_v &= 20, & n_1 &= 599, \\
 S_1 &= \omega_n/25, & b_0 &= 8, \\
 b_{\max} &= 12, & \text{tol}_s &= 0.05,
 \end{aligned}$$

and we consider the following viscosities $v_1 = 75, v_2 = 150, \dots, v_{20} = 1500$.

As in the previous example, we have used $p = 0.5$ and we would like to damp 1% of the undamped eigenfrequencies, which means that $r = 20$, while internal damping is determined using $\alpha = 0.005$. Here, from the similar numerical analysis as in the previous example, we can conclude that T should be larger than 1, and therefore in the continuation of this example we illustrate the efficiency of our approach for $T = 2$. Moreover, the influence of T is significant and with our approach we can efficiently calculate the finite time horizon p -mixed H_2 norm for several values of T , since the major computational cost taken care of in the offline part.

To present comparison with a Lyapunov based approach, we will calculate the finite time horizon p -mixed H_2 norm with new approach and compare it with the Lyapunov based approach that uses the formula (3). Here Lyapunov based approach is implemented in such a way that we solve Lyapunov equation and matrix exponential directly. In particular, in Julia the matrix exponential is calculated using one of the most widely used method based on [25], while the algorithm for solving the Lyapunov equation is based on LAPACK routines that uses direct solvers for Lyapunov equation. For more details see, e.g. [15, 3, 28, 23, 2]. Therefore we can use this value as the exact solution. Figure 3 presents the average relative error for these two approaches for 20 equidistant viscosities, from 75 to 1500, and shows the average relative error for all considered damping positions e_i , where $i = 100, 200, \dots, 1900$. Then, for four different damping positions, that is for $e = e_i$, where $i = 200, 800, 1100, 1600$, Figure 4 shows the relative error for these two approaches for all 20 equidistant viscosities. We can see that our approach for given tolerances results with satisfactory accuracy.

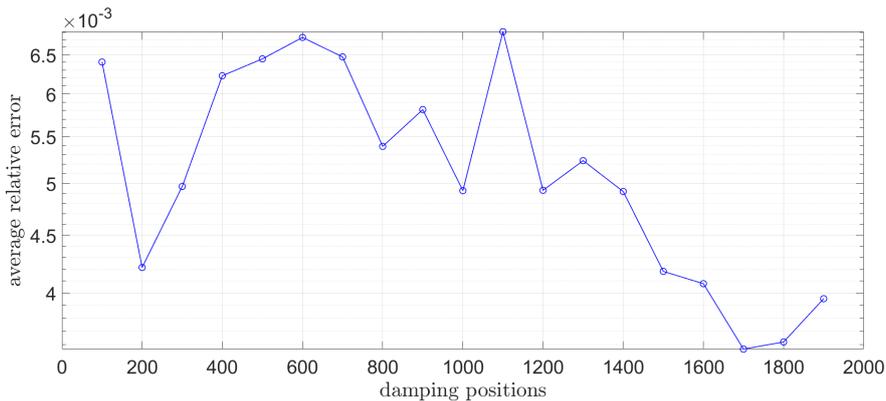


Figure 3: (Example 2) The average relative error for 20 equidistant viscosities from 75 to 1500, for the calculation of the finite time horizon p -mixed H_2 norm using Algorithm 1, compared with the Lyapunov based approach, for 19 different damping positions. The computation is done using the workstation.

For a time comparison, first we have used a laptop to calculate the finite time horizon p -mixed H_2 norm, using Algorithm 1 and using Lyapunov based approach. For four different damping positions given in Figure 4, average acceleration factor is 2.5.

This can be improved by the efficient usage of tensor structures that arise in Algorithm 1 which is illustrated in Figure 5. In particular, Figure 5 presents the time required for the calculation of the finite time horizon p -mixed H_2 norm using Algorithm 1 and the time required for the Lyapunov based approach.

