CLASSIFYING WORD PROBLEMS OF FINITELY GENERATED ALGEBRAS VIA COMPUTABLE REDUCIBILITY

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Abstract. We contribute to a recent research program which aims at revisiting the study of the complexity of word problems, a major area of research in combinatorial algebra, through the lens of the theory of computably enumerable equivalence relations (ceers), which has considerably grown in recent times. To pursue our analysis, we rely on the most popular way of assessing the complexity of ceers, that is via computable reducibility on equivalence relations, and its corresponding degree structure (the c-degrees). On the negative side, building on previous work of Kasymov and Khoussainov, we individuate a collection of c-degrees of ceers which cannot be realized by the word problem of any finitely generated algebra of finite type. On the positive side, we show that word problems of finitely generated semigroups realize a collection of c-degrees which embeds rich structures and is large in several reasonable ways.

1. Introduction

In recent years, computably enumerable (or, simply, c.e.) equivalence relations, often called *ceers* after [10], have been widely studied. One of the reasons motivating this interest lies in the fact that ceers arise naturally in combinatorial algebra as word problems of familiar c.e. algebraic structures like groups, semigroups, rings, and so on. By a *c.e. structure* A, we will mean in this paper a nontrivial algebraic-relational structure for which there exists a *c.e. presentation*, i.e. a structure A_ω of the same type as A but having universe ω, possessing uniformly computable operations, uniformly c.e. relations, and a ceer $=_A$ which is a congruence on A_ω such that $A \simeq A_{\omega/=_A}$, i.e. A is isomorphic with the quotient structure obtained by dividing A_ω by $=_A$. The ceer $=_A$ is called in this case the *word problem* of A (or, rather, of its given c.e. presentation). Selivanov's survey paper [31] (c.e. structures are therein called *positive structures*) and Khoussainov's survey paper [22] are excellent introductions to c.e. structures.

Word problems appeared in mathematics in 1911, when Dehn [6] introduced the word problem for finitely presented groups, with the goal of addressing the topological issue of deciding whether two knots are equivalent. Nowadays, much is known

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about the complexity of word problems for various algebraic structures. Most notably, the Novikov-Boone theorem [29, 5]—one of the most spectacular applications of computability theory to general mathematics—states that the word problem for finitely presented groups is undecidable.

Yet, a basic obstacle towards a full understanding of word problems is that the computability theoretic machinery commonly employed to measure their complexity (e.g., Turing reducibility) is defined for sets, while it is generally acknowledged that many computational facets of word problems emerge only if one interprets them as equivalence relations. For example, if G is a c.e. group, then it is immediate to see that the equivalence classes of $=_G$ are uniformly computably isomorphic with each other. It follows that in a group, the individual word problems (i.e., to decide equality to a word w, as w changes) have all the same complexity. This is not the case for semigroups: Shepherdson [32] proved that from any uniformly c.e. sequence $\{A_i : i \in \omega\}$ of sets, one can construct a finitely presented semigroup S such that the collection of the Turing degrees of the individual word problems of S (i.e. the equivalence classes of $=_S$) contains all the Turing degrees of the various sets A_i.

Hence, to enrich the study of word problems, one shall lift the underlying computability theoretic analysis from sets to equivalence relations. In this direction, it is important to mention that starting from [12], Khoussainov and other authors have conducted a systematic investigation of which c.e. structures have a word problem *coinciding* with a fixed ceer: see also [9, 11]. Particularly important to this line of research is an early result of Kasymov and Khoussainov [19] implying that no ceer whose principal transversal (see Definition 1.6) is hyperimmune can be the word problem of any finitely generated algebra of finite type. Other relevant papers, which investigate how the computability theoretic properties of a ceer affect the algebraic properties of the structures having that ceer as word problem, are [18, 23].

In this paper, we take a slightly different approach, namely we are interested in investigating which ceers can be *identified* in a broader sense with word problems of which c.e. structures, where, rather than mere coincidence, being identified means in this case to lie in the same reducibility degree with respect to some reducibility on equivalence relations which is suitable to measure their relative complexity. Pioneering attempts at this approach can be found for instance in [28, 7]. The most popular reducibility in this sense is the one given by next definition.

Definition 1.1. Given a pair of equivalence relations R, S on ω, we say that R is *computably reducible* (or, simply, c-*reducible*) to S (denoted by $R \leqslant_{\mathrm{c}} S$) if there exists a computable function f such that

$$(\forall x, y)[x \mathrel{R} y \Leftrightarrow f(x) \mathrel{S} f(y)].$$

By means of this reducibility, we identify two equivalence relations R, S if $R \leqslant_{\mathrm{c}} S$ and $S \leqslant_{\mathrm{c}} R$ (denoted by $R \equiv_{\mathrm{c}} S$). The c-*degree* of R is the equivalence class of R under the equivalence relation $\equiv_{\mathrm{c}}$. Given a c.e. structure A, let us say that a ceer R is c-*realized* by A, if R and $=_A$ have the same c-degree; let us also say that a c-degree of ceers is c-*realized* by A if some ceer in the c-degree is c-realized by A (clearly, this is equivalent to saying that all the ceers in the c-degree are c-realized by A). Special attention should be given to the problem of finding families of structures which are c-*complete for the ceers*, namely families of structures such that every ceer R is c-realized by some structure lying in the family.

As aforementioned, it is not difficult to see that the groups are not c-complete for the ceers (observed in [10]; see [7, Fact 3.6] for a proof). On the other hand, it has been shown in [7] that the semigroups are c-complete for the ceers. In fact, for every ceer R there exists a c.e. semigroup S such that the two ceers R and $=_S$ are c-equivalent, in fact they are isomorphic in the category of equivalence relations: recall that two equivalence relations U, V on ω are called *isomorphic* if there is a reduction f of U to V such that the range of f intersects all V-equivalence classes. This implies that the reduction is invertible, i.e. there is a reduction g of V to U such that f ad g invert each other on the equivalence classes, namely $x \; U \; g(f(x))$ and $x \; V \; f(g(x))$, for every number x: this isomorphism relation on equivalence relations has been considered in several papers, including [11, 12, 2]; for a justificaton of the name "isomorphism" given to it, see [8]. However, answering a question raised by Gao and Gerdes in [10], one can build a ceer (see [7]) which is not c-realized by any finitely generated semigroup, thus showing that the finitely generated semigroups are not c-complete for the ceers.

In Section 2 (where, on the negative side, we are interested in describing ceers which cannot be c-realized by finitely generated algebras of finite type) we show (Theorem 2.4) that if R is a "hyperdark" ceer (by Definition 1.4 this means that R has infinitely many equivalence classes and all of its infinite transversals are hyperimmune), then R cannot be c-realized by any finitely generated algebra of finite type. Thus, Theorem 2.4 generalizes to c-realizability the above mentioned result of Kasymov and Khoussainov [19] which, in the terminology of Definition 1.4, states that no hyperdark ceer can coincide with the word problem of any finitely generated algebra of finite type: indeed our proof is a straightforward sharpening of [19], based on the observation (see Lemma 2.3) that, for a ceer, the property that every infinite transversal is hyperimmune is invariant under c-equivalence. In Section 2.2 we give examples of hyperdark ceers: in particular, in Theorem 2.9 we exhibit an example of a hyperdark ceer whose equivalence classes are all finite. On the other hand (Theorem 2.10), we show that ceers having only finite classes are bound to have an infinite transversal which is not hyperhyperimmune (hence, they cannot be "hyperhyperdark", see again Definition 1.4 below).

