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Universal enveloping algebra of a pair of compatible Lie brackets

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Abstract

Applying the Poincaré—Birkhoff—Witt property and the Gröbner—Shirshov bases technique, we find the linear basis of the associative universal enveloping algebra in the sense of V. Ginzburg and M. Kapranov of a pair of compatible Lie brackets. We state that the growth rate of this universal enveloping over n-dimensional compatible Lie algebra equals $n+1$.

Keywords: universal enveloping algebra over an operad, compatible Lie brackets, Gröbner—Shirshov basis, growth rate.

1 Introduction

Hamiltonian pairs (or bihamiltonian structures) play an important role [6, 7, 15] in the theory of integrable systems from mathematical physics. Such structures correspond to pairs of compatible Poisson brackets defined on the same manifold. Two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are said to be compatible if $\alpha\{\cdot,\cdot\}_1+\beta\{\cdot,\cdot\}_2$ is a Poisson bracket for all $\alpha,\beta\in\Bbbk$, where $\Bbbk$ denotes the ground field. In terms of operads, algebras with compatible Poisson brackets form a so called bi-Hamiltonian operad [1, 3].

In the case of linear Poisson brackets, all such structures arise from a pair of compatible Lie brackets. An algebra $\langle L,[\cdot,\cdot]_1,[\cdot,\cdot]_2,+\rangle$ belongs to a variety Lie_2 of pairs of compatible Lie brackets if $\alpha[\cdot,\cdot]_1+\beta[\cdot,\cdot]_2$ is a Lie bracket for all $\alpha,\beta\in\Bbbk$.

A plenty of examples of compatible Lie brackets is presented in [9], the classification results on them see in [16]. In [10], it was shown that every pair of compatible Lie brackets endowed with a common non-degenerate invariant bilinear form produces a rational solution to the classical Yang—Baxter equation. Free algebras with a pair of compatible Lie brackets were studied in [1, 12, 13]. Koszulness of the operad corresponding to a variety Lie_2 was proved in [4].

In [11], the operadic (multiplicative) universal enveloping associative algebra $U_{\mathrm{Lie}_2}(\mathfrak{g})$ of a given algebra $\mathfrak{g}\in\mathrm{Lie}_2$ in the sense of V. Ginzburg and M. Kapranov [8] was considered, and the Poincaré—Birkhoff—Witt (PBW) property for it was proved. By the definition, the associative algebra $U_{\mathrm{Lie}_2}(\mathfrak{g})$ satisfies the following property: the category of modules over $\mathfrak{g}$ and the category of left modules over $U_{\mathrm{Lie}_2}(\mathfrak{g})$ are equivalent.

We find the Gröbner—Shirshov basis of the universal enveloping algebra $U_{\mathrm{Lie}_2}(\mathfrak{g}_0)$ of an algebra $\mathfrak{g}_0$, where $\mathfrak{g}_0$ denotes the vector space $\mathfrak{g}$ with both zero Lie brackets. It allows us, applying the PBW property, to get the linear basis of the algebra $U_{\mathrm{Lie}_2}(\mathfrak{g})$. As a corollary, we compute the (exponential) growth rate of $U_{\mathrm{Lie}_2}(\mathfrak{g})$ when $\mathfrak{g}$ is finite-dimensional.

2 Gröbner—Shirshov basis for $U_{\mathrm{Lie}_2}(\mathfrak{g}_0)$

Due to [11, Corollary 2.11], the operadic universal enveloping associative algebra of a given algebra $\mathfrak{g} \in \mathrm{Lie}_2$ in the sense of V. Ginzburg and M. Kapranov [8] equals

$$U_{\mathrm{Lie}_2}(\mathfrak{g}) = \mathrm{As}\langle X \cup X' \mid xy - yx + [x,y]_1,\, x'y' - y'x' + ([x,y]_2)', \\ xy' - y'x + x'y - yx' + ([x,y]_1)' + [x,y]_2\rangle, \quad (1)$$

where X is a linear basis of $\mathfrak{g}$, X' is a set such that $X \cap X' = \emptyset$ and the map $'\colon X \to X'$, $x \to x'$ is a bijection. It looks like there are misprints in [11, §6.1] while writing the defining relations of $U_{\mathrm{Lie}_2}(\mathfrak{g})$.

