ON NORM AND TIME OPTIMAL CONTROLS FOR SYSTEMS DESCRIBED BY LINEAR PARABOLIC PDES

Marius Tucsnak

ON NORM AND TIME OPTIMAL CONTROLS FOR SYSTEMS DESCRIBED BY LINEAR PARABOLIC PDES

Marius Tucsnak*

Institut de Mathématiques de Bordeaux
Université de Bordeaux
351, Cours de la Libération - F 33 405 TALENCE, France

Abstract. This work proposes a general theory of norm and time optimal control for a class of null controllable systems. We provide sufficient conditions for the existence and uniqueness of optimal controls, the bang-bang property and for the equivalence between norm and time optimal control problems. The obtained abstract theorems allow us to obtain several new results for systems described by parabolic PDEs, with distributed or boundary controls.

1. Introduction

This paper addresses two important questions in control theory:

- Minimizing the energy necessary to steer a linear system to zero, i.e., solve the *norm optimal control problem*.
- Minimizing the time necessary to steer a linear system to zero when the control is constrained to a bounded set. This is the *time optimal control problem*.

The study of norm and time optimal controls for finite dimensional *linear time invariant systems* (designed as LTI's in the remaining part of this work) is a well understood subject. Most of the papers tackling this subject are related to Pontryagyn's maximum principle [4], which is a strong tool, adaptable to nonlinear systems and to other cost functionals. We mention also here the pioneering work on time optimal controls of Bellman, Glicksberg and Gross [3]. According to Fattorini [8, Chapter 1], the approach in [3] "was so elementary that invited to generalizations to infinite dimensional spaces". Indeed, the theory for time optimal control for infinite dimensional LTI's has been initiated in a relatively short time, as reported in Fattorini [9] and Lions [13] and then developed in a series of papers such as Schmidt [19], Mizel and Seidman [17], Wang [24], Phung and Wang [18], Kunisch and Wang [12], Micu, Roventa and Tucsnak [16] or Loheac and Tucsnak [15]. Moreover, Fattorini in [8] and Wang, Wang, Xu and Zhang in [25] devoted complete monographs to this subject.

The main contribution brought in by our work is that it proposes an abstract theory, including the existence and uniqueness of optimal controls, the bang-bang property and the relation between norm and time optimal controls. This theory applies to systems described by parabolic PDEs, for which one of the difficulties is

2020 *Mathematics Subject Classification.* 93B03, 93B05,93B07, 93C15, 35K90, 35Q93.

Key words and phrases. Reachable space, null controllability, bang-bang-property, norm and time optimal functions.

* Author's research has been partially spported the funding by the French ANR Project TRECOS grant ANR-20-CE40-0009 (2021–2024).

that the reachable set may have an empty interior. Our methods allow, in particular, to handle two other difficulties encountered in this type of problem:

- The norm which we aim at minimizing is the L^∞ one with respect to both time and to space variables. Moreover, for the time optimal control problem the constraints are pointwise, thus again involving only L^∞ norms.
- We consider controls which aim at driving the states of the considered systems *exactly to zero*. Unlike in situations when the control aim is to drive the state of system to a neighbourhood of the origin, no maximum principle is available in the literature.

From a methodological view point one of the main novelties is to construct an extension of the dual of an L^∞ null controllable system, such that the extended system is *exactly observable*. This is a surprising fact since, in the usual state space formulation, this dual is only *final state observable*.

The precise statement of our main results, which we choose to give in an abstract setting, requires some notation and preliminaries, so we postpone it to Section 3. However, for reader's convenience, we state below the consequences of our abstract results for a basic example of system described by a parabolic PDE with distributed control. These consequences will be formally established in Section 8, where we also discuss other examples of application of our abstract results.

To formulate this introductory example, let $n \geqslant 2$ be a positive integer and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Let $\mathcal{O}$ be a non-empty open subset of $\partial\Omega$ and let $\chi_{\mathcal{O}}$ be the characteristic function of $\mathcal{O}$. We consider the parabolic equation

$$\frac{\partial z}{\partial t}(t,x) = \Delta z(t,x) \qquad ((t,x) \in (0,\infty) \times \Omega), \tag{1.1}$$

with the boundary conditions

$$z(t,x) = 0 \qquad ((t,x) \in (0,\infty) \times (\partial\Omega \setminus \mathcal{O}))\,, \tag{1.2}$$

$$z(t,x) = u(t,x) \qquad ((t,x) \in (0,\infty) \times \mathcal{O})\,, \tag{1.3}$$

where u is the control function. Moreover, we consider the initial condition

$$z(0,x) = \psi(x) \qquad (x \in \Omega). \tag{1.4}$$

An important result, whose proof is recalled in Section 8, is that the above system is L^∞ null controllable in any time, i.e., that, given $\tau > 0$, for every $\psi \in H^{-1}(\Omega)$ there exists $u \in L^\infty([0,\infty) \times \mathcal{O})$ such that $z(\tau,\cdot) = 0$. This fact enables us to formulate the norm and time optimal control problems for the system described by (1.1), (1.2), (1.3) and (1.4).

More precisely, given $\psi \in L^2(\Omega)$ and $\tau > 0$ the *norm optimal control problem for the system described by* (1.1)-(1.4) consists in determining $\hat{u} \in L^\infty([0,\infty) \times \mathcal{O})$ with

$$\|\hat{u}\|_{L^\infty([0,\infty)\times\mathcal{O})} = \min_{\substack{u \in L^\infty([0,\infty)\times\mathcal{O}) \\ z(\tau,\cdot)=0}} \|u\|_{L^\infty([0,\infty)\times\mathcal{O})}.$$

If the above optimal control problem admits a solution $\hat{u}$ for some $\tau > 0$, we denote by $N^\infty(\tau)$ the optimal cost, i.e., we set $N^\infty(\tau) = \|\hat{u}\|_{L^\infty([0,\infty)\times\mathcal{O})}$.

On the other hand, given $\psi \in L^2(\Omega)$, $M > 0$ and setting

$$\mathcal{U}_M := \left\{u \mid u \in L^\infty((0,\infty) \times \mathcal{O}) \quad \text{s.t. } \|u\|_{L^\infty((0,\infty)\times\mathcal{O})} \leqslant M\right\}.$$

the *time optimal control problem for the system described by* (1.1)-(1.4) consists in determining $(\tau^\infty(M), u_M^\infty) \in (0,\infty) \times \mathcal{U}_M$ such that u_M^∞ drives the initial state

ψ to zero in time $\tau^\infty(M)$, i.e., the state trajectory z_M^∞ associated to u_M^∞ satisfies $z_M^\infty(\tau,\cdot)=0$, and

$$\tau^\infty(M) := \min_{u\in\mathcal{U}_M} \{\tau \mid \exists u \in \mathcal{U}_M \quad \text{with} \quad z(\tau,\cdot)=0\}.$$

Our main result on the system (1.1)-(1.4) is:

Proposition 1.1. *With the above notation, assume that $\partial\Omega$ is of class C^2 or that Ω is a rectangular domain. Then we have:*

(1) *For every $\tau > 0$ and $\psi \in L^2(\Omega) \setminus \{0\}$ the norm optimal control problem for the system (1.1)-(1.4) admits at least one solution. Moreover, this control $\hat{u}$ can be chosen to have the bang-bang property, i.e.,*

$$|\hat{u}(t,x)| = N^\infty(\tau) \qquad ((t,x) \in (0,\tau) \times \mathcal{O} \text{ a.e.}).$$

Finally, the function $\tau \mapsto N^*(\tau)$ is decreasing and $\lim_{\tau\to\infty} N(\tau) = 0$.

(2) *For every $\psi \in L^2(\Omega) \setminus \{0\}$ and $M > 0$, the time optimal control problem for the system (1.1)-(1.4) admits at least one solution $(\tau^\infty(M), u_M^\infty)$. Moreover, u_M^∞ can be selected such that it has the bang-bang property*

$$|u_M^\infty(t,x)| = M \qquad ((t,x) \in [0,\tau^\infty(M)] \times \mathcal{O} \text{ a.e.}).$$

(3) *The functions N^∞, τ^∞ defined via the two assertions above satisfy*

$$N^\infty(\tau^\infty(M)) = M, \qquad \tau^\infty(N^\infty(\tau)) = \tau \qquad (M,\ \tau > 0).$$

Under the above hypothesis, as far as we know, all the results in Proposition 1.1 are new. Adding supplementary assumptions, the second assertion in Proposition 1.1 has been proved in [16] and in Apraiz et al. [2], where the uniqueness of the time optimal control has also been obtained. If we also adopt the supplementary assumption in [2], we obtain:

Proposition 1.2. *With the notation and the assumptions in Proposition 1.1, assume that $\mathcal{O}$ contains a real analytic manifold (in the sense of Definition 5 in [2]). Then the norm and time optimal controls obtained in Proposition 1.1 are unique.*

The remaining part of this work is organised as follows. Section 2 is devoted to notation and some control theoretic background. Our leading assumptions and our main results are then stated in Section 3. In Section 4 we introduce and give the main properties of the L^∞ reachable space and the multiplier space, which play a fundamental role in the present paper. In Section 5 we prove that the dual of the considered systems can be extended to an exactly observable observation system, with the multiplier state constructed in the previous section as state space. Section 6 provides a construction of norm optimal controls via the minimization of an auxiliary functional. Our main abstract results are proved in Section 7. Finally, in Section 8 we describe the applications of our abstract results to systems described by parabolic PDEs.

2. Notation and preliminaries

In this section we introduce some notation which will be constantly used in the remaining part of this paper and we recall some basic facts on well-posed control linear time invariant systems (LTI's). We refer to Weiss [28], Tucsnak and Weiss [22, Chapters 2,3,4] and [23] for detailed information on this subject.

Firstly, we introduce the Hilbert spaces U (the input space) and X (the state space), which will be, for the remaining part of this work, identified with their duals.

Definition 2.1. A *well-posed control LTI system with state space X and input space U* is a couple $\Sigma = (\mathbb{T}, \Phi)$ of families of operators such that

(1) $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is a C^0 semigroup of bounded linear operators on X;

(2) $\Phi = (\Phi_t)_{t\geqslant 0}$ is a family of bounded linear operators from $L^2([0,\infty);U)$ to X such that

$$\Phi_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t\Phi_\tau u + \Phi_t v \qquad (t,\tau \geqslant 0,\ u,v \in L^2([0,\infty);U)), \tag{2.1}$$

where the τ-*concatenation* of two signals u and v, denoted $u \underset{\tau}{\diamondsuit} v$, is the function

$$u \underset{\tau}{\diamondsuit} v = \begin{cases} u(t) & \text{for } \ t \in [0,\tau), \\ v(t-\tau) & \text{for } \ t \geqslant \tau. \end{cases}$$

The maps (Φ_t) are called *input to state maps* and $\operatorname{Ran}\Phi_\tau$ is called the *reachable space in time* τ.

We denote by $A : \mathcal{D}(A) \to X$ the generator of $\mathbb{T}$ and by $\mathbb{T}^*$ the adjoint semigroup. We next denote $\mathcal{D}(A)$ by X_1 and remark that, when endowed with norm

$$\|z_0\|^2_{X_1} = \|z_0\|^2 + \|Az_0\|^2 \qquad (z_0 \in X_1),$$

X_1 is a Hilbert space. Similarly, we denote by X_1^d the Hilbert space obtained by endowing $\mathcal{D}(A^*)$ (the domain of the adjoint A^* of A) with the norm

$$\|z_0\|^2_{X_1^d} = \|z_0\|^2 + \|A^*z_0\|^2 \qquad (z_0 \in X_1^d). \tag{2.2}$$

Let X_{-1} be the dual of X_1^d with respect to the pivot space X, so that $X_1 \subset X \subset X_{-1}$ with continuous and dense embeddings. The duality product, using X as pivot space, between X_{-1} and X_1^d is denoted by $\langle\cdot,\cdot\rangle_{X_{-1},X_1^d}$. We recall that, for each $k \in \{-1,1\}$, the original semigroup $\mathbb{T}$ has a restriction (or an extension) to X_k that is the image of $\mathbb{T}$ through the unitary operator $(\beta I - A)^{-k} \in \mathcal{L}(X, X_k)$, where $\beta \in \rho(A)$ (the resolvent set of A), see [22, Remark 2.10.5]. This restriction (or extension) will be still denoted by $\mathbb{T}$.

Remark 2.1. According to [28] and [23], assumptions (1) and (2) in Definition 2.1 imply that there exists a unique $B \in \mathcal{L}(U, X_{-1})$, called the *control operator* of Σ, such that

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} Bu(\sigma)\,\mathrm{d}\sigma \qquad (\tau \geqslant 0,\ u \in L^2([0,\infty);U)). \tag{2.3}$$

Notice that in the above formula, $\mathbb{T}$ acts on X_{-1} and the integration is carried out in X_{-1}, yielding a result lying in X. This property of B is called *admissibility.* From the above definitions it follows that if $(\mathbb{T},\Phi)$ is a wellposed control LTI system then for all $u \in L^2([0,\infty);U)$, $t \mapsto \Phi_t u$ is a continuous function from $[0,\infty)$ to X. A particular case of interest, which occurs when we consider PDE systems with control distributed in the domain where the PDE holds, is when the control operator B is bounded, i.e., $B \in \mathcal{L}(U,X)$, when the integral in (2.3) is carried out in X.