Current implementation uses the benefit of large number of threads available on the workstation. Thus, it is optimized for better usage of multithreading environment, and this is also

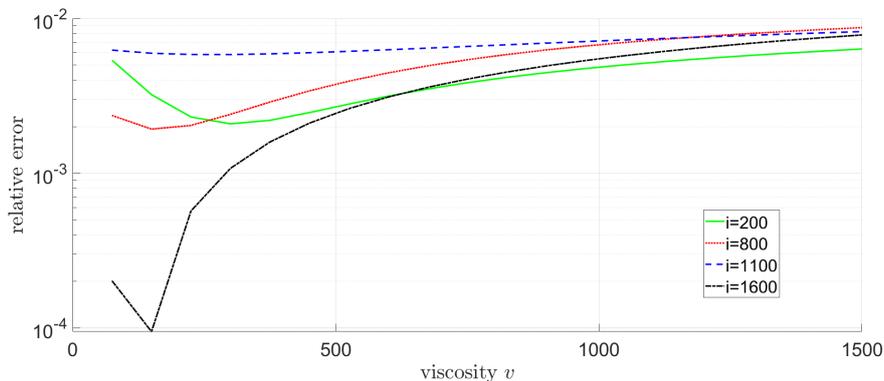


Figure 4: (Example 2) The relative error for the calculation of the finite time horizon p -mixed H_2 norm using Algorithm 1, compared with the Lyapunov based approach, for four different damping positions and 20 equidistant viscosities from 75 to 1500. The computation is done using the workstation.

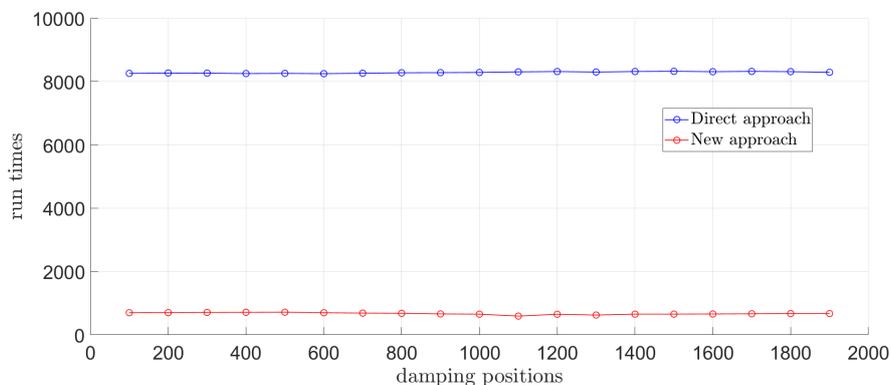


Figure 5: (Example 2) The time required for the calculation of the finite time horizon p -mixed H_2 norm using Algorithm 1 compared to the time required for the Lyapunov based approach, for 19 different damping positions. The computation is done using the workstation.

confirmed in numerical tests. In particular, on the workstation an average acceleration factor for four considered damping positions presented on Figure 4 is 12.2.

Example 3. In this example, we consider the mechanical system shown in Figure 6, consisting of three rows of d masses and $d + 1$ springs which are, on the left-hand side, connected to the fixed base. Springs in each row have the same stiffness equal to k_1 , k_2 and k_3 . On the right-hand side, they are connected to one additional mass, which is connected to the fixed base with a spring of stiffness k_4 . Similar models were considered in the paper [8].

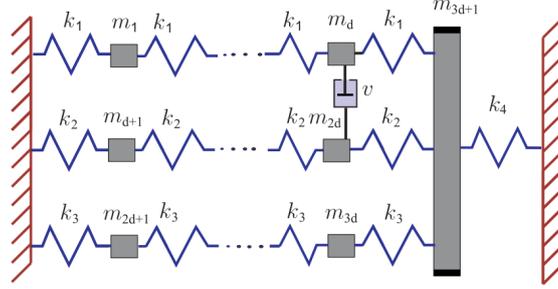


Figure 6: $(3d + 1)$ -mass oscillator with one damper

For such a mechanical system the mathematical model is given by (1), where the mass and stiffness matrices are

$$M = \text{diag}(m_1, m_2, \dots, m_n),$$

$$K = \begin{pmatrix} K_1 & & & -\kappa_1 \\ & K_2 & & -\kappa_2 \\ & & K_3 & -\kappa_3 \\ -\kappa_1 & -\kappa_2 & -\kappa_3 & k_1 + k_2 + k_3 + k_4 \end{pmatrix}, \quad \text{with}$$

$$K_i = k_i \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \quad \kappa_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k_i \end{pmatrix}, \quad i = 1, 2, 3.$$