Assuming that X and X' are primitive elements, $U_{\mathrm{Lie}_2}(\mathfrak{g})$ has a natural Hopf algebra structure.

In [11], the PBW property of $U_{\mathrm{Lie}_2}(\mathfrak{g})$ was proved, it implies that there exists a filtration on $U_{\mathrm{Lie}_2}(\mathfrak{g})$ such that $\mathrm{gr}\, U_{\mathrm{Lie}_2}(\mathfrak{g}) \cong U_{\mathrm{Lie}_2}(\mathfrak{g}_0)$, where $\mathfrak{g}_0$ is a vector space $\mathfrak{g}$ with trivial products $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_2$.

Thus, let us study the algebra $U_{\mathrm{Lie}_2}(\mathfrak{g}_0)$. We may assume that $X = \{f_i \mid i \in I\}$, where I is a well-ordered set. Denote $X' = \{F_i := f_i' \mid i \in I\}$. We define an order on $X \cup X'$ as follows,
— $f_i < f_j$ if $i < j$,
— $F_j < F_i$ if $i < j$,
— $f_i < F_j$ for all $i, j \in I$.

Let us write down the relations

$$f_j f_i - f_i f_j, \quad i < j, \qquad (2)$$
$$F_i F_j - F_j F_i, \quad i < j, \qquad (3)$$
$$F_i w f_k - f_k F_i w - F_k w f_i + f_i F_k w, \quad w \leq i < k, \qquad (4)$$

where $w = 1$ or $w = f_{s_1} \ldots f_{s_t}$ with $s_1 \leq \ldots \leq s_t$ and by $w \leq i$ we mean that $s_t \leq i$.

We recall the main definitions from the theory of Gröbner—Shirshov bases [2, §2.1].

Let $(X, <)$ be a well-ordered set and let X^* denote the set of all words in the alphabet X. Suppose that X^* is well-ordered, moreover, $u < v$ implies $w_1 u w_2 < w_1 v w_2$ for all $w_1, w_2 \in X^*$, such ordering is called monomial. We will use only deg-lex ordering, in which two words first are compared by the degree and then lexicographically. Given a nonzero element f from the free associative algebra $\mathrm{As}(X)$, by $\bar{f}$ we mean its leading word.

Given a monomial ordering $<$ on X^* and two monic polynomials f, g, we define two kinds of compositions:

(i) If w is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $|\bar{f}| + |\bar{g}| > |w|$, then the polynomial $(f,g)_w := fb - ag$ is called the intersection composition of f and g with respect to w.

(ii) If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$, then the polynomial $(f,g)_w := f - agb$ is called the inclusion composition of f and g with respect to w.

Consider $S \subset \mathrm{As}(X)$ such that every $s \in S$ is monic. Take $h \in \mathrm{As}(X)$ and $w \in X^*$. Then h is called trivial modulo (S, w), denoted by $h \to 0 \bmod (S, w)$, if $h = \sum \alpha_i a_i s_i b_i$, where $\alpha_i \in \Bbbk$, $a_i, b_i \in X^*$, and $s_i \in S$ satisfying $a_i s_i b_i < w$.

A monic set $S \subset \mathrm{As}(X)$ is called a Gröbner—Shirshov basis in $\mathrm{As}(X)$ with respect to the monomial ordering $<$ if every composition of polynomials in S is trivial modulo S and the corresponding w.

The Composition Diamond lemma for associative algebras implies that if S is a Gröbner—Shirshov basis in $\mathrm{As}(X)$, then $\mathrm{Irr}(S) = \{u \in X^* \mid u \neq a\bar{s}b,\ s \in S,\ a, b \in X^*\}$ is a linear basis of the algebra $\mathrm{As}\langle X \mid S\rangle$.

Lemma. The set of the relations (2)–(4) forms a Gröbner—Shirshov basis for $U_{\mathrm{Lie}_2}(\mathfrak{g}_0)$.

PROOF. Note that the first series of the defining relations (1) of $U_{\mathrm{Lie}_2}(\mathfrak{g}_0)$ coincides with (2), the second one with (4), and the third series of them multiplied by w on the right gives exactly (4).