Remark 2.2. Given a wellposed control LTI system $(\mathbb{T},\Phi)$ and $\tau > 0$, we frequently use in the remaining part of this work the adjoint Φ_τ^* of the operator Φ_τ. This adjoint is defined, within the remaining part of this work, by identifying X and U with their duals, which means that

$$\langle \Phi_\tau u, z_0\rangle_X = \langle u, \Phi_\tau^* z_0\rangle_{L^2([0,\infty);U)} \qquad (u \in L^2([0,\infty);U)),\ z_0 \in X). \tag{2.4}$$

It is known (see, for instance, [22, Proposition 4.4.1]) that for $z_0 \in \mathcal{D}(A^*)$ and $t > 0$ we have

$$(\Phi_t^* z_0)(\sigma) = \begin{cases} B^*\mathbb{T}^*_{t-\sigma} z_0 & (\sigma \in [0,t]), \\ 0 & (\sigma > t), \end{cases} \tag{2.5}$$

where $B^* \in \mathcal{L}(X_1^d, U)$ is defined by

$$\langle Bv, \varphi\rangle_{X_{-1},X_1^d} = \langle v, B^*\varphi\rangle_U \qquad (v \in U,\ \varphi \in X_1^d).$$

Since for every $t > 0$ we have $\Phi_t^* \in \mathcal{L}(X, L^2([0,\infty); U))$, from (2.5) it follows that for every $t > 0$ there exists $\kappa_t > 0$ such that

$$\int_0^t \|B^*\mathbb{T}_\sigma^*\psi\|_U^2 \, \mathrm{d}\sigma \leqslant \kappa_t^2 \|\psi\|_X^2 \qquad (\psi \in \mathcal{D}(A^*)). \tag{2.6}$$

On the other hand, it is known, see, for instance, [22, Proposition 4.2.5], that the families $\mathbb{T}$ and Φ are the solution operators for the initial value problem

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = z_0. \tag{2.7}$$

This means that for every $\tau > 0$, $z_0 \in X$ and every $u \in L^2([0,\tau]; U)$, the initial value problem (2.7) has a unique solution

$$z \in C([0,\tau]; X) \cap H^1((0,\tau); X_{-1}),$$

given by

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \qquad (t \in [0,\tau]).$$

We also recall the definitions (see, for instance, [22, Sections 4.2 and 11.1]):

Definition 2.2. The wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ is approximatively controllable in time τ if the range of Φ_τ, denoted $\operatorname{Ran}\Phi_\tau$, is dense in X.

The following duality result is a classical tool in the study of approximate controllability of infinite dimensional systems. We refer to [22, Theorem 11.2.1] for the proof and other related issues.

Proposition 2.1. *The wellposed control LTI system $(\mathbb{T}, \Phi)$ is approximatively controllable in time τ if and only if $\operatorname{Ker}\Phi_\tau^* = \{0\}$, where Φ_τ^* has been defined in (2.4). In the particular case when the corresponding control operator satisfies $B \in \mathcal{L}(U, X)$, the system is approximatively controllable in time τ if and only if the only $\psi \in X$ such that $B^*\mathbb{T}_t^*\psi = 0$ for every $t \in [0,\tau]$ is $\psi = 0$.*

We define below a space which plays an important role in the remaining part of this paper.

Definition 2.3. Let $\tau > 0$ and let $\Sigma = (\mathbb{T}, \Phi)$ be a well-posed control LTI system with state space X and control space $U = L^2(\mathcal{O})$, where $\mathcal{O}$ is a compact Riemannian manifold. The L^∞ reachable space in time τ of Σ, denoted $\mathcal{R}_\tau^\infty$ is

$$\mathcal{R}_\tau^\infty = \Phi_\tau\left(L^\infty([0,\infty) \times \mathcal{O})\right). \tag{2.8}$$

It is easily seen that $\mathcal{R}_\tau^\infty$, endowed with the norm

$$\|\xi\|_{\mathcal{R}_\tau^\infty} = \inf_{\substack{u \in L^\infty([0,\infty)\times\mathcal{O}) \\ \Phi_\tau u = \xi}} \|u\|_{L^\infty([0,\tau]\times\mathcal{O})} \qquad (\xi \in \mathcal{R}_\tau^\infty), \tag{2.9}$$

is a normed space.

The controllability type which will be mostly used in this work is:

Definition 2.4. Let $\tau > 0$ and let $\Sigma = (\mathbb{T}, \Phi)$ be a well-posed control LTI system with state pace X and control space $U = L^2(\mathcal{O})$, where $\mathcal{O}$ is a compact Riemannian manifold. The system Σ is said L^∞ null controllable in time τ if $\mathcal{R}_\tau^\infty \supset \operatorname{Ran}\mathbb{T}_\tau$.

We give below a second, less classical, duality result, which we state below in the case when the adjoint semigroup $\mathbb{T}^*$ has a backwards uniqueness property. We refer, for instance, to [16, Proposition 2.5] for a proof of this result.

Proposition 2.2. *Let $\Sigma = (\mathbb{T}, \Phi)$ be a wellposed control LTI system with state space X and control space $U = L^2(\mathcal{O})$, where $\mathcal{O}$ is a compact Riemannian manifold. Assume that $\operatorname{Ker} \mathbb{T}_\tau^* = \{0\}$ for some $\tau > 0$. Then Σ is L^∞ null controllable in time τ if and only if*

$$K_\tau := \inf_{\eta \in X \setminus \{0\}} \frac{\|\Phi_\tau^* \eta\|_{L^1([0,\infty)\times \mathcal{O})}}{\|\mathbb{T}_\tau^* \eta\|_X} > 0. \tag{2.10}$$

Moreover, if (2.10) holds, then for every $\psi \in X$ there exist $u \in L^\infty([0,\infty) \times \mathcal{O})$ with

$$\|u\|_{L^\infty([0,\infty)\times\mathcal{O})} \leqslant K_\tau \|\psi\|_X, \qquad \mathbb{T}_\tau \psi + \Phi_\tau u = 0.$$

Remark 2.3. The conclusion of Proposition 2.2 holds, in particular, if the assumption $\operatorname{Ker} \mathbb{T}_\tau^* = \{0\}$ is replaced by the hypothesis that $\mathbb{T}$ is analytic. Indeed, in this case $\mathbb{T}^*$ is also analytic, which implies that it has the required backwards uniqueness property. Moreover, assuming that $\mathbb{T}$ is analytic, the L^∞ null controllability in time τ of Σ implies its approximate controllability. Indeed, let $\psi \in X$ be such that $(\Phi_\tau^* \psi)(t, x) = 0$ for almost every $t \in [0, \tau]$ and $x \in \mathcal{O}$. Using (2.10) it follows that $\mathbb{T}_\tau^* \psi = 0$. Since $\mathbb{T}^*$ is an analytic semigroup, it follows that $\psi = 0$. The approximate controllability of Σ in time τ follows now from Proposition 2.1.

3. Statement of the main abstract results

In this section we continue to use the notation introduced in the previous one, so that, in particular, the state space X and the input space U are supposed to be Hilbert. We denote by $\Sigma = (\mathbb{T}, \Phi)$ a well-posed control LTI system with state space X and input space U, so that the C^0-semigroup $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ on X and the family of linear operators $\Phi = (\Phi_t)_{t \geqslant 0}$ satisfy the conditions in Definition 2.1.

The main assumptions which will be used in the remaining part of this work are:

[H1] The input space U is given by $U = L^2(\mathcal{O})$, where $\mathcal{O}$ is a compact Riemannian manifold (with or without boundary) of dimension $d \in \mathbb{N}$.

[H2] The semigroup $\mathbb{T}$ is analytic and its generator A has compact resolvents in X.

[H3] The wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies [H1] and it is L^∞ null controllable in any time $\tau > 0$ (in the sense of Definition 2.4).

[H4] The wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies [H1] and if $\tau > 0$, $\psi \in X$ are such that $(\Phi_\tau^* \psi)(t, x) = 0$ for (t, x) in a subset of positive Lebesgue measure of the product manifold $[0, \tau] \times \mathcal{O}$ then $\psi = 0$. We refer, for instance, to Amann and Escher [1, Ch.12] for the definition of the Lebesgue measure on manifolds.

In some of our results below assumptions [H3] and [H4] are replaced by the (stronger) hypothesis:

[H5] The wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies [H1] and for every $\tau > 0$ and every set of positive measure $e \subset [0,\tau] \times \mathcal{O}$ there exists $K_{\tau,e} > 0$ such that

$$K_{\tau,e} \int_e |(\Phi_\tau^* \psi)(t,x)| \, \mathrm{d}x \, \mathrm{d}t \geqslant \|\mathbb{T}_\tau^* \psi\|_X \qquad (\psi \in X). \tag{3.1}$$

In the above inequality $\mathrm{d}x \, \mathrm{d}t$ stands for the Lebesgue measure on the product manifold $[0,\tau] \times \mathcal{O}$, already used to state (H4) above.

Remark 3.1. We note that assumption [H2] above implies that the semigroup $\mathbb{T}^*$ is analytic so that $\operatorname{Ker} \mathbb{T}_t^* = \{0\}$ for every $t \geqslant 0$. From this and Proposition 2.2 it follows that assumption [H5] above implies [H3] and [H4].

With the above notation and assumptions we can state the main optimal control problems considered in this work. The first one is the norm optimal control problem, which can be stated as follows:

The norm optimal control problem $(NP)^\tau$*:* Let $\Sigma = (\mathbb{T}, \Phi)$ be a wellposed control LTI system satisfying [H3]. Given $\psi \in X$, we define $N^\infty : (0,\infty) \to [0,\infty)$ by

$$N^\infty(\tau) := \|\mathbb{T}_\tau \psi\|_{\mathcal{R}_\tau^\infty}.$$

In other words, $N^\infty(\tau)$ is the lower bound of the L^∞ norms of controls steering the system from the initial state ψ to rest in time ψ. Thanks to [H3], N^∞ is clearly well defined and takes values in $[0,\infty)$. Moreover, N^∞ clearly depends on the choice of $\psi \in X$ but, to avoid notational complexity, this dependence will not appear explicitly in what follows.

A norm optimal control at time τ is an input function $\hat{u}$ which steers the initial state ψ rest in time τ and its L^∞ norm is minimal, i.e., such that

$$\Phi_\tau \hat{u} + \mathbb{T}_\tau \psi = 0, \qquad \|\hat{u}\|_{L^\infty([0,\tau]\times\mathcal{O})} = N^\infty(\tau). \tag{3.2}$$

The typical questions to be solved are the existence of such controls and the study of their properties (such as optimality conditions or the bang-bang-property).

Our first result on the norm optimal control problem is:

Theorem 3.1. *Let $\tau > 0$ and let $\Sigma = (\mathbb{T}, \Phi)$ a well-posed control LTI system with state space X and input space U satisfying assumptions [H1]-[H4]. Then for every $\psi \in X \setminus \{0\}$ the norm optimal control problem $(NP)^\tau$ admits at least one solution, in the sense that there exists $\hat{u} \in L^\infty([0,\tau] \times \mathcal{O})$ satisfying and (3.2). Moreover, this control $\hat{u}$ can be chosen to have the bang-bang property, i.e.,*

$$|\hat{u}(t,x)| = N^\infty(\tau) \qquad ((t,x) \in (0,\tau) \times \mathcal{O} \text{ a.e.}). \tag{3.3}$$

Our second result on norm optimal control problems asserts that if assumptions [H3] and [H4] in Theorem 3.1 are replaced by the stronger assumption [H5] (see Remark 3.1) we have:

Theorem 3.2. *Let $\tau > 0$ and let $\Sigma = (\mathbb{T}, \Phi)$ a well-posed control LTI system with state space X and input space U satisfying assumptions [H1], [H2] and [H5]. Then for every $\psi \in X \setminus \{0\}$ the solution of the norm optimal control problem $(NP)^\tau$ is unique and satisfies (3.3).*

Our third abstract result on norm optimal control is:

Theorem 3.3. *Let $\tau > 0$ and let $\psi \in X \setminus \{0\}$. Then, the norm optimal function $N^\infty : (0,\infty) \to (0,\infty)$ is decreasing and continuous. Moreover, we have*

$$\lim_{\tau\to\infty} N^\infty(\tau) = \hat{N} \geqslant 0, \qquad \lim_{\tau\to 0+} N^\infty(\tau) = +\infty, \tag{3.4}$$

with $\hat{N} = 0$ if $\mathbb{T}$ is an exponentially stable semigroup.

