On the equitable vertex arboricity of complete tripartite graphs *

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Abstract

The equitable coloring problem, introduced by Meyer in 1973, has received considerable attention and research. Recently, Wu et al. introduced the concept of equitable (t,k)-tree-coloring, which can be viewed as a generalization of proper equitable t-coloring. The strong equitable vertex k-arboricity of complete bipartite equipartition graphs was investigated in 2013. In this paper, we study the exact value of the strong equitable vertex 3-arboricity of complete equipartition tripartite graphs.

Keywords: equitable coloring, vertex k-arboricity, k-tree-coloring, complete equipartition tripartite graph.

AMS subject classification 2010: 05C05; 05C12; 05C35.

1 Introduction

In this paper, all graphs considered are finite and simple. For a real number x, $\lceil x \rceil$ is the least integer not less than x and $\lfloor x \rfloor$ is the largest integer not larger than x. We use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, edge set, minimum degree and maximum degree of G, respectively. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of

*Research supported by the National Science Foundation of China (Nos. 61164005 and 11161037), the National Basic Research Program of China (No. 2010CB334708) and the Program for Changjiang Scholars and Innovative Research Team in Universities (No. IRT1068), the Research Fund for the Chunhui Program of Ministry of Education of China (No. Z2014022) and the Nature Science Foundation from Qinghai Province (Nos. 2012-Z-943, 2014-ZJ-721 and 2014-ZJ-907).

neighbors of v in G and $d_G(v) = |N_G(v)|$ denote the degree of v. We use $d(v)$ instead of $d_G(v)$ for brevity. For other undefined concepts, we refer the reader to [3].

We associate positive integers $1, 2, \dots, t$ with colors and call f a *t-coloring* of a graph G if f is a mapping from $V(G)$ to $\{1, 2, \dots, t\}$. A t-coloring of G is proper if any two adjacent vertices have different colors. For $1 \leq i \leq t$, let $V_i = \{v \mid f(v) = i\}$. A t-coloring of a graph G is said to be *equitable* if $||V_i| - |V_j|| \leq 1$ for all i and j, that is to say, every color class has size $\lfloor |V(G)|/t \rfloor$ or $\lceil |V(G)|/t \rceil$. A graph G is said to be *properly equitably t-colorable* if G has a proper equitable t-coloring. The smallest number t for which G is properly equitably t-colorable is called *the equitable chromatic number* of G, denoted by $\chi^{=}(G)$.

The equitable coloring problem, introduced by Meyer [8], is motivated by a practical application to municipal garbage collection [10]. In this context, the vertices of the graph represent garbage collection routes. A pair of vertices share an edge if the corresponding routes should not be run on the same day. It is desirable that the number of routes ran on each day be approximately the same. Therefore, the problem of assigning one of the six weekly working days to each route reduces to finding a proper equitable 6-coloring. For more applications such as scheduling, constructing timetables and load balance in parallel memory systems, we refer to [1, 2, 4, 6, 7, 9].

A properly equitably t-colorable graph may admit no proper equitable t'-colorings for some $t' > t$. For example, the complete bipartite graph $H := K_{2m+1,2m+1}$ has no proper equitable $(2m+1)$-colorings, although it satisfies $\chi^{=}(H) = 2$. This fact motivates another interesting parameter for proper equitable coloring. The equitable chromatic threshold of G, denoted by $\chi(G)$, is the smallest integer t such that G has a proper equitable t'-coloring for all $t' \geq t$. This notion was introduced by Fan et al. in [5].

In [5], Fan et al. considered relaxed equitable coloring of graphs. They proved that every graph has a proper equitable $\Delta(G)$-coloring such that each color class induces a forest with maximum degree at most one. On the basis of this research, Wu et al. [11] introduced the notion of equitable (t, k)-tree-coloring, which can be viewed as a generalization of proper equitable t-coloring.

A *(t, k)-tree-coloring* is a t-coloring f of G such that each component of $G[V_i]$ is a tree of maximum degree at most k. An *equitable (t, k)-tree-coloring* is a (t, k)-tree-coloring that is equitable. The *equitable vertex k-arboricity* of a graph G, denoted by $va_k{}^{=}(G)$, is the smallest integer t such that G has an equitable (t, k)-tree-coloring. The *strong equitable vertex k-arboricity* of G, denoted by $va_k{}^{\equiv}(G)$, is the smallest integer t such that G has an equitable (t', k)-tree-coloring for every $t' \geq t$. It is clear that $va_0{}^{=}(G) = \chi^{=}(G)$ and $va_0{}^{\equiv}(G) = \chi^{\equiv}(G)$ for every graph G, and $va_k{}^{=}(G)$ and $va_k{}^{\equiv}(G)$ may vary a lot.

In [11], Wu et al. investigated the strong equitable vertex k-arboricity of complete equipartition bipartite graphs and gave the bounds for $va_1{}^{\equiv}(K_{n,n})$ and $va_\infty{}^{\equiv}(K_{n,n})$. In this paper, we study the strong equitable vertex 3-arboricity of complete equipartition tripartite graphs. In fact, we obtain the exact value of $va_3{}^{\equiv}(K_{n,n,n})$ for most cases.

Theorem 1 *If $n \equiv 3 \ (mod\ 4)$, then $va_3{}^{\equiv}(K_{n,n,n}) \leq 3\lfloor \frac{n+1}{4} \rfloor$.*

Theorem 2 *Let k be a positive integer.*

(i) For $k \equiv 1 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k,4k,4k}) = \frac{12k+3}{5}$.

(ii) For $k \equiv 2 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k,4k,4k}) = \frac{12k+6}{5}$.

(iii) For $k \equiv 3 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k,4k,4k}) = \frac{12k+4}{5}$.

(iv) For $k \equiv 4 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k,4k,4k}) = \frac{12k-3}{5}$.

(v) For $k \equiv 0 \ (mod\ 5)$, if $m \geq 2q+1$ where $q \geq 2$, then $va_3{}^{\equiv}(K_{4k,4k,4k}) \leq 12m-3q-3$; if $m \geq 2q$ where $q \geq 0$, then $va_3{}^{\equiv}(K_{4k,4k,4k}) \leq 12m-3q$.

Theorem 3 *Let k be a positive integer.*

(i) For $k \equiv \ 2(mod\ 5)$, $va_3{}^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k+6}{5}$.

(ii) For $k \equiv 3 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k+9}{5}$.

(iii) For $k \equiv 4 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k+7}{5}$.

(iv) For $k \equiv 0 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k}{5}$.

(v) For $k \equiv 1 \ (mod\ 5)$, if $m \geq 2q+1$ where $q \geq 1$, then $va_3{}^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m-3q$; if $m \geq 2q$ where $q \geq 0$, then $va_3{}^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m-3q+3$.

Theorem 4 *Let k be a positive integer.*

(i) For $k \equiv 3 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) = \frac{12k+9}{5}$.

(ii) For $k \equiv 4 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) = \frac{12k+12}{5}$.

(iii) For $k \equiv 0 \ (mod\ 5)$, $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) = \frac{12k+10}{5}$.

(iv) For $k \equiv 1 \ (mod\ 5)$ $(k \neq 1)$, $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) = \frac{12k+3}{5}$.