The relations (2)–(4) may have only compositions of intersection but not inclusion.

Compositions between (2) and (2) as well as between (3) and (3) are trivial.

Let us compute the composition between (2) and (4). Let $w \leq i < k$ and $l < k$, then

$$F_i w f_k f_l \underset{(2)}{\to} L := F_i w f_l f_k,$$
$$F_i w f_k f_l \underset{(4)}{\to} R := f_k F_i w f_l + F_k w f_i f_l - f_i F_k w f_l.$$

If $l \leq i$, then $L - R \underset{(4),\,(2)}{\to} 0$ for $\widetilde{w} = w f_l$.

Let $i < l$, then

$$\begin{aligned} L \underset{(4)}{\to} f_l F_i w f_k + F_l w f_i f_k - f_i F_l w f_k \underset{(4)}{\to} f_l f_k F_i w + f_l F_k w f_i - \underline{f_l f_i F_k w} \\ + F_l w f_i f_k - f_i f_k F_l w - f_i F_k w f_l + \underline{f_i f_l F_k w}; \end{aligned}$$
$$R \underset{(4)}{\to} f_k f_l F_i w + f_k F_l w f_i - f_k f_i F_l w + F_k w f_i f_l - f_i F_k w f_l.$$

Thus,

$$\begin{aligned} &L - R = f_l F_k w f_i + F_l w f_i f_k - f_k F_l w f_i - F_k w f_i f_l \\ &= (f_l F_k w + F_l w f_k - f_k F_l w - F_k w f_l) f_i + F_l w (f_i f_k - f_k f_i) - F_k w (f_i f_l - f_l f_i) \underset{(4),\,(2)}{\to} 0. \end{aligned}$$

Here it is important that all involved terms are less than the initial word $u = F_i w f_k f_l$.

Now we compute the composition between (3) and (4). Let $w \leq i < k$ and $a < i$. On the one hand, we have

$$\begin{aligned} F_a F_i w f_k \underset{(3)}{\to} F_i F_a w f_k \underset{(2)}{\to} F_i F_a f_k w \underset{(4)}{\to} F_i f_k F_a w + F_i F_k f_a w - F_i f_a F_k w \\ \underset{(4)}{\to} L := f_k F_i F_a w + F_k f_i F_a w - f_i F_k F_a w + F_i F_k f_a w - F_i f_a F_k w. \end{aligned}$$

On the other hand,

$$\begin{aligned} &F_a F_i w f_k \underset{(4)}{\to} F_a f_k F_i w + F_a F_k w f_i - F_a f_i F_k w \\ &\underset{(4)}{\to} R := f_k F_a F_i w + F_k f_a F_i w - \underline{f_a F_k F_i w} + F_a F_k w f_i - f_i F_a F_k w - F_i f_a F_k w + \underline{f_a F_i F_k w}. \end{aligned}$$

Thus,

$$L-R=F_kf_iF_aw+F_iF_kf_aw-F_kf_aF_iw-F_aF_kwf_i$$
$$=F_k(f_iF_aw+F_if_aw-f_aF_iw-F_awf_i)+(F_iF_k-F_kF_i)f_aw-(F_aF_k-F_kF_a)wf_i\underset{(4),(3)}{\rightarrow}0.$$

Finally, note that there are no compositions of intersection between (4) and (4). □

3 Basis of $U_{\mathrm{Lie}_2}(\mathfrak{g})$

Denote by $M(X)$ the set of all (ordered) monomials from $\Bbbk[X]$ including 1. Given $w=f_{j_1}\dots f_{j_n}\in M(X)\setminus\{1\}$, we mean that $j_1\leq\dots\leq j_n$, and for $u=F_{k_1}\dots F_{k_m}\in M(X')\setminus\{1\}$, we mean that $k_m\leq\dots\leq k_1$. Define

$$\lfloor w\rfloor=\max\{j_t\mid t=1,\dots,n\},\quad \lceil u\rceil=\min\{k_t\mid t=1,\dots,m\},$$

i. e., $\lfloor w\rfloor=j_n$ and $\lceil u\rceil=k_m$.

