The 3-forced 2-structures

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Abstract

Given sets S and S', a labeled 2-structure is a function σ from $(S \times S) \setminus \{(s,s) : s \in S\}$ to S'. The set S is called the vertex set of σ and denoted by $V(\sigma)$. The *label set* of σ is the set $L(\sigma)$ of $l \in S'$ such that $l = \sigma(v,w)$ for some $v,w \in V(\sigma)$. Given $W \subseteq V(\sigma)$, the 2-substructure $\sigma_{(\restriction(W\times W)\setminus\{(s,s):s\in W\})}$ of σ is denoted by $\sigma[W]$. The dual $\sigma^\star$ of σ is defined on $V(\sigma^\star) = V(\sigma)$ as follows. For distinct $v,w \in V(\sigma^\star)$, $\sigma^\star(v,w) = \sigma(w,v)$. A labeled 2-structure σ is reversible provided that for $x,x',y,y' \in S$ such that $x \neq y$ and $x' \neq y'$, if $\sigma(x,y) = \sigma(x',y')$, then $\sigma(y,x) = \sigma(y',x')$. We only consider reversible labeled 2-structures whose vertex set is finite.

Let σ and τ be 2-structures such that $V(\sigma) = V(\tau)$. Given $2 \leq k \leq |V(\sigma)|$, σ and τ are k-hemimorphic if for every $W \subseteq V(\sigma)$ such that $|W| \leq k$, $\sigma[W]$ is isomorphic to $\tau[W]$ or $(\tau[W])^\star$. Furthermore, let σ be a 2-structure. Given $2 \leq k \leq |V(\sigma)|$, σ is k-forced if σ and $\sigma^\star$ are the only 2-structures k-hemimorphic to σ. We characterize the 3-forced 2-structures. Lastly, we provide a large class of 4-forced 2-structures.

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1 Introduction

Given a digraph D, the *dual* $D^\star$ of D is obtained from D be reversing all its arcs. A digraph is *self-dual* if it is isomorphic to its dual. Consider two digraphs D and Δ such that $V(D) = V(\Delta)$. Given $2 \leq k \leq |V(D)|$, we say that D and Δ are *k-hemimorphic* if for every $W \subseteq V(D)$ such that $|W| \leq k$, $D[W]$ is isomorphic to $\Delta[W]$ or $(\Delta[W])^\star$. Furthermore, let D be a digraph. Given $2 \leq k \leq |V(D)|$, D is *k-forced* if D and $D^\star$ are the only digraphs k-hemimorphic to D.

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Given a digraph D, a subset M of $V(D)$ is a *module* of D if for any $x, y \in M$ and $v \in V(D) \setminus M$, we have

$$xv \in A(D) \iff yv \in A(D)$$
$$\text{and}$$
$$vx \in A(D) \iff vy \in A(D).$$

For instance, $\varnothing$, $\{v\}$ ($v \in V(D)$), and $V(D)$ are modules of D, called *trivial modules* of D. A digraph D is said to be *prime* if $|V(D)| \geq 3$ and all its modules are trivial.

Gallai [8, 13] proved that prime partial orders are 3-forced. Afterward, Boussaïri et al. [4, 5] established that prime tournaments are 3-forced as well. In general, a prime digraph is not 3-forced. But, a prime digraph becomes 3-forced if all its prime subdigraphs of size 3 are self-dual. We denote by F the digraph $(\{0,1,2\},\{01,10,12\})$. The only prime digraphs of size 3 that are not self-dual are F and $F^\star$. They are called the *flags* (see Definition 31). Boussaïri et al. [4, 5] proved that prime digraphs without flags are 3-forced. Boussaïri [2] characterized the 3-forced prime digraphs. This result was published in [3]. Lastly, given a digraph D, the set of the unordered pairs $\{v, w\}$ of vertices of D such that $|\{vw, wv\} \cap A(D)| = 1$ is denoted by $O(D)$. Boussaïri [2] proved that a prime digraph D such that the graph $(V(D), O(D))$ is connected is 4-forced. Dammak obtained this result as an easy consequence of the main results proved in [6].

Our purpose is to extend the characterization of 3-forced digraphs (see Theorem 8) and the result on 4-forced digraphs above (see Theorem 10) to reversible labeled 2-structures.

Given sets S and S', a labeled 2-structure is a function σ from $(S \times S) \setminus \{(s,s) : s \in S\}$ to S'. The set S is called the *vertex set* of σ and denoted by $V(\sigma)$. The *label set* of σ is the set $L(\sigma)$ of $l \in S'$ such that $l = \sigma(v, w)$ for some $v, w \in V(\sigma)$. A labeled 2-structure σ is *reversible* [7] provided that for $x, x', y, y,' \in S$ such that $x \neq y$ and $x' \neq y'$, we have

$$\text{if } \sigma(x, y) = \sigma(x', y'), \text{ then } \sigma(y, x) = \sigma(y', x'). \tag{1}$$

In what follows, we only consider reversible labeled 2-structures whose vertex set is finite. Given such a 2-structure σ, set $v(\sigma) = |V(\sigma)|$.

For instance, we associate with a digraph D the 2-structure σ_D defined on $V(\sigma_D) = V(D)$ with $L(\sigma_D) \subseteq \mathbb{Z}_4$ as follows. For any $v, w \in V(\sigma_D)$ such that $v \neq w$,

$$\sigma_D(v, w) = \begin{cases} 0 \text{ if } vw, wv \notin A(D), \\ 2 \text{ if } vw, wv \in A(D), \\ 1 \text{ if } vw \in A(D) \text{ and } wv \notin A(D), \\ 3 \text{ if } wv \in A(D) \text{ and } vw \notin A(D). \end{cases}$$

Hence, given distinct $v, w \in V(\sigma_D)$, we have $\sigma_D(v, w) = -\sigma_D(w, v) \pmod 4$.

Let σ be a 2-structure. With each $W \subseteq V(\sigma)$, we associate the *2-substructure* $\sigma[W]$ of σ induced by W defined by

$$\sigma[W] = \sigma_{\restriction (W \times W) \setminus \{(w,w) : w \in W\}}.$$

When $W = V(\sigma) \setminus W'$, $\sigma[W]$ is also denoted by $\sigma - W'$, and by $\sigma - w$ when $W' = \{w\}$.

1.1 Definitions and notations

Definition 1. A 2-structure σ is *constant* if $|L(\sigma)| = 1$. A 2-structure σ is *linear* if there exist distinct $l, l' \in L(\sigma)$ such that $(V(\sigma), \sigma^{-1}(l))$ and $(V(\sigma), \sigma^{-1}(l'))$ are linear orders (in this case, we obtain $L(\sigma) = \{l, l'\}$ and $(V(\sigma), \sigma^{-1}(l')) = (V(\sigma), \sigma^{-1}(l))^{\star}$).

Let σ be a 2-structure. Set

$$\widetilde{L}(\sigma) = \{\{\sigma(v,w), \sigma(w,v)\} : \{v,w\} \in \binom{V(\sigma)}{2}\}.$$

Let $p, q \in \widetilde{L}(\sigma)$. By (1), we have $p = q$ or $p \cap q = \varnothing$. We consider the function

$$\begin{array}{rccl} \widetilde{\sigma} : & \binom{V(\sigma)}{2} & \longrightarrow & \widetilde{L}(\sigma) \\ & \{v,w\} & \longmapsto & \{\sigma(v,w), \sigma(w,v)\}. \end{array}$$

For each $i = 1$ or 2, set

$$\widetilde{L}_i(\sigma) = \{p \in \widetilde{L}(\sigma) : |p| = i\}.$$

Furthermore, set

$$E_i(\sigma) = \{\{v,w\} \in \binom{V(\sigma)}{2} : \widetilde{\sigma}(\{v,w\}) \in \widetilde{L}_i(\sigma)\}.$$

We consider the graph G_σ defined on $V(G_\sigma) = \widetilde{L}_2(\sigma)$ in the following manner. For any $p, q \in V(G_\sigma)$ such that $p \neq q$, $pq \in E(G_\sigma)$ if there exists $W \subseteq V(\sigma)$ satisfying $|W| = 3$, $\sigma[W]$ is prime, and $p, q \in \widetilde{L}(\sigma[W])$. For a digraph D, the edge set of G_{σ_D} is empty because $|\widetilde{L}_2(\sigma_D)| \leq 1$.

