

HILBERT SERIES ASSOCIATED TO SYMPLECTIC QUOTIENTS BY SU_2

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ABSTRACT. We compute the Hilbert series of the graded algebra of real regular functions on the symplectic quotient associated to an SU_2 -module and give an explicit expression for the first nonzero coefficient of the Laurent expansion of the Hilbert series at $t = 1$. Our expression for the Hilbert series indicates an algorithm to compute it, and we give the output of this algorithm for representations of dimension at most 10. Along the way, we compute the Hilbert series of the module of covariants of an arbitrary SL_2 - or SU_2 -module as well its first three Laurent coefficients.

CONTENTS

1. Introduction	1
Acknowledgements	4
2. Background	4
3. The Hilbert Series of the Module of Covariants	5
3.1. The Multivariate Hilbert Series	7
3.2. The Univariate Hilbert Series	9
3.3. The Laurent Coefficients of the Hilbert series	10
4. Hilbert Series of the Graded Algebra of Regular Functions on SU_2 -Symplectic Quotients	16
Appendix A. Hilbert Series in Low Dimensions	17
Appendix B. Visualization of $\gamma_0^{on}(V)$	20
References	20

1. INTRODUCTION

Let (M, ω) be a smooth symplectic manifold with an action of a compact Lie group G preserving the symplectic form ω . Assume that the G -action admits a G -equivariant *moment map* $J : M \rightarrow \mathfrak{g}^*$, where \mathfrak{g}^* denotes the dual space of the Lie algebra \mathfrak{g} of G . In other words, the infinitesimal action of $\xi \in \mathfrak{g}$ is given by the Hamiltonian vector field $\{J_\xi, \cdot\}$ of the function J_ξ obtained by pairing J with ξ . If $0 \in \mathfrak{g}^*$ is a regular value, the zero level $Z := J^{-1}(0)$ is a closed submanifold of M and the *symplectic quotient* $M_0 = Z/G$ (also known as Marsden-Weinstein quotient or Hamiltonian reduction) is a symplectic orbifold. Otherwise, as shown in [39], Z has locally conical singularities and $M_0 = Z/G$ is a *stratified symplectic space*. The strata $(M_0)_{(H)}$ of this stratification are indexed by conjugacy classes (H) of isotropy subgroups $H \subseteq G$ and are given by the set of G -orbits of points whose isotropy group is in (H) . The components of $(M_0)_{(H)}$ are smooth symplectic manifolds. The strata and their interrelationships can be recovered from the Poisson algebra of smooth functions $\mathcal{C}^\infty(M_0) = \mathcal{C}^\infty(M)^G / \mathcal{I}_Z^G$, corresponding to the algebra of *classical observables* of the system. Here \mathcal{I}_Z^G denotes the G -invariant part of the ideal \mathcal{I}_Z of smooth functions vanishing on Z . It has been suggested to view $(M_0, \mathcal{C}^\infty(M_0))$ as a differential space in the sense of Sikorski (cf. [40, 32, 18]). The authors adhere to this philosophy.

Symplectic quotients, or incarnations thereof, frequently arise as moduli spaces in gauge theory. This is due to the fact that the curvature of a connection can be interpreted as a moment map on the cotangent bundle of the space of connections (see e.g. [2, 17]). In many interesting situations, the moduli space is finite-dimensional but exhibits singularities. In this case, it can often be locally understood as a symplectic

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quotient in the sense defined above (see e.g. [23]). For example, the moduli space of flat SU_2 -connections on a Riemann surface has local models that are a symplectic reductions at zero angular momentum [31, 12]. How to deal with the singularities is the subject of ongoing research, and the literature on the topic is extensive.

It is well known since [1, 39] that the symplectic slice theorem can be employed to study the singularities locally. It is therefore natural and convenient to focus our attention on the case when (M, ω) is (\mathbb{C}^n, ω_0) with its standard Kähler structure and the action of G is a unitary representation. In this situation, the moment map is given by homogeneous quadratic polynomials, see Equation (2.1). Typically, invariant theory is used to describe the semialgebraic geometry of M_0 (see e.g. [33, 32, 18, 12]). Closely related to M_0 is its real Zarisky closure $\overline{M_0}^z$. It is the spectrum of the Noetherian \mathbb{R} -algebra

$$\mathbb{R}[M_0] = \mathbb{R}[\mathbb{C}^n]^G / (\mathcal{I}_Z \cap \mathbb{R}[\mathbb{C}^n]^G).$$

This algebra comes with a canonical nonstandard \mathbb{Z}^+ -grading and is a Poisson subalgebra of $\mathcal{C}^\infty(M_0)$. Versions of the complexification $\mathbb{C}[M_0] = \mathbb{R}[M_0] \otimes_{\mathbb{R}} \mathbb{C}$ have been intensely studied in geometric representation theory (see e.g. [35, 21, 15]). For a more careful discussion of the subtleties of the definition of the complex symplectic quotient and the relation to the *2-large* property, see [28].

A fundamental question related to symplectic quotients is the *symplectomorphism problem*, i.e. classifying the singularities that arise in symplectic quotients up to symplectomorphism. The question of when a symplectic quotient is (graded regularly) symplectomorphic to a finite unitary quotient (i.e. a symplectic quotient with G a finite group) has been proposed in [18] and, to some extent, answered in [27]. Considering the symplectomorphism problem the following basic questions arise:

- (A) What are the general traits that all symplectic quotients have in common?
- (B) What are appropriate invariants suitable to distinguish them up to symplectomorphism?

In progress towards answering both of these questions, the Hilbert series of the graded algebra of regular functions $\mathbb{R}[M_0]$ has been instrumental. For a general result addressing question (A), we refer the reader to [28] where it has been shown that symplectic quotients have symplectic singularities. In [29] the Hilbert series of the algebra $\mathbb{R}[M_0]$ of symplectic circle quotients M_0 has been examined, and implications for the graded regular symplectomorphism problem have been discussed. We emphasize that the Hilbert series is, in general, not invariant under regular symplectomorphism. Its main virtues are its accessibility and that it encodes valuable information about the variety $\overline{M_0}^z$ (e.g. dimension, a -invariant, Gorensteinness). We mention that in recent years, Hilbert series calculations have gained considerable attention by people working in supersymmetric Yang-Mills theory as part of the *plethystic program* (see e.g. [19]).

In this paper, we study the Hilbert series of the algebra of regular functions $\mathbb{R}[M_0]$ of the symplectic quotient of a (finite-dimensional) unitary representation V of the group $G = SU_2$. Let $r > 1$ be an integer and let $\Gamma := (\mathbb{Z}^+)^r$ be the semigroup of multi-indices $\mathbf{n} = (n_1, n_2, \dots, n_r)$. Recall that the *Hilbert series* of a Γ -graded vector space $X = \bigoplus_{\mathbf{n} \in \Gamma} X_{\mathbf{n}}$ with $X_{\mathbf{n}}$ finite-dimensional for all $\mathbf{n} \in \Gamma$ is the generating function

$$\begin{aligned} \text{Hilb}_X(\mathbf{t}) &= \text{Hilb}_X(t_1, t_2, \dots, t_r) := \sum_{\mathbf{n} \in \Gamma} \dim(X_{\mathbf{n}}) \mathbf{t}^{\mathbf{n}} \\ &= \sum_{n_1, n_2, \dots, n_r=0}^{\infty} \dim(X_{(n_1, n_2, \dots, n_r)}) t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r} \in \mathbb{Q}[\mathbf{t}] = \mathbb{Q}[t_1, t_2, \dots, t_r]. \end{aligned}$$

If $r = 1$, we refer to $\text{Hilb}_X(\mathbf{t})$ as a *univariate Hilbert series* and set $t := t_1$; otherwise we say it is *multivariate*. A univariate Hilbert series corresponding to grading by total degree can be determined from a multivariate Hilbert series $\text{Hilb}_X(\mathbf{t})$ by specialization $\text{Hilb}_X(t) = \text{Hilb}_X(t, t, \dots, t)$.

The Hilbert series of primary interest for us is the univariate Hilbert series

$$\text{Hilb}_{(G, V)}^{\text{on}}(t) := \text{Hilb}_{\mathbb{R}[M_0]}(t) = \sum_{n=0}^{\infty} \dim(\mathbb{R}[M_0]_n) t^n.$$

We refer to it as the *on-shell Hilbert series*. If the group G is understood we will write simply $\text{Hilb}_V^{\text{on}}(t)$. It is known [27] that $\mathbb{R}[M_0]$ is a graded Gorenstein algebra if the complexified representation $(G_{\mathbb{C}}, V)$ is 2-large (this assumption is not necessary if $G = SU_2$ or the a torus). It then follows that $\text{Hilb}_{(G, V)}^{\text{on}}(t)$ is rational and satisfies the functional equation $\text{Hilb}_{(G, V)}^{\text{on}}(t^{-1}) = (-t)^d \text{Hilb}_{(G, V)}^{\text{on}}(t)$ where d is the Krull dimension of $\overline{M_0}^z$. In order to understand $\text{Hilb}_{(SU_2, V)}^{\text{on}}(t)$, we employ the univariate Hilbert series $\text{Hilb}_{\mathbb{C}[V \oplus V^*]^{SL_2}}(t)$ of the polynomial $SL_2 = (SU_2)_{\mathbb{C}}$ -invariants of the cotangent lifted representation $V \oplus V^*$.

Here we can use the previous work [13] on the univariate Hilbert series

$$\text{Hilb}_{\mathbb{C}[W]^{\text{SL}_2}}(t) = \sum_{n=1}^{\infty} \dim(\mathbb{C}[W]_n^{\text{SL}_2}) t^n$$

of polynomial invariants of a finite-dimensional SL_2 -representation W . In the Laurent expansion

$$\text{Hilb}_{\mathbb{C}[W]^{\text{SL}_2}}(t) = \sum_{n=0}^{\infty} \gamma_n(W)(1-t)^{n-D+3},$$

the pole order equals $D - 3$ (where $D = \dim_{\mathbb{C}} W$) if (SL_2, W) if W is 1-large. Moreover, formulas in terms of Schur functions for the coefficients γ_0, γ_1 and γ_2 have been elaborated and an algorithm computing $\text{Hilb}_{\mathbb{C}[W]^{\text{SL}_2}}(t)$ has been presented and implemented. Many ideas in [13] are adaptations of developments in [29], where $\text{Hilb}_{(G,V)}^{\text{on}}(t)$ in the case of $G = \mathbb{S}^1$ has been examined. Similar considerations have also been used in [14] to examine $\text{Hilb}_{\mathbb{C}[W]^{\mathbb{C}^\times}}(t)$ for a linear representation W of the complex circle \mathbb{C}^\times .

If G is an ℓ -dimensional torus $\mathbb{T}^\ell = (\mathbb{S}^1)^\ell$, the complexification $G_{\mathbb{C}}$ is $\mathbb{T}_{\mathbb{C}}^\ell = (\mathbb{C}^\times)^\ell$. It can be assumed without loss of generality that the representation $(\mathbb{T}_{\mathbb{C}}^\ell, V)$ is stable and faithful. With this assumption, we have the simple relationship

$$\text{Hilb}_{(G, \mathbb{T}^\ell)}^{\text{on}}(t) = (1-t^2)^\ell \text{Hilb}_{\mathbb{C}[V \oplus V^*]^{\mathbb{T}_{\mathbb{C}}^\ell}}(t).$$

If the group G is nonabelian, the relationship is more complicated. We instead systematize the method of [28, Subsection 6.3], i.e. we take the SU_2 -invariant part of the Koszul complex on the moment map. Recall that under mild assumptions (more precisely if $(G_{\mathbb{C}}, V)$ is 1-large), the Koszul complex is a resolution of the algebra of on-shell function [26]. This enables us to express $\text{Hilb}_{(SU_2, V)}^{\text{on}}(t)$ as a linear combination of $\text{Hilb}_{\mathbb{C}[V \oplus V^*]^{\text{SL}_2}}(t)$ and $\text{Hilb}_{(\mathbb{C}[V \oplus V^*] \otimes V_2)^{\text{SL}_2}}(t)$ (see Proposition 2.1). Here V_2 is the irreducible SL_2 -representation given by binary forms of degree 2 (or in physics terminology, spin 1). The space $(\mathbb{C}[V \oplus V^*] \otimes V_2)^{\text{SL}_2}$ can be interpreted as the module of V_2 -covariants, i.e. the space of SL_2 -equivariant linear maps from $\mathbb{C}[V \oplus V^*]$ to V_2 .

The missing piece is the Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ of the module of covariants, and Section 3 is dedicated to elaborating formulas for it. We take the liberty of treating instead of binary forms V_2 of degree two the more general case of binary forms V_L of degree L . If W can be decomposed into irreducibles $W = V_{d_1} \oplus V_{d_2} \oplus \dots \oplus V_{d_r}$, the space $(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}$ carries a natural $(\mathbb{Z}^+)^r$ -grading (see for example [8]). This makes it possible to refine the analysis and look into the multivariate Hilbert series of the covariants

$$(1.1) \quad \text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t_1, t_2, \dots, t_r) = \sum_{n_1, n_2, \dots, n_r=0}^{\infty} \dim_{\mathbb{C}}(\mathbb{C}[W] \otimes V_L)_{(n_1, n_2, \dots, n_r)}^{\text{SL}_2} t_1^{n_1} t_2^{n_2} \dots t_r^{n_r}.$$

The first preparatory result is Proposition 3.1 that presents the multivariate Hilbert series in terms of sums over roots of unity and analytic continuation. Similar formulas have been found by Brion [7] for the algebra of SL_2 -invariant polynomial functions. For algorithms concerning the multivariate and univariate Hilbert series and other approaches to the computation of the Hilbert series of covariants in some cases, see [3, 4, 5, 6]. In addition, Broer has considered the computation of the Hilbert series for SL_2 -covariants, see [10, 9].

In Subsection 3.2 we specialize to the univariate Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ and deduce a formula for it in Theorem 3.4. As a first consequence, we describe an algorithm that we have implemented to compute the covariants. Along with Proposition 2.1, this algorithm enables the computation of $\text{Hilb}_{(SU_2, V)}^{\text{on}}(t)$, and we have produced a list of examples of in Table 1 of Appendix A. In Subsection 3.3 we use Theorem 3.4 to deduce formulas for the first three coefficients $\gamma_{0,L}, \gamma_{1,L}$ and $\gamma_{2,L}$ in the Laurent expansion of $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$, see Theorems 3.7, 3.8 and 3.9.