(v) For $k \equiv 2 \ (mod\ 5)$, if $m \geq 2q+1$ where $q \geq 1$, then $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 12m-3q+3$; if $m \geq 2q$ where $q \geq 0$, then $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 12m-3q+6$.

2 Preliminary

Proposition 1 *The complete ℓ-partite graph $K_{n,\ldots,n}$ has an equitable (t,k)-tree-coloring for every integer t satisfying the following condition:*

$$t = \ell h \ (h \geq 1).$$

Proof. We can easily construct an equitable (t,k)-tree-coloring of $K_{n,\ldots,n}$ by dividing each partite set into h classes equitably and coloring the vertices of each class with one color. ∎

Theorem 5 *Let $K_{n,n,n}$ $(n \geq 3)$ be a complete tripartite graph. Then*

$$va_3{}^{\equiv}(K_{n,n,n}) \leq 3\left\lfloor \frac{n+1}{4} \right\rfloor$$

Proof. By Proposition 1, in order to show $va_3{}^{\equiv}(K_{n,n,n}) \leq 3\lfloor \frac{n+1}{4} \rfloor$, it suffices to prove that $K_{n,n,n}$ has an equitable $(q,3)$-tree-coloring for every q satisfying the following condition: $q \geq 3\lfloor \frac{n+1}{4} \rfloor + 1$ and $3 \nmid q$. Let X,Y and Z be the partite sets of $K_{n,n,n}$.

Let $q = 3a+1$ and $q \geq 3\lfloor \frac{n+1}{4} \rfloor + 1$. Note that $4q - 3n \geq 12\lfloor \frac{n+1}{4} \rfloor + 4 - 3n \geq 12(\frac{n-2}{4}) + 4 - 3n = -2$.

If $q = \frac{3n-2}{4}$, then $4q - 3n = -2$ and $q = 4 + 3t$, $n = 6 + 4t$ $(t = 0, 1, 2, \cdots)$. Let $\{x_1, x_2, x_3, x_4, x_5, x_6\} \subset X$, $\{y_1, y_2, y_3, y_4, y_5, y_6\} \subset Y$ and $\{z_1, z_2, z_3, z_4, z_5, z_6\} \subset Z$. We color x_1, x_2, x_3 and y_1 with 1 and color y_2, y_3, y_4, y_5 and y_6 with 2 and color x_4, x_5, x_6 and z_6 with 3 and color z_1, z_2, z_3, z_4 and z_5 with 4. We divide each of $X\backslash\{x_1, x_2, x_3, x_4, x_5, x_6\}$, $Y\backslash\{y_1, y_2, y_3, y_4, y_5, y_6\}$ and $Z\backslash\{z_1, z_2, z_3, z_4, z_5, z_6\}$ into $\frac{q-4}{3}$ classes equitably and color the vertices of each class with a color in $\{5, \cdots, q\}$. Since $\frac{n-6}{\frac{q-4}{3}} = 4$, the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 4 or 5.

If $\frac{3n-2}{4} < q < \frac{3n+4}{4}$, then $\frac{3n-2}{4} < 3a+1 < \frac{3n+4}{4}$, and hence $n - 2 < 4a < n$. Then $4a = n - 1$, $a = \frac{n-1}{4}$ and $q = \frac{3n+1}{4}$. Therefore, $4q - 3n = 1$ and $q = 4 + 3t$, $n = 5 + 4t$ $(t = 0, 1, 2, \cdots)$. Let $\{x_1, x_2, x_3, x_4, x_5\} \subset X$, $\{y_1, y_2, y_3, y_4, y_5\} \subset Y$ and $\{z_1, z_2, z_3, z_4, z_5\} \subset Z$. We color x_1, x_2, x_3 and y_1 with 1 and color x_4, y_2, y_3 and y_4 with 2 and color x_5, z_1, z_2 and z_3 with 3 and color y_5, z_4 and z_5 with 4. We divide each of $X\backslash\{x_1, x_2, x_3, x_4, x_5\}, Y\backslash\{y_1, y_2, y_3, y_4, y_5\}$ and $Z\backslash\{z_1, z_2, z_3, z_4, z_5\}$ into $\frac{q-4}{3}$ classes equitably and color the vertices of each class with a color in $\{5, \cdots, q\}$. Since $\frac{n-5}{\frac{q-4}{3}} = 4$, it follows that the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 3 or 4.

Suppose $\frac{3n+4}{4} \leq q \leq n$. Let $\{x_1, x_2, x_3, x_4\} \subset X$, $\{y_1, y_2, y_3, y_4\} \subset Y$ and $\{z_1, z_2, z_3, z_4\} \subset Z$. We color x_1, x_2 and y_1 with 1 and color z_1, z_2 and y_2 with 2 and color x_3, x_4 and y_3 with 3 and color y_4, z_3 and z_4 with 4. We divide each of $X\backslash\{x_1, x_2, x_3, x_4\}, Y\backslash\{y_1, y_2, y_3, y_4\}$ and $Z\backslash\{z_1, z_2, z_3, z_4\}$ into $\frac{q-4}{3}$ classes equitably and color the vertices of each class with a color in $\{5, \cdots, q\}$. Since $\left\lfloor \frac{n-4}{\frac{q-4}{3}} \right\rfloor = 3$ and $\left\lceil \frac{n-4}{\frac{q-4}{3}} \right\rceil = 4$, it follows that the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 3 or 4.

Suppose $n + 1 \leq q \leq \frac{3n-1}{2}$. Let $e = xy$ be edge of $K_{n,n,n}$ with $x \in X$ and $y \in Y$. We color x, y with 1 and divide each of $X\backslash\{x\}$, $Y\backslash\{y\}$ and Z into $\frac{q-1}{3}$ classes equitably and color the vertices of each class with a color in $\{2, \ldots, q\}$. One can easily check that the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 2 or 3.

Claim 1: q can not be in $(\frac{3n-1}{2}, \frac{3n+2}{2})$.

Proof of Claim 1: Assume, to the contrary, that there exist a integer q such that q is in

$(\frac{3n-1}{2}, \frac{3n+2}{2})$. Then

$$\frac{3n-1}{2} < 3a+1 < \frac{3n+2}{2}.$$

One can see that $n-1 < 2a < n$, a contradiction. ∎

Suppose $\frac{3n+2}{2} \leq q \leq 3(n-1)+1$. Let $e = xy$ be an edge of $K_{n,n,n}$ with $x \in X$ and $y \in Y$. We color x and y with 1 and divide each of $X \backslash \{x\}, Y \backslash \{y\}$ and Z into $\frac{q-1}{3}$ classes equitably and color the vertices of each class with a color in $\{2, \dots, q\}$. One can easily check that the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 1 or 2.

Let $q = 3a+2$ and $q \geq 3\lfloor \frac{n+1}{4} \rfloor + 2$, we note that $4q - 3n \geq 12\lfloor \frac{n+1}{4} \rfloor + 8 - 3n \geq 12(\frac{n-2}{4}) + 8 - 3n = 2$.