The second type of optimal control question to be studied below concerns time optimal controls and it can be stated as follows:

The time optimal control problem $(TP)^M$: Let $\Sigma = (\mathbb{T}, \Phi)$ a wellposed control LTI system satisfying [H3], let $M > 0$ and denote

$$\mathcal{U}_M := \left\{ u \mid u \in L^\infty((0,\infty)\times\mathcal{O}) \quad \text{s.t. } \|u\|_{L^\infty((0,\infty)\times\mathcal{O})} \leqslant M \right\}.$$

Solving the time optimal control problem $(TP)^M$ consists in determining $(\tau^\infty(M), u_M^\infty) \in (0,\infty)\times\mathcal{U}_M$ such that u_M^∞ drives the initial state ψ to zero in time $\tau^\infty(M)$, i.e.,

$$\mathbb{T}_{\tau^\infty(M)}\psi + \Phi_{\tau^\infty(M)} u_M^\infty = 0,$$

and

$$(TP)^M: \qquad \tau^\infty(M) := \min_{u\in\mathcal{U}_M} \{\tau \mid \exists u \in \mathcal{U}_M \ \text{ with } \ \mathbb{T}_\tau\psi + \Phi_\tau u = 0\}.$$

Here the main questions are the existence of optimal pairs $(\tau^\infty(M), u_M^\infty)$, describing qualitative properties of u_M^∞ (namely bang-bang) and establishing the relation between time and norm optimal control problems.

Our main result on the time optimal control problem $(TP)^M$ is:

Theorem 3.4. *Let $M > 0$ and let $\Sigma = (\mathbb{T}, \Phi)$ be a well-posed control LTI system with state space X and input space U satisfying assumptions [H1]-[H4]. Then, given $\psi \in X \setminus \{0\}$, the time optimal control problem $(TP)^M$ admits at least one solution $(\tau^\infty(M), u_M^\infty)$ if and only if $M > \hat{N}$, where $\hat{N}$ has been defined in (3.4). Moreover,*

$$N^\infty(\tau^\infty(M)) = M \qquad (M > \hat{N}), \tag{3.5}$$

$$\tau^\infty(N^\infty(\tau)) = \tau \qquad (\tau \in (0,\infty)). \tag{3.6}$$

and u_M^∞ can be selected such that it has the bang-bang property

$$|u_M^\infty(t,x)| = M \qquad ((t,x) \in [0,\tau^\infty(M)]\times\mathcal{O} \text{ a.e.}). \tag{3.7}$$

By combining Theorem 3.4 with [16, Propositions 2.5 and 2.6] we immediately obtain:

Corollary 3.1. *Let $M > \hat{N}$, where $\hat{N}$ has been defined in (3.4), and let $\Sigma = (\mathbb{T}, \Phi)$ a well-posed control LTI system with state space X and input space U satisfying assumptions [H1], [H2] and [H5]. Then for every $\psi \in X \setminus \{0\}$ the time optimal control problem $(TP)^M$ admits a unique solution $(\tau^\infty(M), u_M^\infty)$. Moreover, this solution satisfies (3.7).*

4. The L^∞ reachable space and the multiplier space

In this section we continue to use the notation introduced in the previous ones, so that $\Sigma = (\mathbb{T}, \Phi)$ is a well-posed control LTI system with state space X and input space $U = L^2(\mathcal{O})$, where $\mathcal{O}$ is a compact Riemannian manifold. Moreover, we continue to assume that Σ satisfies the assumptions (H1)-(H4) formulated at the beginning of Section 3.

In this section is to describe some of the properties of the L^∞ reachable space of Σ, defined in (2.8) and of the associated multiplier space. The latter, denoted, for every $\tau > 0$, by Z_τ, is defined as the completion of X with respect to the norm

$$\|\eta\|_{Z_\tau} = \int_0^\tau \int_{\mathcal{O}} |(\Phi_t^* \eta)(t, x))| \, \mathrm{d}x \, \mathrm{d}t \qquad (\eta \in X), \tag{4.1}$$

where Φ_t^* has been defined in (2.4). The fact that the right hand side of (4.1) defines a norm on X follows from the approximate controllability in time τ of Σ, see Remark 2.3. We clearly have that Z_τ is a Banach space and that $X \subset Z_\tau$ with continuous and dense embedding. The term *multiplier space* to design Z_τ is inspired to us by the work of Fattorini [10] and it is potentially related to Pontryagin's maximum principle. We refer to [10] for more details on this relationship.

The result below shows that $\mathcal{R}_\tau^\infty$ can be seen as the dual of Z_τ with respect to the pivot space X.

Proposition 4.1. *Let $\tau > 0$ and assume that the wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies assumptions (H1)-(H4) from Section 3. Let Z_τ be the completion of X with respect to the norm (4.1). Then*

$$\mathcal{R}_\tau^\infty = \{\xi \in X \ | \ \text{s.t.} \sup_{\substack{\eta \in X \\ \|\eta\|_{Z_\tau} \leqslant 1}} |\langle \eta, \xi \rangle_X| < \infty\}, \tag{4.2}$$

$$\|\xi\|_{\mathcal{R}_\tau^\infty} = \sup_{\substack{\eta \in X \\ \|\eta\|_{Z_\tau} \leqslant 1}} |\langle \eta, \xi \rangle_X| \qquad (\xi \in \mathcal{R}_\tau^\infty). \tag{4.3}$$

Proof. We first remark that for every $\xi \in \mathcal{R}_\tau^\infty$, there exists a sequence (u_n^ξ) in $L^\infty([0, \tau] \times \mathcal{O})$ such that $\Phi_\tau u_n^\xi = \xi$ for every $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \|u_n^\xi\|_{L^\infty([0,\tau]\times\mathcal{O})} = \|\xi\|_{\mathcal{R}_\tau^\infty}.$$

We next note that for every $\eta \in X$ and $\xi \in \mathcal{R}_\tau^\infty \setminus \{0\}$ we have

$$\begin{aligned}\frac{1}{\|\xi\|_{\mathcal{R}_\tau^\infty}} |\langle \xi, \eta \rangle_X| &= \frac{1}{\|\xi\|_{\mathcal{R}_\tau^\infty}} \left| \langle u_n^\xi, \Phi_\tau^* \eta \rangle_{L^2([0,\tau];U)} \right| \\ &\leqslant \frac{1}{\|\xi\|_{\mathcal{R}_\tau^\infty}} \|u_n^\xi\|_{L^\infty([0,\tau]\times\mathcal{O})} \|\Phi_\tau^* \eta\|_{L^1([0,\tau]\times\mathcal{O})} \qquad (n \in \mathbb{N}).\end{aligned}$$

Passing to the limit when $n \to \infty$ in the last inequality we obtain that for every $\eta \in X$ and $\xi \in \mathcal{R}_\tau^\infty \setminus \{0\}$ we have

$$\frac{1}{\|\xi\|_{\mathcal{R}_\tau^\infty}} |\langle \xi, \eta \rangle_X| \leqslant \|\eta\|_{Z_\tau}.$$

The last estimate clearly implies that

$$\mathcal{R}_\tau^\infty \subset \left\{ \xi \in X \ | \ \text{s.t.} \sup_{\substack{\eta \in X \\ \|\eta\|_{Z_\tau} \leqslant 1}} |\langle \eta, \xi \rangle_X| < \infty \right\},$$

$$\sup_{\|\eta\|_{Z_\tau}\leqslant 1} |\langle \eta, \xi\rangle_X| \leqslant \|\xi\|_{\mathcal{R}_\tau^\infty} \qquad (\xi \in \operatorname{Ran}\Phi_\tau). \tag{4.4}$$

Let now $\xi \in X$ be such that

$$\sup_{\|\eta\|_{Z_\tau}\leqslant 1} |\langle \eta, \xi\rangle_X| = c_{\tau,\xi} < \infty. \tag{4.5}$$

Consider the subspace $\mathcal{X}$ of $L^1([0,\tau]\times\mathcal{O})$ defined by

$$\mathcal{X} = \{\Phi_\tau^*\eta \ \mid \ \eta \in X\}.$$

Consider the linear functional $\mathcal{F}$ on $\mathcal{X}$ defined by

$$\mathcal{F}(\Phi_\tau^*\eta) = -\langle \xi, \eta\rangle_X \qquad (\eta \in X).$$

The fact that this functional is well defined follows from the approximate controllability in time τ of our system Σ. Moreover, using (4.5), it follows that

$$|\mathcal{F}v| \leqslant c_{\tau,\xi}\|\eta\|_{Z_\tau} = c_{\tau,\xi}\ \|v\|_{L^1([0,\tau]\times\mathcal{O})} \qquad (v \in \mathcal{X}).$$

By the Hahn-Banach Theorem, $\mathcal{F}$ can be extended to a bounded linear functional $\widetilde{\mathcal{F}}$ on $L^1([0,\tau]\times\mathcal{O})$ such that

$$|\widetilde{\mathcal{F}}v| \leqslant K_\tau\ \|v\|_{L^1([0,\tau]\times\mathcal{O})} \qquad (v \in L^1([0,\tau]\times\mathcal{O})).$$

By the Riesz representation theorem it follows that there exists $u \in L^\infty([0,\tau]\times\mathcal{O})$ such that $\|u\|_{L^\infty([0,\tau]\times\mathcal{O})} \leqslant c_{\tau,\xi}$ and

$$\int_0^\tau\int_{\mathcal{O}} u(\tau-\sigma,x)\overline{\Phi_\tau^*\eta} + \langle \xi,\eta\rangle_X = 0 \qquad (\eta \in X).$$

From the above formula, it follows that

$$\langle \Phi_\tau u, \eta\rangle_X + \langle \xi,\eta\rangle_X = 0 \qquad (\eta \in X),$$

which implies that

$$\Phi_\tau u = \xi.$$

We have thus shown that

$$\{\xi \in X \ \mid \ \text{s.t.} \sup_{\|\eta\|_{Z_\tau}\leqslant 1} |\langle \eta, \xi\rangle_X| < \infty\} \subset \mathcal{R}_\tau^\infty.$$

The above inclusion and (4.4) imply the conclusion (4.2).

Moreover, we have seen that for every $\xi \in X$ such that $\sup_{\|\eta\|_{Z_\tau}\leqslant 1} |\langle \eta, \xi\rangle_X| < \infty$ (or, equivalently, $\xi \in \mathcal{R}_\tau^\infty$), there exists $u \in L^\infty([0,\tau]\times\mathcal{O})$ such that $\Phi_\tau u = \xi$ and

$$\|u\|_{L^\infty([0,\tau]\times\mathcal{O})} \leqslant \sup_{\|\eta\|_{Z_\tau}\leqslant 1} |\langle \eta, \xi\rangle_X|.$$

Consequently,

$$\|\xi\|_{\mathcal{R}_\tau^\infty} \leqslant \sup_{\|\eta\|_{Z_\tau}\leqslant 1} |\langle \eta, \xi\rangle_X| \qquad (\xi \in \mathcal{R}_\tau^\infty).$$

The last estimate and (4.4) implies our second conclusion (4.3). □

Remark 4.1. The fact that, under the assumptions of Proposition 4.1, $\mathcal{R}_\tau^\infty$ does not depend on $\tau > 0$ can also be proved directly. This can be achieved by following Seidman's proof [21] (see also Kellay, Normand and Tucsnak [11, Proposition 3.4]) of the result asserting that $\operatorname{Ran}\Phi_\tau$ (see Definition 2.2 for the meaning of this notation) is independent of $\tau > 0$.

An obvious consequence of Proposition 4.1 is:

Corollary 4.1. *With the notation and under the assumptions in Proposition 4.1 we have*

$$\|\mathbb{T}_\tau \psi\|_{R_\tau^\infty} = \sup_{\eta\in X\setminus\{0\}} \frac{|\langle \mathbb{T}_\tau\psi, \eta\rangle_X|}{\|\eta\|_{Z_\tau}}. \tag{4.6}$$

The result below also follows from Proposition 4.1:

Corollary 4.2. *With the notation and under the assumptions in Proposition 4.1, $\mathcal{R}_\tau^\infty$, endowed with the norm defined in (2.9), is a Banach space.*

Proof. Let $(\xi_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{R}_\tau^\infty$. This implies, according to (4.3), that for every $\eta \in X$ there exists $l_\eta \in \mathbb{C}$ with

$$\lim_{k\to\infty} \langle \eta, \xi_k\rangle_X = l_\eta, \qquad |l_\eta| \leqslant \sup_{k\in\mathbb{N}} \|\xi_k\|_{\mathcal{R}_\tau^\infty} \|\eta\|_{Z_\tau}. \tag{4.7}$$

The mapping $\eta \mapsto l_\eta$ is obviously linear. Moreover, the above formulas and the continuity of the embedding $X \subset Z_\tau$ imply that there exits a $\xi \in X$ such that

$$l_\eta = \langle \eta, \xi\rangle_X \qquad (\eta \in X). \tag{4.8}$$

Putting together (4.7) and (4.8) it follows that

$$\sup_{\substack{\eta\in X\\ \|\eta\|_{Z_\tau}\leqslant 1}} |\langle \eta, \xi\rangle_X| < \infty,$$

so that $\xi \in \mathcal{R}_\tau^\infty$. We have thus shown that the completion of $\mathcal{R}_\tau^\infty$ with respect to its norm is included in $\mathcal{R}_\tau^\infty$, so that $\mathcal{R}_\tau^\infty$ is indeed a Banach space. □

We are now in a position to state the main results in this section.