Finally, in Section 4 we are able to derive our main result, Theorem 4.2, that expresses the leading coefficient $\gamma_0^{\text{on}}(V)$ of the Laurent expansion of $\text{Hilb}_{(SU_2, V)}^{\text{on}}(t)$ in terms of the weights of the representation using Schur functions. The main difficulty in achieving this result is that there occur certain cancelations in the leading terms of the expansion; see Equation (4.2) and the discussion that follows. For this reason, we are forced to calculate the first three terms in the Laurent expansions of $\text{Hilb}_{(\mathbb{C}[W] \otimes V_2)^{\text{SL}_2}}(t)$ and $\text{Hilb}_{\mathbb{C}[V \oplus V^*]^{\text{SL}_2}}(t)$. The latter calculation was done in the previous work [13]. From the graded Gorensteinness of $\mathbb{R}[M_0]$ and the Gorensteinness of $\mathbb{C}[V \oplus V^*]^{\text{SL}_2}$ we deduce further relations among the Laurent coefficients. For the effects of Gorensteinness on the Laurent expansion of the Hilbert series see our previous article [25].

We draw some empirical conclusions and suggest that $\gamma_0^{on}(V)$ can be seen as a measure of the degree of reducibility of the representation V . In Appendix A, we list $\text{Hilb}_{(\text{SU}_2, V)}^{on}(t)$, γ_0^{on} and γ_2^{on} for $\dim_{\mathbb{R}} M_0$ between 2 and 14 (note that graded Gorensteinness implies $\gamma_1^{on} = 0$ by [24]). Data for higher dimensional cases are available from the authors by request. In Appendix B we provide visualizations of the behavior of γ_0^{on} for $\dim_{\mathbb{R}} M_0$ between 2 and 38.

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2. BACKGROUND

Let G be a compact Lie group and V a unitary G -module. We briefly recall the definition of the corresponding symplectic quotient M_0 and its graded Poisson algebra of regular functions $\mathbb{R}[M_0]$. The reader is referred to [18, Sections 2.1 and 4] or [28, Section 2.2] for more details.

The moment map associated to (G, V) is the regular quadratic map

$$(2.1) \quad J: V \rightarrow \mathfrak{g}^*, \quad v \mapsto (J(v), \xi) := \frac{\sqrt{-1}}{2} \langle v, \xi \cdot v \rangle,$$

where $(J(v), \xi)$ denotes the dual pairing of $J(v) \in \mathfrak{g}^*$ with $\xi \in \mathfrak{g}$, $\langle \cdot, \cdot \rangle$ denotes the Hermitian scalar product, and $\xi \cdot v$ denotes the infinitesimal action. We let Z denote the G -invariant set $J^{-1}(0) \subset V$, informally called the *shell*, and let $M_0 := Z/G$ denote the symplectic quotient. When G is not discrete, $0 \in \mathfrak{g}^*$ is not a regular value of J so that Z is not a manifold. Instead, M_0 is stratified into symplectic manifolds and hence has the structure of a *stratified symplectic space* by [39]. Letting $\mathcal{C}^\infty(V)$ denote the algebra of real-valued smooth functions on V , \mathcal{I}_Z denote the ideal of $\mathcal{C}^\infty(V)$ consisting of those functions that vanish on Z , and $\mathcal{I}_Z^G := \mathcal{C}^\infty(V)^G \cap \mathcal{I}_Z$ the invariant part of \mathcal{I}_Z , the *smooth structure* of M_0 is defined to be the quotient

$$\mathcal{C}^\infty(M_0) := \mathcal{C}^\infty(V)^G / \mathcal{I}_Z^G.$$

The algebra $\mathcal{C}^\infty(M_0)$ inherits a Poisson structure from the usual Poisson bracket on $\mathcal{C}^\infty(V)$, and M_0 along with this Poisson algebra has the structure of a *Poisson differential space*, see [18, Section 4.1].

The *Poisson algebra of regular functions on M_0* , $\mathbb{R}[M_0] = \mathbb{R}[V]^G / (\mathcal{I}_Z \cap \mathbb{R}[V]^G)$, is a Poisson subalgebra of $\mathcal{C}^\infty(M_0)$. It is a \mathbb{Z}^+ -graded Poisson algebra with Poisson bracket being of degree -2 . In the language of [18, Section 4.2], $(\mathbb{R}[M_0], \{ \cdot, \cdot \})$ together with the fundamental polynomial invariants provide a *global chart* for the Poisson differential space M_0 . We let $\text{Hilb}_{\mathbb{R}[V]^G}(t)$ denote the Hilbert series of the graded algebra of real G -invariant polynomials and $\text{Hilb}_{(\mathbb{R}[M_0])}^{on}(t)$ (or simply $\text{Hilb}_V^{on}(t)$ when G is clear from the context) denote the Hilbert series of the graded algebra $\mathbb{R}[M_0]$ of “on-shell” invariants, i.e. real regular functions on M_0 .

Although our primary interest is the real symplectic quotient M_0 by the compact Lie group G and hence the real polynomial invariants, it is often helpful to consider the corresponding complexifications. The action of G on V extends to an action of the complexification $G_{\mathbb{C}}$ on V . If $(V, G_{\mathbb{C}})$ is *1-large* (see [38], [26], or [28] for the definition), then by [26, Corollary 4.3], the quadratic components of J generate a homogeneous real ideal (J) of $\mathbb{R}[V]$ so that $\mathbb{R}[M_0] = \mathbb{R}[V]^G / (J)^G$. Complexifying the underlying real vector space of V yields $V \otimes_{\mathbb{R}} \mathbb{C}$, which is isomorphic to $V \oplus V^*$ as a G -module (or equivalently a $G_{\mathbb{C}}$ -module); we refer to $V \oplus V^*$ as the *cotangent lift* of V . We let $\mu = J \otimes_{\mathbb{R}} \mathbb{C}: V \oplus V^* \rightarrow \mathfrak{g}_{\mathbb{C}}^*$ denote the complexification of the moment map, where $\mathfrak{g}_{\mathbb{C}}^*$ is the Lie algebra of $G_{\mathbb{C}}$, and refer to the subscheme of $V \oplus V^*$ associated to the ideal (μ) as the *complex shell*. The associated *complex symplectic quotient* is defined to be $\text{Spec}(\mathbb{R}[M_0] \otimes_{\mathbb{R}} \mathbb{C})$, which in the case of 1-large $(V, G_{\mathbb{C}})$ coincides with the spectrum of $\mathbb{C}[V \oplus V^*]^{G_{\mathbb{C}}} / (\mu)^{G_{\mathbb{C}}}$. In particular, note that the Hilbert series of the graded algebras $\mathbb{R}[M_0]$ and $\mathbb{C}[V \oplus V^*]^{G_{\mathbb{C}}} / (\mu)^{G_{\mathbb{C}}}$ coincide in the 1-large case. In non-1-large cases, this definition of the complex quotient is not standard; see [28, Section 2.2] for a thorough discussion.

For the rest of the paper, we specialize to the case where $G = \text{SU}_2$ so that $G_{\mathbb{C}} = \text{SL}_2$. Let V be a finite-dimensional unitary SU_2 -module, or equivalently a finite-dimensional SL_2 -module, and assume for simplicity that $V^{\text{SU}_2} = \{0\}$. We use V_d to denote the unique irreducible SU_2 -module of dimension $d+1$ on binary forms of degree d . We assume that (V, SL_2) is 1-large, which by [26, Theorem 3.4] is true for all V not isomorphic to V_1 , $2V_1$, nor V_2 . Then by [26, Lemma 2.1(2)], the components of μ form a regular sequence in $\mathbb{C}[V \oplus V^*]$

so that the Koszul complex of μ is a free resolution of $\mathbb{C}[V \oplus V^*/(\mu)$, see [11, Corollary 1.6.14(b)]. Let $S := \mathbb{C}[V \oplus V^*]$, and then we have an exact sequence

$$0 \longrightarrow S \simeq S \otimes \wedge^3 \mathfrak{sl}_2 \longrightarrow S \otimes \wedge^2 \mathfrak{sl}_2 \longrightarrow S \otimes \mathfrak{sl}_2 \longrightarrow S \longrightarrow S/\mathfrak{sl}_2 \longrightarrow 0,$$

where the elements of \mathfrak{sl}_2 are in degree 2, the elements of $\wedge^2 \mathfrak{sl}_2$ are in degree 4, and the elements of $\wedge^3 \mathfrak{sl}_2$ are in degree 6. As \mathfrak{sl}_2 and $\wedge^2 \mathfrak{sl}_2$ are isomorphic as SL_2 - (or SU_2 -)modules, this exact sequence can be rewritten as

$$0 \longrightarrow S \longrightarrow S \otimes \mathfrak{sl}_2 \longrightarrow S \otimes \mathfrak{sl}_2 \longrightarrow S \longrightarrow S/\mathfrak{sl}_2 \longrightarrow 0.$$

Taking invariants, we have

$$0 \longrightarrow S^{SL_2} \longrightarrow (S \otimes \mathfrak{sl}_2)^{SL_2} \longrightarrow (S \otimes \mathfrak{sl}_2)^{SL_2} \longrightarrow S^{SL_2} \longrightarrow (S/\mathfrak{sl}_2)^{SL_2} \longrightarrow 0.$$

Now, $S^{SL_2} = \mathbb{C}[V \oplus V^*]^{SL_2} = \mathbb{C}[V \oplus V^*]^{SU_2}$, and as $\mathfrak{sl}_2 \simeq V_2$ as SL_2 - or SU_2 -modules, $(S \otimes \mathfrak{sl}_2)^{SL_2}$ is isomorphic to the module of covariants $(\mathbb{C}[V \oplus V^*] \otimes V_2)^{SL_2} = (\mathbb{C}[V \oplus V^*] \otimes V_2)^{SU_2}$. Finally, $(S/\mathfrak{sl}_2)^{SL_2} = \mathbb{R}[M_0] \otimes_{\mathbb{R}} \mathbb{C}$. Therefore, letting $\text{Hilb}_{(\mathbb{C}[V \oplus V^*] \otimes V_2)^{SL_2}}(t)$ denote the Hilbert series of the module of covariants, we have the following.

Proposition 2.1. *Let V be a unitary SU_2 -representation with $V^{SU_2} = \{0\}$ and assume that V is not isomorphic to V_1 , $2V_1$ nor $V_1 \oplus V_2$. Then the on-shell Hilbert series of the graded algebra of regular functions on the symplectic quotient M_0 is given by*

$$(2.2) \quad \text{Hilb}_V^{on}(t) = (1 - t^6) \text{Hilb}_{\mathbb{C}[V \oplus V^*]^{SL_2}}(t) + (t^4 - t^2) \text{Hilb}_{(\mathbb{C}[V \oplus V^*] \otimes V_2)^{SL_2}}(t).$$

Remark 2.2. Note that in each of the non-1-large cases, the symplectic quotients are isomorphic to quotients by finite groups, see [27, Section 5] or [1, 22], and hence easily handled individually. Specifically, the symplectic quotient associated to V_1 is a point, while the (real) symplectic quotients associated to $2V_1$ and V_2 are both isomorphic to $\mathbb{C}/\pm 1$.

In Section 3, we will compute the Hilbert series $\text{Hilb}_{(\mathbb{C}[V \oplus V^*] \otimes W)^{SL_2}}(t)$ of covariants in general as well as the first few Laurent coefficients of this series; note that the case of invariants, i.e. $W = \mathbb{C}$, was treated in [13]. In Section 4, we will apply these results to a computation of the first nonzero Laurent coefficient of $\text{Hilb}_V^{on}(t)$.

3. THE HILBERT SERIES OF THE MODULE OF COVARIANTS

Let W and U be finite-dimensional unitary SU_2 -modules, equivalently finite-dimensional SL_2 -modules. In this section, we compute the Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{SL_2}}(t)$ of the module of covariants $W \rightarrow U$ as well as the first three Laurent coefficients of $\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{SL_2}}(t)$ at $t = 1$.

We use the following notation from [13]. Let $W = \bigoplus_{k=1}^r V_{d_k}$ indicate the decomposition of W into irreducible representations, where V_d denotes the unique irreducible representation of SL_2 of dimension $d + 1$, given by binary forms of degree d . Let $D = \dim_{\mathbb{C}} W = r + \sum_{k=1}^r d_k$. Define

$$\Theta := \{(k, i) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq k \leq r, 0 \leq i \leq d_k\},$$

and for each $(k, i) \in \Theta$, let $a_{k,i} := 2i - d_k$. Set $\mathbf{a}_{\Theta} := (a_{k,i} : (k, i) \in \Theta)$, and then the coordinates of \mathbf{a}_{Θ} are the weights of the representation W . Similarly, let

$$\Lambda := \{(k, i) \in \mathbb{Z} \times \mathbb{Z} : 1 \leq k \leq r, [d_k/2] + 1 \leq i \leq d_k\},$$

let $C := |\Lambda|$, and let $\mathbf{a} := (a_{k,i} : (k, i) \in \Lambda)$; then \mathbf{a} lists the positive weights. When the representation W is not clear from the context, we will use the notation Θ_W , Λ_W , \mathbf{a}_W , etc.

The Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{SL_2}}(t)$ can be expressed as an integral over the Cartan torus \mathbb{T} of SL_2 using the Molien-Weyl formula; see [9, 10]. That is,

$$\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{SL_2}}(t) = \int_{z \in \mathbb{T}} \frac{\chi_U(z) d\mu(z)}{\det_{W^*}(1 - zt)}, \quad |t| < 1,$$

where χ_U is the character associated to U and $\mu(z)$ is a Haar measure on SU_2 such that $\int_{\mathbb{T}} d\mu = 1$. By a minor modification of the proof to the Molien-Weyl formula, the multivariate Hilbert series can be expressed

as

$$\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{\text{SL}_2}}(t_1, \dots, t_r) = \int_{z \in \mathbb{T}} \frac{\chi_U(z) d\mu(z)}{\prod_{k=1}^r \det_{V_k^*}(1 - zt_k)}, \quad |t_k| < 1.$$

This has been used in [41, Equation (13)] for finite groups and [20, Section IV] for the decomposition of a real representation into holomorphic and antiholomorphic parts. It is an immediate consequence that if $U = U_1 \oplus U_2$, we have

$$\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{\text{SL}_2}}(t) = \text{Hilb}_{(\mathbb{C}[W] \otimes U_1)^{\text{SL}_2}}(t) + \text{Hilb}_{(\mathbb{C}[W] \otimes U_2)^{\text{SL}_2}}(t),$$

and the same holds for the multivariate Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{\text{SL}_2}}(t_1, \dots, t_r)$; for this reason, we restrict our attention to the case $U = V_L$ is irreducible with no loss of generality.