We associate with a 2-structure σ the 2-structure $\overrightarrow{\sigma}$ defined on $V(\overrightarrow{\sigma}) = V(\sigma)$ as follows. If $|\widetilde{L}_1(\sigma)| \leq 1$, then $\overrightarrow{\sigma} = \sigma$. Now, suppose that $|\widetilde{L}_1(\sigma)| \geq 2$. Choose $l \in L(\sigma)$ such that $\{l\} \in \widetilde{L}_1(\sigma)$. For any $v, w \in V(\sigma)$ such that $v \neq w$,

$$\overrightarrow{\sigma}(v,w) = \begin{cases} \sigma(v,w) \text{ if } \{v,w\} \in E_2(\sigma), \\ l \text{ if } \{v,w\} \in E_1(\sigma). \end{cases}$$

For instance, consider a digraph D such that there exist $v, v', w, w' \in V(D)$ satisfying $v \neq w$, $v' \neq w'$, $vw, wv \in A(D)$, and $v'w', w'v' \notin A(D)$. We obtain $\widetilde{L}_1(\sigma_D) = \{\{0\}, \{2\}\}$. Now, choose $l = 0$. Consider the oriented graph $\overrightarrow{D}$ defined on $V(\overrightarrow{D}) = V(D)$ by $A(\overrightarrow{D}) = A(D) \setminus A(D^\star)$, where $D^\star$ is the dual of D defined on $V(D^\star) = V(D)$ by $A(D^\star) = \{vw : wv \in A(D)\}$. We obtain

$$\overrightarrow{(\sigma_D)} = \sigma_{\overrightarrow{D}}.$$

Later, in Convention 26, we set out the choice of the label "l" when we consider several 2-structures sharing the same label set.

Definition 2. We associate with a 2-structure σ its *dual* $\sigma^\star$ defined on $V(\sigma^\star) = V(\sigma)$ as follows. For any $v, w \in V(\sigma^\star)$ such that $v \neq w$, $\sigma^\star(v, w) = \sigma(w, v)$.

Consider 2-structures σ and τ such that $V(\sigma) = V(\tau)$. Given $2 \leq k \leq v(\sigma)$, we say that σ and τ are *k-hemimorphic* if for every $W \subseteq V(\sigma)$ such that $|W| \leq k$, $\sigma[W]$ is isomorphic to $\tau[W]$ or $(\tau[W])^\star$.

Given $2 \leq k \leq v(\sigma)$, a 2-structure σ is *k-forced* if σ and $\sigma^\star$ are the only 2-structures that are k-hemimorphic to σ.

Remark 3. For a digraph D, we obtain $(\sigma_D)^\star = \sigma_{(D^\star)}$. Consider digraphs D and Δ such that $V(D) = V(\Delta)$. Given $2 \leq k \leq |V(D)|$, it follows that two digraphs D and Δ are k-hemimorphic if and only if σ_D and σ_Δ are k-hemimorphic. Let D be a digraph. Given $2 \leq k \leq |V(D)|$, we obtain that D is k-forced if and only σ_D is.

1.2 Modular decomposition

For this section, we refer to [8, 13] for graphs, [5] for digraphs, [10] for binary structures, and [7, 11] for 2-structures.

Let σ be a 2-structure. A subset M of $V(\sigma)$ is a *module* of σ if for any $x, y \in M$ and $v \in V(\sigma) \setminus M$, we have $\sigma(x, v) = \sigma(y, v)$. For instance, $\varnothing$, $V(\sigma)$ and $\{v\}$ ($v \in V(\sigma)$) are modules of σ, called the *trivial modules* of σ. A 2-structure σ is *indecomposable* if all its modules are trivial, otherwise it is *decomposable.* A 2-structure σ is *prime* if it is indecomposable with $v(\sigma) \geq 3$.

Let σ be a 2-structure. For disjoint modules M and N of σ, we have $\sigma(x, y) = \sigma(x', y')$ for any $x, x' \in M$ and $y, y' \in M$. This property allows us to define the quotient as follows. A *modular partition* of σ is a partition of $V(\sigma)$ in modules of σ. With a modular partition P of σ, we associate the *quotient* of σ by P defined on $V(\sigma/P) = P$ as follows. For distinct $X, Y \in P$, $(\sigma/P)(X, Y) = \sigma(x, y)$, where $x \in X$ and $y \in Y$.

Let σ be a 2-structure. A subset M of $V(\sigma)$ is a *strong module* of σ if M is a module of σ satisfying: for every module N of σ, if $M \cap N \neq \varnothing$, then $M \subseteq N$ or $N \subseteq M$. For instance, the trivial modules of σ are strong modules too. We denote by $\mathscr{G}(\sigma)$ the family of the strong modules of σ which are maximal under inclusion among the proper strong modules of σ. Gallai's decomposition theorem follows.

Theorem 4. *Given a 2-structure σ such that $v(\sigma) \geq 2$, $\mathscr{G}(\sigma)$ is a modular partition of σ, and the quotient $\sigma/\mathscr{G}(\sigma)$ is constant, linear, or prime.*

1.3 The main results

Since it is not difficult to verify the next fact, we omit its proof.

Fact 5. *Given a 2-structure σ such that $(V(\sigma), E_2(\sigma))$ admits an isolated vertex v, σ is 3-forced if and only if $\sigma - v$ is as well.*

In Section 3, we show the next preliminary result.

Lemma 6. *Given a 2-structure σ, σ is* 3*-forced if and only if $\overrightarrow{\sigma}$ is* 3*-forced.*

Remark 7. The next assumption follows from Fact 5 and Lemma 6. To characterize the 2-structures σ that are 3-forced, we can assume that $\sigma = \overrightarrow{\sigma}$ and $(V(\sigma), E_2(\sigma))$ does not have isolated vertices.

The two main theorems follow.

Theorem 8. *Let σ be a 2-structure. Suppose that $\sigma = \overrightarrow{\sigma}$ and $(V(\sigma), E_2(\sigma))$ does not have isolated vertices. The 2-structure σ is 3-forced if and only if σ satisfies the following three statements*

(T1) G_σ is connected;

(T2) for each $M \in \mathscr{G}(\sigma)$, $E_2(\sigma[M]) = \varnothing$;

(T3) if $|\mathscr{G}(\sigma)| \geq 3$, then $\sigma/\mathscr{G}(\sigma)$ is prime.

Remark 9. Theorem 8 is obtained by Boussaïri [2] for digraphs. Clearly, Statement (T1) is unnecessary for digraphs because it is trivially satisfied. Theorem 8 for digraphs is proved in [3].

Theorem 10. *Let σ be a prime 2-structure. If $(V(\sigma), E_2(\sigma))$ is connected, then σ is 4-forced.*

2 Preliminaries

2.1 Prime 2-structures

The next result is classical in the study of prime 2-structures (for instance, see Ehrenfeucht et al. [7]).

Lemma 11. *Given a prime 2-structure σ, there exists $X \subseteq V(\sigma)$ such that $|X| = 3$ or 4 and $\sigma[X]$ is prime.*

To obtain prime 2-substructure of larger sizes, we use the following subsets.

Notation 12. Let σ be a 2-structure. Given $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime, consider the following subsets of $V(\sigma) \setminus X$

- $\text{Ext}_\sigma(X)$ denotes the set of $v \in V(\sigma) \setminus X$ such that $\sigma[X \cup \{v\}]$ is prime;
- $\langle X \rangle_\sigma$ denotes the set of $v \in V(\sigma) \setminus X$ such that X is a module of $\sigma[X \cup \{v\}]$;

- for each $y \in X$, $X_\sigma(y)$ denotes the set of $v \in V(\sigma) \setminus X$ such that $\{y, v\}$ is a module of $\sigma[X \cup \{v\}]$.