Proposition 4.2. *Assume that the wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies assumptions (H1)-(H4) from Section 3. Then for $h > 0$ and $\eta \in X$ we have*

$$\int_0^{\tau+h}\int_{\mathcal{O}} \left|\left(\Phi_{\tau+h}^*\eta\right)(t,x)\right| \,\mathrm{d}x\,\mathrm{d}t \leqslant \left(1+\kappa_h K_\tau^{-1}\right) \int_0^\tau \int_{\mathcal{O}} |\left(\Phi_\tau^*\eta\right)(t,x)| \,\mathrm{d}x\,\mathrm{d}t, \tag{4.9}$$

where κ_h and K_τ are the constants introduced in (2.6) and (2.10), respectively. Moreover, the spaces Z_τ and $\mathcal{R}_\tau^\infty$ do not depend on $\tau > 0$.

Proof. An obvious density argument shows that if suffices to prove that (4.9) holds for $\eta \in \mathcal{D}(A^*)$. To this aim we note that for every $\eta \in \mathcal{D}(A^*)$ and $\tau,\ h > 0$ we have

$$\int_0^{\tau+h}\int_{\mathcal{O}} |B^*\mathbb{T}_t^*\eta| \,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau\int_{\mathcal{O}} |B^*\mathbb{T}_t^*\eta| \,\mathrm{d}x\,\mathrm{d}t + \int_\tau^{\tau+h}\int_{\mathcal{O}} |B^*\mathbb{T}_t^*\eta| \,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \int_0^\tau\int_{\mathcal{O}} |B^*\mathbb{T}_t^*\eta| \,\mathrm{d}x\,\mathrm{d}t + [h\mu(\mathcal{O})]^{\frac12} \left[\int_\tau^{\tau+h}\int_{\mathcal{O}} |B^*\mathbb{T}_t^*\eta|^2 \,\mathrm{d}x\,\mathrm{d}t\right]^{\frac12}.$$

where $\mu(\mathcal{O})$ is the Lebesgue measure of $\mathcal{O}$. On the other hand,

$$\int_\tau^{\tau+h}\int_{\mathcal{O}} |B^*\mathbb{T}_t^*\eta|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^h\int_{\mathcal{O}} |B^*\mathbb{T}_{t+\tau}^*\eta|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^h\int_{\mathcal{O}} |B^*\mathbb{T}_t^*\mathbb{T}_\tau^*\eta|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \kappa_h^2 \|\mathbb{T}_\tau^*\eta\|_X^2 \qquad (\tau, h > 0,\ \eta \in \mathcal{D}(A^*)),$$

where κ_h (depending only on h) is the constant introduced in (2.6). The above inequality and (2.10) imply that (4.9) holds for $\eta \in \mathcal{D}(A^*)$ and thus for $\eta \in X$.

In order to show that Z_τ does not depend on the choice of $\tau > 0$ it suffices, according to the definition of Z_τ, to prove that, given $\tau_1, \tau_2 > 0$, the corresponding norms defined in (4.1) are equivalent on X. With no loss of generality, assume that $0 < \tau_1 < \tau_2$. Then we obviously have

$$\|\eta\|_{Z_{\tau_1}} \leqslant \|\eta\|_{Z_{\tau_2}} \qquad (\eta \in X).$$

Combining the last estimate and (4.9) yields that $\|\cdot\|_{Z_{\tau_2}}$ and $\|\cdot\|_{Z_{\tau_1}}$ are equivalent on X, so that $Z_{\tau_2} = Z_{\tau_1}$. This equality and (4.2) imply also that $\mathcal{R}^\infty_{\tau_2} = \mathcal{R}^\infty_{\tau_1}$. □

5. On an extension of the adjoint semigroup

In this section we continue to use all the notation introduced in the previous ones. In particular, $\Sigma = (\mathbb{T}, \Phi)$ is a well-posed control LTI system with state space X and input space $U = L^2(\mathcal{O})$, satisfying assumptions (H1)-(H4) formulated at the beginning of Section 3. Moreover, $\mathcal{R}^\infty_\tau$ and Z_τ design the L^∞ reachable space and the multiplier space of Σ at time τ, respectively. We recall that $\mathcal{R}^\infty_\tau$ has been defined in (2.8), that Z_τ is the completion of X with respect to the norm defined in (4.1) and that, as shown in Proposition 4.2, the spaces $\mathcal{R}^\infty_\tau$ and Z_τ are independent of the choice of $\tau > 0$.

The main aim of this section is to show that for every $\tau > 0$ the semigroup $(\mathbb{T}^*_t)_{t\geqslant 0}$ can be extended to a C^0 semigroup on Z_τ. The first step to achieve this goal is the following result:

Proposition 5.1. *Let $\tau > 0$ and assume that the wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies assumptions (H1)-(H4) from Section 3. Then for every $t \in (0, \tau]$ the operator $\mathbb{T}^*_t$ uniquely extends to an operator $\mathbb{S}_t \in \mathcal{L}(Z_\tau, X)$. Moreover, we have*

$$\mathbb{S}_{t+\sigma} = \mathbb{S}_t \mathbb{S}_\sigma \qquad (t, \sigma > 0), \tag{5.1}$$

$$\limsup_{t\to 0+} \|\mathbb{S}_t\|_{\mathcal{L}(Z_\tau)} < \infty, \tag{5.2}$$

$$\|\mathbb{S}_\tau \eta\|_X \leqslant K_\tau \|\eta\|_{Z_\tau} \qquad (\eta \in Z_\tau), \tag{5.3}$$

where K_τ is the constant in (2.10).

Proof. Let $\eta \in Z_\tau$ and let (η_k) be a sequence of X such that $\|\eta_k - \eta\|_{Z_\tau} \to 0$. Then, according to Proposition 4.2, we have that $\eta \in W_t$ and $\eta_k \to \eta$ in Z_t. Using (2.10) it follows that the sequence $(\mathbb{T}^*_t \eta_k)$ converges to some $\gamma_t \in X$. Moreover, it is easy to check that γ_t does not depend on the choice of the approximating sequence (η_k). We can thus define the operator $\mathbb{S}_t$ by setting $\mathbb{S}_t \eta = \gamma_t$. This operator clearly lies in $\mathcal{L}(Z_t, X)$ (thus in $\mathcal{L}(Z_\tau, X)$) and and it extends $\mathbb{T}^*_t$ and the fact that the family $(\mathbb{S}_t)_{t>0}$ satisfies the property (5.1) follows from the corresponding property of $\mathbb{T}^*$.

We next remark that, using the density of $\mathcal{D}(A^*)$ in X, we can assume that the sequence (η_k) introduced at the beginning of this proof takes values in $\mathcal{D}(A^*)$. Since (η_k) is bounded in Z_τ we have that

$$M_{\tau,\eta} := \sup_{k\in\mathbb{N}} \int_0^\tau \int_{\mathcal{O}} |B^* \mathbb{T}^*_\sigma \eta_k| \,\mathrm{d}\, x \mathrm{d}\sigma < \infty. \tag{5.4}$$

It follows that for $0 < t \leqslant \tau$ we have

$$\|S_t\eta\|_{Z_\tau} = \lim_{k\to\infty} \|\mathbb{T}_t^*\eta_k\|_{Z_\tau} = \lim_{k\to\infty} \int_0^\tau \int_{\mathcal{O}} |B^*\mathbb{T}_\sigma^*\mathbb{T}_t^*\eta_k|\,\mathrm{d}x\,\mathrm{d}\sigma \tag{5.5}$$
$$= \lim_{k\to\infty} \int_0^\tau \int_{\mathcal{O}} |B^*\mathbb{T}_{t+\sigma}^*\eta_k|\,\mathrm{d}x\,\mathrm{d}\sigma = \lim_{k\to\infty} \int_t^{\tau+t} \int_{\mathcal{O}} |B^*\mathbb{T}_\sigma^*\eta_k|\,\mathrm{d}x\,\mathrm{d}\sigma$$
$$\leqslant \lim_{k\to\infty} \int_0^{\tau+t} \int_{\mathcal{O}} |B^*\mathbb{T}_\sigma^*\eta_k|\,\mathrm{d}x\,\mathrm{d}\sigma.$$

On the other hand, by combining (4.9) and (5.4) we have

$$\int_0^{\tau+t} \int_{\mathcal{O}} |B^*\mathbb{T}_t^*\eta_k|\,\mathrm{d}x\,\mathrm{d}t \leqslant \left(1 + \kappa_t K_\tau^{-1}\right) M_{\tau,\eta} \qquad (k \in \mathbb{N}),$$

where the constants k_t and K_τ have been defined in (2.10) and (2.6), respectively. The last estimate and (5.5) imply that

$$\limsup_{t\to 0+} \|\mathbb{S}_t\eta\|_{Z_\tau} < \infty \qquad (\tau > 0,\ \eta \in Z_\tau).$$

The conclusion (5.2) follows now by applying the uniform boundedness principle.

Finally, (5.3) follows from (2.10) and the fact that $\mathbb{S}_\tau \in \mathcal{L}(Z_\tau, X)$. □

The main result of this section is:

Theorem 5.1. *Assume that the wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies assumptions (H1)-(H4) from Section 3. Then for every $\tau > 0$ the family $(\mathbb{S}_\sigma)_{\sigma\geqslant 0}$ obtained by setting $\mathbb{S}_0 = \mathbb{I}_{Z_\tau}$ forms a C^0 semigroup on Z_τ.*

Proof. Let $\eta \in Z_\tau$ and $\varepsilon > 0$. Since X is dense in Z_τ, there exists $\eta_1 \in X$ with

$$\|\eta - \eta_1\|_{Z_\tau} \leqslant \frac{\varepsilon}{2}. \tag{5.6}$$

On the other hand, by backwards uniqueness for analytic semigroups, we have that $\operatorname{Ker} \mathbb{T}_\tau = \{0\}$, so that $\operatorname{Ran} \mathbb{T}_\tau^*$ is dense in X. Consequently, denoting by c_τ the norm of the embedding of X into Z_τ, there exists $\eta_2 \in X$ such that

$$\|\mathbb{T}_\tau^*\eta_2 - \eta_1\|_X \leqslant \frac{\varepsilon}{2c_\tau}. \tag{5.7}$$

From the above estimate and (5.6) it follows that $\eta_2 \in X$ satisfies

$$\|\mathbb{S}_\tau\eta_2 - \eta\|_{Z_\tau} = \|\mathbb{T}_\tau^*\eta_2 - \eta\|_{Z_\tau} \leqslant \varepsilon. \tag{5.8}$$

We have thus shown that $\operatorname{Ran} \mathbb{S}_\tau$ is dense in Z_τ.

On the other hand, from Proposition 5.1 we have that $\mathbb{S}_\tau\eta_2 \in X$ so that the strong continuity of the semigroup $\mathbb{T}^*$ on X implies that

$$\lim_{\sigma\to 0+} \|\mathbb{S}_\sigma\mathbb{S}_\tau\eta_2 - \mathbb{S}_\tau\eta_2\|_X = \lim_{\sigma\to 0+} \|\mathbb{T}_\sigma^*\mathbb{S}_\tau\eta_2 - \mathbb{S}_\tau\eta_2\|_X = 0.$$

Consequently, using the continuous imbedding of X into Z_τ, there exists $\delta > 0$ with

$$\|\mathbb{S}_\sigma\mathbb{S}_\tau\eta_2 - \mathbb{S}_\tau\eta_2\|_{Z_\tau} < \varepsilon \qquad (\sigma \in (0, \delta)).$$

The above estimate, combined with (5.2) and with (5.8), yields the existence of $\delta_1 \in (0, \delta)$ such that

$$\|\mathbb{S}_\sigma\eta - \eta\|_{Z_\tau} \leqslant \|\mathbb{S}_\sigma\mathbb{S}_\tau\eta_2 - \mathbb{S}_\tau\eta_2\|_{Z_\tau}$$
$$+ \|\mathbb{S}_\sigma(\mathbb{S}_\tau\eta_2 - \eta) - (\mathbb{S}_\tau\eta_2 - \eta)\|_{Z_\tau} \leqslant 3\varepsilon \qquad (\sigma \in (0, \delta_1)).$$

Since $\eta \in Z_\tau$ and $\varepsilon > 0$ are arbitrary, it follows that indeed $(\mathbb{S}_\sigma)_{\sigma \geqslant 0}$ is a strongly continuous semigroup on Z_τ. □

Using Theorem 5.1 above we can prove that the dual of Σ can be seen, in the sense of inequality (5.12) below, as an exactly observable system, with state space Z_τ and output space $L^1(\mathcal{O})$. More precisely, we have:

Corollary 5.1. *With the assumptions and the notation in Theorem 5.1, let $\tau > 0$. For every $t \in [0, \tau]$ we define the operator Ψ_t by*

$$(\Psi_t \eta)(\sigma) = \begin{cases} B^* \mathbb{S}_{t-\sigma} \eta & (\sigma \in [0, t), \eta \in Z_t), \\ 0 & (\sigma > t, z_0 \in Z_t), \end{cases} \tag{5.9}$$

were $\mathbb{S}$ is the C^0-semigroup introduced in Theorem 5.1. Then

$$\Psi_t \in \mathcal{L}(Z_\tau, L^1([0,\infty) \times \mathcal{O})), \qquad \Psi_t|_X = \Phi_t^* \qquad (t \in (0, \tau]), \tag{5.10}$$

where Φ_t^* has been introduced in (2.5). Moreover, we have

$$\|\eta\|_{Z_t} = \|\Psi_t \eta\|_{L^1([0,\infty)\times\mathcal{O})} \qquad (t \in (0, \tau], \ \eta \in Z_t), \tag{5.11}$$

$$\left(1 + \kappa_{\tau-t} K_t^{-1}\right) \|\Psi_t \eta\|_{L^1([0,\infty)\times\mathcal{O})} \geqslant \|\eta\|_{Z_\tau} \qquad (t \in (0, \tau), \ \eta \in Z_\tau), \tag{5.12}$$

where $\kappa_{\tau-t}$ and K_t are the constants introduced in (2.6) and (2.10), respectively.