On the positive side, in Section 3 we investigate the collection $\mathbf{S}_{f.g.}$ of the c-degrees of ceers which are c-realized by finitely generated semigroups. In this regard, Theorem 2.4 is optimal, since if we drop from hyperimmunity to immunity then c-unrealizability gets lost, as already shown by known results in the literature, including the remarkable theorem of Myasnikov and Osin [27], proving that in fact there exists an infinite finitely generated c.e. group with a word problem in which all infinite transversals are immune. Another useful example (which we sketch in some detail in Example 3.1) of an infinite two-generator c.e. semigroup with a word problem in which all infinite transversals are immune had been exhibited by Hirschfeldt and Khoussainov [15]. The rest of Section 3 investigates the subclass of $\mathbf{S}_{f.g.}$ consisting of the c-degreees of ceers possessing infinite transversals which are not immune. We prove that this subclass of $\mathbf{S}_{f.g.}$ is large in several reasonable ways: for instance, it contains an initial segment of the c-degrees of ceers with infinitely many equivalence classes, which is order-isomorphic with the tree $\omega^{<\omega}$ of finite strings of natural numbers, partially ordered by the prefix relation on strings.

1.1. **Notations and background.** Our main reference for computability theory is Soare's textbook [33], to which the reader is referred for all unexplained notions.

Throughout the paper when we talk about "degrees" without any further specification, we will mean "Turing degrees".

Let **Ceers** denote the collection of all ceers. A ceer is *infinite* if it has infinitely many equivalence classes, it is *finite* otherwise. Let the symbols **Inf**, **Fin**, **FinCl** and **InfCl** denote, respectively, the collection of infinite ceers, the collection of finite ceers, the collection of ceers having only finite equivalence classes, and the collection of ceers having only infinite equivalence classes.

Definition 1.2. The *cylindrification* of a ceer R is the ceer R_∞ given by

$$\langle i, x\rangle \, R_\infty \, \langle j, y\rangle \Leftrightarrow i \; R \; j,$$

where $\langle \cdot, \cdot \rangle$ denotes the Cantor pairing function.

Clearly $R_\infty \in \mathbf{InfCl}$, and $R \equiv_{\mathrm{c}} R_\infty$.

A ceer R is *dark* if $R \in \mathbf{Inf}$ and $\mathrm{Id}_\omega \nleq_{\mathrm{c}} R$, where Id_ω denotes the equality relation on ω. Let the symbol **dark** denote the collection of dark ceers. Dark ceers have been extensively studied in [2]. Being dark for a ceer can be conveniently described using the notion of a transversal for an equivalence relation.

Definition 1.3. If R is an equivalence relation on ω, we say that a set $T \subseteq \omega$ is a *transversal* of R if $x \not{R}\, y$, for every pair of distinct elements $x, y \in T$.

It is now easy to check that a ceer $R \in \mathbf{Inf}$ is dark if and only if it admits no infinite c.e. transversal, or equivalently every infinite transversal of R is *immune*, i.e. it does not contain any infinite c.e. set.[1]

Other stronger immunity notions have been widely considered in classical computability theory, and we briefly recall their definitions. An *array* of sets of natural numbers is a sequence $(X_n)_{n\in\omega}$ of sets of natural numbers. We say that a set $X \subseteq \omega$ *is intersected* by a disjoint array $(X_n)_{n\in\omega}$ (*disjoint* means that $X_n \cap X_m = \varnothing$ if $n \neq m$) if, for all n, $X_n \cap X \neq \varnothing$. An infinite set is *hyperimmune* if it is not intersected by any *strong disjoint array*, i.e. a disjoint array $(F_n)_{n\in\omega}$ of finite sets, presented by their canonical indices: hence $F_n = D_{f(n)}$ for some computable function f. Similarly, an infinite set is *hyperhyperimmune* if it is not intersected by any *weak disjoint array*, i.e. a disjoint array $(F_n)_{n\in\omega}$ of finite sets, presented by their c.e. indices: hence $F_n = W_{f(n)}$ for some computable function f.

In analogy with the definition of a dark ceer, these stronger immunity notions suggest accordingly the following definition.

Definition 1.4. A ceer R is *hyperdark* (respectively, *hyperhyperdark*) if $R \in \mathbf{Inf}$ and all of its infinite transversals are hyperimmune (respectively, hyperhyperimmune).

Let us use the notations **hdark** and **hhdark** to denote, respectively, the collections of hyperdark ceers and hyperhyperdark ceers. Clearly $\mathbf{hhdark} \subseteq \mathbf{hdark} \subseteq \mathbf{dark}$, as hyperhyperimmunity implies hyperimmunity, which in turn implies immunity. Counterexamples witnessing proper inclusions among these classes of ceers can be found by taking suitable *unidimensional* ceers R_X, i.e. ceers of the form $x \; R_X \; y$ if and only if $x, y \in X$ or $x = y$, where X is a given c.e. set, and recalling some well know facts of classical computability theory, which allow to draw the following

[1] We point out that, in the context of c.e. structures (for example in [15] and [23]), the terminology *algorithmically finite* algebra is used to denote a c.e. algebra whose word problem is dark.

conclusions: if X is simple but not hypersimple then $R_X \in \mathbf{dark} \smallsetminus \mathbf{hdark}$, and if X is hypersimple but not hyperhypersimple then $R_X \in \mathbf{hdark} \smallsetminus \mathbf{hhdark}$. Obviously, $R_X \in \mathbf{hhdark}$ if X is hyperhypersimple, hence all these classes are nonempty.

Remark 1.5. In order to distinguish between ceers and their c-degrees, given a class $\mathbf{P}$ of ceers we shall adopt the convention of denoting by $\mathbf{P}_{\mathrm{c}}$ the collection of c-degrees of the members of $\mathbf{P}$.

1.2. On the transversals of a ceer.

We conclude this section with some easy but useful observations about the transversals of ceers, in particular of hyperdark ceers. If R is an equivalence relation on ω, let us denote

$$\mathrm{Tr}(R) := \{T \in 2^\omega : T \text{ is a transversal of } R\}.$$

The following definition points out an important element of $\mathrm{Tr}(R)$.

Definition 1.6. Given an equivalence relation R, its *principal transversal* T_R is the set comprised of the least elements of all R-equivalence classes.

It is immediate to see that if R is a ceer then its principal transversal is co-c.e..

The relevance of the principal transversal in the investigation of hyperdarkness is highlighted by the following observations, where given any infinite set A of numbers, we denote by p_A the *principal function of* A, i.e. the function which enumerates A in order of magnitude.