Identifying \mathbb{T} with the circle $\mathbb{S}^1 \subset \mathbb{C}$, we have

$$d\mu = \frac{(1 - z^2) dz}{2\pi\sqrt{-1}}.$$

Then the character $\chi_{V_L}(z)$ is given by

$$\chi_{V_L}(z) = z^L + z^{L-2} + \dots + z^{2-L} + z^{-L},$$

so that, noting that $W = W^*$ in this case, we have

$$\int_{z \in \mathbb{S}^1} \frac{\chi_{V_L}(z) d\mu(z)}{\det_W(1 - zt)} = \frac{1}{2\pi\sqrt{-1}} \int_{z \in \mathbb{S}^1} \frac{z^{-L} - z^{L+2}}{z \det_W(1 - zt)} dz.$$

We define

$$\Upsilon_{W,\ell}(t) := \frac{1}{2\pi\sqrt{-1}} \int_{z \in \mathbb{S}^1} \frac{z^\ell dz}{z \det_W(1 - zt)}$$

and then $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t) = \Upsilon_{W,-L}(t) - \Upsilon_{W,L+2}(t)$; similarly,

$$\Upsilon_{W,\ell}(t_1, \dots, t_r) := \frac{1}{2\pi\sqrt{-1}} \int_{z \in \mathbb{S}^1} \frac{z^\ell dz}{z \prod_{k=1}^r \det_{V_k^*}(1 - zt_k)},$$

and $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t_1, \dots, t_r) = \Upsilon_{W,-L}(t_1, \dots, t_r) - \Upsilon_{W,L+2}(t_1, \dots, t_r)$. Using the notation established above, we can express

$$(3.1) \quad \Upsilon_{W,\ell}(t) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{S}^1} \frac{z^{\ell-1} dz}{\prod_{d_k \in 2\mathbb{Z}} (1-t) \prod_{(k,i) \in \Lambda} (1-tz^{-a_{k,i}})(1-tz^{a_{k,i}})}$$

and

$$(3.2) \quad \Upsilon_{W,\ell}(t_1, \dots, t_r) = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{S}^1} \frac{z^{\ell-1} dz}{\prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{(k,i) \in \Lambda} (1-t_k z^{-a_{k,i}})(1-t_k z^{a_{k,i}})}.$$

To compute the Hilbert series, we will evaluate the integrals in Equations (3.1) and (3.2) using the methods of [16, Section 4.6.1 and 4.6.4]. This was accomplished in [13] for invariants, i.e. the case $V_L = \mathbb{C}$ is trivial. For arbitrary V_L , we define the quantity

$$(3.3) \quad \nu_{W,L} := L + 1 - \sum_{(k,i) \in \Lambda} a_{k,i}.$$

When $\nu_{W,L} \leq 0$, the computation of $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ follows that of $\text{Hilb}_{\mathbb{C}[W]^{\text{SL}_2}}(t)$ in [13] with little change, while a new residue appears when $\nu_{W,L} > 0$.

We note that the Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{\text{SL}_2}}(t_1, \dots, t_r)$ and $\text{Hilb}_{(\mathbb{C}[W] \otimes U)^{\text{SL}_2}}(t)$ could also be computed using the Clebsch-Gordan decomposition of $\mathbb{C}[W] \otimes U$ and the results of [13]. We found the approach outlined and implemented here to be the most direct.

3.1. The Multivariate Hilbert Series. Our first goal is the computation of the Γ -graded Hilbert Series $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{SL_2}}(t_1, \dots, t_r)$ of the covariants $W \rightarrow V_L$ by evaluating the integral in Equation (3.2). To understand the integrand at $z = 0$, we rewrite it as

$$\frac{z^{\ell-1+\sum_{(k,i) \in \Lambda} a_{k,i}}}{\prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{(k,i) \in \Lambda} (z^{a_{k,i}} - t_k)(1-t_k z^{a_{k,i}})}.$$

Recalling that $a_{k,i} > 0$ for each $(k,i) \in \Lambda$, it is clear that if $\ell = L + 2$, then this expression is holomorphic at $z = 0$. Similarly, if $\ell = -L$, then the numerator is equal to

$$z^{-L-1+\sum_{(k,i) \in \Lambda} a_{k,i}} = z^{-\nu_{W,L}},$$

so the integrand is holomorphic at $z = 0$ if and only if $\nu_{W,L} \leq 0$.

Assume each t_k is fixed with $|t_k| < 1$. As each $a_{k,i} > 0$, the poles inside the unit disk away from $z = 0$ occur when the factors $1 - t_k z^{-a_{k,i}}$ vanish. Hence, the poles are at points of the form $z = \zeta t_k^{1/a_{k,i}}$ where ζ is an $a_{k,i}$ th root of unity and $t_k^{1/a_{k,i}}$ is defined using a fixed branch of the logarithm whose domain includes each of the finitely many points t_k to which it is applied.

Fix a $(K, I) \in \Lambda$ and an $a_{K,I}$ th root of unity ζ_0 , and then we express

$$\begin{aligned} & \frac{z^{\ell-1}}{\prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{(k,i) \in \Lambda} (1-t_k z^{-a_{k,i}})(1-t_k z^{a_{k,i}})} \\ &= \frac{z^{a_{K,I}+\ell-1}}{(z^{a_{K,I}} - t_K)(1-t_K z^{a_{K,I}}) \prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1-t_k z^{-a_{k,i}})(1-t_k z^{a_{k,i}})} \\ &= \frac{z^{a_{K,I}+\ell-1}}{(1-t_K z^{a_{K,I}})(z - \zeta_0 t_K^{1/a_{K,I}}) \prod_{\substack{\zeta^{a_{K,I}}=1 \\ \zeta \neq \zeta_0}} (z - \zeta t_K^{1/a_{K,I}}) \prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1-t_k z^{-a_{k,i}})(1-t_k z^{a_{k,i}})}. \end{aligned}$$

Hence, we have a simple pole at $z = \tau := \zeta_0 t_K^{1/a_{K,I}}$, and the residue at $z = \tau$ is given by

$$\begin{aligned} & \frac{\tau^{a_{K,I}+\ell-1}}{(1-t_K \tau^{a_{K,I}}) \prod_{\substack{\zeta^{a_{K,I}}=1 \\ \zeta \neq \zeta_0}} (\tau - \zeta t_K^{1/a_{K,I}}) \prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1-t_k \tau^{-a_{k,i}})(1-t_k \tau^{a_{k,i}})} \\ &= \frac{\tau^{a_{K,I}+\ell-1}}{(1-t_K^2) \tau^{a_{K,I}-1} \prod_{\substack{\zeta^{a_{K,I}}=1 \\ \zeta \neq 1}} (1-\zeta) \prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1-t_k \tau^{-a_{k,i}})(1-t_k \tau^{a_{k,i}})} \\ &= \frac{\zeta_0^\ell t_K^{\ell/a_{K,I}}}{a_{K,I} (1-t_K^2) \prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1-\zeta_0^{-a_{k,i}} t_k t_K^{-a_{k,i}/a_{K,I}})(1-\zeta_0^{a_{k,i}} t_k t_K^{a_{k,i}/a_{K,I}})}. \end{aligned}$$

Summing over each choice of (K, I) and ζ_0 , we have that when $\nu_{W,L} \leq 0$,

$$\Upsilon_{W,\ell}(t_1, \dots, t_k) = \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} \frac{\zeta^\ell t_K^{\ell/a_{K,I}}}{a_{K,I} (1-t_K^2) \prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1-\zeta^{-a_{k,i}} t_k t_K^{-a_{k,i}/a_{K,I}})(1-\zeta^{a_{k,i}} t_k t_K^{a_{k,i}/a_{K,I}})}.$$

When $\nu_{W,\ell} > 0$, we have in addition a pole at $z = 0$ in the integrand of $\Upsilon_{W,-L}(t_1, \dots, t_r)$,

$$\frac{1}{z^{\nu_{W,L}} \prod_{d_k \in 2\mathbb{Z}} (1-t_k) \prod_{(k,i) \in \Lambda} (z^{a_{k,i}} - t_k)(1-t_k z^{a_{k,i}})}.$$

The residue at $z = 0$ is given by

$$\frac{1}{(\nu_{W,\ell} - 1)! \prod_{d_k \in 2\mathbb{Z}} (1 - t_k)} \frac{d^{\nu_{W,\ell} - 1}}{dz^{\nu_{W,\ell} - 1}} \left(\frac{1}{\prod_{(k,i) \in \Lambda} (z^{a_{k,i}} - t_k)(1 - t_k z^{a_{k,i}})} \right) \Big|_{z=0}.$$

Using the Cauchy product formula for the Maclaurin series of $1/(\prod_{(k,i) \in \Lambda} (z^{a_{k,i}} - t_k)(1 - t_k z^{a_{k,i}}))$ and the series expansions

$$\frac{1}{1 - t_k z^{a_{k,i}}} = \sum_{j=0}^{\infty} t_k^j z^{j a_{k,i}} \quad \text{and} \quad \frac{1}{z^{a_{k,i}} - t_k} = \sum_{j=0}^{\infty} -\frac{z^{j a_{k,i}}}{t_k^{j+1}},$$

the residue is given by summing products of the form $\prod_{(k,i) \in \Lambda} -t_k^{I_{k,i}}/t_k^{J_{k,i}+1}$ where the $I_{k,i} + J_{k,i}$ are coefficients of a linear combination of the $a_{k,i}$ that is equal to $\nu_{W,\ell} - 1$ and the sum is over all such linear combinations. In other words, we can express this residue as

$$\frac{1}{\prod_{d_k \in 2\mathbb{Z}} (1 - t_k)} \sum_S \prod_{(k,i) \in \Lambda} \left(t_k^{I_{k,i}} \left(-\frac{1}{t_k^{J_{k,i}+1}} \right) \right) = \frac{(-1)^C}{\prod_{d_k \in 2\mathbb{Z}} (1 - t_k)} \sum_S \prod_{(k,i) \in \Lambda} t_k^{I_{k,i} - J_{k,i} - 1}$$

where S is the set of nonnegative integer solutions $(I_{k,i}, J_{k,i})_{(k,i) \in \Lambda}$ of the equation $\sum_{(k,i) \in \Lambda} (I_{k,i} + J_{k,i}) a_{k,i} = \nu_{W,\ell} - 1$, i.e.

$$(3.4) \quad S = \left\{ (I_{k,i}, J_{k,i})_{(k,i) \in \Lambda} \in \mathbb{Z}_{\geq 0}^{2C} : \sum_{(k,i) \in \Lambda} (I_{k,i} + J_{k,i}) a_{k,i} = \nu_{W,\ell} - 1 \right\}.$$

Combining the above observations and recalling that the Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes_{\mathbb{V}_L})^{\text{SL}_2}}(t_1, \dots, t_r)$ of the module of covariants is given by $\Upsilon_{W,-L}(t_1, \dots, t_k) - \Upsilon_{W,L+2}(t_1, \dots, t_k)$, we have the following.

Proposition 3.1. *Let $W = \bigoplus_{k=1}^r V_{d_k}$ be an SL_2 -representation with $W^{\text{SL}_2} = \{0\}$ and let $L \in \mathbb{Z}^+$. If $\nu_{W,L} \leq 0$, then the Γ -graded Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes_{\mathbb{V}_L})^{\text{SL}_2}}(t_1, \dots, t_r)$ of the module of covariants $W \rightarrow V_L$ is given by*

$$(3.5) \quad \begin{aligned} & \Upsilon_{W,-L}(t) - \Upsilon_{W,L+2}(t) \\ &= \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}} = 1} \frac{\zeta^{-L} t_K^{-L/a_{K,I}} - \zeta^{L+2} t_K^{(L+2)/a_{K,I}}}{a_{K,I} (1 - t_K^2) \prod_{d_k \in 2\mathbb{Z}} (1 - t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t_k t_K^{-a_{k,i}/a_{K,I}}) (1 - \zeta^{a_{k,i}} t_k t_K^{a_{k,i}/a_{K,I}})}. \end{aligned}$$

If $\nu_{W,L} > 0$, the Hilbert series is given by

$$(3.6) \quad \begin{aligned} & \Upsilon_{W,-L}(t) - \Upsilon_{W,L+2}(t) \\ &= \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}} = 1} \frac{\zeta^{-L} t_K^{-L/a_{K,I}} - \zeta^{L+2} t_K^{(L+2)/a_{K,I}}}{a_{K,I} (1 - t_K^2) \prod_{d_k \in 2\mathbb{Z}} (1 - t_k) \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t_k t_K^{-a_{k,i}/a_{K,I}}) (1 - \zeta^{a_{k,i}} t_k t_K^{a_{k,i}/a_{K,I}})} \\ & \quad - \frac{(-1)^C}{\prod_{d_k \in 2\mathbb{Z}} (1 - t_k)} \sum_S \prod_{(k,i) \in \Lambda} t_k^{I_{k,i} - J_{k,i} - 1}, \end{aligned}$$

where S is defined in Equation (3.4).

Remark 3.2. Using the complete list Θ of weights and approximating \mathbf{a}_{Θ} with real parameters $\mathbf{b}_{\Theta} := (b_{k,i} : (k,i) \in \Theta)$, one can express Equation (3.5) (and hence the first line of Equation (3.6)) more succinctly as

$$\sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}} = 1} \frac{\zeta^{-L} t_K^{-L/a_{K,I}} - \zeta^{L+2} t_K^{(L+2)/a_{K,I}}}{a_{K,I} \prod_{\substack{(k,i) \in \Theta \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t_k t_K^{-a_{k,i}/a_{K,I}})}.$$

Remark 3.3. Let $W = \bigoplus_{k=1}^r V_{d_k}$ be a unitary SL_2 -representation with $W^{SL_2} = \{0\}$ and let $L \in \mathbb{Z}^+$. If each d_k is even and L is odd, then by the Clebsch-Gordan decomposition, the SL_2 -representation $\mathbb{C}[W] \otimes V_L$ has no invariants, and hence $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{SL_2}}(t_1, \dots, t_r) = 0$. This can also easily be concluded from Equations (3.5) and (3.6) by noting that in this case, the terms in the sum over ζ in Equation (3.5) occur in positive and negative pairs, and by a parity argument, the set S in Equation (3.6) is empty.