Define $L=M(X)\cup L'$, where L' consists of all words

$$w_0u_1w_1u_2w_2\dots u_{s-1}w_{s-1}u_sw_s, \tag{5}$$

where

a) $w_i\in M(X)\setminus\{1\}$, $i=1,\dots,s-1$, $w_0,w_s\in M(X)$;

b) $u_i\in M(X')\setminus\{1\}$, $i=1,\dots,s$;

b) $\lfloor w_i\rfloor\leq\lceil u_i\rceil$, $i=1,\dots,s-1$, and $\lfloor w_s\rfloor\leq\lceil u_s\rceil$ or $w_s=1$.

Theorem 1. The set L forms a linear basis of $U_{\mathrm{Lie}_2}(\mathfrak{g})$.

Proof. It follows from Lemma, the Composition Diamond lemma for associative algebras [2, Theorem 1], and the PBW property of $U_{\mathrm{Lie}_2}(\mathfrak{g})$ [11]. □

Given a pair of compatible Lie brackets $\mathfrak{g}$, define L_n as the set of all elements from L of length n. Put $r_n=|L_n|$. The growth rate of $\rho(U_{\mathrm{Lie}_2}(\mathfrak{g}))$ is defined as $\lim\limits_{n\to\infty}\sqrt[n]{r_n}$.

Let us show that such limit always exists. The Fekete's Lemma [14, Lemma 1.2.2] says that given a sequence $\{a_n\}$, $n\geq1$, of real numbers such that $a_{s+t}\leq a_s+a_t$ for all $s,t\in\mathbb{N}$, there exists a limit $\lim\limits_{n\to\infty}\frac{a_n}{n}$. In our case, we have the inequality $r_{s+t}\leq r_sr_t$ for all $s,t\in\mathbb{N}$, since every non-empty subword of the basic element from L lies in L. Hence, it remains to apply the Fekete's Lemma for the sequence $\{\ln r_n\}$.

We are able to compute the growth rate of $U_{\mathrm{Lie}_2}(\mathfrak{g})$ when $\mathfrak{g}$ is finite-dimensional. Firstly, we do it straightforwardly (Theorem 2). Secondly, we derive this result from Lemma 6.1 [11] (Remark 1, suggested by the reviwer). Thirdly, we reprove Theorem 2 with the help of partially commutative algebras and dependence polynomial (Remark 2).

Theorem 2. Let $\mathfrak{g}\in\mathrm{Lie}_2$ and $\dim(\mathfrak{g})=m$. Then the growth rate of $U_{\mathrm{Lie}_2}(\mathfrak{g})$ equals $m+1$.

Proof. Let $X=\{x_1,\dots,x_m\}$ be a basis of $\mathfrak{g}$. Define O_n as a subset of L_n consisting of words starting with $F_i\in X'$. Put $s_n=|O_n|$, assuming that $s_0=1$.

Let us derive the following formula,

$$s_k=\sum_{p=1}^{k-1}p\binom{p+m}{p+1}s_{k-1-p}+\binom{k+m-1}{k}. \tag{6}$$

A word $v \in O_k$ for $k \geq 1$ either consists of only letters from X' ($\binom{k+m-1}{k}$ choices) or v has the form $v = u_1 F_r w_1 v'$, where $r = 1, \dots, m$, $u_1 \in M(X')$, $w_1 \in M(X) \setminus \{1\}$, and $v' \in O_h$ for some h (if $h = 0$, then $v' = 1$). For the latter case, we initially fix the value of $p = |u_1| + |w_1|$ and then consider all cases of $t = |w_1| \geq 1$. Hence, we have

$$s_k = \binom{k+m-1}{k} + \sum_{p=1}^{k-1} \left(\sum_{r=1}^{m} \sum_{t=1}^{p} \binom{t+r-1}{t} \binom{p-t+m-r}{p-t} \right) s_{k-1-p}, \tag{7}$$

where $\binom{t+r-1}{t}$ is responsible for the choice $w_1 \in M(\{x_1, \dots, x_r\}) \setminus \{1\}$ and $\binom{p-t+m-r}{p-t}$ corresponds to the choice of $u_1 \in M(\{x_r, \dots, x_m\}')$.