Lemma 1.7 (Folklore)**.** *If R is an equivalence relation on ω with infinitely many equivalence classes, then for every infinite transversal T of R, the principal function p_T of T* majorizes *the principal function p_{T_R} of the principal transversal, i.e. $p_T(i) \geqslant p_{T_R}(i)$ for every $i \in \omega$.*

Proof. Let T be an infinite transversal of R and for simplicity write $p_{T_R}(i) := m_i$ and $p_T(i) := n_i$. Now, either for every $j < i$ there exists $k < i$ such that $n_j \mathrel{R} m_k$, but then $m_i \leqslant n_i$ since $n_i \notin \bigcup_{k<i}[m_k]_R$; or there exists $j < i$ such that $n_j \not\mathrel{R} m_k$ for every $k < i$, but then $n_j \notin \bigcup_{k<i}[m_k]_R$ hence $m_i \leqslant n_j < n_i$. □

Corollary 1.8. *If $R \in \mathbf{Inf}$ then $R \in \mathbf{hdark}$ if and only if T_R is hyperimmune.*

Proof. A well-known theorem by Kuznecov, Medvedev, and Uspenskii (see [33, Theorem 5.3.3]) states that an infinite set A is not hyperimmune if and only if there exists a computable function majorizing the principal function of A. Therefore if T_R is hyperimmune then so is any infinite transversal of R. □

2. Ceers not c-realized by finitely generated algebras of finite type

Theorem 2.4, the main result of this section, is essentially a consequence of Theorem 2.1 in [19] (see also [11, 12]), with the addition of our Lemma 2.3.

Theorem 2.1. [19] *If $A = (A, F)$ is an infinite c.e. algebra of finite type (i.e., F is a finite set of operations) and the word problem $=_A$ is hyperdark, then every finitely generated subalgebra of A is finite.*

Proof. For later reference we sketch the proof, taken from [12]. Let A be as in the statement of the theorem, and for every $f \in F$ let n_f denote the arity of f. Suppose that $X \subseteq A$ is finite, with $X \neq \emptyset$, but the subalgebra A_X of A, generated by X, is infinite. Define the sequence $(X_i)_{i\in\omega}$ of sets as follows. Let $X_0 := X$; having defined X_i let

$$X_{i+1} := X_i \cup \{y : (\exists f \in F)(\exists \vec{x} \in X_i^{n_f})[y = f(\vec{x})]\}.$$

Clearly each X_i is a finite set of which one can uniformly compute the canonical index, and the union $\bigcup_{i\in\omega} X_i$ gives the universe of A_X. Since A_X is infinite, we have that for every i there exists $y \in X_{i+1}$ such that $y \neq_A z$ for every $z \in X_i$. Thus we can define a sequence $(y_i)_{i\in\omega}$ such that $y_i \in X_{i+1}$ and $y_i \neq_A y_j$ if $i \neq j$, yielding that the set $T = \{y_i : i \in \omega\}$ is a transversal of $=_A$. The function $m(i) := \max(X_{i+1})$ is obviously computable and $\max(\{y_i : i \leqslant n\}) \leqslant m(n)$, for every n. On the other hand, it is clear that $p_T(n) \leqslant \max(\{y_i : i \leqslant n\})$, for every n. Therefore, $=_A$ is not hyperdark, as its infinite transversal T is not hyperimmune. □

Corollary 2.2. *If A is an infinite finitely generated c.e. algebra of finite type then the word problem $=_A$ of A is not hyperdark.*

Proof. Immediate. □

Before proving Theorem 2.4, we observe:

Lemma 2.3. *If $R \in$ **hdark**, $E \in$ **Inf**, and $E \leqslant_{\mathrm{c}} R$, then $E \in$ **hdark**.*

Proof. Suppose that $R \in$ **hdark**, $E \leqslant_{\mathrm{c}} R$, and $E \in$ **Inf** $\smallsetminus$ **hdark**. As $E \equiv_{\mathrm{c}} E_\infty$ and $R \equiv_{\mathrm{c}} R_\infty$, we have that $E_\infty \leqslant_{\mathrm{c}} R_\infty$. It is easy to see that if U, V are ceers with $U \leqslant_{\mathrm{c}} V$ and $V \in$ **InfCl** then $U \leqslant_{\mathrm{c}} V$ via a 1-1 computable function, see for instance [1, Remark 1.2]. Thus, suppose that f_0, f_1 are 1-1 computabe functions reducing $E \leqslant_{\mathrm{c}} E_\infty$ and $E_\infty \leqslant_{\mathrm{c}} R_\infty$, respectively. Let T be an infinite non-hyperimmune transversal of E. Clearly the set $\widehat{T} := (f_1 \circ f_0)[T]$ (i.e. the image of T under the composition $f_1 \circ f_0$) is an infinite transversal of R_∞, and one easily sees that $\widehat{T}$ is not hyperimmune, since, by injectivity, $f_1 \circ f_0$ maps any strong disjoint array intersecting T to a strong disjoint array intersecting $\widehat{T}$. By Lemma 1.7, it follows that the principal transversal T_{R_∞} of R_∞ is not hyperimmune, and thus the principal function $p_{T_{R_\infty}}$ of this transversal is majorized by some computable function g. On the other hand by definition of cylindrification, for every i we have that $p_{T_{R_\infty}}(i) = \langle n_i, 0\rangle$ for some n_i, and the set $\{n_i : i \in \omega\}$ coincides with the principal transversal T_R of R, with principal function $p_{T_R}(i) = n_i$. It immediately follows by Corollary 1.8 that T_R is not hyperimmune, as $p_{T_R}(i) = n_i \leqslant \langle n_i, 0\rangle = p_{T_{R_\infty}}(i) \leqslant g(i)$. □

Theorem 2.4. *If $R \in$ **hdark** then R is not* c*-realized by any finitely generated algebra of finite type.*

Proof. The claim follows by Corollary 2.2, and the fact that, by Lemma 2.3, membership in **hdark** is $\equiv_{\mathrm{c}}$-invariant, i.e. if E, R are ceers with $R \equiv_{\mathrm{c}} E$ then $E \in$ **hdark** if and only if $R \in$ **hdark**. □

2.1. **Π^0_1 classes consisting of infinite transversals.** An easy consequence of Theorem 2.4 is that for every infinite finitely generated c.e. algebra A of finite type there exists a nonempty Π^0_1 class containing only infinite transversals of the word problem of A. Recall that a subset $\mathcal{A}$ of the Cantor space 2^ω is called a Π^0_1 *class* if $\mathcal{A}$ has a Π^0_1 *definition*, i.e. is of the form $\mathcal{A} = \{A \in 2^\omega : (\forall n)R(A, n)\}$, for some decidable predicate $R \subseteq 2^\omega \times \omega$. (The Π^0_1 classes are also known as the *effectively closed* subsets of the Cantor space; it is well-known that a class $\mathcal{A} \subseteq 2^\omega$ is a Π^0_1 class if and only if $\mathcal{A}$ coincides with the collection of the infinite paths of some decidable tree).