We have the following configuration

$$d = 800; \quad k_j = \begin{cases} 800, & j = 1, \\ 600, & j = 2, \\ 700, & j = 3; \end{cases} \quad m_j = \begin{cases} 1000 - 2j, & j = 1, \dots, \frac{d}{2}, \\ j - 200, & j = \frac{d}{2} + 1, \dots, d, \\ j + 100, & j = d + 1, \dots, 2d, \\ n - 2j, & j = 2d + 1, \dots, 3d, \\ 2000, & j = n = 3d + 1. \end{cases}$$

We will consider here one damper with a different damping geometry compared to the previous example. Damper will be located between two masses in different rows and we will consider six

different damping positions. The corresponding vectors e from (28), denoted by e^i are given by

$$(e^i)_j = \begin{cases} 1, & j = i \\ -1, & j = i + d, \quad j = 1, \dots, n, \\ 0, & \text{otherwise} \end{cases} \quad (38)$$

for $i = 20, 320, 620, 920, 1220, 1520$. The internal damping C_{int} is defined as in (2), with $\alpha = 0.005$.

As in previous examples, we consider damping of approximately 1% smallest eigenfrequencies of the system, that is $r = 24$, and parameter p that defines p -mixed H_2 norm is taken to be 0.5. We have calculated an approximation of the finite time horizon p -mixed H_2 norm by using Algorithm 1 with the following initial requirements:

$$\begin{aligned} \text{tol} &= 10^{-7}, & n_t &= 20, \\ n_v &= 20, & n_1 &= 799, \\ S_1 &= \omega_n/15, & b_0 &= 10, \\ b_{\text{max}} &= 14, & \text{tol}_s &= 0.05, \end{aligned}$$

and we consider the following viscosities: $v_1 = 10, v_2 = 210, v_3 = 410, \dots, v_{20} = 3810$. We illustrate the efficiency of our approach for $T = 15$.

To present comparison with a Lyapunov based approach, as in the previous example, we will calculate the finite time horizon p -mixed H_2 norm with the new approach and compare it with the Lyapunov based approach based on the formula (3). For all six damping positions defined by (38), Figure 7 shows the relative error for these two approaches for all 20 viscosities, from 10 to 3810. We can see that our approach for given tolerances results with satisfactory accuracy.

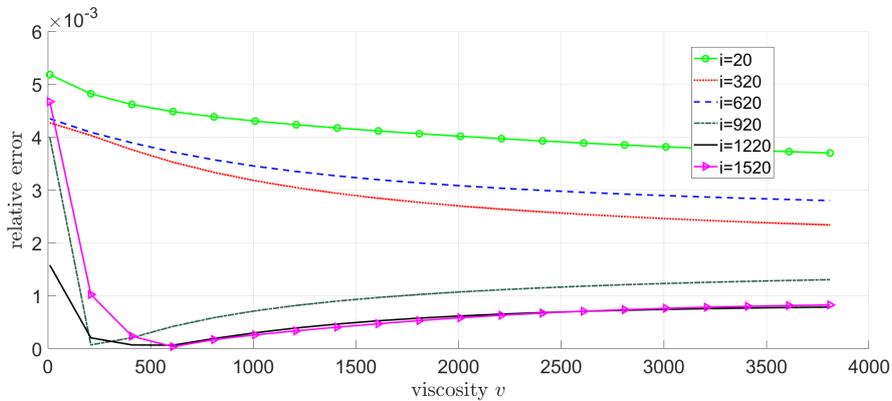


Figure 7: (Example 3) The relative error for the calculation of the finite time horizon p -mixed H_2 norm using Algorithm 1, compared with the Lyapunov based approach, for six different damping positions and 20 equidistant viscosities between 10 and 3810. The computation is done using the workstation.

For a time comparison, Figure 8 presents the time required for calculating the finite time horizon p -mixed H_2 norm using Algorithm 1 and the time required for the Lyapunov based approach.

Here we obtain that an average acceleration factor for six considered damping positions presented on Figure 7 is 5.4.