Proof. We know from Proposition 5.1 that $\mathbb{S}_{\frac{t-\sigma}{2}} \eta \in X$ for every $\eta \in Z_t$ and $\sigma \in (0, t)$. Thus, using the fact that $\mathbb{S}$ extends $\mathbb{T}^*$ and the analyticity of the semigroup $\mathbb{T}^*$ it follows that

$$\mathbb{S}_{t-\sigma} \eta = \mathbb{T}^*_{\frac{t-\sigma}{2}} \mathbb{S}_{\frac{t-\sigma}{2}} \eta \in \mathcal{D}(A^*) \qquad (\sigma \in [0, t), \ \eta \in Z_t),$$

so that (5.9) defines indeed a linear operator on Z_t. Moreover, from the definition of the norm in Z_t it follows that Ψ_t defines an isometry of Z_t. Since the norms in Z_t and Z_τ are equivalent, it follows that Ψ_t satisfies the first condition in (5.10). The fact that the second condition in (5.10) is also satisfied is a direct consequence of the definitions of Ψ_t and Φ_t and of the fact, already used above, that $\mathbb{S}_{\frac{t-\sigma}{2}} \eta \in X$ for every $\eta \in Z_t$ and $\sigma \in (0, t)$.

Due to above facts, formula (5.11) clearly holds for $\eta \in X$ and, by density, for $\eta \in Z_\tau$. Finally (5.12) is a direct consequence of (5.11) and of the inequality (4.9) in Proposition 4.2. □

Remark 5.1. With the assumptions and the notation in Theorem 5.1, we can combine (5.9), (5.11) and and the fact that $\mathbb{S}_\sigma \eta \in \mathcal{D}(A^*)$ for every $\sigma > 0$ and $\eta \in Z_t$, to obtain that

$$\|\eta\|_{Z_t} = \int_0^t \int_{\mathcal{O}} |B^* \mathbb{S}_\sigma \eta| \, \mathrm{d}x \, \mathrm{d}\sigma \qquad (t > 0, \ \eta \in Z_t). \tag{5.13}$$

6. Minimization of an auxiliary functional

In this section we continue to use all the notation and the assumptions recalled at the beginning of Section 5.

Given $\tau > 0$ and $\psi \in X$ we consider we consider the functional $J_\psi^\tau : Z_\tau \to \mathbb{R}$ defined by

$$J_\psi^\tau(\eta) = \frac{1}{2} \|\eta\|_{Z_\tau}^2 + \langle \psi, \mathbb{S}_\tau \eta \rangle_X \qquad (\eta \in Z_\tau), \tag{6.1}$$

where Z_τ is the completion of X with respect to the norm defined in (4.1) and the C^0-semigroup $(\mathbb{S}_\sigma)_{\sigma\geqslant 0}$ has been constructed in Proposition 5.1. This type of functional has been introduced, in the context of approximate controllability of the wave (respectively of the heat) equation, in Lions [14] (respectively Fabre, Puel and Zuazua [7]). The approach in [7] has been adapted for norm and time optimal control problems for heat equations (with point target) in [27] and [26]. Our aim in this section is to prove that the results in [27] and [26] can be generalized to every system satisfying assumptions (H1)-(H4) formulated at the beginning of Section 3.

We first note that we have the result below:

Lemma 6.1. *Assume that the wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies assumptions (H1)-(H4) from Section 3. Let $\tau > 0$, $\psi \in X$ and let J_ψ^τ be the functional defined in (6.1). Then*

$$J_\psi^\tau(\eta) \geqslant \frac{1}{2}\|\eta\|_{Z_\tau}^2 - \|\mathbb{T}_\tau\psi\|_{R_\tau^\infty}\|\eta\|_{Z_\tau} \qquad (\eta \in Z_\tau,\ \psi \in X,\ \tau > 0), \tag{6.2}$$

$$\inf_{\eta\in Z_\tau} J_\psi^\tau(\eta) = -\frac{1}{2}\|\mathbb{T}_\tau\psi\|_{\mathcal{R}_\tau^\infty}^2 \qquad (\tau > 0, \psi \in X). \tag{6.3}$$

Proof. We first note that from Corollary 4.1 it follows that

$$|\langle \mathbb{T}_\tau\psi, \eta\rangle_X| \leqslant \|\mathbb{T}_\tau\psi\|_{R_\tau^\infty}\|\eta\|_{Z_\tau} \qquad (\eta \in X \setminus \{0\}).$$

The above inequality, the fact that for every $\eta \in X$ we have

$$J_\psi^\tau(\eta) = \frac{1}{2}\|\eta\|_{Z_\tau}^2 + \langle \psi, \mathbb{T}_\tau^*\eta\rangle_X\,,$$

and the Cauchy-Schwarz inequality clearly imply (6.2) holds for $\eta \in X$ and thus, by density, for $\eta \in Z_\tau$.

Using next (6.2) and an elementary inequality we obtain that

$$\inf_{\eta\in Z_\tau} J_\psi^\tau(\eta) \geqslant -\frac{1}{2}\|\mathbb{T}_\tau\psi\|_{\mathcal{R}_\tau^\infty}^2 \qquad (\tau > 0, \psi \in X). \tag{6.4}$$

On the other hand, it follows from (4.6) that

$$-\|\mathbb{T}_\tau\psi\|_{R_\tau^\infty} = \inf_{\eta\in X\setminus\{0\}} \frac{\langle \mathbb{T}_\tau\psi, \eta\rangle_X}{\|\eta\|_{Z_\tau}} \qquad (\psi \in X).$$

Consequently, for every $\varepsilon \in (0, \|\mathbb{T}_\tau\psi\|_{R_\tau^\infty})$ there exists $\eta_\varepsilon \in X \setminus \{0\}$ such that

$$\frac{\langle \mathbb{T}_\tau\psi, \eta_\varepsilon\rangle_X}{\|\eta_\varepsilon\|_{Z_\tau}} \leqslant -\|\mathbb{T}_\tau\psi\|_{R_\tau^\infty} + \varepsilon.$$

From the above estimate it follows that for every $\lambda > 0$ we have

$$\begin{aligned}
J_\psi^\tau(\lambda\eta_\varepsilon) = \frac{\lambda^2}{2}\|\eta_\varepsilon\|_{Z_\tau}^2 + \lambda\,\langle \psi, \mathbb{S}_\tau\eta_\varepsilon\rangle_X = \frac{\lambda^2}{2}\|\eta_\varepsilon\|_{Z_\tau}^2 + \lambda\,\langle \mathbb{T}_\tau\psi, \eta_\varepsilon\rangle \\
\leqslant \frac{\lambda^2}{2}\|\eta_\varepsilon\|_{Z_\tau}^2 - \lambda\|\mathbb{T}_\tau\psi\|_{R_\tau^\infty}\|\eta_\varepsilon\|_{Z_\tau} + \lambda\varepsilon\|\eta_\varepsilon\|_{Z_\tau} \\
= \frac{1}{2}\left[\lambda\|\eta_\varepsilon\|_{Z_\tau} - \left(\|\mathbb{T}_\tau\psi\|_{R_\tau^\infty} - \varepsilon\right)\right]^2 - \frac{1}{2}\left(\|\mathbb{T}_\tau\psi\|_{R_\tau^\infty} - \varepsilon\right)^2.
\end{aligned}$$

Taking the infimum with respect to $\lambda > 0$ in both sides of the above we obtain

$$\inf_{\eta\in Z_\tau} J_\psi^\tau(\eta) \leqslant -\frac{1}{2}\left(\|\mathbb{T}_\tau\psi\|_{R_\tau^\infty} - \varepsilon\right)^2.$$

Since the above estimate holds for every $\varepsilon > 0$ it follows that

$$\inf_{\eta \in Z_\tau} J_\psi^\tau(\eta) \leqslant -\frac{\|\mathbb{T}_\tau \psi\|^2_{R_\tau^\infty}}{2}.$$

The last inequality and (6.4) imply the conclusion (6.3). □

A second useful property of J_ψ^τ is given in the result below.

Lemma 6.2. *With the notation and under the assumptions of Lemma 6.1, if $\psi \neq 0$ then 0 is not a minimizer of J_ψ^τ.*

Proof. We use a contradiction argument. Assume that $\psi \neq 0$ and that 0 were the minimizer of J_ψ^τ. Then

$$\frac{J_\psi^\tau(\lambda\eta)}{\lambda} \geqslant 0 \qquad (\lambda > 0, \eta \in X).$$

Passing to the limit for $\lambda \to 0+$ in the above inequality we obtain that $\langle \mathbb{T}_\tau \psi, \eta \rangle \geqslant 0$ for all $\eta \in X$, so that $\mathbb{T}_\tau \psi = 0$. Using backwards uniqueness for analytic semigroups, this implies that $\psi = 0$, which is a contradiction. □

The main result of this section is:

Theorem 6.1. *Assume that the wellposed control LTI system $\Sigma = (\mathbb{T}, \Phi)$ satisfies assumptions (H1)-(H4) from Section 3. Let $\tau > 0$, $\psi \in X$ and let J_ψ^τ be the functional defined in (6.1). Then for every $\psi \in X$ and $\tau > 0$ the functional J_ψ^τ admits at least one minimizer $\hat{\eta} \neq 0$ on Z_τ. Moreover,*

$$J_\psi^\tau(\hat{\eta}) = -\frac{1}{2}\|\hat{\eta}\|^2_{Z_\tau}.$$

Proof. We first recall that the fact that 0 is not a minimizer of J_ψ^τ has been already proved in Lemma 6.2.

We next remark that from (6.3) it follows that there exists a minimizing sequence (η_k) in Z_τ such that

$$\lim_{k\to\infty} J_\psi^\tau(\eta_k) = -\frac{1}{2}\|\mathbb{T}_\tau \psi\|^2_{\mathcal{R}_\tau^\infty}.$$

Since from (6.2) we know that J_ψ^τ is coercive on Z_τ, there is $M > 0$ with

$$\|\eta_k\|_{Z_\tau} \leqslant M \qquad (k \in \mathbb{N}).$$

Using (5.13), the above estimate can be rephrased as

$$\int_0^\tau \int_{\mathcal{O}} |(B^* \mathbb{S}_t \eta_k)(x)| \, \mathrm{d}x \, \mathrm{d}t \leqslant M \qquad (k \in \mathbb{N}).$$

On the other hand, we know that for every $\sigma > 0$ we have $\mathbb{S}_\sigma = \mathbb{T}^*_{\frac{\sigma}{2}} \mathbb{S}_{\frac{\sigma}{2}}$. This fact, combined with the analyticity of the semigroup $\mathbb{T}^*$ and the compactness of the embedding of $\mathcal{D}(A) \subset X$ implies that for every $\sigma > 0$ the operator $\mathbb{S}_\sigma$ is compact from Z_τ to X_1^d (Recall from Section 2 that X_1^d is $\mathcal{D}(A^*)$ endowed with the norm defined in (2.2).) We can thus apply Alaoglu's theorem to obtain that there exists $\hat{\eta} \in Z_\tau$ such that, up to the extraction of a subsequence, we have,

$$\eta_k \to \hat{\eta} \quad \text{in } Z_\tau \ \ \textit{weak}\,*, \tag{6.5}$$

$$\mathbb{S}_\sigma \eta_k \to \mathbb{S}_\sigma \hat{\eta} \quad \text{in } X_1^d \ \text{strongly} \qquad (\sigma > 0). \tag{6.6}$$

From (6.5), (6.6) and the fact that $B^* \in \mathcal{L}(X_1^d, L^1(\mathcal{O}))$ it follows that

$$\int_\delta^\tau \int_{\mathcal{O}} |B^* \mathbb{S}_\sigma \hat{\eta}| \, \mathrm{d}x \, \mathrm{d}\sigma = \lim_{k\to\infty} \int_\delta^\tau \int_{\mathcal{O}} |B^* \mathbb{S}_\sigma \eta_k| \, \mathrm{d}x \, \mathrm{d}\sigma \qquad (\delta > 0). \tag{6.7}$$

Moreover,

$$\lim_{k\to\infty} \int_\delta^\tau \int_{\mathcal{O}} |B^* \mathbb{S}_\sigma \eta_k| \, \mathrm{d}x \, \mathrm{d}\sigma \leqslant \liminf_{k\to\infty} \int_0^\tau \int_{\mathcal{O}} |B^* \mathbb{S}_\sigma \eta_k| \, \mathrm{d}x \, \mathrm{d}\sigma. \tag{6.8}$$

Putting together (6.7) and (6.8) we obtain that

$$\int_0^\tau \int_{\mathcal{O}} |B^* \mathbb{S}_\sigma \hat{\eta}| \, \mathrm{d}x \, \mathrm{d}\sigma \leqslant \liminf_{k\to\infty} \int_0^\tau \int_{\mathcal{O}} |B^* \mathbb{S}_\sigma \eta_k| \, \mathrm{d}x \, \mathrm{d}\sigma.$$

Using (5.13), the above estimate can be rewritten

$$\|\hat{\eta}\|_{Z_\tau} \leqslant \liminf_{k\to\infty} \|\eta_k\|.$$

Combining the last estimate with the definition (6.1) of J_ψ^τ and with (6.6) we obtain

$$J_\psi^\tau(\hat{\eta}) \leqslant \liminf_{k\to\infty} J_\psi^\tau(\eta_k) = \inf_{\eta\in Z_\tau} J_\psi^\tau,$$

so that $\hat{\eta}$ is a global minimizer of J_ψ^τ over Z_τ. □

In order to derive the optimality condition satisfied by the mimimizer $\hat{\eta}$ in Theorem 6.1 we need the following result.