Lemma 2.5. *For every ceer R, $\mathrm{Tr}(R)$ is a nonempty Π^0_1 class of the Cantor space.*

Proof. The claim follows from the observation that if R is a ceer then

$$\mathrm{Tr}(R) = \{T \in 2^\omega : (\forall\, x, y)[x, y \in T \;\&\; x \neq y \Rightarrow x \not R\; y]\},$$

which provides a description of $\mathrm{Tr}(R)$ as a Π^0_1 set since the complement of R is co-c.e. . □

Lemma 2.6. *If A is an infinite finitely generated c.e. algebra of finite type, then $\mathrm{Tr}(=_A)$ contains a nonempty Π^0_1 class of the Cantor space, consisting of infinite non-hyperimmune transversals.*

Proof. Let A be as in the statement of the lemma and let $\{X_i : i \in \omega\}$ be the class of finite sets constructed in the proof of Theorem 2.1, starting with $X_0 := X$, a finite set of generators of A. Consider

$$\mathcal{A} := \mathrm{Tr}(=_A) \cap \{T \in 2^\omega : (\forall i > 0\,)[T \cap X_i \neq \varnothing]\}.$$

By its very definition, all members of $\mathcal{A}$ are infinite and non-hyperimmune, and $\mathcal{A}$ is nonempty because it contains the transversal T built in the proof of Theorem 2.1. □

Remark 2.7. By well-known basis theorems for Π^0_1 classes of the Cantor space (see e.g. [17]) we have that the class $\mathcal{A}$ in the proof of Lemma 2.6 always contains transversals of special computability-theoretic interest, for instance transversals of low Turing degree, and transversals of hyperimmune-free degree (we recall that a set $X \leqslant_{\mathrm{T}} \varnothing'$ is *low* if $X' \equiv_{\mathrm{T}} \varnothing'$, and a set is of *hyperimmune-free degree* if its Turing degree does not contain any hyperimmune set).

In particular we see that the class $\mathcal{A}$ in the proof of Lemma 2.6 contains transversals of hyperimmune-free degree. Of course, none of these transversals can be the principal transversal if $=_A$ is undecidable, since for every undecidable ceer $R \in \mathbf{Inf}$, we have that its principal transversal T_R, being co-c.e. and not decidable, has hyperimmune degree.

2.2. **Hyperdark ceers: some examples.** By the discussion immediately following Definition 1.4 we know that hyperdark ceers do exist, as if X is a hypersimple set then R_X is hyperdark. This gives an example of a hyperdark ceer $R \notin \mathbf{FinCl} \cup \mathbf{InfCl}$. Its cylindrification R_∞ provides an example of a hyperdark ceer $R_\infty \in \mathbf{InfCl}$. We now provide an example lying in $\mathbf{FinCl}$. We first prove the following lemma.

Lemma 2.8. *There exists a ceer $R \in \mathbf{FinCl}$ such that $\varnothing' \leqslant_{\mathrm{T}} T$, for every infinite transversal T of R.*

Proof. The proof will make use of the well-known result (proved by Martin [26], and independently by Tennenbaum [34]) that a sufficient condition for $\varnothing' \leqslant_{\mathrm{T}} A$ is the existence of a function $g \leqslant_{\mathrm{T}} A$ which *dominates* every partial computable function, i.e. for every e there exists a number i_e such that for every $i \geqslant i_e$, if $\varphi_e(i)\downarrow$ then $\varphi_e(i) < g(i)$. Hence, to complete our task, it will be enough to build a ceer R with only finite equivalence classes and such that the function $n \mapsto p_{T_R}(n+1)$ dominates every partial computable function (where we recall that T_R denotes the principal transversal of R, and, given an infinite set A, the symbol p_A denotes the principal function of A). This will show, as argued at the end of the proof, that for every infinite transversal T of R, the function $g(n) := p_T(n+1)$ dominates all partial computable functions, and clearly $g \leqslant_{\mathrm{T}} T$.

Construction. Without loss of generality, we assume that for every e, i, s, if $\varphi_{e,s}(i)\downarrow$ then $\varphi_{e,s}(i) < s$. For every e, s, let

$$f_s(e) := \max\left(\{0\} \cup \{y : (\exists i, j \leqslant e)[\varphi_{i,s}(j)\downarrow= y]\}\right).$$

For all e, s, it holds that $f_s(e) \leqslant f_s(e+1)$ and $f_s(e) \leqslant f_{s+1}(e)$. Moreover, for every e there is a stage u such that, for every $s \geqslant u$, $f_s(e) = f_u(e)$. Hence, $f(e) = \lim_{s\to\infty} f_s(e)$ is well-defined for every e. To achieve our goal, we will try to satisfy, for every e, the requirement

$$\mathcal{R}_e : f(e) < p_{T_R}(e+1),$$

while guaranteeing that each R-equivalence class is finite: this latter goal will be achieved by building R as a ceer yielding a partition of ω in consecutive closed finite intervals. Notice that if $f(e) < p_{T_R}(e+1)$ for every e, then $\varphi_e(i) < p_{T_R}(i+1)$, for every pair of numbers e, i such that $e \leqslant i$, and $\varphi_e(i)\downarrow$.

For the requirements, consider the priority ordering $\mathcal{R}_i < \mathcal{R}_j$, if $i < j$.

We define R in stages, building a uniformly computable sequence $\{R_s\}_{s\in\omega}$ of decidable ceers, such that $R_s \subseteq R_{s+1}$ and $R = \bigcup_{s\in\omega} R_s$. At each stage s, our approximation R_s to R will be an equivalence relation partitioning ω in consecutive closed finite intervals $\{I_{j,s} : j \in \omega\}$, in such a way that the R_s-equivalence of any $x \geqslant s$ is a singleton.

Stage 0. Start up with $I_{j,0} := \{j\}$, for every j. Consequently, $R_0 = \mathrm{Id}_\omega$.

Stage $s+1$. We say that a requirement $\mathcal{R}_e$ *requires attention at stage* $s+1$ if $f_{s+1}(e) > \max(I_{e,s})$. By our assumption on how to approximate the partial computable functions, we may suppose that $f_{s+1}(e) < s+1$. So, at stage $s+1$, see if there is a requirement $\mathcal{R}_e$ with $e \leqslant s$ which requires attention. If not, then go to stage $s+2$, leaving unchanged each I_j. Otherwise, let $\mathcal{R}_e$ be the highest priority requirement which requires attention. Define

$$I_{j,s+1} := \begin{cases} I_{j,s}, & \text{if } j < e,\\ [\min(I_{e,s}), s]\,, & \text{if } j = e,\\ \{s+j-e\}, & \text{if } j > e. \end{cases}$$

We say in this case that $\mathcal{R}_e$ *acts*; clearly, we have that $f_{s+1}(e) \leqslant \max(I_{e,s+1})$ after acting. Notice that every $I_{j,s+1}$ is obtained by collapsing finitely many consecutive intervals into just one interval; clearly $\max(I_{j,s}) \leqslant \max(I_{j,s+1})$, for every j. The ceer R_{s+1} is the ceer corresponding to the new family of intervals $\{I_{j,s+1} : j \in \omega\}$; clearly $R_s \subseteq R_{s+1}$; notice also that the R_{s+1}-equivalence of any $x \geqslant s+1$ is a singleton. Go to the next stage.

Verification. A straightforward argument by induction on the priority of the requirements shows that for every e the requirement $\mathcal{R}_e$ eventually stops requiring attention, the set I_e reaches its limit, and $f(e) \leqslant \max(I_e)$. To see these claims, let s_0 be the least stage such that for all $i < e$, we have that $\mathcal{R}_i$ does not receive attention and I_i does not change at any stage $s > s_0$. Notice that $\mathcal{R}_e$ may require attention at most finitely many times after s_0, as $f_s(e)$ may change only finitely many times, and if $\mathcal{R}_e$ acts at a stage u such that $f_u(e) = f(e)$, then it will never require attention again at any later stage t since $f(e) \leqslant \max(I_{e,u}) \leqslant \max(I_{e,t})$ for every $t \geqslant u$. As a consequence, either $\mathcal{R}_e$ never requires attention at any $s \geqslant s_0$, and thus $I_{e,s} = I_{e,s_0}$ for every $s \geqslant s_0$, or $\mathcal{R}_e$ requires attention for the last time at some $s_1 \geqslant s_0$, giving that $I_{e,s} = I_{e,s_1}$ for every $s \geqslant s_1$. In either case $I_{e,s}$ reaches its limit I_e, and $f(e) \leqslant \max(I_e)$.