The set $\{\mathrm{Ext}_\sigma(X), \langle X \rangle_\sigma\} \cup \{X_\sigma(y) : y \in X\}$ is denoted by $p_{(\sigma,X)}$.

Using the fact that the family of the modules of a 2-structure is weakly partitive (for instance, see Ille and Woodrow [12]), it is not difficult to verify the following claim.

Claim 13. *Given a 2-structure σ, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. The set $p_{(\sigma,X)}$ is a partition of $V(\sigma) \setminus X$.*

Using the subsets described in Notation 12, Ehrenfeucht et al. [7] obtained the following result (see [7, Theorem 6.5]).

Proposition 14. *Given a prime 2-structure σ, consider $X \subseteq V(\sigma)$ such that $\sigma[X]$ is prime. If $|V(\sigma) \setminus X| \geq 2$, then there exist $v, w \in V(\sigma) \setminus X$ such that $v \neq w$ and $\sigma[X \cup \{v, w\}]$ is prime.*

The next result is an immediate consequence of Lemma 11 and Proposition 14.

Corollary 15. *Let σ be a prime 2-structure. If $v(\sigma) \geq 5$, then there exists $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime and $|V(\sigma) \setminus X| \leq 2$.*

The next result follows from Proposition 14.

Lemma 16. *Given a 2-structure σ such that $v(\sigma) \geq 5$, Suppose that there exists $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime. If σ is prime, then there exist elements $y_0, \ldots, y_{v(\sigma)-|X|-1}$ of $V(\sigma) \setminus X$ fulfilling the next three statements*

(S1) $X \cup \{y_0, \ldots, y_{v(\sigma)-|X|-1}\} = V(\sigma)$;

(S2) if $\sigma[X \cup \{y_0\}]$ is decomposable, then there exists $z \in X$ such that $\{z, y_0\}$ is a module of $\sigma[X \cup \{y_0\}]$.

(S3) for each $k \in \{1, \ldots, v(\sigma) - |X| - 2\}$, if $\sigma[X \cup \{y_0, \ldots, y_k\}]$ is decomposable, then the following assertions hold

- *k is even,*
- *$\sigma[X \cup \{y_0, \ldots, y_{k-1}\}]$ is prime,*
- *there exists $z \in X \cup \{y_0, \ldots, y_{k-1}\}$ such that $\{z, y_k\}$ is a module of $\sigma[X \cup \{y_0, \ldots, y_k\}]$.*

In particular, for every $k \in \{0, \ldots, v(\sigma) - |X| - 1\}$, $\sigma[X \cup \{y_0, \ldots, y_k\}]/\mathscr{G}(\sigma[X \cup \{y_0, \ldots, y_k\}])$ is prime.

Proof. It suffices to define y_0 and y_1, and then to pursue by proceeding by induction. By Proposition 14, there exist $v, w \in V(\sigma) \setminus X$ such that $v \neq w$ and $\sigma[X \cup \{v, w\}]$ is prime. If $v, w \in \langle X \rangle_\sigma$ (see Notation 12), then X is a module of $\sigma[X \cup \{v,w\}]$, which contradicts the fact that $\sigma[X \cup \{v,w\}]$ is prime. Consequently, we have $\{v,w\} \setminus \langle X \rangle_\sigma \neq \varnothing$. It follows from Claim 13 that

$$\{v,w\} \cap (\mathrm{Ext}_\sigma(X) \cup (\bigcup_{y \in X} X_\sigma(y))) \neq \varnothing.$$

For instance, assume that $v \in \mathrm{Ext}_\sigma(X) \cup (\bigcup_{y \in X} X_\sigma(y))$. Clearly, for $t \in X_\sigma(y)$, where $y \in X$, we have

$$\mathscr{G}(\sigma[X \cup \{t\}]) = \{\{y,t\}\} \cup \{\{z\} : z \in X \setminus \{y\}\}$$
and
$$\sigma[X \cup \{t\}]/\mathscr{G}(\sigma[X \cup \{t\}]) \text{ is prime.}$$

Hence, we can choose $y_0 = v$ and $y_1 = w$. □

Lemma 17. *Given a 2-structure σ such that $v(\sigma) \geq 5$, if $\sigma/\mathscr{G}(\sigma)$ is prime, then there exists $v \in V(\sigma)$ such that $(\sigma - v)/\mathscr{G}(\sigma - v)$ is prime as well.*

Proof. The result is obvious when σ is decomposable. Hence, suppose that σ is prime. By Corollary 15, there exists $X \subsetneq V(\sigma)$ such that $\sigma[X]$ is prime and $|V(\sigma) \setminus X| \leq 2$. To conclude, it suffices to apply Lemma 16. □

2.2 Critical 2-structures

A prime 2-structure σ is *critical* if for every $v \in V(\sigma)$, $\sigma - v$ is decomposable. Critical digraphs were characterized by Schmerl and Trotter [14] (see also Boudabbous and Ille [1]) .

Notation 18. Given $n \geq 2$, we denote by Σ_{2n} the sets of the 2-structures σ defined on $V(\sigma) = \{0, \ldots, 2n-1\}$ such that $|\widetilde{L}_2(\sigma)| = 2$, $\widetilde{\sigma}(\{0,1\}) \neq \widetilde{\sigma}(\{0,2\})$, and for $p, q \in \{0, \ldots, 2n-1\}$ such that $p < q$, we have

$$\sigma(p,q) = \begin{cases} \sigma(0,1) \text{ if } p \text{ is even and } q \text{ is odd,} \\ \sigma(0,2) \text{ otherwise.} \end{cases} \tag{2}$$

Claim 19. *Given $n \geq 2$, the elements of Σ_{2n} are critical 2-structures.*

Proof. To begin, we prove that the elements of Σ_{2n} are prime. We proceed by induction on $n \geq 2$. When $n = 2$, we verify directly the result. Now, suppose that the elements of Σ_{2n} are prime, where $n \geq 2$. Consider $\sigma \in \Sigma_{2(n+1)}$. Clearly, $\sigma - \{2n, 2n+1\} \in \Sigma_{2n}$. By induction hypothesis, $\sigma - \{2n, 2n+1\}$ is prime. Set

$$X = \{0, \ldots, 2n-1\}.$$

We have $2n+1 \in X_\sigma(2n-1)$ and $2n \in \langle X \rangle_\sigma$ (see Notation 12). Consider a module M of σ such that $|M| \geq 2$. We must show that $M = V(\sigma)$. Since $M \cap X$ is a module of $\sigma[X]$, we have $|M \cap X| \leq 1$ or $X \subseteq M$. For a contradiction, suppose that $|M \cap X| \leq 1$. Since $2n$ and $2n+1$ do not belong to the same block of $p_{(\sigma,X)}$, $M \neq \{2n, 2n+1\}$. Thus, there exists $y \in X$ such that $M \cap X = \{y\}$. Since $|M| \geq 2$, $M \setminus X \neq \varnothing$. Clearly, $(M \setminus X) \subseteq X_\sigma(y)$. By Claim 13, $p_{(\sigma,X)}$ is a partition of $V(\sigma) \setminus X$. It follows that $M = \{2n-1, 2n+1\}$, which is impossible because $\widetilde{\sigma}(\{2n-1, 2n\}) \neq \widetilde{\sigma}(\{2n, 2n+1\})$. Consequently, $X \subseteq M$. Clearly, $(V(\sigma) \setminus M) \subseteq \langle X \rangle_\sigma$. Since $p_{(\sigma,X)}$ is a partition of $V(\sigma) \setminus X$, we obtain $2n+1 \in M$. Furthermore, since $\widetilde{\sigma}(\{2n-1, 2n\}) \neq \widetilde{\sigma}(\{2n, 2n+1\})$, $X \cup \{2n+1\}$ is not a module of σ. It follows that $M = V(\sigma)$.