Lemma 6.3. *With the notation and under the assumptions of Theorem 6.1, the map $j_\tau : Z_\tau \to \mathbb{R}$ defined by*

$$j_\tau(\eta) = \|\eta\|_{Z_\tau} \qquad (\eta \in Z_\tau).$$

Then for every $\tilde{\eta} \in Z_\tau \setminus \{0\}$ we have

$$\lim_{\varepsilon\to 0+} \frac{j_\tau(\tilde{\eta} + \varepsilon\eta) - j_\tau(\tilde{\eta})}{\varepsilon} = \int_0^\tau \int_{\mathcal{O}} \frac{B^* \mathbb{S}_{\tau-\sigma}\tilde{\eta}}{|B^* \mathbb{S}_{\tau-\sigma}\tilde{\eta}|} B^* \mathbb{S}_{\tau-\sigma}\eta \, \mathrm{d}x \, \mathrm{d}\sigma \qquad (\eta \in Z_\tau). \tag{6.9}$$

Proof. We first remark that from our assumptions it follows that the set

$$\mathcal{N} = \{(t, x) \in [0, \tau] \times \mathcal{O} \ \ | \ \ (B^* \mathbb{S}_{\tau-t}\tilde{\eta})(x) = 0\},$$

is of zero measure in $[0, \tau] \times \mathcal{O}$, so that the right hand side of (6.9) is well defined.

We next note that

$$\lim_{\varepsilon\to 0+} \frac{|(B^* \mathbb{S}_{\tau-t}(\tilde{\eta} + \varepsilon\eta))(x)| - |(B^* \mathbb{S}_{\tau-t}\tilde{\eta})(x)|}{\varepsilon} = \frac{(B^* \mathbb{S}_{\tau-t}\tilde{\eta})(x)}{|(B^* \mathbb{S}_{\tau-t}\tilde{\eta})(x)|} (B^* \mathbb{S}_{\tau-t}\eta)(x) \qquad ((t, x) \notin \mathcal{N}, \ \eta \in Z_\tau). \tag{6.10}$$

On the other hand, for every $\varepsilon > 0$, $(t, x) \notin \mathcal{N}$ and $\eta \in Z_\tau$ we have

$$\frac{|(B^* \mathbb{S}_{\tau-t}(\tilde{\eta} + \varepsilon\eta))(x)| - |(B^* \mathbb{S}_{\tau-t}\tilde{\eta})(x)|}{\varepsilon} \leqslant |(B^* \mathbb{S}_{\tau-t}\eta)(x)|. \tag{6.11}$$

Applying (6.11) to $-\eta$ instead of η it follows that

$$\frac{|(B^* \mathbb{S}_{\tau-t}(\tilde{\eta} - \varepsilon\eta))(x)| - |(B^* \mathbb{S}_{\tau-t}\tilde{\eta})(x)|}{\varepsilon} \leqslant |(B^* \mathbb{S}_{\tau-t}\eta)(x)|.$$

Moreover, for every $\varepsilon > 0$, $(t,x) \notin \mathcal{N}$ and $\eta \in Z_\tau$ we have

$$2\left|\left(B^*\mathbb{S}_{\tau-t}\tilde{\eta}\right)(x)\right| \leqslant \left|\left(B^*\mathbb{S}_{\tau-t}(\tilde{\eta}-\varepsilon\eta)\right)(x)\right| + \left|\left(B^*\mathbb{S}_{\tau-t}(\tilde{\eta}+\varepsilon\eta)\right)(x)\right|,$$

or, equivalently

$$-\frac{\left|\left(B^*\mathbb{S}_{\tau-t}(\tilde{\eta}-\varepsilon\eta)\right)(x)\right| - \left|\left(B^*\mathbb{S}_{\tau-t}\tilde{\eta}\right)(x)\right|}{\varepsilon} \leqslant \frac{\left|\left(B^*\mathbb{S}_{\tau-t}(\tilde{\eta}+\varepsilon\eta)\right)(x)\right| - \left|\left(B^*\mathbb{S}_{\tau-t}\tilde{\eta}\right)(x)\right|}{\varepsilon}.$$

Combining the above inequality and (6.11) we obtain that

$$-\left|\left(B^*\mathbb{S}_{\tau-t}\eta\right)(x)\right| \leqslant \frac{\left|\left(B^*\mathbb{S}_{\tau-t}(\tilde{\eta}+\varepsilon\eta)\right)(x)\right| - \left|\left(B^*\mathbb{S}_{\tau-t}\tilde{\eta}\right)(x)\right|}{\varepsilon}, \tag{6.12}$$

for every $\varepsilon > 0$, $(t,x) \notin \mathcal{N}$ and $\eta \in Z_\tau$. Putting together (6.10), (6.11) and (6.12), we can apply the Lebesgue dominated convergence theorem to obtain (6.9). $\square$

By combining Theorem 6.1, Lemma 6.3 and the convexity of J_ψ^τ we obtain:

Corollary 6.1. *With the notation and under the assumptions of Theorem 6.1 any minimizer $\hat{\eta}$ of J_ψ^τ satisfies*

$$\left(\int_0^\tau \int_{\mathcal{O}} |B^*\mathbb{S}_{\tau-\sigma}\hat{\eta}|\,\mathrm{d}x\,\mathrm{d}\sigma\right)\left(\int_0^\tau \int_{\mathcal{O}} \frac{B^*\mathbb{S}_{\tau-\sigma}\hat{\eta}}{|B^*\mathbb{S}_{\tau-\sigma}\hat{\eta}|} B^*\mathbb{S}_{\tau-\sigma}\eta\,\mathrm{d}x\,\mathrm{d}\sigma\right) + \langle\psi, \mathbb{S}_\tau\eta\rangle_X = 0 \qquad (\eta \in Z_\tau). \tag{6.13}$$

Conversely, any $\hat{\eta} \in Z_\tau$ satisfying (6.13) is a minimizer of J_ψ^τ over Z_τ.

7. Proof of the main results

This section is devoted to the proof of the main abstract results which have been stated in Section 3. We begin by those concerning norm optimal control problems, i.e., of Theorems 3.1, 3.2 and 3.3.

Proof of Theorem 3.1. Let $\hat{\eta}$ be a minimizer of J_ψ^τ, whose existence has been proved in Theorem 6.1. Define $\gamma \in L^1([0,\tau]\times\mathcal{O})$ by

$$\gamma(t,x) = \left(B^*\mathbb{S}_{\tau-t}\widehat{\eta}\right)(x) \qquad (t \in [0,\tau],\ x \in \mathcal{O}). \tag{7.1}$$

Using assumption (H4), it follows that

$$\gamma(t,x) \neq 0 \qquad ((t,x) \in (0,\tau)\times\mathcal{O} \text{ a.e.}). \tag{7.2}$$

Let $\hat{u}$ be the control function defined by

$$\hat{u}(t,x) = \left(\int_0^\tau \int_{\mathcal{O}} |\gamma(\sigma,y)|\,\mathrm{d}y\,\mathrm{d}\sigma\right)\frac{\gamma(t,x)}{|\gamma(t,x)|} \qquad ((t,x) \in (0,\tau)\times\mathcal{O} \text{ a.e.}). \tag{7.3}$$

We show below that $\hat{u}$ is a norm optimal control solving $(NP)^\tau$. To this aim, we first remark that from (7.2) it follows that (7.3), with γ defined in (7.1), defines indeed a function in $L^\infty((0,\tau)\times\mathcal{O})$. We next remark that (6.13) implies that,

$$\mathbb{T}_\tau\psi + \Phi_\tau\hat{u} = 0,$$

i.e., the control $\hat{u}$ steers the initial data ψ to zero in time τ.

To prove the optimality of $\widehat{u}$, let $u \in L^\infty([0,\tau]\times\mathcal{O})$ be such that

$$\mathbb{T}_\tau\psi + \Phi_\tau u = 0.$$

Then

$$\int_0^\tau \langle\hat{u}(\sigma), B^*\mathbb{T}_{\tau-\sigma}^*\hat{\eta}\rangle_U\,\mathrm{d}\sigma = \int_0^\tau \langle u(\sigma), B^*\mathbb{T}_{\tau-\sigma}^*\hat{\eta}\rangle_U\,\mathrm{d}\sigma = -\langle\psi, \mathbb{S}_\tau\hat{\eta}\rangle_X.$$

The above formula and (7.3) (with γ given by (7.1)) imply that

$$\left(\int_0^\tau \int_{\mathcal{O}} \left|\left(B^*\mathbb{S}^*_{\tau-\sigma}\widehat{\eta}\right)(y)\right| \,\mathrm{d}y\,\mathrm{d}\sigma\right)^2 \leqslant \|u\|_{L^\infty([0,\tau]\times\mathcal{O})} \int_0^\tau \int_{\mathcal{O}} \left|\left(B^*\mathbb{S}^*_{\tau-\sigma}\widehat{\eta}\right)(y)\right| \,\mathrm{d}y\,\mathrm{d}\sigma,$$

which clearly implies the optimality of $\hat{u}$. Moreover, (7.3) obviously implies that $\hat{u}$ has the announced bang-bang property, which ends the proof. □

We prove next our second result on norm optimal control problems.

Proof of Theorem 3.2. Assume, by contradiction, that there exists a norm optimal control u not satisfying (3.3). Then there would exist an

$$\varepsilon \in \left(0, \frac{N^\infty(\tau)}{2}\right) \tag{7.4}$$

and a set of positive measure $e \subset [0,\tau]\times\mathcal{O}$, such that

$$|u(t,x)| \leqslant N^\infty(\tau) - \varepsilon \qquad ((t,x)\in e).$$

Let

$$\delta = \frac{\varepsilon}{K_{\tau,e'}\|\psi\|_X + \varepsilon} \in (0,1), \tag{7.5}$$

where

$$e' = \{(\tau - t, x) \ \mid \ (t,x)\in e\},$$

and $K_{\tau,e'}$ is a constant ensuring that (3.1), with e' instead of e, holds. From [H5] [16, Proposition 2.5] and (7.5), it follows that there is a $v_\delta \in L^\infty([0,\tau]\times\mathcal{O})$ with

$$\delta\mathbb{T}_\tau\psi + \Phi_\tau v_\delta = 0, \qquad \operatorname{supp} v_\delta \subset e, \tag{7.6}$$

$$\|v_\delta\|_{L^\infty(e)} \leqslant \delta K_{\tau,e'}\|\psi\|_X = \frac{\varepsilon K_{\tau,e'}}{K_{\tau,e'}\|\psi\|_X + \varepsilon}\|\psi\|_X \leqslant \varepsilon. \tag{7.7}$$

Denoting by χ_e the characteristic function of e we set

$$u_\delta = (1-\delta)u + \chi_e v_\delta. \tag{7.8}$$

Then from (7.6) and the fact that $\mathbb{T}_\tau\psi + \Phi_\tau u = 0$ it follows that

$$\mathbb{T}_\tau\psi + \Phi_\tau u_\delta = 0. \tag{7.9}$$

Moreover, (7.7), (7.8) and (7.4), imply that

$$|u_\delta(t,x)| = \begin{cases} (1-\delta)|u(t,x)| \leqslant (1-\delta)N^\infty(\tau) & (t,x)\notin e, \\ (1-\delta)|u(t,x)| + |v_\delta(t,x)| \leqslant \left(1-\frac{\delta}{2}\right)N^\infty(\tau) & (t,x)\in e, \end{cases}$$

so that $\|u_\delta\|_{L^\infty([0,\tau]\times\mathcal{O})} \leqslant \left(1-\frac{\delta}{2}\right)N^\infty(\tau)$. This fact and (7.9) contradict the definition of $N^\infty(\tau)$, so that every norm optimal control u satisfies

$$|u(t,x)| = N^\infty(\tau) \qquad ((t,x)\in(0,\tau)\times\mathcal{O} \text{ a.e.}), \tag{7.10}$$