Suppose $\frac{3n+2}{4} \leq q \leq n$. Let $\{x_1, x_2\} \subset X, \{y_1, y_2\} \subset Y$ and $\{z_1, z_2\} \subset Z$. We color x_1, x_2 and y_1 with 1 and color y_2, z_1 and z_2 with 2 and divide each of $X \backslash \{x_1, x_2\}, Y \backslash \{y_1, y_2\}$ and $Z \backslash \{z_1, Z_2\}$ into $\frac{q-2}{3}$ classes equitably and color the vertices of each class with a color in $\{3, \dots, q\}$. One can easily check that the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 3 or 4.

Suppose $n \leq q \leq \frac{3n-2}{2}$. Let $\{x_1, x_2\} \subset X, \{y_1, y_2\} \subset Y$ and $\{z_1, z_2\} \subset Z$. We color x_1, x_2 and y_1 with 1 and color y_2, z_1 and z_2 with 2 and divide each of $X \backslash \{x_1, x_2\}, Y \backslash \{y_1, y_2\}$ and $Z \backslash \{z_1, Z_2\}$ into $\frac{q-2}{3}$ classes equitably and color the vertices of each class with a color in $\{3, \dots, q\}$. One can easily check that the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 2 or 3.

Claim 2: q can not be in $(\frac{3n-2}{2}, \frac{3n+1}{2})$.

Proof of Claim 2: Assume, to the contrary, that there exist a q such that q is in $(\frac{3n-2}{2}, \frac{3n+1}{2})$. Then,

$$\frac{3n-2}{2} < 3a+2 < \frac{3n+1}{2}.$$

We find that $n-2 < 2a < n-1$, a contradiction. ∎

Suppose $\frac{3n+1}{2} \leq q \leq 3(n-1)+2$. Let $e = xy$ be an edge of $K_{n,n,n}$ with $x \in X$ and $y \in Y$. Let z be a vertex of Z. We color x and y with 1 and color z with 2 and divide each of $X \backslash \{x\}, Y \backslash \{y\}$ and $Z \backslash \{z\}$ into $\frac{q-2}{3}$ classes equitably and color the vertices of each class with a color in $\{3, \dots, q\}$. One can easily check that the resulting coloring is an equitable $(q,3)$-tree-coloring of $K_{n,n,n}$ with the size of each color class being 1 or 2. ∎

3 Main results

We now in a position to give our main results.

3.1 The strong equitable vertex 3-arboricity of $K_{4k+3,4k+3,4k+3}$

From Theorem 5, we can give a proof of Theorem 1.

Proof of Theorem 1: From Theorem 5, we have $va_3^{\equiv}(K_{n,n,n}) \leq 3\lfloor \frac{n+1}{4} \rfloor$. ∎

Note that $K_{3,3,3}$ can attain the upper bound of Theorem 1. One can check that $va_3^{\equiv}(K_{3,3,3}) = 3$.

3.2 The strong equitable vertex 3-arboricity of $K_{4k,4k,4k}$

We investigate the strong equitable vertex 3-arboricity of the complete tripartite graph $K_{4k,4k,4k}$.

The following upper bound of $K_{4k,4k,4k}$ can be proved easily.

Proposition 2 *If $k \geq 5m$ $(m \geq 0)$ and $k \neq 0$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq 3k - 3m$.*

Proof. We prove the proposition by induction on m. If $m = 0$, then the result holds by Theorem 5. Assume $m \geq 1$. Since $k \geq 5m > 5(m-1)$, by the induction hypothesis and Proposition 1, we need to prove that $K_{4k,4k,4k}$ has an equitable $(3k-3m+1, 3)$-tree-coloring and an equitable $(3k-3m+2, 3)$-tree-coloring.

Divide X into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, k-m+1\}$. Divide Y into $k-m$ classes equitably and color the vertices of each class with a color in $\{k-m+2, \cdots, 2k-2m+1\}$. Divide Z into $k-m$ classes equitably and color the vertices of each class with a color in $\{2k-2m+2, \cdots, 3k-3m+1\}$. It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(3k-3m+1, 3)$-tree-coloring with the size of each color class being 4 or 5.

Divide X into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, k-m+1\}$. Divide Y into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{k-m+2, \cdots, 2k-2m+2\}$. Divide Z into $k-m$ classes equitably and color the vertices of each class with a color in $\{2k-2m+3, \cdots, 3k-3m+2\}$. It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(3k-3m+2, 3)$-tree-coloring with the size of each color class being 4 or 5. ∎

The following corollaries are immediate.

Corollary 1 *Let k be a positive integer.*

(i) If $k \equiv 1$ (mod 5), then $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k+3}{5}$.

(ii) If $k \equiv 2$ (mod 5), then $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k+6}{5}$.

(iii) If $k \equiv 3$(mod 5), then $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k+9}{5}$.

(iv) If $k \equiv 4$ (mod 5), then $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k+12}{5}$.

Lemma 1 *If $k \equiv 1 (mod\ 5)$, then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k+3}{5}$.*

Proof. Let $k = 5m+1$. We only need to show that $K_{4k,4k,4k}$ has no equitable $(12m+2, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m+2, 3)$-tree-coloring of $K_{4k,4k,4k}$. Then the size of every color class in c is at least 5 because $\lfloor \frac{12k}{12m+2} \rfloor = \lfloor \frac{60m+12}{12m+2} \rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 12$$

$$c_5 + c_6 = 12m + 2.$$

We have the unique solution $c_5 = 12m$, $c_6 = 2$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k,4k,4k}$, it follows that $K_{4k,4k,4k}$ has no equitable$(12m+2, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k+3}{5}$. ∎

Lemma 2 *If $k \equiv 2 (mod\ 5)$, then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k+6}{5}$.*

Proof. Let $k = 5m+2$. We only need to show that $K_{4k,4k,4k}$ has no equitable $(12m+5, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m+5, 3)$-tree-coloring of $K_{4k,4k,4k}$. Then the size of every color class in c is at least 4 because $\lfloor \frac{12k}{12m+5} \rfloor = \lfloor \frac{60m+24}{12m+5} \rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 24$$

$$c_4 + c_5 = 12m + 5.$$

We have the unique solution $c_4 = 1$, $c_5 = 12m + 4$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k,4k,4k}$, it follows that $K_{4k,4k,4k}$ has no equitable$(12m + 5, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k+6}{5}$. ∎

Lemma 3 *If $k \equiv 3 (mod\ 5)$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k+4}{5}$.*

Proof. Form (iii) of Corollary1, we have $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k+9}{5}$. Let $k = 5m + 3$. We only need to show that $K_{4k,4k,4k}$ has an equitable $(12m+8, 3)$-tree-coloring. Then the size of every color class is at least 4 because $\lfloor \frac{12k}{12m+8} \rfloor = \lfloor \frac{60m+36}{12m+8} \rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 36$$

$$c_4 + c_5 = 12m + 8.$$

We have the unique solution $c_4 = 4$, $c_5 = 12m + 4$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k,4k,4k}$, there are $4m + 2$ color classes containing exactly 5 vertices in some partite set of $K_{4k,4k,4k}$ and there are $4m + 1$ color classes containing exactly 5 vertices in other partite sets of $K_{4k,4k,4k}$. Since there are $20m + 12$ vertices in every partite set of $K_{4k,4k,4k}$, there are 2 vertices of color class containing exactly 4 vertices in some partite set and there are 7 vertices of color class containing exactly 4 vertices in other partite sets. In this case, $K_{4k,4k,4k}$ has an equitable $(12m + 8, 3)$-tree-coloring. ∎