Lastly, we verify that elements of Σ_{2n} are critical. Given $n \geq 2$, consider $\sigma \in \Sigma_{2n}$. We have

- $\{2, \ldots, 2n-1\}$ is a module of $\sigma - 0$;
- $\{0, \ldots, 2n-3\}$ is a module of $\sigma - (2n-1)$;
- for each $p \in \{1, \ldots, 2n-2\}$, $\{p-1, p+1\}$ is a module of $\sigma - p$.

Therefore, σ is critical. □

Example 20. Let $n \geq 2$. We consider the graph G_{2n} defined on $V(G_{2n}) = \{0, \ldots, 2n-1\}$ by

$$E(G_{2n}) = \{2i(2j+1) : i \leq j \in \{0, \ldots, n-1\}\}.$$

The graph G_{2n} is critical. Furthermore, G_{2n} is a comparability graph. We consider the unique transitive orientation Q_{2n} of G_{2n} such that $0 <_{Q_{2n}} 1$. The complement of G_{2n} is a comparability graph too. We consider also the unique transitive orientation R_{2n} of the complement of G_{2n} such that $0 <_{R_{2n}} 2$. The partial orders Q_{2n} and R_{2n} are critical. We consider the 2-structure γ_{2n} defined on $V(\gamma_{2n}) = \{0, \ldots, 2n-1\}$ with $L(\gamma_{2n}) = \mathbb{Z}_5 \setminus \{0\}$ as follows. For distinct $v, w \in V(\gamma_{2n})$,

$$\gamma_{2n}(v,w) = \begin{cases} 1 \text{ if } v <_{Q_{2n}} w, \\ 4 \text{ if } w <_{Q_{2n}} v, \\ 2 \text{ if } v <_{R_{2n}} w, \\ 3 \text{ if } w <_{R_{2n}} v. \end{cases}$$

Clearly, γ_{2n} satisfies (2). Furthermore, $\widetilde{\gamma_{2n}}(\{0,1\}) = \{1,4\}$ and $\widetilde{\gamma_{2n}}(\{0,2\}) = \{2,3\}$. Hence, we have $\widetilde{\gamma_{2n}}(\{0,1\}) \neq \widetilde{\gamma_{2n}}(\{0,2\})$ and $|\widetilde{L}_2(\gamma_{2n})| = 2$. Therefore, $\gamma_{2n} \in \Sigma_{2n}$. It follows from Claim 19 that γ_{2n} is critical.

Lastly, suppose for a contradiction that there exists $X \subsetneq V(\gamma_{2n})$ such that $|X|$ is odd and $\gamma_{2n}[X]$ is prime. Since $V(\gamma_{2n})$ is even, it follows from Proposition 14 applied several times that there exists $v \in V(\gamma_{2n})$ such that $\gamma_{2n} - v$ is prime, which contradicts the fact that γ_{2n} is critical. Consequently, for every $X \subsetneq V(\gamma_{2n})$ such that $\gamma_{2n}[X]$ is prime, we have $|X|$ is even. Thus, for every $X \subsetneq V(\gamma_{2n})$ such that $|X| = 3$, $\gamma_{2n}[X]$ is decomposable. It follows that $E(G_{\gamma_{2n}}) = \varnothing$, and hence $G_{\gamma_{2n}}$ is disconnected. Therefore, γ_{2n} does not satisfy Statement (T1) of Theorem 8.

3 The 3-hemimorphy

The next result is obvious.

Fact 21. *A 2-structure σ is 2-forced if and only if* $|E_2(\sigma)| \leq 1$.

Consider 2-hemimorphic 2-structures σ and τ. We associate to σ and τ the graph $\mathcal{E}(\sigma,\tau)$ defined on $V(\sigma)$ as follows. For distinct $v,w \in V(\sigma)$, $vw \in E(\mathcal{E}(\sigma,\tau))$ if $\sigma(v,w) = \tau(v,w)$ and $\{v,w\} \in E_2(\sigma)$. We associate also to σ and τ the graph $\mathcal{D}(\sigma,\tau)$ defined on $V(\sigma)$ as follows. For distinct $v,w \in V(\sigma)$, $vw \in E(\mathcal{D}(\sigma,\tau))$ if $\sigma(v,w) \neq \tau(v,w)$ (and hence $\{v,w\} \in E_2(\sigma)$). Note that

$$\begin{cases} \mathcal{D}(\sigma,\tau) = \mathcal{E}(\sigma,\tau^\star), \\ \text{and} \\ E(\mathcal{D}(\sigma,\tau)) \cup E(\mathcal{E}(\sigma,\tau)) = E_2(\sigma). \end{cases} \tag{3}$$

Thus,

$$\text{if } E_1(\sigma) = \varnothing, \text{ then } \mathcal{E}(\sigma,\tau) \text{ is the complement of } \mathcal{D}(\sigma,\tau). \tag{4}$$

Definition 22. Let σ be a 2-structure. Given $W \subseteq V(\sigma)$, the 2-structure $\mathrm{Inv}(\sigma,W)$ is defined on $V(\mathrm{Inv}(\sigma,W)) = V(\sigma)$ as follows. Given distinct $v,w \in V(\mathrm{Inv}(\sigma,W))$,

$$\mathrm{Inv}(\sigma,W)(v,w) = \begin{cases} \sigma(w,v) \text{ if } v,w \in W \\ \sigma(v,w) \text{ otherwise.} \end{cases}$$

It is easy to show the next fact.

Fact 23. *Let σ be a 2-structure. For a module M of σ, σ and $\mathrm{Inv}(\sigma,M)$ are 3-hemimorphic.*

Fact 24. *Given 3-hemimorphic 2-structures σ and τ such that $v(\sigma) = 3$. If $E(\mathcal{D}(\sigma,\tau))$ and $E(\mathcal{E}(\sigma,\tau))$ are nonempty, then there exist distinct $v,w \in V(\sigma)$ such that*

- *$\{v,w\}$ is a module of σ and τ;*
- *$E(\mathcal{D}(\sigma,\tau)) = \{vw\}$ or $E(\mathcal{E}(\sigma,\tau)) = \{vw\}$;*
- *$E_1(\sigma) = \varnothing$ (so $\mathcal{E}(\sigma,\tau)$ is the complement of $\mathcal{D}(\sigma,\tau)$ by (4)).*

Proof. We can assume that $V(\sigma) = \{0,1,2\}$, $01 \in E(\mathcal{D}(\sigma,\tau))$, and $02 \in E(\mathcal{E}(\sigma,\tau))$. If $|\widetilde{L}(\sigma)| = 3$, then the only isomorphism from σ onto τ or $\tau^\star$ is $\mathrm{Id}_{\{0,1,2\}}$, which is impossible because $E(\mathcal{D}(\sigma,\tau))$ and $E(\mathcal{E}(\sigma,\tau))$ are nonempty. It follows that $|\widetilde{L}(\sigma)| \leq 2$.

To begin, suppose that $\widetilde{\sigma}(\{0,1\}) = \widetilde{\sigma}(\{0,2\})$. If $\widetilde{\sigma}(\{0,1\}) \neq \widetilde{\sigma}(\{1,2\})$, then one among σ or τ is prime whereas the other one is decomposable. It follows that $\widetilde{\sigma}(\{0,1\}) = \widetilde{\sigma}(\{1,2\})$. Hence, σ and τ are linear. Moreover, we obtain that $\{1,2\}$ is a module of σ or τ. For instance, assume that $\{1,2\}$ is a module of σ.

By exchanging τ and $\tau^\star$ if necessary, we can assume that $12 \in E(\mathcal{D}(\sigma,\tau))$. We obtain that $\{0,2\}$ is a module of σ and τ. Furthermore, $E(\mathcal{E}(\sigma,\tau)) = \{02\}$.