3.2. The Univariate Hilbert Series. The univariate Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{SL_2}}(t)$ is evidently given by the substitution $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{SL_2}}(t) = \text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{SL_2}}(t, \dots, t)$ in Equations (3.5) and (3.6). However, in Equations (3.5), this yields

$$\sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} \frac{\zeta^{-L} t^{-L/a_{K,I}} - \zeta^{L+2} t^{(L+2)/a_{K,I}}}{a_{K,I} (1-t^2) (1-t)^e \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t^{(a_{K,I} - a_{k,i})/a_{K,I}}) (1 - \zeta^{a_{k,i}} t^{(a_{K,I} + a_{k,i})/a_{K,I}})},$$

where e denotes the number of even d_k . When two of the d_k have the same parity, this expression fails to be defined; for example, for terms where $\zeta = 1$ and $a_{K,I} = a_{k,i}$, we have $1 - \zeta^{-a_{k,i}} t^{(a_{K,I} - a_{k,i})/a_{K,I}} = 0$. However, it was explained in [13, Section 3.2] for the case $L = 0$ that the singularities that occur in these degenerate cases are removable; an identical procedure applies in this case and was also applied in [29, Section 3.2] and [14, Theorem 3.3]. Note that no such singularities arise in the additional terms in Equation (3.6).

Specifically, suppose a is a positive value of $a_{i,j}$ that occurs N times, and let $\Lambda^a := \{(k,i) \in \Lambda \mid a_{k,i} \neq a\}$. We choose x_j for $j = 1, \dots, N$ and $x_{k,i}$ for $(k,i) \in \Lambda^a$ to be distinct elements of the interior of the unit disk. One then considers the integral

$$\frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{S}^1} \frac{z^{\ell-1} dz}{(1-t)^e \prod_{j=1}^N (1-x_j z^{-a})(1-x_j z^a) \prod_{(k,i) \in \Lambda^a} (1-t x_{k,i}^{-a_{k,i}})(1-x_{k,i} z^{a_{k,i}})},$$

which can be computed as in Subsection 3.1. The singularities appear in the form of factors $x_p - x_q$ in the denominator, and the sum of residues for a fixed ζ is divisible by the full Vandermonde determinant $\prod_{1 \leq p < q \leq N} x_p - x_q$. It is then easy to verify that the numerator is an alternating polynomial in the x_j , implying that these singularities are removable. Therefore, approximating the $a_{k,i}$ with distinct real parameters $b_{k,i}$ and letting $\mathbf{b} := (b_{k,i} : (k,i) \in \Lambda)$, we can express $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{SL_2}}(t)$ as the limit $\mathbf{b} \rightarrow \mathbf{a}$ as follows.

Theorem 3.4. *Let $W = \bigoplus_{k=1}^r V_{d_k}$ be a unitary SL_2 -representation with $W^{SL_2} = \{0\}$ and let $L \in \mathbb{Z}^+$. If $\nu_{W,L} \leq 0$, then the Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{SL_2}}(t)$ of the module of covariants $W \rightarrow V_L$ is given by*

$$(3.7) \quad \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} \frac{\zeta^{-L} t^{-L/b_{K,I}} - \zeta^{L+2} t^{(L+2)/b_{K,I}}}{b_{K,I} (1-t^2) (1-t)^e \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t^{(b_{K,I} - b_{k,i})/b_{K,I}}) (1 - \zeta^{a_{k,i}} t^{(b_{K,I} + b_{k,i})/b_{K,I}})}.$$

If $\nu_{W,L} > 0$, the Hilbert series is given by

$$(3.8) \quad \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} \frac{\zeta^{-L} t^{-L/b_{K,I}} - \zeta^{L+2} t^{(L+2)/b_{K,I}}}{b_{K,I} (1-t^2) (1-t)^e \prod_{\substack{(k,i) \in \Lambda \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t^{(b_{K,I} - b_{k,i})/b_{K,I}}) (1 - \zeta^{a_{k,i}} t^{(b_{K,I} + b_{k,i})/b_{K,I}})} \\ + \frac{(-1)^C}{(1-t)^e} \sum_S \prod_{(k,i) \in \Lambda} t^{I_{k,i} - J_{k,i} - 1},$$

where S is defined in Equation (3.4).

Remark 3.5. As in the case of Remark 3.2, we can express Equation (3.7) and the first line of Equation (3.8) more succinctly as

$$\lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} \frac{\zeta^{-L} t^{-L/b_{K,I}} - \zeta^{L+2} t^{(L+2)/b_{K,I}}}{b_{K,I} \prod_{\substack{(k,i) \in \Theta \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t^{(b_{K,I} - b_{k,i})/b_{K,I}})}.$$

As we will see below, expressing the Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ as a limit as in Theorem 3.4 will allow us to compute the first few Laurent coefficients in general. Moreover, the expressions in Theorem 3.4 can be used to determine an algorithm to compute the Hilbert series. In the *generic* case, i.e. when the weights $a_{k,i}$ are distinct (which implies that $r \leq 2$ and, when $r = 2$, the d_i have opposite parities), the limit is unnecessary. The algorithm is accomplished by using the operator U_a on formal power series that assigns to $F(t) = \sum_{i=0}^{\infty} F_i t^i$ the series

$$(U_a F)(t) := \sum_{i=0}^{\infty} F_i a^i t^i.$$

When $F(t)$ is the power series of a rational function, it is easy to see that

$$(U_a F)(t) = \frac{1}{a} \sum_{\zeta^a=1} F(\zeta \sqrt[a]{t}),$$

so that U_a can be used to compute the sums over roots of unity in Equations (3.7) and (3.8). In the *degenerate* case, where the $a_{k,i}$ are not distinct and so that the limit is required, a partial fraction decomposition can be used to remove the singularities before applying a similar procedure.

This algorithm was described in detail for the case $L = 0$ in [13, Section 6]. The case of Equation (3.7) has only slight modifications, so we refer the reader to that reference for more details. For the case of Equation (3.8), one need only compute the second line, the residue at $z = 0$, which is easy to implement directly. We have implemented this algorithm on *Mathematica* [42], and it is available from the authors upon request.

3.3. The Laurent Coefficients of the Hilbert series. In this section, we compute the first three Laurent coefficients of the Hilbert series $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ of covariants. First, let us establish some notation. Recall that D denotes the dimension of W , and $\mathbb{C}[W]^{\text{SL}_2}$ has Krull dimension $3 - D$ unless (W, SL_2) fails to be 1-large, i.e. unless W is isomorphic to V_1 , $2V_1$, or V_2 , see [38, Remark 9.2(3)] and [26, Theorem 3.4]. We use $\gamma_{m,L}(W)$ for the Laurent coefficients, so that

$$\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t) = \sum_{m=0}^{\infty} \gamma_{m,L}(W) (1-t)^{m - \dim \mathbb{C}[W]^{\text{SL}_2}}.$$

We will often abbreviate $\gamma_{m,L}(W)$ as $\gamma_{m,L}$ when W is clear from the context. Following [13], we let $\gamma_m(W) := \gamma_{m,0}(W)$ denote the Laurent coefficients of the invariants. Note in particular that we index the $\gamma_{m,L}(W)$ to match the degrees of the $\gamma_m(W)$, i.e. $\gamma_{m,L}(W)$ will denote the degree $3 - D + m$ coefficient even if this implies $\gamma_{0,L}(W) = 0$. We will see below that the pole order of $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ is bounded by that of $\mathbb{C}[W]^{\text{SL}_2}$, which was also observed in [9].

In order to take advantage of the computations of the γ_m in [13], we use a slightly different formulation of the Hilbert series than that given in Theorem 3.4. Define the function

$$H_{W,K,I,\zeta}^{\ell}(\mathbf{b}_{\Theta}, t) = \frac{t^{\ell/b_{K,I}} \zeta^{\ell} (1 - \zeta^2 t^{2/b_{K,I}})}{b_{K,I} \prod_{\substack{(k,i) \in \Theta \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t^{(b_{K,I} - b_{k,i})/b_{K,I}})}.$$

By telescoping the numerator in the sum over ℓ , it is easy to see that when $\nu_{W,L} \leq 0$, the Hilbert series is given by

$$\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t) = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} H_{W,K,I,\zeta}^{2\ell-L}(\mathbf{b}_{\Theta}, t).$$

Note that $H_{W,K,I,\zeta}^0(\mathbf{b}_{\Theta}, t)$ was denoted $H_{W,K,I,\zeta}(\mathbf{b}_{\Theta}, t)$ in [13]. Using this notation,

$$H_{W,K,I,\zeta}^{\ell}(\mathbf{b}_{\Theta}, t) = t^{\ell/b_{K,I}} \zeta^{\ell} H_{W,K,I,\zeta}(\mathbf{b}_{\Theta}, t),$$

so that we can express the Hilbert series as

$$(3.9) \quad \text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t) = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} t^{(2\ell-L)/b_{K,I}} \sum_{\zeta^{a_{K,I}}=1} \zeta^{2\ell-L} H_{W,K,I,\zeta}(\mathbf{b}_{\Theta}, t).$$

In particular, note that as $t^{(2\ell-L)/b_{K,I}}$ is holomorphic at $t = 1$, the pole orders of the covariants are bounded by the pole orders of the invariants as noted above. Let $\gamma_m(W, K, I, \zeta)$ denote the contribution to γ_m of the term $H_{W,K,I,\zeta}(\mathbf{b}_\Theta, t)$, i.e. the coefficient of degree $3 - D + m$.

Remark 3.6. The computation of the $\gamma_{m,L}$ for $m \leq 2$ can be treated uniformly except for a handful of low-dimensional W , which can easily be computed individually using the algorithm described in Section 3.2. For the representations V_1 and V_2 , the pole order at $t = 1$ is not equal to $D - 3$; see [13, Table 1]. Other exceptions arise because a term $H_{W,K,I,\zeta}^\ell(\mathbf{b}_\Theta, t)$ with $\zeta \neq \pm 1$ contributes to $\gamma_{m,L}$ for $m \leq 2$, which only occurs in small dimensions, or because the terms $H_{W,K,I,\zeta}^\ell(\mathbf{b}_\Theta, t)$ fail to have enough factors in the denominator for the general arguments to apply. We refer the reader to [13, Section 4.1] for a careful discussion of the reason for each exception; as we use many of the computations from that reference, the reasoning remains the same. Here, we only summarize that the exceptions for the computation of $\gamma_{0,L}$ are V_d for $d = 1, 2, 3, 4$ and $2V_1$; the exceptions for the computation of $\gamma_{1,L}$ are V_d for $d = 1, 2, 3, 4, 2V_1, V_1 \oplus V_2$, and $2V_2$ (see below); and the exceptions for the computation of $\gamma_{2,L}$ are V_d for $d = 1, 2, 3, 4, 5, 6, 8, 2V_1, V_1 \oplus V_2, V_1 \oplus V_3, V_1 \oplus V_4, 2V_2, V_2 \oplus V_3, V_2 \oplus V_4, 2V_3$, and $2V_4$. For our primary interest in computing the first Laurent coefficient of $\text{Hilb}_V^{on}(t)$, only five of these exceptions are relevant and given in Table 1; see Section 4.

We may also ignore the additional sum that appears in Equation (3.8). Note that this sum has a pole at $t = 1$ of order of e , and hence contributes to $\gamma_{m,L}$ if and only if $D \leq e + m + 3$. For $m \leq 2$, one easily checks that this sum contributes to $\gamma_{m,L}$ only in cases on the list of exceptions for $\gamma_{m,L}$ above; this is the reason we included $2V_2$ on the exception list for $\gamma_{1,L}$, which was not an exception for γ_1 in [13, Theorem 1.1].

Using the series expansion

$$\begin{aligned} t^{(2\ell-L)/b_{K,I}} &= 1 + \frac{L-2\ell}{b_{K,I}}(1-t) + \frac{(L-2\ell)(b_{K,I}+L-2\ell)}{2b_{K,I}^2}(1-t)^2 \\ &\quad + \frac{(L-2\ell)(b_{K,I}+L-2\ell)(2b_{K,I}+L-2\ell)}{6b_{K,I}^3}(1-t)^3 \\ &\quad + \frac{(L-2\ell)(b_{K,I}+L-2\ell)(2b_{K,I}+L-2\ell)(3b_{K,I}+L-2\ell)}{24b_{K,I}^4}(1-t)^4 \\ &\quad + \frac{(L-2\ell)(b_{K,I}+L-2\ell)(2b_{K,I}+L-2\ell)(3b_{K,I}+L-2\ell)(4b_{K,I}+L-2\ell)}{120b_{K,I}^5}(1-t)^5 + \dots \end{aligned}$$

and Equation (3.9), we can express $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ as the product

$$(3.10) \quad \begin{aligned} \text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t) &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \left(1 + \frac{L-2\ell}{b_{K,I}}(1-t) + \frac{(L-2\ell)(b_{K,I}+L-2\ell)}{2b_{K,I}^2}(1-t)^2 + \dots \right) \\ &\quad \sum_{\zeta^{\alpha_{K,I}}=1} \zeta^{2\ell-L} \left(\gamma_0(W, K, I, \zeta)(1-t)^{3-D} + \gamma_1(W, K, I, \zeta)(1-t)^{4-D} + \gamma_2(W, K, I, \zeta)(1-t)^{5-D} + \dots \right). \end{aligned}$$

We will use this expression to compute the Laurent coefficients of $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ and express them in terms of Schur polynomials.

For an integer partition $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{Z}^n$ with $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n \geq 0$, let $s_\rho(\mathbf{x})$ denote the corresponding Schur polynomial in the variables $\mathbf{x} = (x_1, \dots, x_n)$, i.e.

$$s_\rho(\mathbf{x}) = \frac{\det \left(x_i^{\rho_j + n - i} \right)}{\det \left(x_i^{n - i} \right)}.$$

See [34, I.3] or [36, Section 4.6] for more details. Note that we will sometimes for convenience refer to $s_\rho(\mathbf{x})$ where the condition $\rho_1 \geq \rho_2 \geq \dots \geq \rho_n \geq 0$ does not hold; these are defined in the same way but may yield Laurent-Schur polynomials or vanish. We will often use the shorthand $s_m(\mathbf{x})$ to denote the Schur polynomial $s_{m, n-2, n-3, \dots, 1, 0}(\mathbf{x})$.