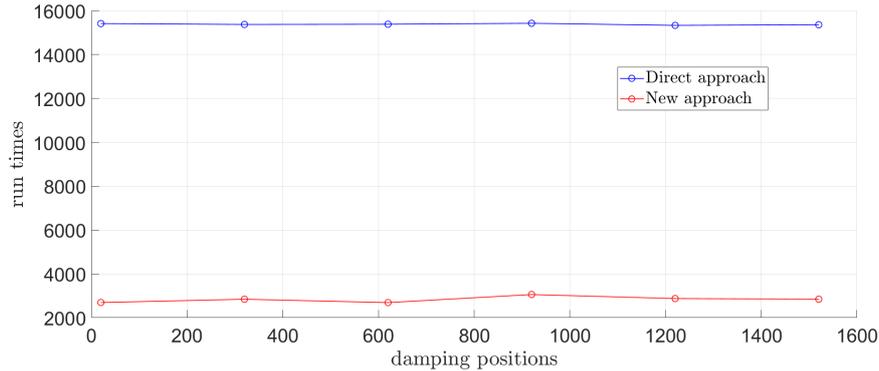


Figure 8: (Example 3) The time required for calculating the finite time horizon p -mixed H_2 norm using Algorithm 1 compared to the time required for the Lyapunov based approach, for six different damping positions. The computation is done using the workstation.

A limitation of the proposed method is that with this method, one can not efficiently reach very high accuracies. More accurate approximations may be significantly costlier, which makes this method not feasible for specific applications where high accuracy is needed. However, regarding typical vibrational systems, it is unrealistic to expect that the viscosity needs to be calculated with an accuracy greater than 10^{-3} , as typical damping devices can not be that precisely calibrated. Thus, with the proposed method, we can obtain satisfactory accuracy for damping purposes with a significant time speed-up with respect to the standard approaches.

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6. Conclusions

We considered a control problem for damped vibrational systems where the performance measure of the system is chosen to be p -mixed H_2 norm over the finite time horizon. The algorithm presented in the paper offers an efficient calculation of this norm in the case when the system is dependent on one parameter and the number of inputs or outputs of the system is significantly smaller than the order of the system. This approach can be extended to the case of the multi-parameter setting as well as for other parameter-dependent systems where an efficient calculation of finite time horizon H_2 norm is needed. In future work we will extend this approach also to the optimal control problems for damped vibrational systems and also other problems in control theory.

A. Proofs of Propositions 1, 2 and 3

Proof of Proposition 1. For matrix $L(S)$, we have

$$\begin{aligned} L(s) &= \text{diag}(l_1(s), \dots, l_n(s)) \\ &= \text{diag}\left(\frac{1}{\omega_1^2 - s^2 + is\nu\omega_1}, \dots, \frac{1}{\omega_n^2 - s^2 + is\nu\omega_n}\right) = F(s) - iG(s). \end{aligned}$$

For $1 \leq j \leq r$ we have

$$\begin{bmatrix} I + sDG(s) & -sDF(s) \\ sDF(s) & I + sDG(s) \end{bmatrix} \begin{bmatrix} x_j^{\Re} \\ x_j^{\Im} \end{bmatrix} = \begin{bmatrix} D\Omega F(s)e_j \\ -D\Omega G(s)e_j \end{bmatrix}.$$

For $n+1 \leq j \leq n+r$ we have

$$\begin{bmatrix} I + sDG(s) & -sDF(s) \\ sDF(s) & I + sDG(s) \end{bmatrix} \begin{bmatrix} x_j^{\Re} \\ x_j^{\Im} \end{bmatrix} = \begin{bmatrix} e_{j-n} \\ 0 \end{bmatrix}.$$

It is easy to see from the construction of $G(s)$ and the assumption on D that the matrix $I + sDG(s)$ is always non-singular. If we multiply this systems from the left by the matrix

$$\begin{bmatrix} (I + sDG(s))^{-1} & 0 \\ -sDF(s)(I + sDG(s))^{-1} & I \end{bmatrix},$$

we obtain the stated result. □

Proof of Proposition 2. First, by using the Sherman–Morrison–Woodbury formula (see, e.g., [23]) we obtain

$$(I + sDG(s))^{-1} = I - sU \left(\frac{1}{\gamma} + sU^\top G(s)U \right)^{-1} U^\top G(s).$$