Having shown that for every e, the interval I_e reaches its limit and $f(e) \leqslant \max(I_e)$, it is now straightforward to conclude that $f(e) < \min(I_{e+1}) = p_{T_R}(e+1)$. By Lemma 1.7, if T is any infinite transversal of R, then $p_{T_R}(e+1) \leqslant p_T(e+1)$, and thus the function $g(e) := p_T(e+1)$ dominates all partial computable functions. □

Theorem 2.9. $\mathbf{hdark} \cap \mathbf{FinCl} \neq \varnothing$.

Proof. We first show that if $R \in \mathbf{Inf}$ is a ceer such that $\varnothing' \leqslant_{\mathrm{T}} T$ for every infinite transversal T of R, then $R \in \mathbf{hdark}$. Clearly, for such an R we have $R \in \mathbf{non\text{-}low}$, where **non-low** denotes the class of infinite ceers possessing no low infinite transversal. On the other hand, one can show that $\mathbf{non\text{-}low} \subseteq \mathbf{hdark}$. To see this, assume that U is an infinite ceer such that $U \notin \mathbf{hdark}$, thus U possesses an infinite transversal T which is not hyperimmune as witnessed by a strong disjoint array $(D_{f(n)})_{n\in\omega}$, and consider the class of sets

$$\mathcal{A} := \mathrm{Tr}(U) \cap \{X \in 2^\omega : (\forall n)[X \cap D_{f(n)} \neq \varnothing]\}.$$

It is easy to see that $\mathcal{A}$ is a Π^0_1 class of the Cantor space, as by Lemma 2.5 $\mathcal{A}$ is the intersection of two Π^0_1 classes. Moreover, by the very definition of $\mathcal{A}$, it is clear that all members of $\mathcal{A}$ must be infinite, and $\mathcal{A} \neq \varnothing$ since $T \in \mathcal{A}$. Therefore, by the Low Basis Theorem for Π^0_1 classes (see e.g. [33]), $\mathcal{A}$ contains a low member, that is an infinite low transversal of U, thus $U \notin \mathbf{non\text{-}low}$. By contrapositive, this shows that $\mathbf{non\text{-}low} \subseteq \mathbf{hdark}$.

The theorem now follows from Lemma 2.8. □

Theorem 2.9 exhibits a ceer $R \in \mathbf{FinCl}$ such that every infinite transversal of R is of hyperhyperimmune degree. In fact, our example is built so that every infinite transversal of R computes $\varnothing'$, and thus it is of hyperhyperimmune degree by [16, Corollary 4.2]. On the other hand at least one of the infinite transversals of R is not hyperhyperimmune, because our next theorem shows that hyperhyperdarkness becomes an empty notion, when considering only ceers in **FinCl**.

Theorem 2.10. $\mathbf{hhdark} \cap \mathbf{FinCl} = \varnothing$.

Proof. Suppose that $R \in \mathbf{FinCl}$. We are going to show that there exists a transversal $T \in \mathrm{Tr}(R)$ and a weak disjoint array $(F_n)_{n\in\omega}$ which intersects T, so that T is not hyperhyperimmune. Throughout the proof we refer to some fixed computable approximation $\{R_s : s \in \omega\}$ to R, namely a sequence of uniformly decidable equivalence relations $\{R_s\}_{s\in\omega}$, such that $R_0 := \mathrm{Id}_\omega$, $R_s \subseteq R_{s+1}$, and $R := \bigcup_{s\in\omega} R_s$: every ceer has such an approximation, see e.g. [1, Lemma 1.4]. We construct $(F_n)_{n\in\omega}$ in stages, so that at stage s we define $F_{n,s}$ for every n, and $(F_{n,s})_{n,s\in\omega}$ is a strong array which provides a uniformly computable approximation to the desired weak array $(F_n)_{n\in\omega}$, where $F_n := \bigcup_{s\in\omega} F_{n,s}$.

At stage 0 let $F_{n,0} := \varnothing$ for every n. At stage $s+1$, let n be the least number such that $F_{n,s} \subseteq \bigcup_{i<n}[F_{i,s}]_{R_s}$ (such an n exists since all but finitely many $F_{m,s}$ are

empty). Pick the least fresh number x (thus x does not lie in any $F_{m,s}$) and define $F_{n,s+1} := F_{n,s} \cup \{x\}$; for all $m \neq n$ let $F_{m,s+1} := F_{m,s}$.

Verification. The array $(F_n)_{n\in\omega}$ is clearly disjoint. An easy inductive argument shows: for every n there exists a least stage s_n such that $F_{n,s} = F_{n,s_n}$ for all $s > s_n$ (we use here that $R \in \mathbf{FinCl}$); for all n, $F_n \smallsetminus \bigcup_{m<n}[F_m]_R \neq \emptyset$. Therefore $(F_n)_{n\in\omega}$ is a weak disjoint array, such that for every n, the set $T_n := F_n \smallsetminus \bigcup_{m<n}[F_m]_R$ is nonempty. For every n, pick $t_n := \min(T_n)$. Then $T := \{t_n : n \in \omega\}$ is an infinite transversal of R, which is not hyperimmune since it is intersected by the disjoint weak array $(F_n)_{n\in\omega}$. □

3. Ceers c-realized by finitely generated semigroups

In the previous section we have isolated a class of infinite ceers (the hyperdark ceers) which are not c-realized by any finitely generated algebra of finite type. If we restrict our attention to finitely generated semigroups, and denote by $\mathbf{S}_{f.g.}$ the structure of c-degrees of ceers which are c-realized by word problems of finitely generated semigroups, it follows by the results of the previous section that $\mathbf{S}_{f.g.} \cap \mathbf{hdark}_c = \emptyset$. In the present section we will face the opposite problem, trying to individuate classes of c-degrees of ceers which lie in $\mathbf{S}_{f.g.}$. It is clear that every finite c-degree is in $\mathbf{S}_{f.g.}$ since every finite ceer is c-realized by some finite finitely presented group. Therefore, we will confine ourselves to discuss how large the class $\mathbf{S}^{\infty}_{f.g.} := \mathbf{S}_{f.g.} \cap \mathbf{Inf}_c$ is, i.e. the class consisting of the c-degrees in $\mathbf{S}_{f.g.}$ containing infinite ceers, or equivalently of the c-degrees of ceers realized by word problems of infinite finitely generated semigroups.

We will work over the free semigroup on two generators: hence, let us consider the alphabet $X := \{a, b\}$ and denote by X^+ the set of nonempty finite words of elements of X, with λ denoting the empty word, and the binary operation on X^+ given by the concatenation of words. It is well known that X^+ together with this binary operation is the free semigroup on two generators. In the rest of this discussion, X^+ will be identified by coding with the set of natural numbers ω.