Theorem 3.7. *Let $W = \bigoplus_{k=1}^r V_{d_k}$ be a unitary SL_2 -representation with $W^{\text{SL}_2} = \{0\}$ and let $L \in \mathbb{Z}^+$, and assume that W is not isomorphic to V_d for $d \leq 4$ nor $2V_1$. If at least one d_k is odd, the degree $3 - D$*

coefficient $\gamma_{0,L}$ of the Laurent series of $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ is given by

$$\gamma_{0,L} = (L+1)\gamma_0 = \frac{(L+1)s_\rho(\mathbf{a})}{s_\delta(\mathbf{a})},$$

where $\rho = (C-3, C-3, C-3, C-4, \dots, 1, 0)$ and $\delta = (C-1, C-2, \dots, 1, 0)$. If all d_k are even, then

$$\gamma_{0,L} = \frac{(1+(-1)^L)(L+1)\gamma_0}{2} = \frac{(1+(-1)^L)(L+1)s_\rho(\mathbf{a})}{s_\delta(\mathbf{a})}.$$

Note in particular that when all d_k are even and L is odd, $\gamma_{0,L} = 0$; in fact $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t) = 0$ in this case by Remark 3.3.

Proof. From Equation (3.9), we have

$$\gamma_{0,L} = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} \zeta^{2\ell-L} \gamma_0(W, K, I, \zeta).$$

As explained in [13, Section 4.1], excluding the noted exceptions, if any d_k are odd, then $\gamma_0(W, K, I, \zeta) = 0$ unless $\zeta = 1$, so that

$$\gamma_0 = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \gamma_0(W, K, I, 1).$$

If all d_k are even, then $\gamma_0(W, K, I, -1) = \gamma_0(W, K, I, 1)$ and all $\gamma_0(W, K, I, \zeta) = 0$ for $\zeta \neq \pm 1$, i.e.

$$\gamma_0 = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} (\gamma_0(W, K, I, 1) + \gamma_0(W, K, I, -1)) = \lim_{\mathbf{b} \rightarrow \mathbf{a}} 2 \sum_{(K,I) \in \Lambda} \gamma_0(W, K, I, 1).$$

Hence, when at least one d_k is odd,

$$\gamma_{0,L} = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \gamma_0(W, K, I, 1) = \sum_{\ell=0}^L \gamma_0 = (L+1)\gamma_0.$$

When all d_k are even,

$$\begin{aligned} \gamma_{0,L} &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} (\gamma_0(W, K, I, 1) + (-1)^{2\ell-L} \gamma_0(W, K, I, -1)) \\ &= \frac{1}{2} \sum_{\ell=0}^L \gamma_0 + (-1)^L \gamma_0 = \frac{(1+(-1)^L)(L+1)\gamma_0}{2}. \end{aligned}$$

The expressions for γ_0 in terms of Schur polynomials are given in [13, Theorem 1.1], completing the proof. \square

Theorem 3.8. *Let $W = \bigoplus_{k=1}^r V_{d_k}$ be a unitary SL_2 -representation with $W^{\text{SL}_2} = \{0\}$ and $d_1 \leq d_2 \leq \dots \leq d_r$, and assume that W is not isomorphic to V_d for $d \leq 4$, $2V_1$, $V_1 \oplus V_2$, nor $2V_2$. Let $L \in \mathbb{Z}^+$. If at least one d_k is odd, the degree $4-D$ coefficient $\gamma_{1,L}$ of the Laurent series of $\text{Hilb}_{(\mathbb{C}[W] \otimes V_L)^{\text{SL}_2}}(t)$ is given by*

$$\gamma_{1,L} = (L+1)\gamma_1 = \frac{3(L+1)s_\rho(\mathbf{a})}{2s_\delta(\mathbf{a})},$$

where $\rho = (C-3, C-3, C-3, C-4, \dots, 1, 0)$ and $\delta = (C-1, C-2, \dots, 1, 0)$. If all d_k are even, then

$$\gamma_{1,L} = \frac{(1+(-1)^L)(L+1)\gamma_1}{2} = \frac{3(1+(-1)^L)(L+1)s_\rho(\mathbf{a})}{2s_\delta(\mathbf{a})}.$$

Proof. From Equation (3.10), we express

$$\gamma_{1,L} = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \sum_{\zeta^{a_{K,I}}=1} \zeta^{2\ell-L} \left(\frac{L-2\ell}{b_{K,I}} \gamma_0(W, K, I, \zeta) + \gamma_1(W, K, I, \zeta) \right).$$

We consider three cases.

Case I: Assume that at least two d_k are odd or one $d_k > 1$ is odd. Then for $m = 0, 1$, we have $\gamma_m(W, K, I, \zeta) = 0$ unless $\zeta = 1$ so that

$$\gamma_m = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K, I) \in \Lambda} \gamma_m(W, K, I, 1), \quad m = 0, 1.$$

Hence,

$$\begin{aligned} \gamma_{1,L} &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K, I) \in \Lambda} \left(\frac{L-2\ell}{b_{K,I}} \gamma_0(W, K, I, 1) + \gamma_1(W, K, I, 1) \right) \\ &= (L+1)\gamma_1 + \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K, I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}} \sum_{\ell=0}^L (L-2\ell) \\ &= (L+1)\gamma_1, \end{aligned}$$

as $\sum_{\ell=0}^L (L-2\ell) = 0$.

Case II: Assume that all d_k are even. Then for $m = 0, 1$, we have $\gamma_0(W, K, I, 1) = \gamma_0(W, K, I, -1)$ and all other $\gamma_0(W, K, I, \zeta) = 0$. Therefore,

$$\gamma_m = \lim_{\mathbf{b} \rightarrow \mathbf{a}} 2 \sum_{(K, I) \in \Lambda} \gamma_m(W, K, I, 1), \quad m = 0, 1,$$

and

$$\begin{aligned} \gamma_{1,L} &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K, I) \in \Lambda} (1 + (-1)^{2\ell-L}) \left(\frac{L-2\ell}{b_{K,I}} \gamma_0(W, K, I, 1) + \gamma_1(W, K, I, 1) \right) \\ &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} (1 + (-1)^{2\ell-L})(L+1) \sum_{(K, I) \in \Lambda} \gamma_1(W, K, I, 1) + \sum_{(K, I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}} \sum_{\ell=0}^L (L-2\ell) \\ &= \frac{(1 + (-1)^L)(L+1)\gamma_1}{2}, \end{aligned}$$

again as $\sum_{\ell=0}^L (L-2\ell) = 0$.

Case III: Assume $d_1 = 1$ and d_k is even for $k > 1$. Then $\gamma_0(W, K, I, \zeta) = 0$ for $\zeta \neq 1$, $\gamma_1(W, K, I, \zeta) = 0$ for $\zeta \neq \pm 1$, but $\gamma_1(W, K, I, 1)$ and $\gamma_1(W, K, I, -1)$ are not necessarily equal. We compute

$$\begin{aligned} \gamma_{1,L} &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K, I) \in \Lambda} \left(\frac{L-2\ell}{b_{K,I}} \gamma_0(W, K, I, 1) + \gamma_1(W, K, I, 1) + (-1)^{2\ell-L} \gamma_1(W, K, I, -1) \right) \\ &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \left(\sum_{(K, I) \in \Lambda} \gamma_1(W, K, I, 1) + (-1)^L \gamma_1(W, K, I, -1) \right) + \sum_{(K, I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}} \sum_{\ell=0}^L (L-2\ell) \\ &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} (L+1) \left(\sum_{(K, I) \in \Lambda} \gamma_1(W, K, I, 1) + (-1)^L \sum_{(K, I) \in \Lambda} \gamma_1(W, K, I, -1) \right). \end{aligned}$$

Now, in the proof of [13, Theorem 5.5], it is demonstrated that $\sum_{(K, I) \in \Lambda} \gamma_1(W, K, I, -1) = 0$, hence

$$\gamma_{1,L} = \lim_{\mathbf{b} \rightarrow \mathbf{a}} (L+1) \left(\sum_{(K, I) \in \Lambda} \gamma_1(W, K, I, 1) \right) = (L+1)\gamma_1. \quad \square$$

Theorem 3.9. Let $W = \bigoplus_{k=1}^r V_{d_k}$ be a unitary SL_2 -representation with $W^{SL_2} = \{0\}$ and $d_1 \leq d_2 \leq \dots \leq d_r$, and assume that W is not isomorphic to V_d for $d = 1, 2, 3, 4, 5, 6, 8, 2V_1, V_1 \oplus V_2, V_1 \oplus V_3, V_1 \oplus V_4, 2V_2, V_2 \oplus V_3, V_2 \oplus V_4, 2V_3$, nor $2V_4$. Let $L \in \mathbb{Z}^+$.

If at least two d_k are odd or one $d_k > 1$ is odd, the degree 5-D coefficient $\gamma_{2,L}$ of the Laurent series of $\text{Hilb}_{\mathbb{C}[W] \otimes_{V_L} SL_2}(t)$ is given by

$$\gamma_{2,L} = (L+1)\gamma_2 - \frac{L(L+1)(L+2)s_{\rho'}(\mathbf{a})}{6s_{\delta}(\mathbf{a})},$$

where $\rho' = (C-3, C-4, C-4, C-4, C-5, \dots, 1, 0)$ and $\delta = (C-1, C-2, \dots, 1, 0)$. If all d_k are even, then

$$\gamma_{2,L} = (1 + (-1)^L) \left(\frac{L+1}{2} \gamma_2 - \frac{L(L+1)(L+2)s_{\rho'}(\mathbf{a})}{6s_{\delta}(\mathbf{a})} \right).$$

If $d_1 = 1$ and each d_k is even for $k > 1$, then

$$\gamma_{2,L} = (L+1) \left(\frac{42s_{\rho}(\mathbf{a}) + s_{\rho'}(\mathbf{a})(P_2(\mathbf{a}) - 8 - 4L(L+2))}{24s_{\delta}(\mathbf{a})} + (-1)^L \frac{s_{C-4, C-4, C-4, C-5, \dots, 1, 0}(\mathbf{a}_1)}{4s_{C-2, C-3, C-4, \dots, 1, 0}(\mathbf{a}_1)} \right),$$

where $\rho = (C-3, C-3, C-3, C-4, \dots, 1, 0)$, P_2 is the quadratic power sum, and \mathbf{a}_1 denotes \mathbf{a} with entry $a_{1,1}$ removed.

Proof. Again using Equation (3.10), we can express $\gamma_{2,L}$ as

$$\lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \sum_{\zeta^{\mathbf{a}_{K,I}} = 1} \zeta^{2\ell-L} \left(\frac{(L-2\ell)(b_{K,I} + L - 2\ell)}{2b_{K,I}^2} \gamma_0(W, K, I, \zeta) + \frac{L-2\ell}{b_{K,I}} \gamma_1(W, K, I, \zeta) + \gamma_2(W, K, I, \zeta) \right).$$

As explained in [13, Section 4.1], except for the listed exceptions, we have that $\gamma_0(W, K, I, \zeta) = \gamma_1(W, K, I, \zeta) = \gamma_2(W, K, I, \zeta) = 0$ unless $\zeta = \pm 1$. We consider three cases.

Case I: Assume that at least two d_k are odd or one $d_k > 1$ is odd. Then $\gamma_i(W, K, I, \zeta) = 0$ for $i = 0, 1, 2$ unless $\zeta = 1$, so that

$$\gamma_m = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \gamma_m(W, K, I, 1), \quad m = 0, 1, 2.$$

Therefore,

$$\begin{aligned} \gamma_{2,L} &= \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \left(\frac{(L-2\ell)(b_{K,I} + L - 2\ell)}{2b_{K,I}^2} \gamma_0(W, K, I, 1) + \frac{L-2\ell}{b_{K,I}} \gamma_1(W, K, I, 1) + \gamma_2(W, K, I, 1) \right) \\ &= (L+1)\gamma_2 + \frac{L(L+1)(L+2)}{6} \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}^2}. \end{aligned}$$

We are able to compute the sum over $(K, I) \in \Lambda$ using the computations of [13]. First, using Proposition 4.2 of that reference,

$$\sum_{(K,I) \in \Lambda} \gamma_0(W, K, I, 1) = \sum_{(K,I) \in \Lambda} \frac{b_{K,I}^{D-3} - 2b_{K,I}^{D-4} - \sum_{\substack{(\kappa, \lambda) \in \Theta \setminus \\ \{(K,I)\}}} b_{K,I}^{D-4} b_{\kappa, \lambda}}{\prod_{\substack{(k,i) \in \Theta \setminus \\ \{(K,I)\}}} (b_{K,I} - b_{k,i})}.$$

Dividing each term by $b_{K,I}^2$ and using the functions

$$\Sigma_{R,S}(\mathbf{b}_{\Theta}) := \sum_{(K,I) \in \Lambda} \sum_{\substack{(\kappa, \lambda) \in \Theta \setminus \\ \{(K,I)\}}} \frac{b_{K,I}^R b_{\kappa, \lambda}^S}{\prod_{\substack{(k,i) \in \Theta \setminus \\ \{(K,I)\}}} (b_{K,I} - b_{k,i})} \quad \text{and} \quad \Sigma_R(\mathbf{b}_{\Theta}) := \sum_{(K,I) \in \Lambda} \frac{b_{K,I}^R}{\prod_{\substack{(k,i) \in \Theta \setminus \\ \{(K,I)\}}} (b_{K,I} - b_{k,i})}$$

defined in [13, Section 5], we have

$$\sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}^2} = \Sigma_{D-5}(\mathbf{b}) - 2\Sigma_{D-6}(\mathbf{b}) - \Sigma_{D-6,1}(\mathbf{b}).$$

Then by [13, Lemma 5.2], specifically Equations (34) and (35), we have

$$\sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}^2} = \frac{2s_{D-e-C-5}(\mathbf{b}) - 2s_{D-e-C-6}(\mathbf{b}) - P_1(\mathbf{b}_{\Theta})s_{D-e-C-6}(\mathbf{b}) + s_{D-e-C-5}(\mathbf{b})}{s_{\delta}(\mathbf{b})},$$

where P_1 denotes the power sum of degree 1, and we recall the shorthand $s_m(\mathbf{b})$ denotes $s_{m,C-2,C-3,\dots,1,0}(\mathbf{b})$. As the entries of \mathbf{b}_Θ are either 0 or occur in positive and negative pairs, $P_1(\mathbf{b}_\Theta) = 0$; using this and the fact that $D - e - C = C$,

$$\sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}^2} = \frac{s_{C-5}(\mathbf{b}) - s_{C-6}(\mathbf{b})}{s_\delta(\mathbf{b})}.$$

The non-standard Schur polynomial $s_{C-5}(\mathbf{a})$ is defined in terms of the alternant associated to $\delta + (C - 5, C - 2, C - 3, \dots, 1, 0) = (2C - 6, 2C - 4, 2C - 6, \dots, 2, 0)$ and hence vanishes. Rewriting the non-standard Schur polynomial $s_{C-6}(\mathbf{b}) = s_{C-6,C-2,C-3,\dots,1,0}(\mathbf{b})$ in standard form by permuting columns yields $s_{C-6}(\mathbf{b}) = s_{C-3,C-4,C-4,C-4,C-5,\dots,1,0}(\mathbf{b})$. Applying these observations completes the proof in this case.