Note that $U^\top G(s)U \in \mathbb{R}$ so the inverse on the right hand side is trivial to solve,

$$(I + sDG(s))^{-1} = I - \frac{s\gamma}{1 + s\gamma g} UU^\top G(s).$$

With this, (26) and (27) become

$$\begin{aligned} & \begin{bmatrix} I & -\frac{s\gamma}{1 + s\gamma g(s)} UU^\top F(s) \\ 0 & s^2\gamma^2 UU^\top F(s) \left(I - \frac{s\gamma}{1 + s\gamma g(s)} UU^\top G(s) \right) UU^\top F(s) + I + s\gamma UU^\top G(s) \end{bmatrix} \begin{bmatrix} x_j^{\Re} \\ x_j^{\Im} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\gamma}{1 + s\gamma g(s)} UU^\top \Omega F(s) e_j \\ -\left(\frac{s\gamma^2 f(s)}{1 + s\gamma g(s)} UU^\top \Omega F(s) + \gamma UU^\top \Omega G(s) \right) e_j \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} & \begin{bmatrix} I & -\frac{s\gamma}{1 + s\gamma g(s)} UU^\top F(s) \\ 0 & s^2\gamma^2 UU^\top F(s) \left(I - \frac{s\gamma}{1 + s\gamma g(s)} UU^\top G(s) \right) UU^\top F(s) + I + s\gamma UU^\top G(s) \end{bmatrix} \begin{bmatrix} x_j^{\Re} \\ x_j^{\Im} \end{bmatrix} \\ &= \begin{bmatrix} \left(I - \frac{s\gamma}{1 + s\gamma g(s)} UU^\top G(s) \right) e_{j-n} \\ -(s\gamma UU^\top F(s) + s^2\gamma^2 f(s) UU^\top G(s)) e_{j-n} \end{bmatrix}. \end{aligned}$$

Note that since we know that (26) and (27) have unique solutions, it follows that the (2, 2) entry in the matrix on the left hand side is non-singular. As the (2, 2) entry in the matrix on the left hand side is

$$\begin{aligned} & s^2\gamma^2 UU^\top F(s) \left(I - \frac{s\gamma}{1 + s\gamma g(s)} UU^\top G(s) \right) UU^\top F(s) + I + s\gamma UU^\top G(s) \\ &= I + s\gamma U \left(U^\top G(s) + s\gamma U^\top F(s) \left(I - \frac{s\gamma}{1 + s\gamma g(s)} UU^\top G(s) \right) UU^\top F(s) \right) \\ &= I + s\gamma U \left(U^\top G(s) + \frac{s\gamma f(s)}{1 + s\gamma g(s)} U^\top F(s) \right), \end{aligned}$$

we have

$$\begin{aligned}
& \left(I + s\gamma U \left(U^\top G(s) + \frac{s\gamma f(s)}{1 + s\gamma g(s)} U^\top F(s) \right) \right)^{-1} \\
&= I - sU \left(\frac{1}{\gamma} + s \left(U^\top G(s) + \frac{s\gamma f(s)}{1 + s\gamma g(s)} U^\top F(s) \right) U \right)^{-1} \times \\
& \quad \times \left(U^\top G(s) + \frac{s\gamma f(s)}{1 + s\gamma g(s)} U^\top F(s) \right) \\
&= I - sU \left(\frac{1}{\gamma} + sg(s) + \frac{s^2\gamma f(s)^2}{1 + s\gamma g(s)} \right)^{-1} \left(U^\top G(s) + \frac{s\gamma f(s)}{1 + s\gamma g(s)} U^\top F(s) \right) \\
&= I - \frac{s\gamma}{(1 + s\gamma g(s))^2 + (s\gamma f(s))^2} \left((1 + s\gamma g(s))UU^\top G(s) + s\gamma f(s)UU^\top F(s) \right).
\end{aligned}$$

Now, the formulae (29)-(33) follow by direct calculation. \square

Proof of Proposition 3. First, for $1 \leq k \leq n$ we obtain

$$(\mathrm{i}s - A_0^\top)^{-1} e_k = \begin{bmatrix} (\nu\omega_k + \mathrm{i}s)l_k(s)e_k \\ \omega_k l_k(s)e_k \end{bmatrix},$$

and for $n+1 \leq k \leq 2n$ we obtain

$$(\mathrm{i}s - A_0^\top)^{-1} e_k = \begin{bmatrix} -\omega_{k-n}l_{k-n}(s)e_{k-n} \\ \mathrm{i}s l_{k-n}(s)e_{k-n} \end{bmatrix},$$

where $l_k(s) = f_k(s) - \mathrm{i}g_k(s)$, for $k = 1, \dots, n$.

