# Sound and complete axiomatisations
# of coalgebraic language equivalence*

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Coalgebras provide a uniform framework to study dynamical systems, including several types of automata. In this paper, we make use of the coalgebraic view on systems to investigate, in a uniform way, under which conditions calculi that are sound and complete with respect to behavioral equivalence can be extended to a coarser coalgebraic language equivalence, which arises from a generalised powerset construction that determinises coalgebras. We show that soundness and completeness are established by proving that expressions modulo axioms of a calculus form the rational fixpoint of the given type functor. Our main result is that the rational fixpoint of the functor $FT$, where $T$ is a monad describing the branching of the systems (e.g. non-determinism, weights, probability etc.), has as a quotient the rational fixpoint of the "determinised" type functor $\bar{F}$, a lifting of $F$ to the category of $T$-algebras. We apply our framework to the concrete example of weighted automata, for which we present a new sound and complete calculus for weighted language equivalence. As a special case, we obtain non-deterministic automata, where we recover Rabinovich's sound and complete calculus for language equivalence.

## 1. INTRODUCTION

State-based structures of various kinds are used to model the behavior of phenomena in many different fields of computer science, mathematics, and more recently of biology and physics. So the theories of state based systems, their specification, semantics and logical descriptions are topics at the heart of theoretical computer science.

A major step forward, in the last years was the realisation that a vast majority of state-based systems can be uniformly described as instances of the general notion of coalgebra. For an endofunctor $F$ on a category $\mathcal{A}$, an $F$-coalgebra is a pair $(X, f)$,

where $X$ is an object of $\mathcal{A}$ representing the state space and $f\colon X \to FX$ is an arrow of $\mathcal{A}$ defining the observations and transitions of the states. The strength of coalgebraic modelling lies in the fact that the type $F$ of the system determines a standard notion of equivalence called $F$-behavioral equivalence and a canonical domain of behavior, the so-called *final* coalgebra, into which any $F$-coalgebra is mapped by a unique homomorphism that identifies all equivalent states.

The coalgebraic perspective on state-based systems has recently been proved very relevant by the development of a number of generalised calculi of *regular expressions* admitting Kleene theorems and equipped with sound and complete equational logics, which are expressive enough to characterise the behavioral equivalence of *all* finite state coalgebras uniformly for an inductively defined class of type functors on sets. This includes Mealy automata [Bonsangue et al. 2008], automata whose type is given by Kripke polynomial functors [Silva et al. 010a], automata for the so-called quantitative functors [Silva et al. 2011] (e. g., weighted automata, Segala systems and Pnüeli-Zuck systems) and closed stream circuits [Milius 2010]. This line of work generalises Kleene's classical theorem [Kleene 1956] as well as work on sound and complete expression calculi (e. g. [Salomaa 1966], see also [Kozen 1994]). A key result for the generalization is that soundness and completeness is equivalent to proving that generalized regular expressions for $F$-coalgebras modulo the axioms and rules of the calculus form a final locally finite coalgebra $\varrho F$; equivalently, this is the initial iterative $F$-algebra of [Adámek et al. 2006].

The above calculi axiomatize $F$-behavioral equivalence, which, for a functor $F$ preserving weak pullbacks, is equivalent to bisimilarity. However, bisimilarity is a very fine grained equivalence, and one is often interested in a coarser trace or language equivalence. In this paper we will present a general methodology to extend sound and complete calculi with respect to behavioral equivalence to sound and complete calculi for a new *coalgebraic language equivalence*.

As one approach to this equivalence it has recently been shown [Silva et al. 2010] that the classical powerset construction, which transforms a non-deterministic automaton into a deterministic one, providing language semantics to the former, can be extended to a large class of systems, coalgebras for a given type functor, which includes probabilistic and weighted automata. The aforementioned paper models systems as the composite of a functor type $F$ and a monad $T$, which encodes the non-determinism or probabilities that one wants to *determinise*. The determinised coalgebra is actually a coalgebra in the category of Eilenberg-Moore algebras for the monad $T$. We will call the equivalence obtained by this construction, that is, the $F$-behavioral equivalence in the category of $T$-algebras, coalgebraic language equivalence. For example, the construction above applied to non-deterministic automata yields a deterministic automaton in the category of join-semilattices. Coalgebraic language equivalence corresponds to ordinary language equivalence, while $FT$-behavioral equivalence is just ordinary bisimilarity. More interestingly, the construction also applies to weighted automata, in which case the determinisation is an automaton in the category of vector spaces (assuming the weights are elements of a field). Coalgebraic language equivalence corresponds to weighted language equivalence, while $FT$-behavioral equivalence is just weighted bisimilarity of [Buchholz 2008].

The contributions of our paper are twofold and we explain them in the following subsections.

## 1.1. Finitary Coinduction

Firstly, we will develop a mathematical theory of finitary coinduction w.r.t. coalgebraic language equivalence. Our theory builds on [Adámek et al. 2006] which provides the foundations for a theory of finitary coinduction w.r.t. ordinary behavioral equivalence,

and we extend here the first steps for the new theory taken (in a very special case) by the second author in [Milius 2010].

We start by systematically studying coalgebras for endofunctors $F$ having a lifting $\bar{F}$ to the category of (Eilenberg-Moore) algebras for a monad $T$. To begin with, in Section 3, we study the relationship between the final coalgebras $\nu F$ and $\nu(FT)$ as well as the between the *rational fixpoints* $\varrho(FT)$ for $FT$ and $\varrho\bar{F}$ for the lifting of $F$ to $T$-algebras.