To show the uniqueness, let u and v be two norm optimal controls. Setting $w = \frac{1}{2}(u+v)$ it follows that w is also a norm optimal control. Since any norm optimal control satisfies (7.10), we have $|u(t,x)| = |v(t,x)| = |w(t,x)|$ a.e. in $[0,\tau]\times\mathcal{O}$. If $u(t,x)\neq v(t,x)$ in a set of positive measure $\tilde{e}\subset[0,\tau]\times\mathcal{O}$ then

$$0 = u(t,x) + v(t,x) = 2w(t,x) \qquad ((t,x)\in\tilde{e}),$$

which contradicts the bang-bang property of w. We thus have that $u(t,x) = v(t,x)$ in $[0,\tau]\times\mathcal{O}$ almost everywhere, which ends our proof. □

Proof of Theorem 3.3. To prove that N^∞ is decreasing we assume, by contradiction, that there exist τ_1, τ_2, with $0 < \tau_1 < \tau_2$, such that

$$N^\infty(\tau_2) \geqslant N^\infty(\tau_1) > 0. \tag{7.11}$$

Let u^* be the extension of the norm optimal control $\hat{u}_1$, solution of $(NP)^{\tau_1}$, to $(0, \tau_2)$ by setting $u^*(t, \cdot) = 0$ for $t \in (\tau_1, \tau_2)$. We thus have

$$\|u^*\|_{L^\infty((0,\tau_2)\times\mathcal{O})} = N^\infty(\tau_1) \text{ and } \mathbb{T}_{\tau_2}\psi + \Phi_{\tau_2} u^* = 0. \tag{7.12}$$

On the other hand, due to a classical duality argument (see, for instance, [16, Proposition 2.6]), (2.10) implies that there exists $v \in L^\infty((0, \tau_2) \times \mathcal{O})$, supported in $[\tau_1, \tau_2] \times \mathcal{O}$, such that

$$\mathbb{T}_{\tau_2}\psi + \Phi_{\tau_2} v = 0 \text{ and } \|v\|_{L^\infty((0,\tau_2)\times\mathcal{O})} \leqslant K_{\tau_2-\tau_1}\|\mathbb{T}_{\tau_1}\psi\|_X,$$

where K_t has been defined in (2.10).

We next choose $\lambda_0 \in (0, 1)$ such that

$$\lambda_0 K_{\tau_2-\tau_1}\|\mathbb{T}_{\tau_1}\psi\|_X < N^\infty(\tau_1), \tag{7.13}$$

and define a new control $\widetilde{u}$ on $(0, \tau_2) \times \mathcal{O}$ by

$$\widetilde{u} = \chi_{[0,\tau_1)}(1 - \lambda_0)u_1^* + \chi_{[\tau_1,\tau_2]}\lambda_0 v, \tag{7.14}$$

where χ_I stands for the characteristic function of the interval I. We clearly have that

$$\Phi_{\tau_2}\widetilde{u} + \mathbb{T}_{\tau_2}\psi = 0,$$

which implies that $\|\widetilde{u}\|_{L^\infty((0,\tau_2)\times\mathcal{O})} \geqslant N^\infty(\tau_2)$. Meanwhile, it follows from (7.14), (7.12) and (7.13) that $\|\widetilde{u}\|_{L^\infty((0,\tau_2)\times\mathcal{O})} < N^\infty(\tau_1)$. The above two inequalities contradict the assumption (7.11). Consequently $\tau \mapsto N^\infty(\tau)$ is decreasing on $(0, \infty)$.

We next show that $\tau \mapsto N^\infty(\tau)$ is right continuous. To this end, let $(\tau_n)_{n\in\mathbb{N}}$ be an arbitrary decreasing sequence in (τ, ∞) with $\lim_{n\to\infty} \tau_n = \tau$. For each $n \in \mathbb{N}$ we denote by u_n^* be the extension of the norm optimal control of $(NP)^{\tau_n}$ to $(0, \infty)$, obtained by setting $u_n^* = 0$ on $(\tau_n, \infty) \times \mathcal{O}$. It is then clear that

$$\mathbb{T}_{\tau_n}\psi + \Phi_{\tau_n} u^* = 0 \text{ and } \|u_n^*\|_{L^\infty((0,\infty)\times\mathcal{O})} = N^\infty(\tau_n) \leqslant N^\infty(\tau),$$

Since $(u_n^*)_{n\in\mathbb{N}}$ is bounded in $L^\infty((0, \infty) \times \mathcal{O})$, there exists a subsequence, denoted in the same way, and a control $g \in L^\infty((0, \infty) \times \mathcal{O})$ such that

$$u_n^* \to g \quad \text{weakly} * \text{ in } L^\infty((0,\infty)\times\mathcal{O}), \qquad \mathbb{T}_{\tau_n}\psi + \Phi_{\tau_n} u_n^* \to \mathbb{T}_\tau\psi + \Phi_\tau g \quad \text{weakly in } X.$$

Thus, we have

$$N^\infty(\tau) \leqslant \|g\|_{L^\infty((0,\tau)\times\mathcal{O})} \leqslant \liminf_{n\to\infty} \|u_n^*\|_{L^\infty((0,\tau)\times\mathcal{O})} = \lim_{n\to\infty} N^\infty(\tau_n) \leqslant N^\infty(\tau).$$

Hence, $\tau \mapsto N^\infty(\tau)$ is right continuous.

We now prove that the map $\tau \mapsto N^\infty(\tau)$ is left continuous. To this aim, we consider an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ in $(0, \tau)$ with $\lim_{n\to\infty} \tau_n = \tau$. Since we have seen that N^∞ is decreasing, we have that $N^\infty(\tau) \leqslant N(\tau_n) \leqslant N(\tau_1)$ for every $n \in \mathbb{N}$. It follows that

$$\lim_{n\to\infty} N^\infty(\tau_n) = N^\infty(\tau) + \delta, \tag{7.15}$$

for some $\delta \geqslant 0$. We show below that assuming that $\delta > 0$ leads to a contradiction. Indeed, let u^* be a norm optimal control of $(NP)^\tau$. This means that

$$\mathbb{T}_\tau\psi + \Phi_\tau u^* = 0 \text{ and } \|u^*\|_{L^\infty((0,\infty)\times\mathcal{O})} = N^\infty(\tau). \tag{7.16}$$

Using the continuity of the map $t \mapsto \mathbb{T}_t\psi + \Phi_t u^*$ it follows that there exists a natural number n_1 such that

$$\|(\mathbb{T}_{\tau_n}\psi + \Phi_{\tau_n} u^*)\|_X \leqslant \frac{\delta}{2K_{\tau_1}} \qquad (n > n_1),$$

with K_{τ_1} introduced in (2.10). This, along with (2.10) and (7.16), yields that for every $n \in \mathbb{N},\ n > n_1$ there exists $v_n \in L^\infty([0,\tau_n] \times \mathcal{O})$ such that

$$\mathbb{T}_{\tau_n}\psi + \Phi_{\tau_n}(u^* + v_n) = 0, \qquad \|v_n\|_{L^\infty([0,\tau_n]\times\mathcal{O})} \leqslant \frac{\delta}{2}.$$

Denoting $u_n = u^* + v_n$, the last formula and (7.16) imply that

$$\mathbb{T}_{\tau_n}\psi + \Phi_{\tau_n} u_n = 0, \qquad \|u_n\|_{L^\infty(\mathcal{O}\times[0,\tau_n])} \leqslant N^\infty(\tau) + \frac{\delta}{2} \qquad (n > n_1),$$

which clearly contradicts (7.15). Hence, $\tau \mapsto N^\infty(\tau)$ is indeed left continuous. We have thus shown that N^∞ is continuous on $(0,\infty)$.

From the just established monotonicity and continuity properties of N^∞ it obviously follows that there exists $\hat{N} \geqslant 0$ such that the first equality in (3.4) holds.

To prove the second equality in (3.4) we use a contradiction argument, by assuming that there exists a decreasing sequence of positive numbers (τ_n), with $\tau_n \to 0$, such that the sequence $(N^\infty(\tau_n))$ is bounded. Due to the monotonicity of N^∞, the sequence $(N^\infty(\tau_n))$ is increasing, thus there exists $\overline{N} > 0$ such that $\lim_{n\to\infty} N^\infty(\tau_n) = \overline{N}$. According to Theorem 3.1 it follows that for every $n \in \mathbb{N}$ there exists $u_n \in L^\infty([0,\infty) \times \mathcal{O})$ such that

$$\mathbb{T}_{\tau_n}\psi + \Phi_{\tau_n} u_n = 0, \qquad \|u_n\|_{L^\infty([0,\infty)\times\mathcal{O})} = N^\infty(\tau_n). \tag{7.17}$$

On the other hand,

$$\|\Phi_{\tau_n} u_n\|_X \leqslant \kappa_{\tau_1}\|u_n\|_{L^2([0,\tau_n]\times\mathcal{O})} \qquad (n \in \mathbb{N}),$$

where κ_{τ_1} is a constant satisfying (2.6) for $t = \tau_1$. The last inequality and the second equality in (7.17) imply that

$$\|\Phi_{\tau_n} u_n\|_X \leqslant \kappa_{\tau_1}\sqrt{\tau_n} N^\infty(\tau_n) \leqslant \kappa_{\tau_1}\sqrt{\tau_n}\,\overline{N} \to 0.$$

The above estimate allows us to passing to the limit in the first equality in (7.17) to obtain that $\psi = 0$, which contradicts one of our leading assumptions. This ends the proof of the second equality in (3.4) and thus of our theorem. □

We end this section by proving our results on time optimal control and on the relation between time and norm optimal control problems. More precisely, we give below the proof of Theorem 3.4.

Proof of Theorem 3.4. If $M \leqslant \hat{N}$ then, according to Theorem 3.3 we have that

$$M < N^\infty(\tau) \qquad (\tau > 0).$$

The above estimate implies that there is no $\tau > 0$ for which one can find $u \in \mathcal{U}_M$ steering the initial stat $\psi \in X \setminus \{0\}$ to zero in time τ, so that the time optimal control problem $(TP)^M$ has no solution.

If $M > \hat{N}$ then, according to Theorem 3.3, there is a $\tau^*(M) \in (0,\infty)$ such that

$$N^\infty(\tau^*(M)) = M. \tag{7.18}$$

Moreover, we can use Theorem 3.1 to deduce the existence of $u_M^* \in \mathcal{U}_M$ such that

$$\mathbb{T}_{\tau^*(M)}\psi + \Phi_{\tau^*(M)} u_M^* = 0,$$

$$|u_M^*(t,x)| = M \qquad ((t,x) \in [0,\tau^*(M)] \times \mathcal{O}).$$

We claim that $(\tau^*(M), u_M^*)$ is a solution of the time optimal control problem $(TP)^M$. Indeed, assume, by contradiction, that there is $\tau \in (0, \tau^*(M))$ and $u \in \mathcal{U}_M$ with

$$\mathbb{T}_\tau \psi + \Phi_\tau u = 0.$$

In this case we would have $N^\infty(\tau) \leqslant M$. This, together with (7.18), contradicts the fact that N^* is decreasing, which has been established in Theorem 3.3. We can thus conclude that if we set $\tau^\infty(M) = \tau^*(M)$ and $u_M^\infty = u_M^*$ then $(\tau^\infty(M); u_M^\infty)$ is a solution of $(TP)^M$, satisfying (3.5) and the bang-bang property (3.7). Moreover, from (3.5) it follows that

$$N^\infty(\tau^\infty(N^\infty(\tau))) = N^\infty(\tau) \qquad (\tau \in (0,\infty)).$$

The latter, along with the continuity and monotonicity of N^∞, leads to (3.6). □

8. Applications to systems described by parabolic PDEs

In this section we prove Propositions 1.1 and 1.2 and we discuss some other PDE applications of our main results stated in Section 3.

Proof of Proposition 1.1. It is well known (see, for instance, [22, Sections 10.6, 10.7]) that equations (1.1)-(1.3) determine a well-posed control LTI (in the sense of Definition 2.1), with the following choice of spaces and operators:

- The state space is $X = H^{-1}(\Omega)$ and the control space is $U = L^2(\mathcal{O})$.
- $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is the C^0 semigroup on X generated by the Dirichlet Laplacian (with domain $H_0^1(\Omega)$).
- $\Phi = (\Phi_t)_{t\geqslant 0}$ is the family of bounded linear operators from $L^2([0,\infty); U)$ to X defined by (2.1), where B is defined, by duality, by the formula

$$B^* g = \frac{\partial(A^{-1}g)}{\partial\nu}|_{\mathcal{O}} \qquad (g \in H_0^1(\Omega)), \tag{8.1}$$

and $\frac{\partial}{\partial\nu}$ is the outward normal derivation operator on $\partial\Omega$.