3.1. **Finitely generated semigroups with dark word problem.**

First, we remark that, for the case of semigroups, Theorem 2.4 is optimal, in the sense that, while the word problem of any finitely generated semigroup cannot be c-equivalent to a hyperdark ceer, there exist two-generator semigroups with dark word problem. The following example is taken from Hirschfeldt and Khoussainov [15] (see, in particular, Lemmas 2.2, 2.3, 2.4 and Theorem 3.7).

Example 3.1. [15] Let us first introduce some terminology and notation. Given words $x, y \in X^+$, we say that y is a *subword* of x if there are words $u_1, u_2 \in X^+ \cup \{\lambda\}$ such that $x = u_1 y u_2$. Similarly, $y \in X^+$ is a *subword* of an infinite sequence $f \in X^\omega$ if there exist a word $u \in X^+ \cup \{\lambda\}$ and an infinite sequence $g \in X^\omega$ such that $f = uyg$. We say that a word or an infinite sequence α *avoids* a finite word y whenever α does not contain y as a subword. Finally, for every $l \in \omega$, let us denote by $X^{\leqslant l}$ and $X^{\geqslant l}$ the set of words on the alphabet X of length, respectively, at most l and at least l.

For $Z \subseteq X^+$, let

$$\overline{Z} := \{u \in X^+ : \ (\exists z \in Z)(\exists u_1, u_2 \in X^+ \cup \{\lambda\})[u = u_1 z u_2]\}.$$

In other words, $\overline{Z}$ is the set of words in X^+ containing a word of Z as a subword. Notice that $Z \subseteq \overline{Z}$. It is easy to see that the unidimensional ceer $R_{\overline{Z}}$ is a congruence of X^+ for every set $Z \subseteq X^+$: hence, for any c.e. set $Z \subseteq X^+$, we get a finitely generated c.e. semigroup $S_Z = X^+/_{R_{\overline{Z}}}$.

By the discussion following Definition 1.4 we know that a unidimensional ceer R_X is dark if and only if X is simple. Therefore, in order to get that S_Z is infinite with a dark word problem, it suffices to build a simple set $Z \subseteq X^+$ such that $\overline{Z}$ is coinfinite: this ensures that $\overline{Z}$ is simple, being a coifinite c.e. superset of a simple set. This can be achieved using the following result by Miller [30, Corollary 2.2]: If a set $Y \subseteq X^+$ contains, for each i, at most one word of length $i+5$ and no words of length $\leqslant 4$, then there is an infinite sequence $f \in X^\omega$ such that f avoids all the words in Y.

So suppose that $Z \subseteq X^+$ is a set such that $Z \cap X^{\leqslant k+4}$ contains at most k elements for every number k. From this and using the fact that avoiding a word implies avoiding all its extensions, it is easy to build a set $Y \subseteq X^+$, such that Y contains exactly a string of length $i+5$ for every number i, it contains no string of length $\leqslant 4$, and if a string avoids all words in Y then it avoids all words in Z as well. By Miller's result there is an infinite sequence $f \in X^\omega$ avoiding Y, and thus there are infinitely many finite words avoiding Z, implying that $\overline{Z}$ is coinfinite.

Therefore, to complete our example it remains only to show that there exists a simple set $Z \subseteq X^+$ such that for every k, Z contains at most k words of length $\leqslant k+4$. This follows along the lines of the standard Post's construction of a simple set (see, e.g. [33], Theorem 5.2.3): Given a standard numbering $\{W_i : i \in \omega\}$ of the c.e. subsets of X^+, we let

$$Z = \{u \in X^+ : (\exists i)(\exists s)[u \in W_{i,s+1} \cap X^{\geqslant i+5} \text{ and } W_{i,s} \cap X^{\geqslant i+5} = \varnothing]\}.$$

In other words, we enumerate each set W_i until a word u of length at least $i+5$ appears: whenever this happens, we enumerate u into Z, and we do not put any more elements from W_i into Z. Notice that Z is simple as every infinite c.e. set W_i must contain words of arbitrary length. Moreover, by definition of Z, a word $u \in Z \cap X^{\leqslant k+4}$ must have been taken from a set W_i with $i \leqslant k-1$, which ensures that Z contains at most k such words.

As already anticipated in the introduction, we also notice that the above example has been strikingly strengthened by Myasnikov and Osin in [27], where it is even built a finitely generated c.e. group with dark word problem.

3.2. **Finitely generated semigroups with non-dark word problem.** We partition X^+, the set of nonempty words on the alphabet $\{a, b\}$, as follows:

(1) $C := \{ab^ia : i \neq 0\}$ (where for any string u, we denote by u^i the string obtained by concatenating i times u with itself). We refer to the elements of C as *coding words*.
(2) C_+ consists of the words in X^+ which properly contain coding words as subwords, i.e.

$$C_+ := \{w \in X^+ \smallsetminus C : (\exists v, v' \in X^+ \cup \{\lambda\})(\exists u \in C)[w = vuv']\}.$$

(3) $C_- := X^+ \smallsetminus (C \cup C_+)$.

Observe that the sets C_-, C, and C_+ are computable, infinite, and they partition X^+.

Next, given any ceer R, let $S(R)$ be the two-generator semigroup presented by

$$S(R) := \langle X \mid \{ab^{i+1}a =_{S(R)} ab^{j+1}a : i \mathrel{R} j\} \cup \{v =_{S(R)} w : v, w \in C_+\}\rangle.$$

Thus, the $=_{S(R)}$-closure of the set C is partitioned in classes, with representatives $ab^i a$ for each $i \neq 0$, and $ab^i a =_{S(R)} ab^j a$ if and only if $(i-1) \mathrel{R} (j-1)$. The $=_{S(R)}$-closure $[C_+]_{=_{S(R)}}$ of C_+ consists of just an equivalence class, say the $=_{S(R)}$-equivalence class of $aaba$. Finally the set $[C_-]_{=_{S(R)}}$ consists of an infinite bunch of singletons (namely the singletons of words which avoid the words of the form $ab^i a$ with $i \neq 0$).

Recall that the *uniform join* $U \oplus V$ of two equivalence relations U, V on ω is the equivalence relation on ω defined as $U \oplus V := \{(2x, 2y) : x \mathrel{U} y\} \cup \{(2x+1, 2y+1) : x \mathrel{V} y\}$.

Lemma 3.2. *$=_{S(R)}$ is c-equivalent (in fact, isomorphic, as defined in the paragraph following Definition 1.1) with $R \oplus \mathrm{Id}_\omega$, where we recall that Id_ω denotes the equality relation on ω.*

Proof. We recall the notion of the *restriction* of a ceer R to a nonempty c.e. set W, see for instance [3]: fix a computable surjection $\pi : \omega \to W$, and define $R{\restriction}W$ to be the ceer

$$x \mathrel{R{\restriction}W} y \Leftrightarrow \pi(x) \mathrel{R} \pi(y).$$

It is immediate to see that, up to $\equiv_{\mathrm{c}}$, $R{\restriction}W$ does not depend on the chosen computable surjection.