For $1 \leq j \leq r$ and $1 \leq k \leq n$ we obtain

$$\begin{aligned}
e_j^\top (\mathrm{i}s - A^\top)^{-1} e_k &= (\nu\omega_k + \mathrm{i}s)l_k(s)e_j^\top e_k + \omega_k l_k(s) (x_j^2)^\top e_k \\
&= (\nu\omega_k + \mathrm{i}s)(f_k(s) - \mathrm{i}g_k(s))e_j^\top e_k + \omega_k(f_k(s) - \mathrm{i}g_k(s)) (x_j^2)^\top e_k \\
&= (\nu\omega_k f_k(s) + sg_k(s))\delta_{j,k} + \omega_k \left(f_k(s)(x_j^{\Re})_k + g_k(s)(x_j^{\Im})_k \right) \\
& \quad + \mathrm{i} \left((sf_k(s) - \nu\omega_k g_k(s))\delta_{j,k} + \omega_k \left(f_k(s)(x_j^{\Im})_k - g_k(s)(x_j^{\Re})_k \right) \right),
\end{aligned}$$

for $n+1 \leq j \leq n+r$ and $1 \leq k \leq n$ we obtain

$$\begin{aligned}
e_j^\top (\mathrm{i}s - A^\top)^{-1} e_k &= \omega_k l_k(s) (x_j^2)^\top e_k = \omega_k(f_k(s) - \mathrm{i}g_k(s)) (x_j^2)^\top e_k \\
&= \omega_k \left(f_k(s)(x_j^{\Re})_k + g_k(s)(x_j^{\Im})_k \right) + \mathrm{i}\omega_k \left(f_k(s)(x_j^{\Im})_k - g_k(s)(x_j^{\Re})_k \right),
\end{aligned}$$

for $1 \leq j \leq r$ and $n+1 \leq k \leq 2n$ we obtain

$$\begin{aligned}
e_j^\top (\mathrm{i}s - A^\top)^{-1} e_k &= -\omega_{k-n}l_{k-n}(s)e_j^\top e_{k-n} + \mathrm{i}s l_{k-n}(s) (x_j^2)^\top e_{k-n} \\
&= -\omega_{k-n}(f_{k-n}(s) - \mathrm{i}g_{k-n}(s))e_j^\top e_{k-n} + \mathrm{i}s(f_{k-n}(s) - \mathrm{i}g_{k-n}(s)) (x_j^2)^\top e_{k-n} \\
&= -\omega_{k-n}f_{k-n}(s)\delta_{j,k-n} + sg_{k-n}(s)(x_j^{\Re})_{k-n} - sf_{k-n}(s)(x_j^{\Im})_{k-n} \\
& \quad + \mathrm{i} \left(\omega_{k-n}g_{k-n}(s)\delta_{j,k-n} + sf_{k-n}(s)(x_j^{\Re})_{k-n} + sg_{k-n}(s)(x_j^{\Im})_{k-n} \right),
\end{aligned}$$

and for $n + 1 \leq j \leq n + r$ and $n + 1 \leq k \leq 2n$ we obtain

$$\begin{aligned} e_j^\top (is - A^\top)^{-1} e_k &= is l_{k-n}(s) (x_j^2)^\top e_{k-n} = is(f_{k-n}(s) - ig_{k-n}(s)) (x_j^2)^\top e_{k-n} \\ &= s \left(g_{k-n}(s) (x_j^{\Re})_{k-n} - f_{k-n}(s) (x_j^{\Im})_{k-n} \right) \\ &\quad + is \left(f_{k-n}(s) (x_j^{\Re})_{k-n} + g_{k-n}(s) (x_j^{\Im})_{k-n} \right). \end{aligned}$$

Now we take into account that the inner integral has a real value (see, e.g., [13]), so we just need to calculate its real part. Thus, the function h_{jk} defined by (21) satisfies

$$h_{jk}(t, s) = 2 \cos st \cdot \Re(e_j^\top (is - A^\top)^{-1} e_k).$$

The result now follows directly by taking real parts in the formulae given above. \square

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