Case II: Assume that each d_k is even. Then for $i = 0, 1, 2$, $\gamma_i(W, K, I, \zeta) = 0$ unless $\zeta = \pm 1$, and $\gamma_i(W, K, I, 1) = \gamma_i(W, K, I, -1)$.

$$\gamma_m = \lim_{\mathbf{b} \rightarrow \mathbf{a}} 2 \sum_{(K,I) \in \Lambda} \gamma_m(W, K, I, 1), \quad m = 0, 1, 2.$$

Hence $\gamma_{2,L}$ is given by

$$\begin{aligned} & \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} (1 + (-1)^{2\ell-L}) \left(\frac{(L-2\ell)(b_{K,I} + L - 2\ell)}{2b_{K,I}^2} \gamma_0(W, K, I, 1) + \frac{L-2\ell}{b_{K,I}} \gamma_1(W, K, I, 1) + \gamma_2(W, K, I, 1) \right) \\ &= (1 + (-1)^L) \left(\frac{L+1}{2} \gamma_2 + \frac{L(L+1)(L+2)}{6} \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}^2} \right). \end{aligned}$$

This is identical to the previous case except for the $1 + (-1)^L$ prefactor and the change to the coefficient of γ_2 , but the remaining argument is identical.

Case III: Assume that $d_1 = 1$ and d_k is even for $k > 1$. Then $\gamma_0(W, K, I, \zeta) = 0$ for $\zeta \neq 1$, and for $m = 1, 2$, $\gamma_m(W, K, I, \zeta) = 0$ for $\zeta \neq \pm 1$, yet $\gamma_m(W, K, I, 1)$ and $\gamma_m(W, K, I, -1)$ are not necessarily equal. Hence we can express $\gamma_{2,L}$ as

$$\begin{aligned} & \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{\ell=0}^L \sum_{(K,I) \in \Lambda} \sum_{\zeta^a, I=1} \zeta^{2\ell-L} \left(\frac{(L-2\ell)(b_{K,I} + L - 2\ell)}{2b_{K,I}^2} \gamma_0(W, K, I, \zeta) + \frac{L-2\ell}{b_{K,I}} \gamma_1(W, K, I, \zeta) + \gamma_2(W, K, I, \zeta) \right) \\ &= (L+1) \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \left(\gamma_2(W, K, I, 1) + (-1)^L \gamma_2(W, K, I, -1) \right) \\ & \quad + \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \frac{1}{b_{K,I}} \left(\gamma_1(W, K, I, 1) + (-1)^L \gamma_1(W, K, I, -1) \right) \sum_{\ell=0}^L (L-2\ell) \\ & \quad + \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{2b_{K,I}^2} \sum_{\ell=0}^L (L-2\ell)(b_{K,I} + L - 2\ell) \\ &= (L+1) \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \left(\gamma_2(W, K, I, 1) + (-1)^L \gamma_2(W, K, I, -1) \right) + \frac{L(L+1)(L+2)}{6} \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}^2}. \end{aligned}$$

From the proof of [13, Theorem 5.8], we have

$$\sum_{(K,I) \in \Lambda} \gamma_2(W, K, I, 1) = \frac{42s_{C-3,C-3,C-3,C-4,\dots,1,0}(\mathbf{a}) + s_{C-3,C-4,C-4,C-4,C-5,\dots,1,0}(\mathbf{a})(P_2(\mathbf{a}) - 8)}{24s_\delta(\mathbf{a})},$$

and

$$\sum_{(K,I) \in \Lambda} \gamma_2(W, K, I, -1) = \frac{s_{C-4,C-4,C-4,C-5,\dots,1,0}(\mathbf{a}_1)}{4s_{C-2,C-3,C-4,\dots,1,0}(\mathbf{a}_1)},$$

and from the computations for Case I, we have

$$\sum_{(K,I) \in \Lambda} \frac{\gamma_0(W, K, I, 1)}{b_{K,I}^2} = -\frac{s_{C-3,C-4,C-4,C-4,C-5,\dots,1,0}(\mathbf{a})}{2s_\delta(\mathbf{a})}.$$

Combining these observations completes the proof. \square

4. HILBERT SERIES OF THE GRADED ALGEBRA OF REGULAR FUNCTIONS ON SU_2 -SYMPLECTIC QUOTIENTS

In this section, we use the computation of the Laurent coefficients of the Hilbert series $\text{Hilb}_{(\mathbb{C}[V \oplus V^*] \otimes V_2)^{SL_2}}(t)$ in the previous section, as well as the Laurent coefficients of $\text{Hilb}_{\mathbb{C}[V \oplus V^*]^{SL_2}}(t)$ computed in [13], to determine an explicit expression for the first nonzero Laurent coefficient $\gamma_0^{on}(V)$ of the Hilbert series $\text{Hilb}_V^{on}(t)$ of the graded algebra $\mathbb{R}[M_0]$ of regular functions on the symplectic quotient M_0 associated to V .

First, we note the following. If (V, SL_2) is 1-large, i.e. not isomorphic to V_1 , $2V_1$, nor $V_1 \oplus V_2$, then $\nu_{V,2} \leq 0$, see Equation (3.3). Hence, Proposition 2.1 and Theorem 3.4 yield the following.

Corollary 4.1. *Let V be a unitary SU_2 -representation with $V^{SU_2} = \{0\}$ and assume that V is not isomorphic to V_1 , $2V_1$ nor $V_1 \oplus V_2$. Then the on-shell Hilbert series of the graded algebra of regular functions on the symplectic quotient M_0 is given by*

$$(4.1) \quad \text{Hilb}_V^{on}(t) = \lim_{\mathbf{b} \rightarrow \mathbf{a}} \sum_{(K,I) \in \Lambda_{V \oplus V^*}} \sum_{\zeta^{a_{K,I}}=1} \frac{(1-t^6)(1-\zeta^2 t^{2/b_{K,I}}) + (t^4-t^2)(\zeta^{-2} t^{-2/b_{K,I}} - \zeta^4 t^{4/b_{K,I}})}{b_{K,I} \prod_{\substack{(k,i) \in \Theta_{V \oplus V^*} \setminus \\ \{(K,I)\}}} (1 - \zeta^{-a_{k,i}} t^{(b_{K,I} - b_{k,i})/b_{K,I}})},$$

where we recall that $\Theta_{V \oplus V^*}$ and $\Lambda_{V \oplus V^*}$ denote the sets of weights (respectively positive weights) for the cotangent lifted representation $V \oplus V^*$.

Using Corollary 4.1, the algorithm described in Section 3.2 to compute the Hilbert series $\text{Hilb}_{(\mathbb{C}[V] \otimes V_L)^{SL_2}}(t)$ of covariants can as well be applied to compute the Hilbert series $\text{Hilb}_V^{on}(t)$ of $\mathbb{R}[M_0]$. This amounts to applying the covariant algorithm twice to compute both $\text{Hilb}_{\mathbb{C}[V \oplus V^*]^{SL_2}}(t)$ and $\text{Hilb}_{(\mathbb{C}[V \oplus V^*] \otimes V_2)^{SL_2}}(t)$.

Now, recall from Section 3.3 that $\gamma_m(V \oplus V^*)$ denotes the Laurent coefficients of $\mathbb{C}[V \oplus V^*]^{SL_2}$ and $\gamma_{m,2}(V \oplus V^*)$ denotes the Laurent coefficients of the covariants $V \oplus V^* \rightarrow V_2$. Assume (V, SL_2) is 1-large so that $\dim(\mathbb{C}[V]^{SL_2}) = D - 3$, see [38, Remark (9.2)(3)]. Using the Kempf-Ness homeomorphism [37], the corresponding symplectic M_0 quotient has real dimension $2D - 6$. Note that the algebra of invariants $\mathbb{C}[V \oplus V^*]^{SL_2}$ has dimension $2D - 3$ so that $\gamma_0(V \oplus V^*)$ and $\gamma_{0,2}(V \oplus V^*)$ occur in degree $2D - 3$.

Using Proposition 2.1, the Laurent expansion of $\text{Hilb}_V^{on}(t)$ begins as given below, where each $\gamma_{m,L} = \gamma_{m,L}(V \oplus V^*)$ and $\gamma_m = \gamma_m(V \oplus V^*)$:

$$(4.2) \quad (6\gamma_0 - 2\gamma_{0,2})(1-t)^{4-2D} + (-15\gamma_0 + 5\gamma_{0,2} + 6\gamma_1 - 2\gamma_{1,2})(1-t)^{5-2D}$$

$$(4.3) \quad + (20\gamma_0 - 4\gamma_{0,2} - 15\gamma_1 + 5\gamma_{1,2} + 6\gamma_2 - 2\gamma_{2,2})(1-t)^{6-2D}$$

$$(4.4) \quad + (-15\gamma_0 + \gamma_{0,2} + 20\gamma_1 - 4\gamma_{1,2} - 15\gamma_2 + 5\gamma_{2,2} + 6\gamma_3 - 2\gamma_{3,2})(1-t)^{7-2D}$$

$$+ (6\gamma_0 - 15\gamma_1 + \gamma_{1,2} + 20\gamma_2 - 4\gamma_{2,2} - 15\gamma_3 + 5\gamma_{3,2} + 6\gamma_4 - 2\gamma_{4,2})(1-t)^{8-2D} + \dots$$

As the pole order of $\text{Hilb}_V^{on}(t)$ is equal to the dimension of M_0 , the first two of these coefficients (4.2) are zero. One also checks using the expressions in Theorems 3.7 and 3.8 that this is the case, i.e. that

$$6\gamma_0(V \oplus V^*) - 2\gamma_{0,2}(V \oplus V^*) = -15\gamma_0(V \oplus V^*) + 5\gamma_{0,2}(V \oplus V^*) + 6\gamma_1(V \oplus V^*) - 2\gamma_{1,2}(V \oplus V^*) = 0.$$

Let $\sigma_V = 2$ if each d_k is even and 1 otherwise, and note that $\sigma_V = \sigma_{V \oplus V^*}$. Recall that $\mathbf{a}_{V \oplus V^*}$ denotes the vector of positive weights of the representation $V \oplus V^*$ so that $\mathbf{a}_{V \oplus V^*}$ is two copies of \mathbf{a}_V concatenated. Let $\hat{\delta} = (2C - 1, 2C - 2, \dots, 1, 0)$, $\hat{\rho} = (2C - 3, 2C - 3, 2C - 3, 2C - 4, \dots, 1, 0)$, and $\hat{\rho}' = (2C - 3, 2C - 4, 2C - 4, 2C - 4, 2C - 5, \dots, 1, 0)$, i.e. the respective partitions δ , ρ , and ρ' used in Section 3 corresponding to the representation $V \oplus V^*$. We have the following.

Theorem 4.2. *Let $V = \bigoplus_{k=1}^r V_{d_k}$ be a unitary SU_2 -representation with $V^{SU_2} = \{0\}$ and assume that V is not isomorphic to V_d for $d = 1, 2, 3, 4$ nor $V_1 \oplus V_2$. Then the first nonzero Laurent coefficient $\gamma_0^{on}(V)$ of the Hilbert series $\text{Hilb}_V^{on}(t)$ of the graded algebra $\mathbb{R}[M_0]$ of regular functions on the symplectic quotient M_0 is given by*

$$(4.5) \quad \gamma_0^{on}(V) = 8\gamma_0(V \oplus V^*) + \frac{8\sigma_V s_{\rho'}(\mathbf{a}_{V \oplus V^*})}{s_{\delta}(\mathbf{a}_{V \oplus V^*})} = \frac{8\sigma_V (s_{\rho}(\mathbf{a}_{V \oplus V^*}) + s_{\rho'}(\mathbf{a}_{V \oplus V^*}))}{s_{\delta}(\mathbf{a}_{V \oplus V^*})}.$$

Proof. Note that for all cases under consideration, (V, SL_2) is 1-large, and $V \oplus V^*$ is not on the lists of low-dimensional cases excluded in Theorems 3.7, 3.8, and 3.9. We then have by these three theorems that

$$\begin{aligned} \gamma_{0,2}(V \oplus V^*) &= 3\gamma_0(V \oplus V^*), & \gamma_{1,2}(V \oplus V^*) &= 3\gamma_1(V \oplus V^*), & \text{and} \\ \gamma_{2,2}(V \oplus V^*) &= 3\gamma_2(V \oplus V^*) - \frac{4\sigma_V s_{\hat{\rho}}(\mathbf{a}_{V \oplus V^*})}{s_{\hat{\delta}}(\mathbf{a}_{V \oplus V^*})}. \end{aligned}$$

The expression for $\gamma_0^{on}(V)$ in (4.3) then reduces to

$$\gamma_0^{on}(V) = 20\gamma_0(V \oplus V^*) - 12\gamma_0(V \oplus V^*) - 15\gamma_1(V \oplus V^*) + 15\gamma_1(V \oplus V^*) + 6\gamma_2(V \oplus V^*) - 2\gamma_{2,2}(V \oplus V^*).$$

Equation (4.5) then follows using the expressions for the $\gamma_m(V \oplus V^*)$ given in [13, Theorem 1.1]. \square