Lemma 4 *If $k \equiv 3(mod\ 5)$, then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k+4}{5}$.*

Proof. Let $k = 5m+3$. We only need to show that $K_{4k,4k,4k}$ has no equitable $(12m+7, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m + 7, 3)$-tree-coloring of $K_{4k,4k,4k}$. Then the size of every color class in c is at least 5 because $\lfloor \frac{12k}{12m+7} \rfloor = \lfloor \frac{60m+36}{12m+7} \rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 36$$

$$c_5 + c_6 = 12m + 7.$$

We have the unique solution $c_5 = 12m + 6$, $c_6 = 1$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k,4k,4k}$, it follows that $K_{4k,4k,4k}$ has no equitable$(12m + 7, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k+4}{5}$. ∎

Lemma 5 *If $k \equiv 4(mod\ 5)$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k-3}{5}$.*

Proof. Form (iv) of Corollary1, we have $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k+12}{5}$. Let $k = 5m + 4$. We need to show that $K_{4k,4k,4k}$ has an equitable $(12m + 11, 3)$-tree-coloring and an equitable $(12m + 10, 3)$-tree-coloring.

If $K_{4k,4k,4k}$ has an equitable $(12m + 11, 3)$-tree-coloring , then the size of every color class is at least 4 because $\lfloor \frac{12k}{12m+11} \rfloor = \lfloor \frac{60m+48}{12m+11} \rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 48$$

$$c_4 + c_5 = 12m + 11.$$

We have the unique solution $c_4 = 7$, $c_5 = 12m + 4$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k,4k,4k}$, there are $4m + 2$ color classes containing exactly 5 vertices in some partite set of $K_{4k,4k,4k}$ and there are $4m + 1$ color classes containing exactly 5 vertices in other partite sets of $K_{4k,4k,4k}$. Since there are $20m + 16$ vertices in every partite set of $K_{4k,4k,4k}$, there are 6 vertices of color class containing exactly 4 vertices in some partite set and there are 11 vertices of color class containing exactly 4 vertices in other partite sets. In this case, $K_{4k,4k,4k}$ has an equitable $(12m + 11, 3)$-tree-coloring.

We can prove that $K_{4k,4k,4k}$ has an equitable $(12m+10, 3)$-tree-coloring using an similar argument.

Form the above argument and Proposition 1, we prove that $va_3^{\equiv}(K_{4k,4k,4k}) \leq \frac{12k-3}{5}$. ∎

Lemma 6 *If $k \equiv 4 (mod\ 5)$, then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k-3}{5}$.*

Proof. Let $k = 5m+4$. We only need to show that $K_{4k,4k,4k}$ has no equitable $(12m+8, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m + 8, 3)$-tree-coloring of $K_{4k,4k,4k}$. Then the size of every color class in c is at least 5 because $\lfloor \frac{12k}{12m+8} \rfloor = \lfloor \frac{60m+48}{12m+8} \rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 48$$

$$c_5 + c_6 = 12m + 8.$$

We have the unique solution $c_5 = 12m$, $c_6 = 8$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k,4k,4k}$, it follows that $K_{4k,4k,4k}$ has no equitable$(12m+8, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k,4k,4k}) \geq \frac{12k-3}{5}$. ∎

In the following, we consider the remaining case $k \equiv 0 (mod\ 5)$ and give the proof of (v) of Theorem 2.

Lemma 7 *If $k = 5m$ and $m \geq 2$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq 12m - 3$.*

Proof. By Proposition 2, $va_3^{\equiv}(K_{4k,4k,4k}) \leq 12m$. So we need to prove that $K_{4k,4k,4k}$ has an equitable $(12m - 2, 3)$-tree-coloring and an equitable $(12m - 1, 3)$-tree-coloring by Proposition 1.

Divide X into $4m-1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m - 1\}$. Divide Y into $4m - 1$ classes equitably and color the vertices of each class with a color in $\{4m, \cdots, 8m-2\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m-1, \cdots, 12m-2\}$. It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(12m-2, 3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m-1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m-1\}$. Divide Y into $4m$ classes equitably and color the vertices of each class with a color in $\{4m, \cdots, 8m-1\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m, \cdots, 12m-1\}$. It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(12m-1, 3)$-tree-coloring with the size of each color class being 5 or 6. ∎

Lemma 8 *If $k = 5m$ and $m \geq 4$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq 12m - 6$.*

Proof. Similarly to the proof of Lemma 5, we can prove it. ∎

Lemma 9 *If $k = 5m$ and $m \geq 2q + 1$ where $q \geq 2$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq 12m - 3q - 3$.*

Proof. We prove this theorem by induction on q. Suppose $q = 2$. It is equivalent to prove that if $m \geq 5$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq 12m - 9$. Since $m \geq 5 > 4$, it follows Lemma 6 from $va_3^{\equiv}(K_{4k,4k,4k}) \leq 12m - 6$. We need to prove that $K_{4k,4k,4k}$ has an equitable $(12m-8, 3)$-tree-coloring and an equitable $(12m-7, 3)$-tree-coloring by Proposition 1.

Divide X into $4m-3$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m-3\}$. Divide Y into $4m-3$ classes equitably and color the vertices of each class with a color in $\{4m-2, \cdots, 8m-6\}$. Divide Z into $4m-2$ classes equitably and color the vertices of each class with a color in $\{8m-5, \cdots, 12m-8\}$. It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(12m-8, 3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m-3$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m-3\}$. Divide Y into $4m-2$ classes equitably and color the vertices of each class with a color in $\{4m-2, \cdots, 8m-5\}$. Divide Z into $4m-2$ classes equitably and color the vertices of each class with a color in $\{8m-4, \cdots, 12m-7\}$. It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(12m-7, 3)$-tree-coloring with the size of each color class being 5 or 6.

Suppose $q \geq 3$. Since $m \geq 2q + 1 > 2(q-1) + 1$, by the induction hypothesis and Proposition 1, we need to prove that $K_{4k,4k,4k}$ has an equitable $(12m-3q-1, 3)$-tree-coloring and an equitable $(12m-3q-2, 3)$-tree-coloring.

Divide X into $4m-q-1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m-q-1\}$. Divide Y into $4m-q-1$ classes equitably and color the vertices of each class with a color in $\{4m-q, \cdots, 8m-2q-2\}$. Divide Z into $4m-q$ classes equitably and color the vertices of each class with a color in $\{8m-2q-1, \cdots, 12m-3q-2\}$.