In [17, Theorem 1.4] the formula

$$\sum_{i=0}^{m} \frac{\binom{m}{i}}{\binom{n+m}{p+i}} = \frac{n+m+1}{n+1} \cdot \frac{1}{\binom{n}{p}}$$

for $0 \leq m$ and $0 \leq p \leq n$ was stated. Applying it, we derive

$$\begin{aligned}
\sum_{t=0}^{p} \binom{t+a}{a} \binom{p-t+b}{b} &= \sum_{t=0}^{p} \frac{(t+a)!}{t!a!} \frac{(p-t+b)!}{(p-t)!b!} \\
&= \frac{1}{a!b!} \sum_{t=0}^{p} \frac{(t+a)!(p-t+b)!}{t!(p-t)!} \frac{p!}{p!} \frac{(p+a+b)!}{(p+a+b)!} = \binom{p+a+b}{p, a, b} \sum_{t=0}^{p} \frac{\binom{p}{t}}{\binom{p+a+b}{t+a}} \\
&= \frac{p+a+b+1}{a+b+1} \binom{p+a+b}{p, a, b} \Big/ \binom{a+b}{a} = \binom{p+a+b+1}{p}.
\end{aligned}$$

Substituting this equality for $a = r-1$ and $b = m-r$ in (7), we get

$$\begin{aligned}
&\sum_{r=1}^{m} \sum_{t=1}^{p} \binom{p-t+m-r}{p-t} \binom{t+r-1}{t} \\
&\qquad = \sum_{r=1}^{m} \left(\sum_{t=0}^{p} \binom{p-t+m-r}{p-t} \binom{t+r-1}{t} - \binom{p+m-r}{p} \right) \\
&\qquad = \sum_{r=1}^{m} \left(\binom{p+m}{p} - \binom{p+m-r}{p} \right) = m \binom{p+m}{p} - \binom{p+m}{p+1} = p \binom{p+m}{p+1},
\end{aligned}$$

and we have proved the formula (6).

Let us prove that $s_n = m(m+1)^{n-1}$ for any $n \geq 1$ by induction on n. The case $n = 1$ is trivial, we have $O_1 = \{F_1, \dots, F_m\}$ and $s_1 = m$.

Suppose that we have proved that $s_n = m(m+1)^{n-1}$ for all $n < k$, where $2 \leq k$. By (6), we get

$$\begin{aligned}
s_k &= \sum_{p=1}^{k-2} p \binom{p+m}{p+1} s_{k-1-p} + (k-1) \binom{k+m-1}{k} + \binom{k+m-1}{k} \\
&= \sum_{p=1}^{k-2} p \frac{m(m+1)\dots(m+p)}{(p+1)!} m(m+1)^{k-2-p} + \frac{m(m+1)\dots(m+k-1)}{(k-1)!}
\end{aligned}$$

$$= m\left(\sum_{p=1}^{k-2} p\frac{((m+1)-1)(m+1)^{k-2-p}}{(p+1)!}\sum_{l=0}^{p}\begin{bmatrix}p\\ l\end{bmatrix}(m+1)^l + \sum_{l=0}^{k-1}\frac{\begin{bmatrix}k-1\\ l\end{bmatrix}}{(k-1)!}(m+1)^l\right),$$

here $\begin{bmatrix}p\\ l\end{bmatrix}$ denotes the (unsigned) Stirling number of the first kind.

The coefficient A_q of s_k/m by $(m+1)^q$, $q = 0, \ldots, k-1$, equals

$$\begin{aligned}
A_q = &\sum_{p=k-q-1}^{k-2}\frac{p}{(p+1)!}\begin{bmatrix}p\\ q+p+1-k\end{bmatrix} - \sum_{p=k-q-2}^{k-2}\frac{p}{(p+1)!}\begin{bmatrix}p\\ q+p+2-k\end{bmatrix} + \frac{\begin{bmatrix}k-1\\ q\end{bmatrix}}{(k-1)!}\\
&= \sum_{p=k-q-1}^{k-2}\left(\frac{p}{(p+1)!}\begin{bmatrix}p\\ q+p+1-k\end{bmatrix} - \frac{(p-1)}{p!}\begin{bmatrix}p-1\\ q+p+1-k\end{bmatrix}\right)\\
&\qquad + \frac{\begin{bmatrix}k-1\\ q\end{bmatrix}}{(k-1)!} - \frac{(k-2)\begin{bmatrix}k-2\\ q\end{bmatrix}}{(k-1)!}.
\end{aligned} \tag{8}$$