Finally, we observe that the value of $\gamma_{3,2}(V \oplus V^*)$ is determined by Equation (4.4) and quantities computed above. Because the algebra of on-shell regular functions $\mathbb{R}[M_0]$ is graded Gorenstein by [28, Theorem 1.3], we have by [24, Corollary 1.8] that $\gamma_1^{on} = 0$; see also [25, Theorem 1.1]. Then using the expression for γ_1^{on} in (4.4) along with the facts that $\gamma_1(V \oplus V^*) = 3\gamma_0(V \oplus V^*)/2$ and $\gamma_3(V \oplus V^*) = 5(\gamma_2(V \oplus V^*) - \gamma_0(V \oplus V^*)) / 2$ from [13, Theorem 1.1], we have

$$-15\gamma_0(V \oplus V^*) + 15\gamma_2(V \oplus V^*) - \frac{20\sigma_V s_{\hat{\rho}}(\mathbf{a}_{V \oplus V^*})}{s_{\hat{\delta}}(\mathbf{a}_{V \oplus V^*})} - 2\gamma_{3,2}(V \oplus V^*) = 0,$$

i.e.

$$\gamma_{3,2}(V \oplus V^*) = \frac{-15}{2}\gamma_0(V \oplus V^*) + \frac{15}{2}\gamma_2(V \oplus V^*) - \frac{10\sigma_V s_{\hat{\rho}}(\mathbf{a}_{V \oplus V^*})}{s_{\hat{\delta}}(\mathbf{a}_{V \oplus V^*})}.$$

We conclude with some empirical observations. The data indicate that for fixed dimension of M_0 , the irreducible representation has the smallest or second smallest γ_0^{on} (second to $V_k \oplus V_1$ in some cases when $\dim_{\mathbb{R}} M_0 = 0 \pmod{4}$), while the largest γ_0^{on} comes from the representation with the highest reducibility. If $\dim_{\mathbb{R}} M_0 = 2 \pmod{4}$ the latter is kV_1 where $k = \frac{3}{2} + \frac{1}{4} \dim_{\mathbb{R}} M_0$, while for $\dim_{\mathbb{R}} M_0 = 0 \pmod{4}$ it is $kV_1 \oplus V_2$ with $k = \frac{1}{4} \dim_{\mathbb{R}} M_0$. This suggests that γ_0^{on} measures the degree of reducibility of the representation. The observation that for kV_1 the denominator of γ_0 is a power of 2 can be justified using Equation (4.5), as $s_{\hat{\delta}}$ is a product of sums of pairs of variables. Among all representations with fixed $\dim_{\mathbb{R}} M_0$ the irreducible representation has by far the most intricate Hilbert series.

We also note that, with the exception of the non-1-large representations V_2 and $2V_1$, which correspond to graded regularly symplectomorphic symplectic quotients (see Remark 2.2), the $\gamma_0^{on}(V)$ are distinct for each V with $\dim_{\mathbb{R}} M_0 \leq 38$. This in particular implies that there are no graded regular symplectomorphisms among these cases. Whether this is the case in arbitrary dimension, and in particular whether $\gamma_0^{on}(V)$ determines V for 1-large V , will be considered in a future work.

APPENDIX A. HILBERT SERIES IN LOW DIMENSIONS

For the benefit of those readers interested in explicit descriptions of the algebra $\mathbb{R}[M_0]$, we present in Table 1 the Hilbert series $\text{Hilb}_V^{on}(t)$ of $\mathbb{R}[M_0]$ for symplectic quotients corresponding to SU_2 -modules of dimension at most 10. The first three cases V_1 , V_2 , and $2V_1$ are determined using the identification of M_0 with orbifolds, see [27, Section 5] or [1, 22]. The other cases were computed using the algorithm based on Corollary 4.1 described in Section 4, which has been implemented on *Mathematica* [42] and is available from the authors upon request. The time to compute $\text{Hilb}_V^{on}(t)$ a PC varies widely even for representations of the same dimension; the most time-consuming example presented here was $3V_1 \oplus V_3$, which took 107 minutes, while V_9 was computed in 29 seconds and $5V_1$ was computed in less than 2 seconds.

When space prohibits the ordinary expression of a rational function, we use the following abbreviated notation for the numerator. The expression $\{a_0, a_1, a_2, \dots, a_k; n\}$ indicates the palindromic polynomial of total degree n that begins

$$a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k + \dots$$

Note that this could either refer to

$$a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k + a_{k-1} t^{k+1} + a_{k-2} t^{k+2} + \dots + a_0 t^n$$

or

$$a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k + a_k t^{k+1} + a_{k-1} t^{k+2} + \dots + a_0 t^n;$$

this ambiguity is removed by indicating the value of n .

V	$\dim_{\mathbb{R}} M_0$	$\text{Hilb}_V^{on}(t)$	γ_0^{on}	γ_2^{on}
$\dim_{\mathbb{C}} V = 2$				
V_1	0	1	1	0
$\dim_{\mathbb{C}} V = 3$				
V_2	2	$\frac{1+t^2}{(1-t^2)^2}$	$\frac{1}{2}$	$\frac{1}{8}$
$\dim_{\mathbb{C}} V = 4$				
$2V_1$	2	$\frac{1+t^2}{(1-t^2)^2}$	$\frac{1}{2}$	$\frac{1}{8}$
V_3	2	$\frac{1+t^4}{(1-t^2)(1-t^4)}$	$\frac{1}{4}$	$\frac{5}{16}$
$\dim_{\mathbb{C}} V = 5$				
V_4	4	$\frac{1+t^2+2t^3+t^4+t^6}{(1-t^2)^2(1-t^3)^2}$	$\frac{1}{6}$	$\frac{1}{8}$
$V_1 \oplus V_2$	4	$\frac{1+2t^2+2t^3+2t^4+t^6}{(1-t^2)^2(1-t^3)^2}$	$\frac{2}{9}$	$\frac{11}{108}$
$\dim_{\mathbb{C}} V = 6$				
$3V_1$	6	$\frac{1+9t^2+9t^4+t^6}{(1-t^2)^6}$	$\frac{5}{16}$	$\frac{3}{64}$
$2V_2$	6	$\frac{1+4t^2+4t^4+t^6}{(1-t^2)^6}$	$\frac{5}{32}$	$\frac{11}{128}$
$V_1 \oplus V_3$	6	$\frac{1+12t^4+13t^6+13t^8+12t^{10}+t^{14}}{(1-t^2)^2(1-t^4)^4}$	$\frac{13}{256}$	$\frac{27}{1024}$
V_5	6	$\frac{\{1,0,0,0,2,0,1,0,14,0,13,0,29,0,16; 28\}}{(1-t^2)(1-t^4)(1-t^6)^2(1-t^8)^2}$	$\frac{17}{2304}$	$\frac{113}{27648}$
$\dim_{\mathbb{C}} V = 7$				
$2V_1 \oplus V_2$	8	$\frac{\{1,0,5,14,13,22,34; 12\}}{(1-t^2)^4(1-t^3)^4}$	$\frac{1}{9}$	$\frac{13}{324}$
$V_2 \oplus V_3$	8	$\frac{\{1,0,2,2,9,15,24,36,44,57,64; 21\}}{(1-t^2)^2(1-t^3)^2(1-t^4)(1-t^5)^3}$	$\frac{127}{4500}$	$\frac{721}{54000}$
$V_1 \oplus V_4$	8	$\frac{\{1,0,2,0,3,6,16,12,19,18; 18\}}{(1-t^2)^2(1-t^3)^4(1-t^5)^2}$	$\frac{34}{2025}$	$\frac{221}{24300}$
V_6	8	$\frac{\{1, 0, 2, 0, 5, 0, 17, 8, 38, 25, 71, 64, 120, 125, 177, 195, 240, 252, 299, 295, 316; 41\}}{(1-t^2)(1-t^4)^3(1-t^6)(1-t^9)(1-t^{10})^2}$	$\frac{5}{768}$	$\frac{17}{5120}$
$\dim_{\mathbb{C}} V = 8$				
$4V_1$	10	$\frac{1+18t^2+65t^4+65t^6+18t^8+t^{10}}{(1-t^2)^{10}}$	$\frac{21}{128}$	$\frac{21}{512}$
$V_1 \oplus 2V_2$	10	$\frac{\{1,0,5,8,24,28,48,44; 14\}}{(1-t^2)^6(1-t^3)^4}$	$\frac{17}{324}$	$\frac{167}{7776}$
$2V_1 \oplus V_3$	10	$\frac{\{1,0,2,0,59,0,89,0,340,0,240; 20\}}{(1-t^2)^5(1-t^4)^5}$	$\frac{611}{16384}$	$\frac{1107}{65536}$
$2V_3$	10	$\frac{\{1,0,1,0,21,0,35,0,130,0,100; 20\}}{(1-t^2)^5(1-t^4)^5}$	$\frac{119}{8192}$	$\frac{215}{32768}$
$V_2 \oplus V_4$	10	$\frac{\{1,0,0,6,13,8,19,28; 14\}}{(1-t^2)^6(1-t^3)^4}$	$\frac{61}{2592}$	$\frac{353}{31104}$
$V_1 \oplus V_5$	10	$\frac{\{1, 0, 1, 0, 11, 0, 68, 0, 286, 0, 746, 0, 1820, 0, 3451, 0, 5733, 0, 8042, 0, 9993, 0, 10532; 44\}}{(1-t^2)(1-t^4)^3(1-t^6)^4(1-t^8)^2}$	$\frac{5903}{884736}$	$\frac{36461}{10616832}$
V_7	10	$\frac{\{1, 0, 1, 0, 4, 0, 17, 0, 100, 0, 301, 0, 967, 0, 2333, 0, 5291, 0, 10464, 0, 19436, 0, 32516, 0, 51410, 0, 74928, 0, 103252, 0, 132793, 0, 162204, 0, 185681, 0, 202349, 0, 207442; 76\}}{(1-t^4)(1-t^6)^2(1-t^8)^2(1-t^{10})^3(1-t^{12})^2}$	$\frac{1087769}{663552000}$	$\frac{6309547}{7962624000}$
$\dim_{\mathbb{C}} V = 9$				
$3V_1 \oplus V_2$	12	$\frac{\{1,0,12,34,62,158,297,366,486,580; 18\}}{(1-t^2)^6(1-t^3)^6}$	$\frac{853}{11664}$	$\frac{1211}{46656}$

V	$\dim_{\mathbb{R}} M_0$	$\text{Hilb}_V^{on}(t)$	γ_0^{on}	γ_2^{on}
$3V_2$	12	$\frac{\{1,0,9,14,30,48,44; 12\}}{(1-t^2)^{12}}$	$\frac{31}{512}$	$\frac{51}{2048}$
$V_1 \oplus V_2 \oplus V_3$	12	$\frac{\{1, 0, 3, 15, 43, 106, 247, 510, 959, 1662, 2674, 3983, 5578, 7281, 8962, 10378, 11329, 11644; 34\}}{(1-t^2)^2(1-t^3)^3(1-t^4)^3(1-t^5)^3(1-t^6)}$	$\frac{6617}{288000}$	$\frac{20803}{2073600}$
$2V_1 \oplus V_4$	12	$\frac{\{1, 0, 7, 0, 27, 64, 177, 308, 619, 1036, 1692, 2618, 3715, 4950, 6311, 7664, 8632, 9348, 9614; 36\}}{(1-t^2)^2(1-t^3)^4(1-t^5)^4(1-t^6)^2}$	$\frac{6497}{455625}$	$\frac{12773}{1822500}$
$V_3 \oplus V_4$	12	$\frac{\{1, 0, 3, 2, 16, 31, 96, 196, 419, 739, 1285, 2018, 3106, 4453, 6190, 8114, 10251, 12290, 14195, 15628, 16604, 16888; 42\}}{(1-t^2)(1-t^3)^4(1-t^4)^3(1-t^7)^3}$	$\frac{104081}{13891500}$	$\frac{112691}{33339600}$
$V_2 \oplus V_5$	12	$\frac{\{1, 0, 4, 4, 18, 33, 103, 227, 527, 1088, 2201, 4159, 7564, 13162, 22088, 35778, 56103, 85378, 126257, 181801, 255208, 349731, 468381, 613621, 787245, 989611, 1220152, 1476055, 1753528, 2046090, 2346792, 2646000, 2934505, 3201233, 3436754, 3630791, 3776029, 3865470, 3895992; 76\}}{(1-t^5)^3(1-t^6)^2(1-t^7)^3(1-t^8)^2(1-t^{12})^2}$	$\frac{4785211}{889056000}$	$\frac{27426803}{10668672000}$
$V_1 \oplus V_6$	12	$\frac{\{1, 0, 4, 0, 13, 14, 63, 116, 295, 564, 1161, 2020, 3683, 5916, 9678, 14566, 21837, 30762, 42930, 56848, 74413, 93114, 114990, 136452, 159818, 180478, 201079, 216702, 230366, 237126, 241006; 60\}}{(1-t^4)^2(1-t^5)^4(1-t^6)^3(1-t^7)^2(1-t^{12})}$	$\frac{244439}{79380000}$	$\frac{2897051}{1905120000}$
V_8	12	$\frac{\{1, 0, 3, 3, 9, 18, 43, 84, 179, 326, 604, 1015, 1706, 2655, 4082, 5914, 8367, 11262, 14751, 18428, 22410, 26071, 29490, 32017, 33793, 34264; 50\}}{(1-t^3)(1-t^4)^3(1-t^5)^3(1-t^6)^3(1-t^7)^2}$	$\frac{32909}{18144000}$	$\frac{36809}{43545600}$
$\dim_{\mathbb{C}} V = 10$				
$5V_1$	14	$\frac{\{1,0,31,0,231,0,595,0; 14\}}{(1-t^2)^{14}}$	$\frac{429}{4096}$	$\frac{495}{16384}$
$2V_1 \oplus 2V_2$	14	$\frac{\{1,0,10,32,98,220,488,860,1366,1836,2253,2376; 22\}}{(1-t^2)^6(1-t^3)^8}$	$\frac{29}{729}$	$\frac{5}{324}$
$3V_1 \oplus V_3$	14	$\frac{\{1,0,9,0,179,0,762,0,3375,0,6834,0,12999,0,13524; 28\}}{(1-t^2)^7(1-t^4)^7}$	$\frac{30921}{1048576}$	$\frac{52659}{4194304}$
$2V_2 \oplus V_3$	14	$\frac{\{1, 0, 7, 9, 51, 134, 366, 784, 1593, 2947, 5199, 8400, 12830, 18152, 24504, 31023, 37472, 42613, 46145, 47252; 38\}}{(1-t^2)^4(1-t^3)^3(1-t^4)(1-t^5)^5(1-t^6)}$	$\frac{15991}{1012500}$	$\frac{81041}{12150000}$
$V_1 \oplus 2V_3$	14	$\frac{\{1,0,0,0,93,0,286,0,1569,0,2758,0,5901,0,5530; 28\}}{(1-t^2)^7(1-t^4)^7}$	$\frac{13373}{1048576}$	$\frac{23199}{4194304}$
$V_1 \oplus V_2 \oplus V_4$	14	$\frac{\{1,0,1,10,29,68,156,268,446,724,1015,1214,1406,1500; 26\}}{(1-t^2)^6(1-t^3)^6(1-t^5)^2}$	$\frac{761}{72900}$	$\frac{2789}{583200}$
$2V_4$	14	$\frac{\{1,0,2,14,17,24,92,154,161,234,306; 20\}}{(1-t^2)^8(1-t^3)^6}$	$\frac{71}{7776}$	$\frac{125}{31104}$
$2V_1 \oplus V_5$	14	$\frac{\{1, 0, 6, 0, 46, 0, 454, 0, 2849, 0, 12140, 0, 43131, 0, 127076, 0, 315389, 0, 673304, 0, 1260139, 0, 2076447, 0, 3042040, 0, 3982739, 0, 4675695, 0, 4928416; 60\}}{(1-t^2)(1-t^4)^5(1-t^6)^6(1-t^8)^2}$	$\frac{1167229}{191102976}$	$\frac{127411}{42467328}$
$V_3 \oplus V_5$	14	$\frac{\{1, 0, 0, 0, 40, 0, 235, 0, 1536, 0, 6245, 0, 22073, 0, 62288, 0, 153198, 0, 322982, 0, 604168, 0, 1000931, 0, 1491320, 0, 1998930, 0, 2427434, 0, 2672013, 0; 62\}}{(1-t^2)^2(1-t^4)^4(1-t^6)^4(1-t^8)^4}$	$\frac{1793899}{452984832}$	$\frac{9710395}{5435817984}$
$V_2 \oplus V_6$	14	$\frac{\{1, 0, 3, 0, 35, 34, 195, 318, 899, 1580, 3412, 5788, 10695, 17170, 28357, 43056, 65617, 94006, 134421, 182888, 246941, 320684, 411834, 511564, 628197, 748348, 880927, 1009232, 1141267, 1258738, 1370691, 1456394, 1528351, 1566260, 1584434; 68\}}{(1-t^2)^3(1-t^4)^5(1-t^6)^4(1-t^{10})^2}$	$\frac{4463}{829440}$	$\frac{25219}{9953280}$