It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(12m-3q-2,3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m-q-1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m-q-1\}$. Divide Y into $4m-q$ classes equitably and color the vertices of each class with a color in $\{4m-q,\cdots,8m-2q-1\}$. Divide Z into $4m-q$ classes equitably and color the vertices of each class with a color in $\{8m-2q,\cdots,12m-3q-1\}$. It is easy to check that the resulting coloring of $K_{4k,4k,4k}$ is an equitable $(12m-3q-1,3)$-tree-coloring with the size of each color class being 5 or 6. ∎

Lemma 10 *If $k=5m$ and $m\geq 2q$, where $q\geq 0$, then $va_3^{\equiv}(K_{4k,4k,4k})\leq 12m-3q$.*

Proof. When $q=0,1,2$, the result holds by Proposition 2, Lemma 5 and Lemma 6. If $q\geq 3$, then $q-1\geq 2$. Since $m\geq 2q>2(q-1)+1$, it follows that $va_3^{\equiv}(K_{4k,4k,4k})\leq 12m-3(q-1)-3=12m-3q$. ∎

Proof of Theorem 2: Suppose $k\equiv 1(mod\ 5)$. From (i) of Corollary 1 and Lemma 1, $va_3^{\equiv}(K_{4k,4k,4k})=\frac{12k+3}{5}$. Suppose $k\equiv 2(mod\ 5)$. From (ii) of Corollary1 and Lemma 2, $va_3^{\equiv}(K_{4k,4k,4k})=\frac{12k+6}{5}$. Suppose $k\equiv 3(mod\ 5)$. From Lemma 3 and Lemma 4, $va_3^{\equiv}(K_{4k,4k,4k})=\frac{12k+4}{5}$. Suppose $k\equiv 4(mod\ 5)$. From Lemma 5 and Lemma 6, $va_4^{\equiv}(K_{4k,4k,4k})=\frac{12k-3}{5}$. Suppose $k\equiv 0(mod\ 5)$. From Lemma 9, if $k=5m$ and $m\geq 2q+1$ where $q\geq 2$, then $va_4^{\equiv}(K_{4k,4k,4k})\leq 12m-3q-3$. From Lemma 10, if $k=5m$ and $m\geq 2q$ where $q\geq 0$, then $va_4^{\equiv}(K_{4k,4k,4k})\leq 12m-3q$. ∎

3.3 The strong equitable vertex 3-arboricity of $K_{4k+1,4k+1,4k+1}$

We investigate the strong equitable vertex 3-arboricity of the complete tripartite graph $K_{4k+1,4k+1,4k+1}$.

The following upper bound of $K_{4k+1,4k+1,4k+1}$ can be proved easily.

Proposition 3 *If $k\geq 5m+1$ where $m\geq 0$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\leq 3k-3m$.*

Proof. We prove the proposition by induction on m. If $m=0$, the result holds by Theorem 5.

Suppose $m\geq 1$. Since $k\geq 5m+1>5(m-1)+1$, by the induction hypothesis and Proposition 1, we need to prove that $K_{4k+1,4k+1,4k+1}$ has an equitable $(3k-3m+1,3)$-tree-coloring and an equitable $(3k-3m+2,3)$-tree-coloring.

Divide X into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,k-m+1\}$. Divide Y into $k-m$ classes equitably and color the vertices of each class with a color in $\{k-m+2,\cdots,2k-2m+1\}$. Divide Z into $k-m$ classes equitably and color the vertices of each class with a color in $\{2k-2m+2,\cdots,3k-3m+1\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(3k-3m+1,3)$-tree-coloring with the size of each color class being 4 or 5.

Divide X into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,k-m+1\}$. Divide Y into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{k-m+2,\cdots,2k-2m+2\}$. Divide Z into $k-m$ classes equitably and color the vertices of each class with a color in $\{2k-2m+3,\cdots,3k-3m+2\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(3k-3m+2,3)$-tree-coloring with the size of each color class being 4 or 5. ∎

The following corollaries are immediate.

Corollary 2 *(i) If $k\equiv\ 2(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\leq\frac{12k+6}{5}$.*

(ii) If $k\equiv\ 3(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\leq\frac{12k+9}{5}$.

(iii) If $k\equiv\ 4(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\leq\frac{12k+12}{5}$.

(iv) If $k\equiv\ 0(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\leq\frac{12k+15}{5}$.

Lemma 11 *If $k\equiv 2(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\geq\frac{12k+6}{5}$.*

Proof. Let $k=5m+2$. We only need to show that $K_{4k+1,4k+1,4k+1}$ has no equitable $(12m+5,3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m+5,3)$-tree-coloring of $K_{4k+1,4k+1,4k+1}$. Then the size of every color class in c is exactly 5 because $\frac{12k+3}{12m+5}=\frac{60m+25}{12m+5}=5$. However, the size of every partite sets is $20m+9$, which is not divisible by 5, a contradiction. ∎

Lemma 12 *If $k\equiv 3(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\geq\frac{12k+9}{5}$.*

Proof. Let $k=5m+3$. We only need to show that $K_{4k+1,4k+1,4k+1}$ has no equitable $(12m+8,3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m+8,3)$-tree-coloring of $K_{4k+1,4k+1,4k+1}$. Then the size of every color class in c is at least 4 because $\lfloor\frac{12k+3}{12m+8}\rfloor=\lfloor\frac{60m+39}{12m+8}\rfloor=4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i=4,5$. Then we have the following two equations:

$$4c_4+5c_5=60m+39$$

$$c_4+c_5=12m+8.$$

We have the unique solution $c_4=1$, $c_5=12m+7$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k+1,4k+1,4k+1}$, it follows that $K_{4k+1,4k+1,4k+1}$ has no equitable$(12m+8,3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1})\geq\frac{12k+9}{5}$. ∎

Lemma 13 *If $k \equiv 4(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq \frac{12k+7}{5}$.*

Proof. Form (iii) of Corollary2, we have $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq \frac{12k+12}{5}$. Let $k = 5m+4$. We only need to show that $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m + 11, 3)$-tree-coloring. Then the size of every color class is at least 4 because $\lfloor \frac{12k+3}{12m+11} \rfloor = \lfloor \frac{60m+51}{12m+11} \rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 51$$

$$c_4 + c_5 = 12m + 11.$$

We have the unique solution $c_4 = 4$, $c_5 = 12m+7$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k+1,4k+1,4k+1}$, there are $4m+3$ color classes containing exactly 5 vertices in some partite set of $K_{4k+1,4k+1,4k+1}$ and there are $4m + 2$ color classes containing exactly 5 vertices in other partite sets of $K_{4k+1,4k+1,4k+1}$. Since there are $20m + 17$ vertices in every partite set of $K_{4k+1,4k+1,4k+1}$, there are 2 vertices of color class containing exactly 4 vertices in some partite set and there are 7 vertices of color class containing exactly 4 vertices in other partite sets. In this case, $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m + 11, 3)$-tree-coloring. ∎

Lemma 14 *If $k \equiv 4(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \geq \frac{12k+7}{5}$.*

Proof. Let $k = 5m + 4$. We only need to show that $K_{4k+1,4k+1,4k+1}$ has no equitable $(12m + 10, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m + 10, 3)$-tree-coloring of $K_{4k+1,4k+1,4k+1}$. Then the size of every color class in c is at least 5 because $\lfloor \frac{12k+3}{12m+10} \rfloor = \lfloor \frac{60m+51}{12m+10} \rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 51$$

$$c_5 + c_6 = 12m + 10.$$

We have the unique solution $c_5 = 12m + 9$, $c_6 = 1$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k+1,4k+1,4k+1}$, it follows that $K_{4k+1,4k+1,4k+1}$ has no equitable$(12m+10, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \geq \frac{12k+7}{5}$. ∎

Lemma 15 *If $k \equiv 0(mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq \frac{12k}{5}$.*

Proof. Form (iv) of Corollary2, we have $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq \frac{12k+15}{5}$. Let $k = 5m+5$. We need to show that $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m + 14, 3)$-tree-coloring and an equitable $(12m + 13, 3)$-tree-coloring.