We check below that this well-posed control LTI satisfies assumptions (H1)-(H4).

Assumption (H1) is obviously satisfied. To check (H2) it suffices to remark that A is self-adjoint in X, thus it generates an analytic semigroup and that, due to the Rellich-Kondrachov theorem, the embedding $\mathcal{D}(A) \subset X$ is compact.

The prove that (H3) holds, we denote by $(e_k)_{k\in\mathbb{N}}$ an orthonormal (in $L^2(\Omega)$) basis formed of eigenvectors of the $-A$ and by $(\lambda_k)_{k\in\mathbb{N}}$ the corresponding sequence of eigenvalues, which is positive, nondecreasing and with $\lambda_k \to \infty$. Moreover, for $\lambda > 0$ we define

$$\mathcal{E}_\lambda f = \sum_{\lambda_j < \lambda} \langle f, e_j \rangle_{L^2(\Omega)} e_j \qquad (f \in L^2(\Omega)).$$

According to Remark 5 from [2] the domain Ω is locally star-shaped. We can thus apply Theorem 3 from [2] to conclude that we have the spectral Lebeau-Robbiano inequality, asserting that if B_r, with $r > 0$, is a ball contained in Ω, then there exist a positive constant $N = N(\Omega, r)$ such that

$$\|\mathcal{E}_\lambda f\|_{L^2(\Omega)} \leqslant N \exp(N\sqrt{\lambda}) \|\mathcal{E}_\lambda f\|_{L^2(B_r)} \qquad (f \in L^2(\Omega)).$$

Combining the above inequality with Remark 13 in [2] we conclude that indeed our system satisfies assumption (H3).

In order to check that (H4) is also satisfied we follow a procedure inspired by the proof of Corollary 2.3 in Weck and Schmidt [20]. More precisely, with the sequences (λ_k) and (e_k) defined above, we denote by $(\varphi_k)_{k\geqslant 1}$ the sequence defined by $\varphi_k = \sqrt{\lambda_k} e_k$ for every $k \in \mathbb{N}$. Then (φ_k) is an orthonormal basis in X formed of eigenvectors of A and $\mathbb{T}_t f = \sum_{k\in\mathbb{N}} \exp(-\lambda_k t)\langle f, \varphi_k\rangle_X \varphi_k$ for every $f \in X$. Using next (2.5) it follows that for every $f \in \mathcal{D}(A^*)$, $t \in [0,\tau]$ and $x \in \mathcal{O}$ we have

$$(\Phi_\tau^* f)(t,x) = \sum_{k\in\mathbb{N}} \exp(-\lambda_k(\tau - t))\langle f, \varphi_k\rangle_X (B^*\varphi_k)(x). \tag{8.2}$$

On the other hand, using (2.6) with $\psi = \varphi_k$ it follows that the sequence $(\|B^*\varphi_k\|_U)$ is bounded. Using this fact it easily follows that (8.2) holds (with convergence in $L^2([0,\tau];U)$) for all $f \in X$. Moreover, denoting by (μ_l) the increasing sequence formed by the distinct eigenvalues of $-A$, for $f \in X$, $t \in [0,\tau]$ and $x \in \mathcal{O}$ we have

$$(\Phi_\tau^* f)(t,x) = \sum_{l\in\mathbb{N}} \exp(-\mu_l(\tau - t))(B^* f_l)(x), \tag{8.3}$$

where

$$f_l = \sum_{\lambda_k = \mu_l} \langle f, \varphi_k\rangle_X \varphi_k. \tag{8.4}$$

Assume next that $f \in X$ is such that $(\Phi_\tau^* f)(t,x) = 0$ for (t,x) in a subset e of positive Lebesgue measure in $[0,\tau]\times\mathcal{O}$. Let S be the subset of $x \in \mathcal{O}$ for which the corresponding section

$$e_x = \{t \in [0,\tau] \ \mid \ (t,x) \in e\}$$

has positive one dimensional Lebesgue measure. By Fubini's theorem S is a subset of positive Lebesgue measure of $\mathcal{O}$. Moreover, from (8.3) it follows that the map

$$t \mapsto \sum_{l\in\mathbb{N}} \exp(-\mu_l(\tau - \sigma))(B^* f_l)(x) \qquad (x \in S, t \in [0,\tau]),$$

vanishes for t in a set of positive measure. Elementary facts about Dirichlet series imply that $(B^* f_l)(x) = 0$ for every $l \in \mathbb{N}$ and $x \in S$. Since f_l is an eigenvector of $-A$ associated to the eigenvalue μ_l and formula (8.1) it follows that

$$\frac{\partial f_l}{\partial \nu}(x) = 0 \qquad (l \in \mathbb{N}, x \in S).$$

We have seen that S is of positive measure, so that we can apply standard strong unique continuation for elliptic operators (see [20, Corollary 2.2] for details) to obtain that $f_l = 0$ for every $l \in \mathbb{N}$. Going back to (8.4) we obtain that $f = 0$.

We have thus shown that the wellposed LTI control system described by (1.1)-(1.3) satisfies (H1)-(H4). All the conclusions announced in our proposition follow now by the application of Theorems 3.1-3.4 and the remark that, due the exponential stability of $\mathbb{T}$, the constant $\hat{N}$ in (3.4) vanishes. □

We are now in a position to prove Proposition 1.2.

Proof of Proposition 1.2. Due to Theorem 2 in [2], the system described by (1.1)-(1.3) satisfies (H5), so that the conclusions follow by applying Corollary 3.1. □

Our abstract results in Section 3 are applicable to other PDE systems with various control operators. We think, for instance, to the Stokes system with distributed control, for which they imply, in particular, the results in Chaves-Silva, Souza and

Zhang [6]. To avoid excessive lengthening of this work, we do not detail this application, choosing to end this paper with an example for a system described by a variable coefficients parabolic equation. This example generalizes the results obtained in [2] for the constant coefficients case.

Example 8.1. *Let M be a C^∞ bounded Riemannian manifold (with or without boundary) and let $\mathcal{O} \subset M$ be an open non empty submanifold of M. We assume that the metric tensor g of M is a globally Lipschitz function of $x \in M$.*

We denote $X = L^2(M)$ and by A the Beltrami Laplacian on $L^2(M)$, with Dirichlet boundary conditions. More precisely, $\mathcal{D}(A) = H^2(M) \cap H_0^1(M)$ and

$$A\varphi = \frac{1}{\sqrt{\det g}}\operatorname{div}\left(\sqrt{\det g}\, g^{-1}\nabla\varphi\right) \qquad (\varphi \in \mathcal{D}(A)).$$

We also consider the control space $U = L^2(\mathcal{O})$ and define the control operator $B \in \mathcal{L}(U, X)$ by

$$Bu = u\chi_0 \qquad (u \in U),$$

where χ_O is the characteristic function of $\mathcal{O}$. With the above choice of spaces and operators, we consider the control LTI system described by

$$\dot{z} = Az + Bu.$$

This system is clearly well-posed since $B \in \mathcal{L}(U, X)$. Moreover, assumption (H1) in Section 3 is obviously satisfied. The fact that (H2) holds for the considered system follows from the fact that A is skew-adjoint and the compactness of the embedding $\mathcal{D}(A) \subset X$. Moreover, from Burq and Moyano [5, Theorem 2] *it follows that our system also satisfies assumption [H5] from Section 3. We can thus apply Theorem 3.2 to conclude that for every $\tau > 0$ the associated norm optimal control problem admits a unique solution $\hat{u}$, which satisfies $|\hat{u}(t,x)| = N^\infty(\tau)$ for almost every $(t,x) \in (0,\tau) \times \mathcal{O}$. We can also apply Theorem 3.4 and Corollary 3.1 to assert that for every $M > 0$ the associated time optimal control problem admits a unique solution $(\tau^\infty(M), u_M^\infty)$, satisfying $|u_M^\infty(t,x)| = M$ for almost every $(t,x) \in [0, \tau^\infty(M)] \times \mathcal{O}$. Finally, N^∞ and τ^∞ satisfy (3.5) and (3.6).*

Acknowledgements. The author warmly thanks NYU Shanghai for its kind hospitality during the spring term of 2024, creating the conditions to achieve this work. The author is grateful to Sylvain Ervedoza, Sorin Micu and Gengsheng Wang for their careful reading of the manuscript and their useful suggestions.

References

[1] H. Amann and J. Escher, *Analysis. III*, Birkhäuser Verlag, Basel, 2009.

[2] J. Apraiz, L. Escauriaza, G. Wang and C. Zhang, Observability inequalities and measurable sets, *J. Eur. Math. Soc. (JEMS)*, **16** (2014), 2433–2475.

[3] R. Bellman, I. Glicksberg and O. Gross, On the "bang-bang" control problem, *Quart. Appl. Math.*, **14** (1956), 11–18.

[4] V. G. Boltyanskiĭ, R. V. Gamkrelidze and L. S. Pontryagin, On the theory of optimal processes, *Dokl. Akad. Nauk SSSR*, **110** (1956), 7–10.

[5] N. Burq and I. Moyano, Propagation of smallness and control for heat equations, *Journal of the European Mathematical Society*, **25** (2022), 1349–1377.

[6] F. W. Chaves-Silva, D. A. Souza and C. Zhang, Observability inequalities on measurable sets for the Stokes system and applications, *SIAM J. Control and Optimization*, **58** (2020), 2188–2205.

[7] C. Fabre, J.-P. Puel and E. Zuazua, Approximate controllability of the semilinear heat equation, *Proc. Royal Society of Edinburgh Section A: Mathematics*, **125** (1995), 31–61.

[8] H. O. Fattorini, *Infinite Dimensional Linear Control Systems. The Time Optimal and Norm Optimal Control Problems*, Elsevier, Amsterdam, 2005.
[9] H. O. Fattorini, Boundary control systems, *SIAM J. Control and Optim.*, **6** (1968), 349–385.
[10] H. Fattorini, Time and norm optimal controls: a survey of recent results and open problems, *Acta Mathematica Scientia*, **31** (2011), 2203–2218.
[11] K. Kellay, T. Normand and M. Tucsnak, Sharp reachability results for the heat equation in one space dimension, *Analysis & PDE*, **15** (2022), 891–920.
[12] K. Kunisch and L. Wang, Time optimal control of the heat equation with pointwise control constraints, *ESAIM Control Optim. Calc. Var.*, **19** (2013), 460–485.
[13] J.-L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Avant propos de P. Lelong, Dunod, Paris, 1968.
[14] J.-L. Lions, Remarks on approximate controllability, *Journal d'Analyse Mathématique*, **59** (1992), 103–116.
[15] J. Lohéac and M. Tucsnak, Maximum principle and bang-bang property of time optimal controls for Schrodinger-type systems, *SIAM J.Control and Optimization*, **51** (2013), 4016–4038.
[16] S. Micu, I. Roventa and M. Tucsnak, Time optimal boundary controls for the heat equation, *Journal of Functional Analysis*, **263** (2012), 25–49.
[17] V. J. Mizel and T. I. Seidman, An abstract bang-bang principle and time-optimal boundary control of the heat equation, *SIAM J. Control Optim.*, **35** (1997), 1204–1216.
[18] K. D. Phung and G. Wang, An observability estimate for parabolic equations from a measurable set in time and its applications, *J. Eur. Math. Soc. (JEMS)*, **15** (2013), 681–703.
[19] E. G. Schmidt, The "bang-bang" principle for the time-optimal problem in boundary control of the heat equation, *SIAM J. Control and Optimization*, **18** (1980), 101–107.
[20] E. G. Schmidt and N. Weck, On the boundary behavior of solutions to elliptic and parabolic equations—with applications to boundary control for parabolic equations, *SIAM J. Control and Optimization*, **16** (1978), 593–598.
[21] T. I. Seidman, Time-invariance of the reachable set for linear control problems, *J. Math. Anal. Appl.*, **72** (1979), 17–20.
[22] M. Tucsnak and G. Weiss, *Observation and control for operator semigroups*, Birkhäuser Verlag, Basel, 2009.
[23] M. Tucsnak and G. Weiss, Well-posed systems—the LTI case and beyond, *Automatica*, **50** (2014), 1757–1779.
[24] G. Wang, L^∞-null controllability for the heat equation and its consequences for the time optimal control problem, *SIAM J. Control Optim.*, **47** (2008), 1701–1720.
[25] G. Wang, L. Wang, Y. Xu and Y. Zhang, *Time optimal control of evolution equations*, Birkhäuser/Springer, Cham, 2018.
[26] G. Wang, Y. Xu and Y. Zhang, Attainable subspaces and the bang-bang property of time optimal controls for heat equations, *SIAM J. Control and Optimization*, **53** (2015), 592–621.
[27] G. Wang and E. Zuazua, On the equivalence of minimal time and minimal norm controls for internally controlled heat equations, *SIAM J. Control and Optim.*, **50** (2012), 2938–2958.
[28] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control Optim.*, **27** (1989), 527–545.

Email address: marius.tucsnak@u-bordeaux.fr