Now, suppose that $\widetilde{\sigma}(\{0,1\}) \neq \widetilde{\sigma}(\{0,2\})$. Since $|\widetilde{L}(\sigma)| \leq 2$, we can assume that $\widetilde{\sigma}(\{0,1\}) = \widetilde{\sigma}(\{1,2\})$. For a contradiction, suppose that $\{0,2\}$ is not a module of σ. Since σ and τ are both prime or not, we obtain $12 \in E(\mathcal{D}(\sigma,\tau))$. It is not difficult to verify that σ is isomorphic neither to τ nor to $\tau^\star$. Consequently, $\{0,2\}$ is a module of σ. Hence, τ is decomposable. It follows that $E(\mathcal{E}(\sigma,\tau)) = \{02\}$, and hence $\{0,2\}$ is a module of τ. □

The next fact is an immediate consequence of Fact 24.

Fact 25. *Given a 2-structure σ such that $v(\sigma) = 3$, if σ is prime, then σ is 3-forced.*

It is not difficult to show the next result. Nevertheless, we have to adopt the following convention.

Convention 26. Let σ and τ be 2-hemimorphic 2-structures. We have $\widetilde{L}(\sigma) = \widetilde{L}(\tau)$, and hence $\widetilde{L}_1(\sigma) = \widetilde{L}_1(\tau)$. Suppose that $|\widetilde{L}_1(\sigma)| \geq 2$. We choose the same $l \in L(\sigma) \cap L(\tau)$ such that $\{l\} \in \widetilde{L}_1(\sigma) \cap \widetilde{L}_1(\tau)$ to define $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$. Otherwise, $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$ would not be 2-hemimorphic.

Lemma 27. *Given 2-hemimorphic 2-structures σ and τ, σ and τ are 3-hemimorphic if and only if $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$ are 3-hemimorphic too.*

Now, we are ready to prove Lemma 6 (with Convention 26).

Proof of Lemma 6. To begin, suppose that σ is 3-forced. Consider a 2-structure τ which is 3-hemimorphic to $\overrightarrow{\sigma}$. There exists a unique 2-structure ρ such that σ and ρ are 2-hemimorphic and $\overrightarrow{\rho} = \tau$. By Lemma 27, σ and ρ are 3-hemimorphic. Since σ is 3-forced, we obtain $\sigma = \tau$ or $\tau^\star$. It follows that $\overrightarrow{\sigma} = \overrightarrow{\rho}$ or $(\overrightarrow{\rho})^\star$.

Conversely, suppose that $\overrightarrow{\sigma}$ is 3-forced. Consider a 2-structure τ which is 3-hemimorphic to σ. By Lemma 27, $\overrightarrow{\sigma}$ and $\overrightarrow{\tau}$ are 3-hemimorphic. Since $\overrightarrow{\sigma}$ is 3-forced, we obtain $\overrightarrow{\sigma} = \overrightarrow{\tau}$ or $(\overrightarrow{\tau})^\star$. Since σ and τ are 2-hemimorphic, we have $\sigma = \tau$ or $\tau^\star$. □

Example 28. Consider a set S admitting a partition $P = \{X_0, \dots, X_{n-1}\}$, where $n \geq 4$. We consider the partial order O defined on S satisfying

- P is a modular partition of O;
- O/P is the linear order $X_0 < X_1 < \dots < X_{n-1}$;
- for each $i \in \{0, \dots, n-1\}$, the arc set of $O[X_i]$ is empty.

Moreover, we consider the partiel order Q defined on S satisfying

- P is a modular partition of Q;
- O/P is the linear order $X_2 < X_0 < X_3 < \dots \leq X_{n-1} < X_1$;

- for each $i \in \{0, \ldots, n-1\}$, the arc set of $Q[X_i]$ is empty.

Clearly, σ_O and σ_Q are 3-hemimorphic. It is not difficult to verify that P is also a modular partition of $\mathcal{D}(\sigma_O, \sigma_Q)$ and $\mathcal{E}(\sigma_O, \sigma_Q)$ such that

- $E(\mathcal{D}(\sigma_O, \sigma_Q)/P) = \{X_0X_2\} \cup \{X_1X_i : i \in \{2, \ldots, n-1\}\}$;
- $\mathcal{E}(\sigma_O, \sigma_Q)/P$ is the complement of $\mathcal{D}(\sigma_O, \sigma_Q)/P$.

It follows that $\mathcal{D}(\sigma_O, \sigma_Q)$ and $\mathcal{E}(\sigma_O, \sigma_Q)$ are both connected. Hence, σ_O is not 3-forced. It is not difficult to verify that $P = \mathscr{G}(\sigma_O)$ and $\sigma_O/\mathscr{G}(\sigma_O)$ is linear; whence the necessity of Statement (T3) in Theorem 8. Furthermore, $X_0 \cup X_1$ is a module of σ_O such that $E_2(\sigma[X_0 \cup X_1]) \neq \varnothing$; whence the necessity of Statement (T4) in Theorem 42.

Example 29. Given $n \geq 2$, consider the critical 2-structures γ_{2n} introduced in Example 20. We consider also the 2-structure ρ_{2n} defined on $V(\rho_{2n}) = \{0, \ldots, 2n-1\}$ in the following manner. For any $p, q \in \{0, \ldots, 2n-1\}$ such that $p < q$,

$$\rho_{2n}(p,q) = \begin{cases} \gamma_{2n}(q,p) \text{ if } p \text{ is even and } q \text{ is odd,} \\ \gamma_{2n}(p,q) \text{ otherwise.} \end{cases} \tag{5}$$

By Claim 19, ρ_{2n} is critical. It is not difficult to verify that γ_{2n} and ρ_{2n} are 3-hemimorphic. Nevertheless, by considering $\{0,1,2,3\}$, we see that γ_{2n} and ρ_{2n} are not 4-hemimorphic. Clearly, we have

$$E(\mathcal{D}(\gamma_{2n}, \rho_{2n})) = E(G_{2n}) \text{ (see Example 20).}$$

By (3), $\mathcal{E}(\gamma_{2n}, \rho_{2n})$ is the complement of $\mathcal{D}(\gamma_{2n}, \rho_{2n})$. It follows that $\mathcal{D}(\gamma_{2n}, \rho_{2n})$ and $\mathcal{E}(\gamma_{2n}, \rho_{2n})$ are both connected. Hence, γ_{2n} is not 3-forced (see Problem 41). As seen in Example 20, γ_{2n} does not satisfy Statement (T1) of Theorem 8; whence the necessity of Statement (T1) in Theorem 8.

Fact 30. *Let σ and τ be two 3-hemimorphic 2-structures. Suppose that $v(\sigma) = 4$. If $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$ are both connected, and if $\sigma/\mathscr{G}(\sigma)$ is prime, then there exists an isomorphism φ from σ onto an element of Σ_4 (see Notation 18), which is also an isomorphism from $\mathcal{D}(\sigma, \tau)$ onto G_4 (see Example 20).*

Proof. Since $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$ are connected, $K_{1,3}$ and K_3 do not embed into $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$. It follows that $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$ are isomorphic to P_4. Hence, $\mathcal{D}(\sigma, \tau)$ is the complement of $\mathcal{E}(\sigma, \tau)$. Up to isomorphism, we can assume that $V(\sigma) = \{0,1,2,3\}$ and $\mathcal{D}(\sigma, \tau) = G_4$. Since $01, 03 \in E(\mathcal{D}(\sigma, \tau))$ and $13 \in E(\mathcal{E}(\sigma, \tau))$, it follows from Fact 24 that $\{1,3\}$ is a module of $\sigma[\{0,1,3\}]$. Hence, we have $\sigma(0,1) = \sigma(0,3)$. Similarly, we have $\sigma(0,3) = \sigma(2,3)$ and $\sigma(0,2) = \sigma(1,2) = \sigma(1,3)$. Therefore, σ satisfies (2). If $\widetilde{\sigma}(\{0,1\}) = \widetilde{\sigma}(\{0,2\})$, then σ is linear, which contradicts the fact that $\sigma/\mathscr{G}(\sigma)$ is prime. It follows that $\widetilde{\sigma}(\{0,1\}) \neq \widetilde{\sigma}(\{0,2\})$. Thus, $\sigma \in \Sigma_4$. □

Definition 31. Recall that a digraph is a *flag* [9] if it is isomorphic to the digraph $(\{0,1,2\},\{01,10,12\})$. A simple generalization to 2-structures follows. A 2-structure σ defined on $V(\sigma)=\{0,1,2\}$ is a *flag* if $|\widetilde{L}_1(\sigma)|=2$ and $|\widetilde{L}_2(\sigma)|=1$.