If $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m+14,3)$-tree-coloring , then the size of every color class is at least 4 because $\lfloor\frac{12k+3}{12m+14}\rfloor = \lfloor\frac{60m+63}{12m+14}\rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 63$$

$$c_4 + c_5 = 12m + 14.$$

We have the unique solution $c_4 = 7$, $c_5 = 12m+7$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k+1,4k+1,4k+1}$, there are $4m+3$ color classes containing exactly 5 vertices in some partite set of $K_{4k+1,4k+1,4k+1}$ and there are $4m+2$ color classes containing exactly 5 vertices in other partite sets of $K_{4k+1,4k+1,4k+1}$. Since there are $20m+21$ vertices in every partite set of $K_{4k+1,4k+1,4k+1}$, there are 6 vertices of color class containing exactly 4 vertices in some partite set and there are 11 vertices of color class containing exactly 4 vertices in other partite sets. In this case, $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m+14,3)$-tree-coloring.

We can prove that $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m+13,3)$-tree-coloring using an similar argument.

Form the above argument and Proposition 1, we prove that $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq \frac{12k}{5}$. ∎

Lemma 16 *If $k \equiv 0 (mod\ 5)$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \geq \frac{12k}{5}$.*

Proof. Let $k = 5m+5$. We only need to show that $K_{4k+1,4k+1,4k+1}$ has no equitable $(12m+11,3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m+8,3)$-tree-coloring of $K_{4k+1,4k+1,4k+1}$. Then the size of every color class in c is at least 5 because $\lfloor\frac{12k+3}{12m+11}\rfloor = \lfloor\frac{60m+63}{12m+11}\rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 63$$

$$c_5 + c_6 = 12m + 11.$$

We have the unique solution $c_5 = 12m+3$, $c_6 = 8$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k+1,4k+1,4k+1}$, it follows that $K_{4k+1,4k+1,4k+1}$ has no equitable$(12m+11,3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \geq \frac{12k}{5}$. ∎

In the following, we consider the remaining case $k \equiv 1 (mod\ 5)$ and give the proof of (v) of Theorem 3.

Lemma 17 *If $k = 5m+1$ and $m \geq 2$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m$.*

Proof. By Proposition 3 and Proposition 1, $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m+3$. It suffices to prove that $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m+2,3)$-tree-coloring and an equitable $(12m+1,3)$-tree-coloring.

Divide X into $4m+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m+1\}$. Divide Y into $4m+1$ classes equitably and color the vertices of each class with a color in $\{4m+2,\cdots,8m+2\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m+3,\cdots,12m+2\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(12m+2,3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m+1\}$. Divide Y into $4m$ classes equitably and color the vertices of each class with a color in $\{4m+2,\cdots,8m+1\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m+2,\cdots,12m+1\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(12m+1,3)$-tree-coloring with the size of each color class being 5 or 6. ∎

Lemma 18 *If $k=5m+1$ and $m \geq 2q+1$ where $q \geq 1$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m-3q$.*

Proof. We prove the theorem by induction on q. When $q=1$, it is equivalent to prove that if $m \geq 3$, then $va_3^{\equiv}(K_{4k,4k,4k}) \leq 12m$. Since $m \geq 3 > 2$, we have $va_4^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m$ by Lemma 13. It suffices to prove that $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m-1,3)$-tree-coloring and an equitable $(12m-2,3)$-tree-coloring by Proposition 1.

Divide X into $4m-1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m-1\}$. Divide Y into $4m$ classes equitably and color the vertices of each class with a color in $\{4m,\cdots,8m-1\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m,\cdots,12m-1\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(12m-1,3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m-1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m-1\}$. Divide Y into $4m-1$ classes equitably and color the vertices of each class with a color in $\{4m,\cdots,8m-2\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m-1,\cdots,12m-2\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(12m-2,3)$-tree-coloring with the size of each color class being 5 or 6. The result holds for $q=1$.

Suppose $q \geq 2$. Since $m \geq 2q+1 > 2(q-1)+1$, by the induction hypothesis and Proposition 1, we need to prove that $K_{4k+1,4k+1,4k+1}$ has an equitable $(12m-3q+1,3)$-tree-coloring and an equitable $(12m-3q+2,3)$-tree-coloring.

Divide X into $4m-q+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m-q+1\}$. Divide Y into $4m-q$ classes equitably and color the vertices of each class with a color in $\{4m-q+2, \cdots, 8m-2q+1\}$. Divide Z into $4m-q$ classes equitably and color the vertices of each class with a color in $\{8m-2q+2, \cdots, 12m-3q+1\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(12m-3q+1, 3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m-q+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m-q+1\}$. Divide Y into $4m-q+1$ classes equitably and color the vertices of each class with a color in $\{4m-q+2, \cdots, 8m-2q+2\}$. Divide Z into $4m-q$ classes equitably and color the vertices of each class with a color in $\{8m-2q+3, \cdots, 12m-3q+2\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k=1}$ is an equitable $(12m-3q+2, 3)$-tree-coloring with the size of each color class being 5 or 6. ∎

Lemma 19 *If $k = 5m+1$ and $m \geq 2q$ where $q \geq 0$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m-3q+3$.*

Proof. When $q = 0, 1$, the result holds by Proposition 3 and Lemma 13. If $q \geq 3$, then $q-1 \geq 2$. Since $m \geq 2q > 2(q-1)+1$, it follows that $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m-3(q-1) = 12m-3q+3$. ∎

Proof of Theorem 3: Suppose $k \equiv 2$ $(mod\ 5)$. From (i) of Corollary 2 and Lemma 11, $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k+6}{5}$. Suppose $k \equiv 3$ $(mod\ 5)$. From (ii) of Corollary 2 and Lemma 12, $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k+9}{5}$. Suppose $k \equiv 4$ $(mod\ 5)$. From Lemma 13 and Lemma 14, $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k+7}{5}$. Suppose $k \equiv 0$ $(mod\ 5)$. From Lemma 15 and Lemma 16, $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) = \frac{12k}{5}$. Suppose $k \equiv 1$ $(mod\ 5)$. From Lemma 18, if $k = 5m+1$ and $m \geq 2q+1$ where $q \geq 1$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k=1}) \leq 12m-3q$. From Lemma 19, if $k = 5m+1$ and $m \geq 2q$ where $q \geq 0$, then $va_3^{\equiv}(K_{4k+1,4k+1,4k+1}) \leq 12m-3q+3$. ∎

3.4 The strong equitable vertex 3-arboricity of $K_{4k+2,4k+2,4k+2}$

We investigate the strong equitable vertex 3-arboricity of the complete tripartite graph $K_{4k+2,4k+2,4k+2}$.