V	$\dim_{\mathbb{R}} M_0$	$\text{Hilb}_V^{on}(t)$	γ_0^{on}	γ_2^{on}
$V_1 \oplus V_7$	14	$\{1, 0, 2, 0, 16, 0, 120, 0, 949, 0, 4484, 0, 19061, 0, 66638, 0,$ $206241, 0, 563855, 0, 1399730, 0, 3161375, 0, 6596301, 0,$ $12755465, 0, 23052381, 0, 39054709, 0, 62358940, 0,$ $94039452, 0, 134429968, 0, 182432050, 0, 235602923,$ $0, 289833599, 0, 340185063, 0, 381119164, 0,$ $407976216, 0, 417274692; 100\}$ $\frac{\quad}{(1-t^4)(1-t^6)^4(1-t^8)^4(1-t^{10})^3(1-t^{12})^2}$	$\frac{2423496049}{1528823808000}$	$\frac{184571731}{244611809280}$
V_9	14	$\{1, 0, 1, 0, 10, 0, 42, 0, 334, 0, 1566, 0, 6958, 0, 25277, 0, 83391,$ $0, 244771, 0, 662241, 0, 1652020, 0, 3858520, 0, 8466785, 0,$ $17599687, 0, 34772336, 0, 65630156, 0, 118662007, 0,$ $206217754, 0, 345216158, 0, 558013931, 0, 872420918,$ $0, 1321591127, 0, 1942389147, 0, 2773434697, 0,$ $3851193503, 0, 5206038355, 0, 6856598397, 0,$ $8805135769, 0, 11032270863, 0, 13494344349, 0,$ $16121552188, 0, 18820073941, 0, 21475826372, 0,$ $23962298579, 0, 26149487910, 0, 27915236756,$ $0, 29155789630, 0, 29795890397, 0; 154\}$ $\frac{\quad}{(1-t^8)^2(1-t^{10})^3(1-t^{12})^4(1-t^{14})^3(1-t^{16})^2}$	$\frac{62728171711}{116530348032000}$	$\frac{68283510691}{279672835276800}$

Table 1: Hilbert series of $\mathbb{R}[M_0]$ for symplectic quotients M_0 corresponding to low-dimensional SU_2 -modules V .

APPENDIX B. VISUALIZATION OF $\gamma_0^{on}(V)$

Here, we give three graphs of the values of $\gamma_0^{on}(V)$ to illustrate its dependence on the weights of V , the number of irreducible subrepresentations of V , and dimension of M_0 . In Figure 1, the horizontal axis is the dimension of M_0 and the vertical axis is $-\log \gamma_0^{on}(V)$. In Figure 2, the horizontal axes are the dimension of M_0 and the sum of the positive weights in V , while the vertical axis is $-\log \gamma_0^{on}(V)$. In Figure 3, the horizontal axes are the dimension of M_0 and the number of irreducible subrepresentations of V , and the vertical axis is again $-\log \gamma_0^{on}(V)$. All three graphs plot the value of $\gamma_0^{on}(V)$ for every V such that $V^{SU_2} = \{0\}$ and $\dim M_0 \leq 38$.

These values were computed with *Mathematica* [42] using the expression in Theorem 4.2. The Schur polynomials were computed by expressing them in terms of elementary symmetric polynomials using the Jacobi-Trudi identities [36, Theorem 4.5.1], which is much faster than via the definition as a quotient of alternants.

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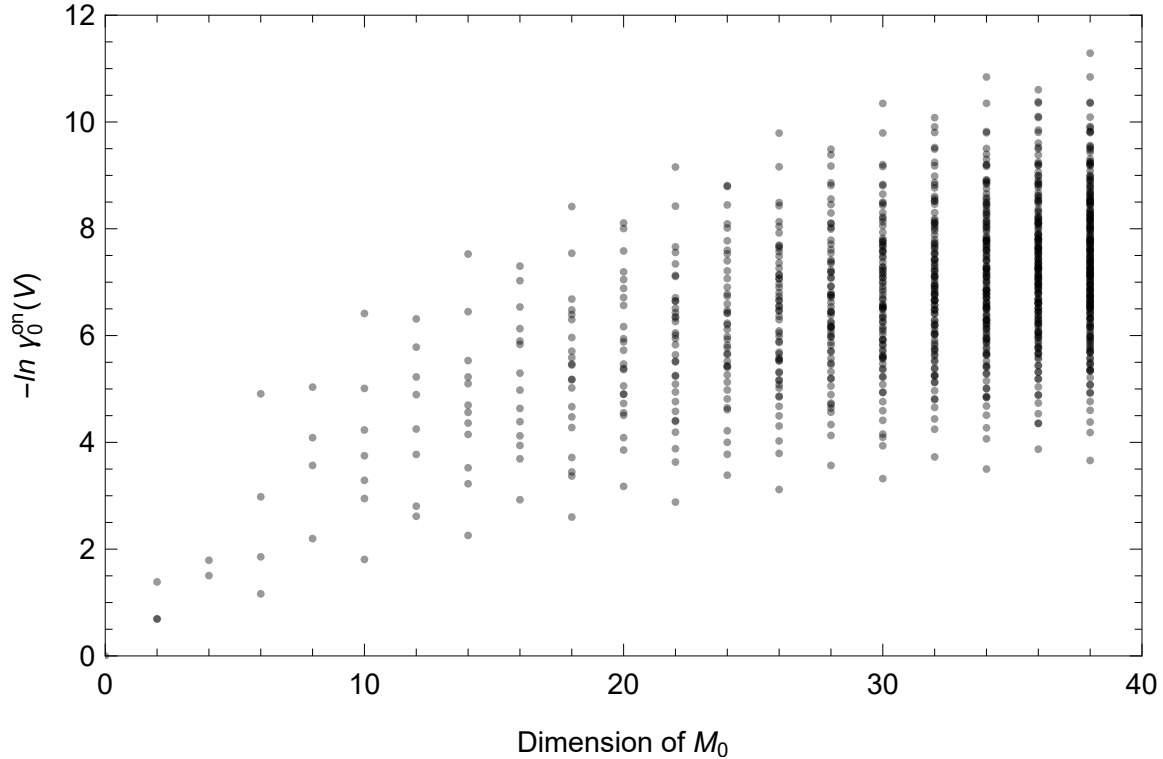


FIGURE 1. A plot of the values of $-\ln \gamma_0^{on}(V)$ vs. $\dim_{\mathbb{R}} M_0$. Includes all V with $V^{SU_2} = \{0\}$ and $\dim_{\mathbb{R}} M_0 \leq 38$. For fixed $\dim_{\mathbb{R}} M_0$, the largest or second largest $-\gamma_0^{on}$ comes from the irreducible representation. The smallest value of $-\gamma_0^{on}$ comes from the representation with $kV_1 \oplus V_2$ if $\dim_{\mathbb{R}} M_0$ is divisible by four and kV_1 otherwise.

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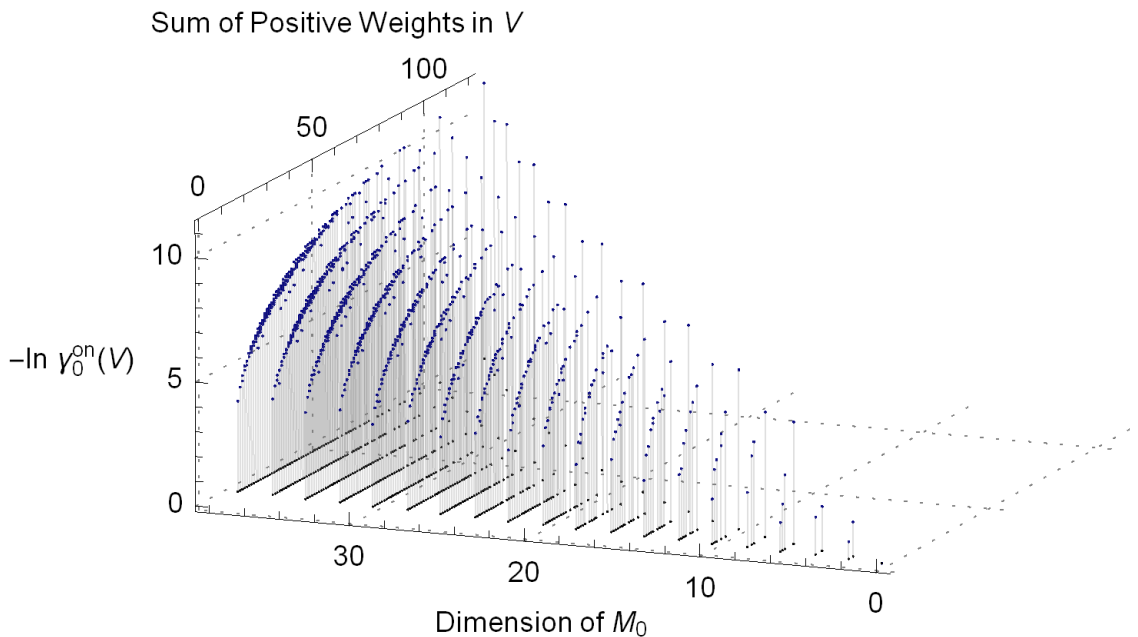


FIGURE 2. A plot of the values of $-\ln \gamma_0^{\text{on}}(V)$ vs. the sum of the positive weights in V (i.e. elements of Λ_V) and $\dim_{\mathbb{R}} M_0$. Includes all V with $V^{\text{SU}_2} = \{0\}$ and $\dim_{\mathbb{R}} M_0 \leq 38$.

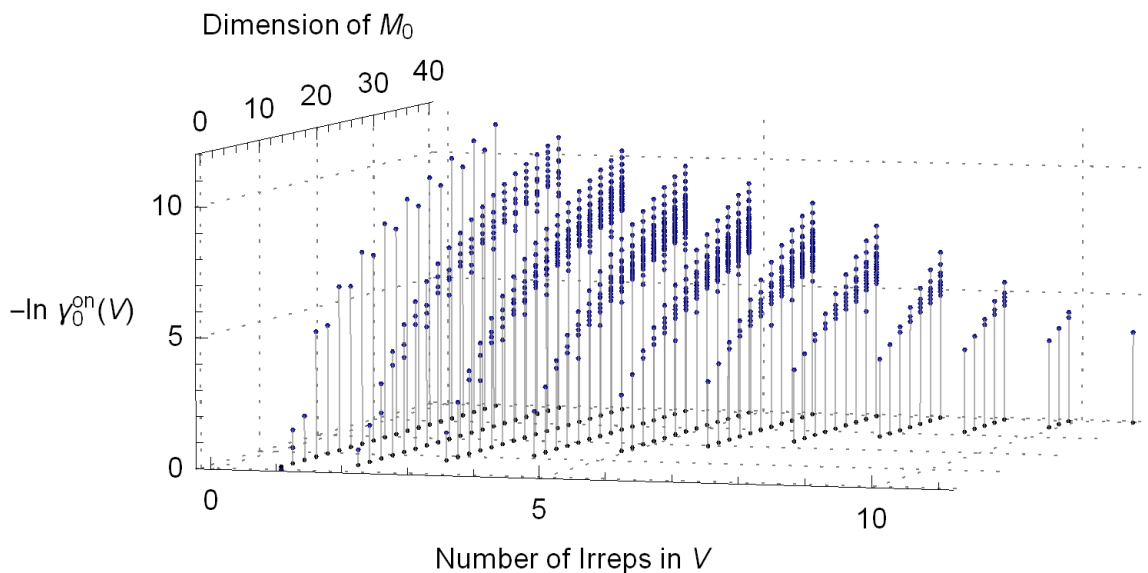


FIGURE 3. A plot of the values of $-\ln \gamma_0^{\text{on}}(V)$ vs. the number of irreducible summands in V (i.e. r) and $\dim_{\mathbb{R}} M_0$. Includes all V with $V^{\text{SU}_2} = \{0\}$ and $\dim_{\mathbb{R}} M_0 \leq 38$.

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