Hagendorf and Lopez [9] showed the next result for digraphs. The generalization to 2-structures is not difficult. Its proof uses mainly Fact 24.

Fact 32. *Let σ and τ be two 3-hemimorphic 2-structures. Suppose that σ does not contain flags. Each component of $\mathcal{D}(\sigma,\tau)$ (or of $\mathcal{E}(\sigma,\tau)$) is a module of σ and τ.*

Lemma 33. *Let σ and τ be two 3-hemimorphic 2-structures. Suppose that $v(\sigma)\geq 5$. If $\mathcal{D}(\sigma,\tau)$ and $\mathcal{E}(\sigma,\tau)$ are both connected and if $\sigma/\mathscr{G}(\sigma)$ is prime, then the following two statements hold*

1. *there exits $v\in V(\sigma)$ such that $(\sigma-v)/\mathscr{G}(\sigma-v)$ is prime;*
2. *for each $v\in V(\sigma)$ such that $(\sigma-v)/\mathscr{G}(\sigma-v)$ is prime, $\mathcal{D}(\sigma-v,\tau-v)$ and $\mathcal{E}(\sigma-v,\tau-v)$ are both connected.*

Proof. We begin with the following observation. Consider 3-hemimorphic 2-structures γ and ρ. By Lemma 27, $\overrightarrow{\gamma}$ and $\overrightarrow{\tau}$ are 3-hemimorphic. We have $\mathcal{D}(\gamma,\rho)=\mathcal{D}(\overrightarrow{\gamma},\overrightarrow{\rho})$ and $\mathcal{E}(\gamma,\rho)=\mathcal{E}(\overrightarrow{\gamma},\overrightarrow{\rho})$. Moreover, suppose that the graph $(V(\gamma),E_2(\gamma))$ is connected. It follows from Theorem 4 that $\gamma/\mathscr{G}(\gamma)$ is linear or prime. Similarly, it follows from Theorem 4 that $\overrightarrow{\gamma}/\mathscr{G}(\overrightarrow{\gamma})$ is linear or prime. Clearly, $\gamma/\mathscr{G}(\gamma)$ is linear if and only if $\overrightarrow{\gamma}/\mathscr{G}(\overrightarrow{\gamma})$ is as well. Therefore, $\gamma/\mathscr{G}(\gamma)$ is prime if and only if $\overrightarrow{\gamma}/\mathscr{G}(\overrightarrow{\gamma})$ is as well.

Consequently, we can assume that $\sigma=\overrightarrow{\sigma}$ and $\tau=\overrightarrow{\tau}$. In particular, σ and τ do not contain flags.

The first statement follows from Lemma 17. For the second statement, consider $v\in V(\sigma)$ such that $(\sigma-v)/\mathscr{G}(\sigma-v)$ is prime. For a contradiction, suppose that $\mathcal{D}(\sigma-v,\tau-v)$ and $\mathcal{E}(\sigma-v,\tau-v)$ are not both connected. By exchanging $\sigma-v$ and $(\sigma-v)^\star$ if necessary, we can assume that $\mathcal{D}(\sigma-v,\tau-v)$ is disconnected. By Fact 32, the components of $\mathcal{D}(\sigma-v,\tau-v)$ are modules of $\sigma-v$ and $\tau-v$. Since $\mathcal{E}(\sigma,\tau)$ is connected, there exists $x\in V(\sigma)\smallsetminus\{v\}$ such that

$$xv\in E(\mathcal{E}(\sigma,\tau)).$$

Denote by C the component of $\mathcal{D}(\sigma-v,\tau-v)$ containing x. Furthermore, since $(\sigma-v)/\mathscr{G}(\sigma-v)$ is not linear, there exists a component D of $\mathcal{D}(\sigma-v,\tau-v)$ such that $D\neq C$ and $\sigma(z,x)\neq\sigma(v,x)$ for $z\in D$. Lastly, recall that $\mathcal{D}(\sigma,\tau)$ is connected. By considering a shortest path in $\mathcal{D}(\sigma,\tau)$ from v to an element of D, we obtain $z\in D$ such that

$$zv\in E(\mathcal{D}(\sigma,\tau)).$$

It follows from Fact 24 that $\sigma[\{x,z,v\}]$ and $\tau[\{x,z,v\}]$ are not 3-hemimorphic, which contradicts the fact that σ and τ are 3-hemimorphic. Consequently, $\mathcal{D}(\sigma-v,\tau-v)$ and $\mathcal{E}(\sigma-v,\tau-v)$ are both connected. ☐

The next result is an immediate consequence of Fact 30 and Lemma 33.

Corollary 34. *Let σ and τ be two 3-hemimorphic 2-structures. Suppose that $v(\sigma) \geq 4$. If $\mathcal{D}(\sigma,\tau)$ and $\mathcal{E}(\sigma,\tau)$ are both connected and if $\sigma/\mathscr{G}(\sigma)$ is prime, then there exits $W \subseteq V(\sigma)$ such that $\sigma[W]$ and $\tau[W]$ are isomorphic to an element of Σ_4 (see Notation 18).*

Remark 35. Consider 3-hemimorphic digraphs D and Δ. The definitions of the modular partition $\mathscr{G}(D)$ and of the graphs $\mathcal{D}(D,\Delta)$ and $\mathcal{E}(D,\Delta)$ in [5] are similar to those given for 2-structures here. It is easy to see that $\mathscr{G}(D) = \mathscr{G}(\sigma_D)$, $\mathcal{D}(D,\Delta) = \mathcal{D}(\sigma_D,\sigma_\Delta)$, and $\mathcal{E}(D,\Delta) = \mathcal{E}(\sigma_D,\sigma_\Delta)$. Boussaïri et al. [4, 5] proved the following. If $D/\mathscr{G}(D)$ is prime, then $\mathcal{D}(D,\Delta)$ and $\mathcal{E}(D,\Delta)$ are not both connected.

The next result follows from Corollary 34 and Fact 32.

Corollary 36. *Let σ be a 2-structure such that $v(\sigma) \geq 4$. Suppose that σ contains neither flags nor elements of Σ_4. If σ is prime, then σ is 3-forced.*

Proof. Let τ be a 2-structure, which is 3-hemimorphic to σ. Clearly, $\sigma/\mathscr{G}(\sigma)$ is prime because σ is prime. Since σ does not contain elements of Σ_4, it follows from Corollary 34 that $\mathcal{D}(\sigma,\tau)$ and $\mathcal{E}(\sigma,\tau)$ are not both connected. By exchanging σ and $\sigma^\star$ if necessary, we can assume that $\mathcal{D}(\sigma,\tau)$ is disconnected. Let C be any component of $\mathcal{D}(\sigma,\tau)$. Since σ does not contain flags, it follows from Fact 32 that $V(C)$ is a module of σ. Lastly, since σ is prime, $V(C)$ is a singleton. Consequently, the edge set of $\mathcal{D}(\sigma,\tau)$ is empty, which means $\sigma = \tau$. □

Remark 37. For digraphs, Boussaïri et al.. [4, 5] obtained the analogous result without forbidding the elements of Σ_4. Indeed, they proved that a prime digraph without flags is 3-forced. As seen in Example 29, it is also necessary to forbid the elements of Σ_4 for 2-structures.