The following upper bound of $K_{4k+2,4k+2,4k+2}$ can be proved easily.

Proposition 4 *If $k \geq 5m+2$ where $m \geq 0$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 3k-3m$.*

Proof. We prove the proposition by induction on m. If $m = 0$, the result holds by Theorem 5.

Suppose $m \geq 1$. Since $k \geq 5m+2 > 5(m-1)+2$, by the induction hypothesis and Proposition 1, we need to prove that $K_{4k+2,4k+2,4k+2}$ has an equitable $(3k-3m+1, 3)$-tree-coloring and an equitable $(3k-3m+2, 3)$-tree-coloring.

Divide X into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, k-m+1\}$. Divide Y into $k-m$ classes equitably and color the vertices of each class with a color in $\{k-m+2, \cdots, 2k-2m+1\}$. Divide Z into $k-m$ classes equitably and color the vertices of each class with a color in $\{2k-2m+2, \cdots, 3k-3m+1\}$. It is easy to check that the resulting coloring of $K_{4k+2,4k+2,4k+2}$ is an equitable $(3k-3m+1, 3)$-tree-coloring with the size of each color class being 4 or 5.

Divide X into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, k-m+1\}$. Divide Y into $k-m+1$ classes equitably and color the vertices of each class with a color in $\{k-m+2, \cdots, 2k-2m+2\}$. Divide Z into $k-m$ classes equitably and color the vertices of each class with a color in $\{2k-2m+3, \cdots, 3k-3m+2\}$. It is easy to check that the resulting coloring of $K_{4k+2,4k+2,4k+2}$ is an equitable $(3k-3m+2, 3)$-tree-coloring with the size of each color class being 4 or 5. ∎

The following corollaries are immediate.

Corollary 3 *(i) If $k \equiv 3$ (mod 5), then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+9}{5}$.*

(ii) If $k \equiv 4$ (mod 5), then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+12}{5}$.

(iii) If $k \equiv 0$ (mod 5), then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+15}{5}$.

(iv) If $k \equiv 1$ (mod 5), then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+18}{5}$.

Lemma 20 *If $k \equiv 3(mod\ 5)$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+9}{5}$.*

Proof. Let $k = 5m+3$. We only need to show that $K_{4k+2,4k+2,4k+2}$ has no equitable $(12m+8, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m+8, 3)$-tree-coloring of $K_{4k+2,4k+2,4k+2}$. Then the size of every color class in c is at least 5 because $\lfloor \frac{12k+6}{12m+8} \rfloor = \lfloor \frac{60m+42}{12m+8} \rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 42$$

$$c_5 + c_6 = 12m + 8.$$

We have the unique solution $c_5 = 12m + 6$, $c_6 = 2$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k+2,4k+2,4k+2}$, it follows that $K_{4k+2,4k+2,4k+2}$ has no equitable$(12m+8, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+9}{5}$. ∎

Lemma 21 *If $k \equiv 4(mod\ 5)$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+12}{5}$.*

Proof. Let $k = 5m+4$. We only need to show that $K_{4k+2,4k+2,4k+2}$ has no equitable $(12m+11, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m+11, 3)$-tree-coloring of $K_{4k+2,4k+2,4k+2}$. Then the size of every color class in c is at least 4 because $\lfloor \frac{12k+6}{12m+11} \rfloor = \lfloor \frac{60m+54}{12m+11} \rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 54$$

$$c_4 + c_5 = 12m + 11.$$

We have the unique solution $c_4 = 1$, $c_5 = 12m + 10$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k+2,4k+2,4k+2}$, it follows that $K_{4k+2,4k+2,4k+2}$ has no equitable$(12m+11, 3)$-tree-coloring satisfying the above conditions. Then $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+12}{5}$. ∎

Lemma 22 *If $k \equiv 0(mod\ 5)$, then $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+10}{5}$.*

Proof. Form (iii) of Corollary3, we have $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+15}{5}$. Let $k = 5m+5$. We only need to show that $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 14, 3)$-tree-coloring. Then the size of every color class is at least 4 because $\lfloor\frac{12k+6}{12m+14}\rfloor = \lfloor\frac{60m+66}{12m+14}\rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 66$$

$$c_4 + c_5 = 12m + 14.$$

We have the unique solution $c_4 = 4$, $c_5 = 12m + 10$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k+2,4k+2,4k+2}$, there are $4m+4$ color classes containing exactly 5 vertices in some partite set of $K_{4k+2,4k+2,4k+2}$ and there are $4m + 3$ color classes containing exactly 5 vertices in other partite sets of $K_{4k+2,4k+2,4k+2}$. Since there are $20m+22$ vertices in every partite set of $K_{4k+2,4k+2,4k+2}$, there are 2 vertices of color class containing exactly 4 vertices in some partite set and there are 7 vertices of color class containing exactly 4 vertices in other partite sets. In this case, $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 14, 3)$-tree-coloring. ∎

Lemma 23 *If $k \equiv 0(mod\ 5)$, then $va_3{}^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+10}{5}$.*

Proof. Let $k = 5m + 5$. We only need to show that $K_{4k+2,4k+2,4k+2}$ has no equitable $(12m + 13, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m + 13, 3)$-tree-coloring of $K_{4k+2,4k+2,4k+2}$. Then the size of every color class in c is at least 5 because $\lfloor\frac{12k+6}{12m+13}\rfloor = \lfloor\frac{60m+66}{12m+13}\rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 66$$

$$c_5 + c_6 = 12m + 13.$$

We have the unique solution $c_5 = 12m + 12$, $c_6 = 1$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k+2,4k+2,4k+2}$, it follows that $K_{4k+2,4k+2,4k+2}$ has no equitable$(12m+13, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+10}{5}$. ∎

Lemma 24 *If $k \equiv 1(mod\ 5)$ $(k \neq 1)$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+3}{5}$.*

Proof. Form (iv) of Corollary3, we have $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+18}{5}$. Let $k = 5m+6$. We need to show that $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 17, 3)$-tree-coloring and an equitable $(12m + 16, 3)$-tree-coloring.

If $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 17, 3)$-tree-coloring , then the size of every color class is at least 4 because $\lfloor\frac{12k+6}{12m+17}\rfloor = \lfloor\frac{60m+78}{12m+17}\rfloor = 4$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 4, 5$. Then we have the following two equations:

$$4c_4 + 5c_5 = 60m + 78$$

$$c_4 + c_5 = 12m + 17.$$

We have the unique solution $c_4 = 7$, $c_5 = 12m + 10$. Since each color class containing exactly 5 vertices must appear in some partite set of $K_{4k+2,4k+2,4k+2}$, there are $4m+4$ color classes containing exactly 5 vertices in some partite set of $K_{4k+2,4k+2,4k+2}$ and there are $4m + 3$ color classes containing exactly 5 vertices in other partite sets of $K_{4k+2,4k+2,4k+2}$. Since there are $20m+26$ vertices in every partite set of $K_{4k+2,4k+2,4k+2}$, there are 6 vertices of color class containing exactly 4 vertices in some partite set and there are 11 vertices of color class containing exactly 4 vertices in other partite sets. In this case, $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 17, 3)$-tree-coloring.