By pairwise disjointness of the $=_{S(R)}$-closures of the computable sets C, C_+, C_- which yield a partition of ω, we have that

$$=_{S(R)} \equiv_{\mathrm{c}} \big((=_{S(R)} {\restriction} C) \oplus (=_{S(R)} {\restriction} C_+) \oplus (=_{S(R)} {\restriction} C_-)\big).$$

On the other hand, it is easily seen that $=_{S(R)} {\restriction} C \equiv_{\mathrm{c}} R$, $=_{S(R)} {\restriction} C_+ \equiv_{\mathrm{c}} \mathrm{Id}_1$ (where Id_1 is the ceer with just one equivalence class) and $=_{S(R)} {\restriction} C_- \equiv_{\mathrm{c}} \mathrm{Id}_\omega$. Hence $=_{S(R)} \equiv_{\mathrm{c}} R \oplus \mathrm{Id}_1 \oplus \mathrm{Id}_\omega$, and from $\mathrm{Id}_\omega \equiv_{\mathrm{c}} \mathrm{Id}_1 \oplus \mathrm{Id}_\omega$ we conclude

$$=_{S(R)} \equiv_{\mathrm{c}} R \oplus \mathrm{Id}_\omega\,.$$

Finally it is easy to see that all the bi-equivalences $\equiv_{\mathrm{c}}$ mentioned in the proof are in fact isomorphisms of equivalence relations, so $=_{S(R)}$ is isomorphic with $R \oplus \mathrm{Id}_\omega$. □

Theorem 3.3. *If R is a ceer such that $R \equiv_{\mathrm{c}} R \oplus \mathrm{Id}_\omega$, then there is a two-generator semigroup S such that $=_S \equiv_{\mathrm{c}} R$.*

Proof. Given R, take $S := S(R)$. □

Remark 3.4. Notice that $=_{S(R)} \in$ **Inf** $\smallsetminus$ **dark**. The ceers in **Inf** $\smallsetminus$ **dark** are frequently called *light*, and their class is denoted as **light**, see e.g. [2].

3.3. **How large is $\mathbf{S}_{f.g.}$?** We conclude this section with a few remarks which suggests that the class of ceers which are c-realized by finitely generated semigroups is unexpectedly large.

Theorem 3.3 allows indeed to show that $\mathbf{S}^\infty_{f.g.}$ (in fact the subclass of $\mathbf{S}^\infty_{f.g.}$ consisting of the c-degrees of light ceers) embeds rich structures. First, recall the reducibility notion of the following definition, where, for every $n \geqslant 1$, we fix a ceer Id_n with exactly n equivalence classes.

Definition 3.5. [2] For ceers R and S, $R \leqslant_{\mathcal{I}} S$ if and only if $R \leqslant_{\mathrm{c}} S \oplus \mathrm{Id}_n$, for some number $n \geqslant 1$.

It is known (see [2, 4]) that the structure of the $\mathcal{I}$-degrees of dark ceers, denoted as $\mathbf{dark}_{/\mathcal{I}}$, is in a reasonable sense as complicated as possible: its first order theory is computably isomorphic with first order arithmetic.

Corollary 3.6. $\mathbf{dark}_{/\mathcal{I}}$ *embeds into* $\mathbf{S}^{\infty}_{f.g.}$.

Proof. Let R and S be any pair of dark ceers. Andrews and Sorbi [2, Theorem 6.2] proved that $R \equiv_{\mathcal{I}} S$ if and only if $R \oplus \mathrm{Id}_\omega \equiv_{\mathrm{c}} S \oplus \mathrm{Id}_\omega$. This implies that the map

$$\iota : R \mapsto R \oplus \mathrm{Id}_\omega$$

induces an embedding of $\mathbf{dark}_{/\mathcal{I}}$ into $\mathbf{light}_{/\mathcal{I}}$, see [4, Lemma 6.2], where of course $\mathbf{light}_{/\mathcal{I}}$ denotes the $\mathcal{I}$-degrees of light ceers. To conclude, it suffices to note that, by Theorem 3.3, the image of the embedding induced by ι is contained in $\mathbf{S}^{\infty}_{f.g.}$. □

Finally, we recall that (as proved in [3]) one can embed the tree $(\omega^{<\omega}, \subseteq)$ of finite strings of natural numbers, partially ordered by the prefix relation on strings, as an initial segment of $\mathbf{Inf}_{\mathrm{c}}$.

Corollary 3.7. *The tree* $(\omega^{<\omega}, \subseteq)$ *embeds as an initial segment of* $\mathbf{Inf}_{\mathrm{c}}$ *in such a way that the range of the embedding is included in* $\mathbf{S}^{\infty}_{f.g.}$.

Proof. The proof follows by Theorem 3.3 and by the fact that [3, Corollary 3.1] shows that the tree $\omega^{<\omega}$ can be embedded as an initial segment of $\mathbf{Inf}_{\mathrm{c}}$ in such a way that the range of the embedding is included in the c-degrees of the ceers R such that $R \equiv_{\mathrm{c}} R \oplus \mathrm{Id}_\omega$. □

Notice that the embeddings mentioned in the proofs of both Corollary 3.6 and Corollary 3.7 take as images c-degrees of light ceers, in particular of ceers of the form $R \oplus \mathrm{Id}_\omega$.

3.4. **What about finitely presented semigroups?** It might be worth noticing that if $R \in \mathbf{FinCl}$ and R is undecidable then $S(R)$, as described above, is not finitely presentable. Indeed, Litvinceva [25] showed that, if S is a finitely presented semigroup such that $=_S$ has only finitely many infinite equivalence classes, then $=_S$ is decidable. Now, if $R \in \mathbf{FinCl}$ and R is undecidable then $=_{S(R)}$ has only one infinite equivalence class, namely $[C_+]_{=_{S(R)}}$, but on the other hand $=_{S(R)}$ is undecidable, as $R \leqslant_{\mathrm{c}} =_{S(R)}$.

If one looks only for finitely generated semigroups S such that $=_S \in \mathbf{FinCl}$, then a slight modification of the proof of Lemma 3.2 shows how to build a finitely generated semigroup S, such that $=_S \in \mathbf{FinCl}$ and $R \leqslant_{\mathrm{c}} =_S$, starting from any ceer $R \in \mathbf{FinCl}$. For this, it is enough to take $S := \langle X \mid \{ab^{i+1}a =_S ab^{j+1}a : iRj\}\rangle$. To show that $=_S \in \mathbf{FinCl}$, one easily sees that if $v \in C_+$ and $u_1, \dots, u_n$ are the subwords of v of the form $ab^{i_j+1}a$ for some number i_j with $1 \leqslant j \leqslant n$, occurring in distinct places of v, and k_j is the number of elements in the R-class of i_j, then the cardinality of the $=_S$-class of v equals the product $k_1 \cdot k_2 \cdot \ldots \cdot k_n$. Once again, by [25] if R is undecidable then no such S is finitely presentable.

4. Future research

The research program of locating which c-degrees of ceers are c-realized by word problems of suitable algebras is vast and, to our knowledge, plenty of questions remain, untouched. We conclude by listing a few research lines that may inspire future work.

(1) Let $\mathbf{S}_{f.p.}$ and $\mathbf{S}_{c.e.}$ denote the c-degrees of ceers which are c-realized by the word problems of, respectively, finitely presented semigroups and c.e. semigroups. As aforementioned in the introduction, $\mathbf{S}_{c.e.}$ coincides with $\mathbf{Ceers}_{\mathrm{c}}$ and, by Theorem 2.4 above, $\mathbf{S}_{c.e.} \smallsetminus \mathbf{S}_{f.g.}$ contains every hyperdark c-degree. Then, it is natural to ask if $\mathbf{S}_{f.g.} \smallsetminus \mathbf{S}_{f.p.}$ is also nonempty. If this is the case, one may try to compare $\mathbf{S}_{f.g.}$ and $\mathbf{S}_{f.p.}$ with respect to the initial segments of $\mathbf{Inf}_{\mathrm{c}}$ that they realize in the sense of Corollary 3.7.