4 Proofs of Theorems 8 and 10

Proposition 38. *Let σ be a 2-structure. Suppose that $\sigma = \overrightarrow{\sigma}$ and $(V(\sigma), E_2(\sigma))$ does not have isolated vertices. If σ is 3-forced, then σ satisfies Statements (T1), (T2), and (T3) (see Theorem 8).*

Proof. To begin, suppose that G_σ is disconnected. Let C be a component of G_σ. We consider the 2-structure τ defined on $V(\tau) = V(\sigma)$ as follows. Given distinct $v, w \in V(\tau)$,

$$\tau(v,w) = \begin{cases} \sigma(w,v) \text{ if } \widetilde{\sigma}(\{v,w\}) \in V(C) \\ \sigma(v,w) \text{ otherwise.} \end{cases}$$

We verify that σ and τ are 3-hemimorphic. Clearly, σ and τ are 2-hemimorphic. Let $W \subseteq V(\sigma)$ satisfying $|W| = 3$. If $\widetilde{L}_2(\sigma[W]) \subseteq V(C)$, then $\tau[W] = (\sigma[W])^\star$. Furthermore, if $\widetilde{L}_2(\sigma[W]) \cap V(C) = \varnothing$, then $\tau[W] = (\sigma[W])$. Therefore, suppose that there exist $p \in \widetilde{L}_2(\sigma[W]) \cap V(C)$ and $q \in (\widetilde{L}_2(\sigma[W]) \setminus V(C))$. By definition of G_σ, $\sigma[W]$ is decomposable. It follows that $\widetilde{L}_2(\sigma[W]) = \{p, q\}$. Consider distinct $v, w \in W$ such that $\{v, w\}$ is a nontrivial module of $\sigma[W]$. If $\widetilde{\sigma}(\{v, w\}) = p$, then $\tau[W]$ is isomorphic to $\sigma[W]$. Moreover, if $\widetilde{\sigma}(\{v, w\}) = q$, then $\tau[W]$ is isomorphic to $(\sigma[W])^\star$. Consequently, σ and τ are 3-hemimorphic. It follows that σ is not 3-forced. In what follows, we suppose that Statement (T1) holds.

To continue, suppose that there exists $M \in \mathscr{G}(\sigma)$ such that $E_2(\sigma[M]) \neq \varnothing$. By Fact 23, σ and $\text{Inv}(\sigma, M)$ are 3-hemimorphic. Since $E_2(\sigma[M]) \neq \varnothing$, $\text{Inv}(\sigma, M) \neq \sigma$. Let $v \in V(\sigma) \setminus M$. Since $(V(\sigma), E_2(\sigma))$ does not have isolated vertices, there exists $w \in V(\sigma) \setminus \{v\}$ such that $\{v, w\} \in E_2(\sigma)$. Since $\{v, w\} \setminus M \neq \varnothing$, $\text{Inv}(\sigma, M) \neq \sigma^\star$. It follows that σ is not 3-forced. In what follows, we suppose that Statement (T2) holds.

Lastly, suppose that $|\mathscr{G}(\sigma)| \geq 3$ but $\sigma/\mathscr{G}(\sigma)$ is not prime. By Theorem 4, $\sigma/\mathscr{G}(\sigma)$ is constant or linear. Let $M \in \sigma/\mathscr{G}(\sigma)$. Consider $v \in M$. Since $(V(\sigma), E_2(\sigma))$ does not have isolated vertices, there exists $w \in V(\sigma) \setminus \{v\}$ such that $\{v, w\} \in E_2(\sigma)$. Furthermore, since Statement (T2) holds, $w \notin M$. It follows that $\sigma/\mathscr{G}(\sigma)$ is not constant, so $\sigma/\mathscr{G}(\sigma)$ is linear. Let M_0 and M_1 be the first two elements of $\sigma/\mathscr{G}(\sigma)$. Clearly, $M_0 \cup M_1$ is a module of σ. By Fact 23, σ and $\text{Inv}(\sigma, M_0 \cup M_1)$ are 3-hemimorphic. For $v_0 \in M_0$ and $v_1 \in M_1$, we have $\{v_0, v_1\} \in E_2(\sigma)$, and hence $\text{Inv}(\sigma, M) \neq \sigma$. Since $|\mathscr{G}(\sigma)| \geq 3$, there exists $v \in V(\sigma) \setminus (M_0 \cup M_1)$. We obtain $\{v_0, v\} \in E_2(\sigma)$, and hence $\text{Inv}(\sigma, M) \neq \sigma^\star$. It follows that σ is not 3-forced. Consequently, Statement (T3) holds. □

Proposition 39. *Let σ be a prime 2-structure. Suppose that $\sigma = \overrightarrow{\sigma}$. If σ satisfies Statement (T1), then σ is 3-forced.*

Proof. Consider a 2-structure τ which is 3-hemimorphic to σ. To begin, suppose that $|\widetilde{L}_2(\sigma)| = 1$. Since $\sigma = \overrightarrow{\sigma}$, σ does not contain flags. Furthermore, since $|\widetilde{L}_2(\sigma)| = 1$, σ does not contain elements of Σ_4. It follows from Corollary 36 that $\tau = \sigma$ or $\sigma^\star$.

Now, suppose that $|\widetilde{L}_2(\sigma)| \geq 2$. Since G_σ is connected by Statement (T1), there exists $X \subseteq V(\sigma)$ such that $|X| = 3$ and $\sigma[X]$ is prime. By Lemma 16, there exist elements $y_0, \dots, y_{v(\sigma)-4}$ of $V(\sigma) \setminus X$ such that Statements (S1), (S2), and (S3) hold. For a contradiction, suppose that $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$ are both connected. By applying several times Lemma 33, we obtain that $\mathcal{D}(\sigma[X \cup \{y_0\}], \tau[X \cup \{y_0\}])$ and $\mathcal{E}(\sigma[X \cup \{y_0\}], \tau[X \cup \{y_0\}])$ are both connected. By Fact 30, $\sigma[X \cup \{y_0\}]$ is isomorphic to an element of Σ_4 (see Notation 18). By Claim 19, $\sigma[X \cup \{y_0\}]$ is critical, which is impossible because $\sigma[X]$ is prime. Consequently, $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$ are not both connected. By exchanging τ and $\tau^\star$ if necessary, we can assume that $\mathcal{D}(\sigma, \tau)$ is disconnected. As shown at the end of the proof of Corollary 36, we obtain $\sigma = \tau$. □

The next result is an immediate consequence of Propositions 38 and 39.

Corollary 40. *Let σ be a prime 2-structure. Suppose that $\sigma = \overrightarrow{\sigma}$. The 2-structure σ is 3-forced if and only if σ satisfies Statement (T1).*

Problem 41. A 2-structure σ is said to be *asymmetric* if $\widetilde{L}_1(\sigma) = \varnothing$. Describe the structural form of a prime asymmetric 2-structure σ such that there exists a 2-structure τ satisfying

- σ and τ are 3-hemimorphic;
- $\mathcal{D}(\sigma,\tau)$ and $\mathcal{E}(\sigma,\tau)$ are both connected.

As seen in Example 29, γ_{2n} is such a 2-structure.

Proof of Theorem 8. To begin, suppose that σ is 3-forced. By Proposition 38, σ satisfies Statements (T1), (T2), and (T3). Conversely, suppose that σ satisfies Statements (T1), (T2), and (T3). If σ is prime, then it follows from Proposition 39 that σ is 3-forced. Hence, suppose that σ is decomposable. Consider a 2-structure τ which is 3-hemimorphic to σ. Since σ satisfies Statement (T2), it follows from Fact 24 that $\mathscr{G}(\sigma)$ is a modular partition of τ as well.