Applying the identity $\begin{bmatrix}n+1\\ l\end{bmatrix} = n\begin{bmatrix}n\\ l\end{bmatrix} + \begin{bmatrix}n\\ l-1\end{bmatrix}$, we compute for $p > 0$

$$\begin{aligned}
&p\begin{bmatrix}p\\ q+p+1-k\end{bmatrix} - (p-1)(p+1)\begin{bmatrix}p-1\\ q+p+1-k\end{bmatrix}\\
&\quad = p\begin{bmatrix}p\\ q+p+1-k\end{bmatrix} - (p+1)\begin{bmatrix}p\\ q+p+1-k\end{bmatrix} + (p+1)\begin{bmatrix}p-1\\ q+p-k\end{bmatrix}\\
&\quad = -\begin{bmatrix}p\\ q+p+1-k\end{bmatrix} + (p+1)\begin{bmatrix}p-1\\ q+p-k\end{bmatrix}.
\end{aligned}$$

When $p = 0$, we may apply the same formula only adding the summand equal to the Kronecker delta $\delta_{q,k-1}$, since $\begin{bmatrix}0\\ 0\end{bmatrix} = 1$.

Therefore, we have

$$\begin{aligned}
A_q &= \frac{\begin{bmatrix}k-2\\ q-1\end{bmatrix}}{(k-1)!} + \delta_{q,k-1} + \sum_{p=k-q-1}^{k-2}\frac{1}{(p+1)!}\left(-\begin{bmatrix}p\\ q+p+1-k\end{bmatrix} + (p+1)\begin{bmatrix}p-1\\ q+p-k\end{bmatrix}\right)\\
&= \frac{\begin{bmatrix}k-2\\ q-1\end{bmatrix}}{(k-1)!} + \delta_{q,k-1} - \frac{\begin{bmatrix}k-2\\ q-1\end{bmatrix}}{(k-1)!} = \delta_{q,k-1}.
\end{aligned}$$

Thus, $s_k = m\sum_{q=0}^{k-1} A_q(m+1)^q = m(m+1)^{k-1}$, as required.

Now, we express

$$r_n = \sum_{l=0}^{n}\binom{l+m-1}{l}s_{n-l}, \tag{9}$$

this formula counts the leftmost position $(l+1)$ such that the $(l+1)$st letter of a given element $z \in L_n$ is from X'. The value $l = n$ means that $z \in M(X)$.

By (9), we get

$$r_n = \sum_{l=0}^{n}\binom{l+m-1}{l}s_{n-l} = \binom{n+m-1}{n} + m\sum_{l=0}^{n-1}\binom{l+m-1}{l}(m+1)^{n-l-1}. \tag{10}$$

On the one hand, $r_n \geq s_n = m(m+1)^{n-1}$ and so, $\rho(U_{\mathrm{Lie}_2}(\mathfrak{g})) \geq m+1$.

On the other hand,

$$r_n \leq \binom{n+m-1}{n} + \sum_{l=0}^{n-1} \binom{l+m-1}{l} (m+1)^{n-l} = \sum_{l=0}^{n} \binom{l+m-1}{l} (m+1)^{n-l}$$
$$= \sum_{l=0}^{n} \frac{m(m+1)\dots(m+l-1)}{l!} (m+1)^{n-l} \leq \sum_{l=0}^{n} (m+1)^n = (n+1)(m+1)^n,$$

which implies $\rho(U_{\mathrm{Lie}_2}(\mathfrak{g})) \leq m+1$. Hence, $\rho(U_{\mathrm{Lie}_2}(\mathfrak{g})) = m+1$. □

Remark 1 (suggested by the reviwer). Let $\mathcal{P} = \cup_{n\geq 1}\mathcal{P}(n)$ be a symmetric operad. One may assign to it the generating series $\chi_{\mathcal{P}}(p_1, p_2, \dots) = \sum_{n\geq 1} \chi_{S_n}(\mathcal{P}(n))$, where $\chi_{S_n}(V) = \sum_{\rho\vdash n} \frac{p_\rho}{z_\rho} \mathrm{tr}_V(\rho)$ is a symmetric function associated with the corresponding S_n-character of the symmetric group given in the basis of $p_k = \sum_i x_i^k$.