(2) More generally, it may be interesting to move the focus from semigroups to other algebraic varieties $\mathcal{V}$, and to investigate which c-degrees of ceers are c-realized by members of $\mathcal{V}$, and in particular by finitely presented or finitely generated members of $\mathcal{V}$. The case of groups appears to be natural and challenging at the same time.

(3) The literature is also rich of papers studying presentations of structures but using coceers instead of ceers (a *coceer* is an equivalence relation whose complement is c.e.), and investigating which structures have a "word problem" coinciding with a given coceer: see for instance [24, 20, 13]. A more recent variation ([21, 14]) studies which random structures have a "word problem" coinciding with a given ceer or coceer. In all these cases, it would be interesting to generalize these approaches to c-realizability, aiming at describing those structures having word problems which are c-realizable (not just merely coinciding) with given ceers or coceers.

References

[1] U. Andrews, S. Badaev, and A. Sorbi. A survey on universal computably enumerable equivalence relations. In A. Day, M. Fellows, N. Greenberg, B. Khoussainov, A. Melnikov, and F. Rosamond, editors, *Computability and Complexity. Essays Dedicated to Rodney G. Downey on the Occasion of His 60th Birthday*, pages 418–451. Springer, 2017.

[2] U. Andrews and A. Sorbi. Joins and meets in the structure of ceers. *Computability*, 8(3-4):193–241, 2019.

[3] U. Andrews and A. Sorbi. Initial segments of the degrees of ceers. *J. Symbolic Logic*, 87(3):1260––1283, 2022.

[4] Uri Andrews, Noah Schweber, and Andrea Sorbi. The theory of ceers computes true arithmetic. *Ann. Pure Appl. Logic*, 171(8):102811, 2020.

[5] W. W. Boone. The word problem. *Ann. of Math.*, 70(2):207–265, 1959.

[6] M. Dehn. Über unendliche diskontinuierliche gruppen. *Mathematische Annalen*, 71(1):116–144, 1911.

[7] V. Delle Rose, L. San Mauro, and A. Sorbi. Word problems and ceers. *MLQ Math. Log. Q.*, 66(3):341–354, 2020.

[8] V. Delle Rose, L. San Mauro, and A. Sorbi. The category of equivalence relations. *Algebra and Logic*, 60(5):295–307, 2021. Russian Original Vol. 60, No. 5, September-October, 2021.

[9] E. Fokina, B. Khoussainov, P. Semukhin, and D. Turetsky. Linear orders realized by c.e. equivalence relations. *J. Symb. Log.*, 81(2):463–482, 2016.

[10] S. Gao and P. Gerdes. Computably enumerable equivalence relations. *Studia Logica*, 67(1):27–59, 2001.

[11] A. Gavruskin, S. Jain, B. Khoussainov, and F. Stephan. Graphs realised by r.e. equivalence relations. *Ann. Pure Appl. Logic*, 165(7-8):1263–1290, 2014.

[12] A. Gavryushkin, B. Khoussainov, and F. Stephan. Reducibilities among equivalence relations induced by recursively enumerable structures. *Theoret. Comput. Sci.*, 612:137–152, 2016.
[13] M. T. Godziszewski. *Formal Theories of Truth and Nonstandard Models of Arithmetic*. PhD thesis, University of Warsaw, 2022.
[14] M. Harrison-Trainor, B. Khoussainov, and D. Turetsky. Effective aspects of algorithmically random structures. *Computability*, 8(3-4):359–375, 2019.
[15] D. R. Hirschfeldt and B. Khoussainov. Finitely presented expansions of computably enumerable semigroups. *Algebra and logic*, 51(5):435–444, 2012.
[16] C. G. Jockusch, Jr. The degrees of hyperhyperimmune sets. *J. Symbolic Logic*, 34:489–493, 1969.
[17] C. G. Jockusch, Jr. and R. I. Soare. Degrees of members of Π^0_1 classes. *Pacific J. Math.*, 40:605–616, 1972.
[18] N. Kh. Kasymov. Algebras with residually finite positively presented expansions. *Algebra i Logika*, 26(6):715–730, 771, 1987.
[19] N. Kh. Kasymov and B. M. Khoussainov. Finitely generated enumerable and absolutely locally finite algebras. *Vychisl. Sistemy*, (116, Prikladn. Logika):3–15, 162, 1986.
[20] N. Kh. Kasymov and A. S. Morozov. On the definability of linear orders over negative equivalences. *Algebra Logika*, 55(1):37–57, 2016.
[21] B. Khoussainov. A quest for algorithmically random infinite structures. In *Proceedings of the Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages Article No. 56, 9. ACM, New York, 2014.
[22] B. Khoussainov. A journey to computably enumerable structures (tutorial lectures). In F. Manea, R. G. Miller, and D. Nowotka, editors, *Sailing Routes in the World of Computation, 14th Conference on Computability in Europe, CiE 2018 Kiel, Germany, July 30 August 3, 2018 Proceedings*, volume 10936 of *LNCS*, pages 1–19. Springer, Cham, 2018.
[23] B. Khoussainov and A. Miasnikov. Finitely presented expansions of groups, semigroups, and algebras. *Trans. Amer. Math. Soc.*, 366(3):1455–1474, 2014.
[24] B. Khoussainov, T. Slaman, and P. Semukhin. Π^0_1-presentations of algebras. *Arch. Math. Logic*, 45(6):769—-781, 2006.
[25] Z. K. Litvinceva. The complexity of individual identity problems in semigroups. *Algebra and Logic*, 9:172–199, 1970.
[26] D. A. Martin. Classes of recursively enumerable sets and degrees of unsolvability. *Z. Math. Logik Grundlagen Math.*, 12:295–310, 1966.
[27] A. Myasnikov and D. Osin. Algorithmically finite groups. *J. Pure Appl. Algebra*, 215(11):2789–2796, 2011.
[28] A. Nies and A. Sorbi. Calibrating word problems of groups via the complexity of equivalence relations. *Mathematical Structures in Computer Science*, pages 1–15, 2018.
[29] P. S. Novikov. On the algorithmic unsolvability of the word problem in group theory. *Trudy Mat. Inst. Steklov.*, 44:143, 1955. (Russian).
[30] Miller J. S. Two notes on subshifts. *Proceedings of the American Mathematical Society*, 140(5):1617–1622, 2012.
[31] V. Selivanov. Positive structures. In S. B. Cooper and S. S. Goncharov, editors, *Computability and Models*, pages 321–350. Springer, New York, 2003.
[32] J. C. Shepherdson. Machine configuration and word problems of given degree of unsolvability. *Z. Math. Logik Grundlagen Math.*, 11:149–175, 1965.
[33] R. I. Soare. *Turing computability*. Theory and Applications of Computability. Springer-Verlag, Berlin, 2016. Theory and applications.
[34] S. Tennenbaum. Degree of unsolvability and the rate of growth of functions. In *Proc. Sympos. Math. Theory of Automata (New York, 1962)*, pages 71–73. Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y., 1963.

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