To continue, we show that for each $M \in \mathscr{G}(\sigma)$, there exists $N \in \mathscr{G}(\sigma) \setminus \{M\}$ such that

$$\{x,y\} \in E_2(\sigma) \tag{6}$$

for any $x \in M$ and $y \in N$. Let $M \in \mathscr{G}(\sigma)$. Consider $v \in M$. Since $(V(\sigma), E_2(\sigma))$ does not have isolated vertices, there exists $w \in V(\sigma) \setminus \{v\}$ such that $\{v,w\} \in E_2(\sigma)$. Furthermore, since Statement (T2) holds, $w \notin M$. By denoting by N the element of $\mathscr{G}(\sigma)$ containing w, we obtain that (6) holds.

We conclude as follows. First, suppose that $|\mathscr{G}(\sigma)| = 2$. It follows from (6) that $\sigma/\mathscr{G}(\sigma)$ is linear. Since σ and τ are 3-hemimorphic, we have $E_2(\tau[M]) = \varnothing$ for each $M \in \mathscr{G}(\sigma)$. Lastly, since $\mathscr{G}(\sigma)$ is a modular partition of τ, we obtain $\tau = \sigma$ or $\sigma^\star$.

Second, suppose that $|\mathscr{G}(\sigma)| \geq 3$. Since σ satisfies Statement (T3), $\sigma/\mathscr{G}(\sigma)$ is prime. Consider the set $\mathcal{T}$ of $X \subseteq V(\sigma)$ such that $|X \cap M| = 1$ for each $M \in \mathscr{G}(\sigma)$. For every $X \in \mathcal{T}$, we verify that

$$\tau[X] = \sigma[X] \text{ or } (\sigma[X])^\star. \tag{7}$$

Let $X \in \mathcal{T}$. Since $\sigma/\mathscr{G}(\sigma)$ is prime, $\sigma[X]$ is prime. Since σ satisfies Statements (T1), (T2), and (T3), $\sigma[X]$ satisfies Statement (T1). Since $\sigma[X]$ and $\tau[X]$ are 3-hemimorphic too, it follows from Proposition 39 that $\tau[X] = \sigma[X]$ or $(\sigma[X])^\star$. Consequently, (7) holds. Finally, by (6), there exist distinct $M,N \in \mathscr{G}(\sigma)$ such that $\{x,y\} \in E_2(\sigma)$ for any $x \in M$ and $y \in N$. Since $\mathscr{G}(\sigma)$ is a modular partition of τ, we obtain

$$\text{for any } x \in M \text{ and } y \in N,\ xy \in E(\mathcal{D}(\sigma,\tau))$$

or

$$\text{for any } x \in M \text{ and } y \in N,\ xy \in E(\mathcal{E}(\sigma,\tau)).$$

By exchanging τ and $\tau^\star$ if necessary, we can assume that $xy \in E(\mathcal{E}(\sigma,\tau))$ for any $x \in M$ and $y \in N$. For every $X \in \mathcal{T}$, it follows from (7) that $\tau[X] = \sigma[X]$. We obtain $\sigma = \tau$. $\square$

An equivalent statement of Theorem 8 follows.

Theorem 42. *Let σ be a 2-structure. Suppose that $\sigma = \overrightarrow{\sigma}$ and $(V(\sigma), E_2(\sigma))$ does not have isolated vertices. The 2-structure σ is 3-forced if and only if the following two statements hold*

(T1) G_σ is connected;

(T4) for each nontrivial module M of σ, $E_2(\sigma[M]) = \varnothing$.

We conclude with the proof of Theorem 10.

Proof of Theorem 10. Suppose that σ is prime and $(V(\sigma), E_2(\sigma))$ is connected. If $v(\sigma) = 3$, then it suffices to apply Fact 25. Hence, suppose that $v(\sigma) \geq 4$. Consider a 2-structure τ which is 4-hemimorphic to σ. We have to show that $\tau = \sigma$ or $\sigma^\star$.

For a contradiction, suppose that $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$ are both connected. By applying several times Lemma 33, we obtain $X \subseteq V(\sigma)$ satisfying

- $|X| = 4$;
- $\sigma[X]/\mathscr{G}(\sigma[X])$ is prime;
- $\mathcal{D}(\sigma[X], \tau[X])$ and $\mathcal{E}(\sigma[X], \tau[X])$ are both connected.

By Fact 30, there exists an isomorphism φ from $\sigma[X]$ onto an element of Σ_4 (see Notation 18), which is also an isomorphism from $\mathcal{D}(\sigma[X], \tau[X])$ onto G_4 (see Example 20). As already noted in Example 29, $\sigma[X]$ is never isomorphic to either $\tau[X]$ or $(\tau[X])^\star$, which contradicts the fact that σ and τ are 4-hemimorhic. Consequently, $\mathcal{D}(\sigma, \tau)$ and $\mathcal{E}(\sigma, \tau)$ are not both connected.

By exchanging σ and $\sigma^\star$ if necessary, we can assume that $\mathcal{D}(\sigma, \tau)$ is disconnected. Consider a component C of $\mathcal{D}(\sigma, \tau)$. We verify that $V(C)$ is a module of σ. Let $v \in V(\sigma) \setminus V(C)$. We distinguish the following two cases.

1. There exists $x \in V(C)$ such that $\{x, v\} \in E_2(\sigma)$. Since $v \notin V(C)$, we have $xv \in E(\mathcal{E}(\sigma, \tau))$. Let $y \in V(C)$ such that $xy \in E(\mathcal{D}(\sigma, \tau))$. We prove that

$$\sigma(x, v) = \sigma(y, v).$$

It follows from Fact 24 that either $\{x, y\}$ is a module of $\sigma[\{x, y, v\}]$ and $\tau[\{x, y, v\}]$ or $\{x, v\}$ is a module of $\sigma[\{x, y, v\}]$ and $\tau[\{x, y, v\}]$. If $\{x, v\}$ is a module of $\sigma[\{x, y, v\}]$ and $\tau[\{x, y, v\}]$, then $yv \in E(\mathcal{D}(\sigma, \tau))$, which contradicts the fact that C is a component of $\mathcal{D}(\sigma, \tau)$. Hence, $\{x, y\}$ is a module of $\sigma[\{x, y, v\}]$ and $\tau[\{x, y, v\}]$. Thus, we have $\sigma(x, v) = \sigma(y, v)$. Now, by considering $z \in V(C)$ such that $yz \in E(\mathcal{D}(\sigma, \tau))$, we obtain in the same manner that $\sigma(y, v) = \sigma(z, v)$, and hence $\sigma(x, v) = \sigma(z, v)$. Since C is a component of $\mathcal{D}(\sigma, \tau)$, we obtain $\sigma(x, v) = \sigma(t, v)$ for every $t \in V(C)$.

2. For every $x \in V(C)$, $\{x, v\} \in E_1(\sigma)$. Since $(V(\sigma), E_2(\sigma))$ is connected, there exists $y \in V(C)$ and $w \in V(\sigma) \setminus V(C)$ such that $\{y, w\} \in E_2(\sigma)$. By the first case above, we have $\sigma(y, w) = \sigma(t, w)$ for every $t \in V(C)$. Let $z \in V(C)$ such that $yz \in E(\mathcal{D}(\sigma, \tau))$. We have $\sigma(y, w) = \sigma(z, w)$ and $zw \in E(\mathcal{E}(\sigma, \tau))$. Since σ and τ are 4-hemimorphic, there exists an isomorphism f from $\sigma[\{y, z, v, w\}]$ onto $\tau[\{y, z, v, w\}]$ or $(\tau[\{y, z, v, w\}])^\star$. Since v is the unique element of $\{y, z, v, w\}$ such that

$$|\{t \in \{y, z, v, w\} : \{t, v\} \in E_1(\sigma)\}| \geq 2,$$

we have $f(v) = v$. We obtain $f = (yz)$ or (ywz). In both cases, we have $\sigma(y, v) = \sigma(z, v)$. We conclude as in the first case above by using the fact that C is a component of $\mathcal{D}(\sigma, \tau)$.

It follows that $V(C)$ is a module of σ. Since σ is prime, $V(C)$ is a singleton. Consequently, the edge set of $\mathcal{D}(\sigma, \tau)$ is empty, which means $\sigma = \tau$. □

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