In Lemma 6.1 of [11], it was stated that

$$\chi_{U^0_{\mathrm{Lie}_2}} = \frac{\sum_{k\geq 1} h_k}{1 - \sum_{k\geq 1} p_k}. \tag{11}$$

Calculating the generating series for dimensions of $U_{\mathrm{Lie}_2}(\mathfrak{g})$, when $\dim(\mathfrak{g}) = m$, corresponds to the calculation of the plethystic substitution of the polynomial $f(t) = mt$ into (11). Plethysm with the numerator creates the generating function for the dimensions of the symmetric algebra $S(\mathfrak{g})$, since $\sum_{k\geq 1} h_k$ is the character of the operad Com. Plethysm with the denominator gives

$$\frac{1}{1 - \sum\limits_{k\geq 1} p_k} \circ (mt) = \frac{1}{1 - \sum\limits_{k\geq 1} mt^k} = \frac{1}{1 - \frac{mt}{1-t}} = \frac{1-t}{1-(m+1)t}, \tag{12}$$

so, one gets precisely the generating function for the numbers $s_k = m(m+1)^{k-1}$. Altogether, r_n counts all words of the form $w_0\gamma$, where $w_0 \in M(X)$ and $\gamma \in O_k$ for appropriate k. Thus, we get the formula (9), and Theorem 2 follows.

Moreover, we have confirmed the conjecture of A. Khoroshkin [11, §6.1] of isomorphism

$$U_{\mathrm{Lie}_2^0}(V) \cong S(V) \otimes FL(V)$$

of Schur functors while projecting on the subspace of words of length n. Due to Theorem 1, we may present $FL(V)$ as the tensor algebra $T(G(V))$, where $G(V)$ is the quotient of $S(V \oplus V)$ by the ideal $\mathrm{Span}(v'u - u'v \mid u, v \in V)$. Here we identify $(a, b) \in V \oplus V$ with $a' + b$.

Remark 2. Theorem 2 may be obtained with the help of partially commutative algebras. Given a graph $G(V, E)$, an associative algebra $\mathrm{As}(G) = \mathrm{As}\langle V \mid ab = ba, (a, b) \in E\rangle$ is called a partially commutative algebra. Dependence polynomial [5] of a graph G is defined as $D(G, x) = 1 + \sum\limits_{k=1}^{\omega(G)} (-1)^k c_k(G) x^k$, where $c_i(G)$ denotes a number of distinct cliques in G of the size i and $\omega(G)$ equals the size of a maximum clique in G.

Consider a graph $G(V,E)$ with $V = \{f_1,\dots,f_m\}\cup\{F_1,\dots,F_m\}$ and

$$E = \{(f_i,f_j) \mid i<j\}\cup\{(F_i,F_j)\mid i<j\}\cup\{(F_i,f_j)\mid i<j\}.$$

Note that the set of all basic words from L coincides with the set of all pairwise distinct words in $\mathrm{As}(G)$. It is known [5] that the generating function of $\mathrm{As}(G)$ equals to $1/D(G,x)$. Since

$$D(G,x) = \sum_{i=0}^{m}\binom{m}{i}(-1)^i(i+1)x^i = (1-x)^{m-1}(1-(m+1)x),$$

we conclude that the growth rate of $U_{\mathrm{Lie}_2}(\mathfrak{g})$ equals $m+1$. The fact that $\dfrac{1}{(1-x)^{m-1}(1-(m+1)x)}$ is a generating function of $U_{\mathrm{Lie}_2}(\mathfrak{g})$ follows from (12). Indeed, it is enough to recall that the generating function of the free commutative m-generated algebra equals $1/(1-x)^m$.

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