We can prove that $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 16, 3)$-tree-coloring using an similar argument.

Form the above argument and Proposition 1, we prove that $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq \frac{12k+3}{5}$. ∎

Lemma 25 *If $k \equiv 1(mod\ 5$ $(k \neq 1)$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+3}{5}$.*

Proof. Let $k = 5m + 6$. We only need to show that $K_{4k+2,4k+2,4k+2}$ has no equitable $(12m + 14, 3)$-tree-coloring. Assume, to the contrary, that c is an equitable $(12m + 14, 3)$-tree-coloring of $K_{4k+2,4k+2,4k+2}$. Then the size of every color class in c is at least 5 because $\lfloor\frac{12k+6}{12m+14}\rfloor = \lfloor\frac{60m+78}{12m+14}\rfloor = 5$.

Let c_i denote the number of those color classes such that each color class contains exactly i vertices, where $i = 5, 6$. Then we have the following two equations:

$$5c_5 + 6c_6 = 60m + 78$$

$$c_5 + c_6 = 12m + 14.$$

We have the unique solution $c_5 = 12m + 6$, $c_6 = 8$. Since each color class containing exactly 5 or 6 vertices must appear in some partite set of $K_{4k+2,4k+2,4k+2}$, it follows that $K_{4k+2,4k+2,4k+2}$ has no equitable$(12m+14, 3)$-tree-coloring satisfying the above conditions. Then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \geq \frac{12k+3}{5}$. ∎

In the following, we consider the remaining case $k \equiv 2(mod\ 5)$ and give the proof of (v) of Theorem 4.

Lemma 26 *If $k = 5m + 2$ and $m \geq 2$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 12m + 3$.*

Proof. By Proposition 4 and Proposition 1, $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 12m+6$. So we need to prove that $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 4, 3)$-tree-coloring and an equitable $(12m + 5, 3)$-tree-coloring.

Divide X into $4m+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m+1\}$. Divide Y into $4m+1$ classes equitably and color the vertices of each class with a color in $\{4m+2, \cdots, 8m+2\}$. Divide Z into $4m+2$ classes equitably and color the vertices of each class with a color in $\{8m + 3, \cdots, 12m + 4\}$. It is easy to check that the resulting coloring of $K_{4k+2,4k+2,4k+2}$ is an equitable $(12m+4, 3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m+1\}$. Divide Y into $4m+2$ classes equitably and color the vertices of each class with a color in $\{4m+2, \cdots, 8m+3\}$. Divide Z into $4m+2$ classes equitably and color the vertices of each class with a color in $\{8m + 4, \cdots, 12m + 5\}$. It is easy to check that the resulting coloring of $K_{4k+2,4k+2,4k+2}$ is an equitable $(12m+5, 3)$-tree-coloring with the size of each color class being 5 or 6. ∎

Lemma 27 *If $k = 5m + 2$ and $m \geq 2q + 1$ where $q \geq 1$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 12m - 3q + 3$.*

Proof. We prove this theorem by induction on q. When $q = 1$, it is equivalent to prove that if $m \geq 3$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 12m$. Since $m \geq 3 > 2$, it follows from Lemma 20 that $va_3^{\equiv}(K_{4k+2,4k+2,4k+2}) \leq 12m + 3$. We need to prove that $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m + 1, 3)$-tree-coloring and an equitable $(12m + 2, 3)$-tree-coloring by Proposition 1.

Divide X into $4m+1$ classes equitably and color the vertices of each class with a color in $\{1, 2, \cdots, 4m + 1\}$. Divide Y into $4m$ classes equitably and color the vertices of each class with a color in $\{4m+2,\cdots,8m+1\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m+2,\cdots,12m+1\}$. It is easy to check that the resulting coloring of $K_{4k+1,4k+1,4k+1}$ is an equitable $(12m+1,3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m+1\}$. Divide Y into $4m+1$ classes equitably and color the vertices of each class with a color in $\{4m+2,\cdots,8m+2\}$. Divide Z into $4m$ classes equitably and color the vertices of each class with a color in $\{8m+3,\cdots,12m+2\}$. It is easy to check that the resulting coloring of $K_{4k+2,4k+2,4k+2}$ is an equitable $(12m+2,3)$-tree-coloring with the size of each color class being 5 or 6.

Suppose $q\geq 2$. Since $m\geq 2q+1>2(q-1)+1$, by the induction hypothesis and Proposition 1, it suffices to prove that $K_{4k+2,4k+2,4k+2}$ has an equitable $(12m-3q+5,3)$-tree-coloring and an equitable $(12m-3q+4,3)$-tree-coloring.

Divide X into $4m-q+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m-q+1\}$. Divide Y into $4m-q+1$ classes equitably and color the vertices of each class with a color in $\{4m-q+2,\cdots,8m-2q+2\}$. Divide Z into $4m-q+2$ classes equitably and color the vertices of each class with a color in $\{8m-2q+3,\cdots,12m-3q+4\}$. It is easy to check that the resulting coloring of $K_{4k+2,4k+2,4k+2}$ is an equitable $(12m-3q+4,3)$-tree-coloring with the size of each color class being 5 or 6.

Divide X into $4m-q+1$ classes equitably and color the vertices of each class with a color in $\{1,2,\cdots,4m-q+1\}$. Divide Y into $4m-q+2$ classes equitably and color the vertices of each class with a color in $\{4m-q+2,\cdots,8m-2q+3\}$. Divide Z into $4m-q+2$ classes equitably and color the vertices of each class with a color in $\{8m-2q+4,\cdots,12m-3q+5\}$. It is easy to check that the resulting coloring of $K_{4k+2,4k+2,4k+2}$ is an equitable $(12m-3q+5,3)$-tree-coloring with the size of each color class being 5 or 6.
∎

Lemma 28 *If $k=5m+2$ and $m\geq 2q$ where $q\geq 0$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})\leq 12m-3q+6$.*

Proof. When $q=0,1$, the result holds by Proposition 4 and Lemma 20. If $q\geq 2$, then $q-1\geq 1$. Since $m\geq 2q>2(q-1)+1$, it follows that $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})\leq 12m-3(q-1)+3=12m-3q+6$. ∎

Proof of Theorem 4: Suppose $k\equiv 3(mod 5)$. From (i) of Corollary 3 and Lemma 20, $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})=\frac{12k+9}{5}$. Suppose $k\equiv 4(mod 5)$. From (ii) of Corollary3 and Lemma 21, $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})=\frac{12k+12}{5}$. Suppose $k\equiv 0(mod 5)$. From Lemma 22 and Lemma 23, $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})=\frac{12k+10}{5}$. Suppose $k\equiv 1(mod 5)$. From Lemma 24 and Lemma 25, $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})=\frac{12k+5}{5}$. Suppose $k\equiv 2(mod\ 5)$. From Lemma 27, if $k=5m+2$ and $m\geq 2q+1$, where $q\geq 1$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})\leq 12m-3q+3$. From Lemma 28, if $k=5m+2$ and $m\geq 2q$, where $q\geq 0$, then $va_3^{\equiv}(K_{4k+2,4k+2,4k+2})\leq$

$12m - 3q + 6$. ∎

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