Intuitively, one should think of $\nu(FT)$ and $\nu F$ as the behaviors of *all* systems modulo bisimilarity and language equivalence, respectively. Furthermore, $\varrho(FT)$ and $\varrho\bar{F}$ are the behaviors of all *finite state* systems modulo bisimilarity and language equivalence, respectively.

We prove that for every finitary endofunctor $H$, $\varrho H$ is the final locally finitely presentable coalgebra. It is also a fixpoint of $H$, and the inverse of its structure map yields the initial iterative algebra for $H$ in the sense of [Adámek et al. 2006]. The latter characterisation gives a precise connection of the work in [Silva et al. 010a; Silva et al. 2011; Milius 2010] with the classical work on iterative algebras by Nelson [1983] and Elgot's iterative theories [Elgot 1975] (see also [Bloom and Ésik 1993b]). In our setting we use a well-known coalgebraic construction of $\varrho H$: it is the colimit of all finite $H$-coalgebras. Here we consider $H = FT$ as above, and we prove (see Corollary 3.35) that the rational fixpoint $\varrho\bar{F}$ of the lifting $\bar{F}$ of $F$ to $T$-algebras is also a colimit of finite $FT$-coalgebras but with a different set of connecting morphisms in the corresponding diagram: in lieu of homomorphisms between coalgebras $X \to FTX$ one uses $\bar{F}$-coalgebra homomorphisms on the corresponding *determinisations* $TX \to FTX$. As our main result we establish the relationships between the four mentioned fixpoints of $F$ and $FT$ as summarised by the following commutative square among $T$-algebras (see Theorem 3.41):

$$\begin{array}{ccc} \varrho(FT) & \rightarrowtail & \nu(FT) \\ \downarrow & & \downarrow \\ \varrho\bar{F} & \rightarrowtail & \nu F \end{array} \tag{1.1}$$

This diagram shows that the rational fixpoints of $FT$ and $\bar{F}$ are, as expected, subcoalgebras of the respective final coalgebras (horizontal maps). This makes precise the above intuition that the rational fixpoints are the behaviors of finite state systems; in fact, behavioral equivalent states of finite state systems are identified by the unique coalgebra homomorphisms into $\varrho(FT)$ and $\varrho\bar{F}$, respectively. Furthermore, the final coalgebra for $F$ is a quotient of the final coalgebra for $FT$ and this quotient restricts to the respective rational fixpoints (vertical maps). This means that $FT$-behavioral equivalence implies coalgebraic language equivalence (i. e. $F$-behavioral equivalence), our abstract version of the well-known fact that bisimilarity implies language (or trace) equivalence. All these results hold whenever (a) finitely generated $T$-algebras are closed under kernel pairs, (b) $T$ is a finitary monad and (c) $F$ is a finitary functor preserving weak pullbacks and having a lifting to the category of $T$-algebras. Examples of algebras satisfying the above condition (a) include join-semilattices, Abelian groups, vector spaces, semimodules for Noetherian semirings, but e.g. not groups.
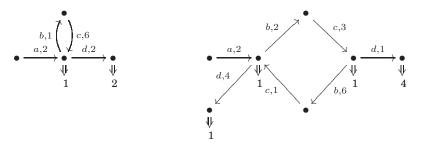
## 1.2. Expression Calculi for Coalgebraic Language Equivalence

In Section 4 we apply our results from Section 3 to obtain an abstract Kleene's theorem (Theorem 4.2) and soundness and completeness results (Theorems 4.4 and 4.5), and we show that in our setting it is possible to extend a given calculus for behavioral
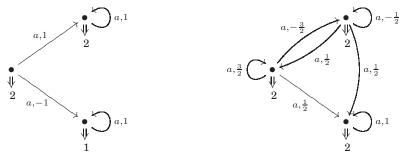
equivalence to one for coalgebraic language equivalence. Here we work without concrete syntax; the results collect those parts of the soundness and completeness proofs that are generic, so that for concrete calculi one saves work. We prove that showing soundness and completeness of a concrete calculus is equivalent to proving that the syntactic expressions modulo the axioms and rules of the calculus form the rational fixpoint $\varrho\bar{F}$ for the lifting of the functor $F$ to $T$-algebras.

Then we apply our abstract results to the monad $V$ of free semimodules for a Noetherian semiring $\mathbb{S}$ and the functor $FX = \mathbb{S} \times X^A$, where $A$ is a finite input alphabet, and we show how to obtain a sound and complete calculus for the language equivalence of weighted automata in Section 5, and, as a special case, of non-deterministic automata in Section 6.

Weighted automata were introduced by Schützenberger [1961], see also [Droste et al. 2009]. For example, take the following two weighted automata over the alphabet $A = \{a, b, c, d\}$ with weight over the semiring of integers (output values in the states are represented with a double arrow, when omitted they are zero):

We will see in Section 2.2 that they are coalgebras for the composition of the functor functor $FX = \mathbb{S} \times X^A$ with the the monad $V$ of free semimodules for the semiring $\mathbb{S}$ of integers. What is interesting is that the leftmost states of these automata are not bisimilar, but they recognise the same weighted language. Namely, the language that associates with each word $a(bc)^n$ the weight $2 \cdot 6^n$, with $a(bc)^n d$ the weight $2 \cdot 6^n \cdot 4$ and with any other word weight zero. We will provide an algebraic proof of this equivalence in the sequel. To give upfront the reader a feeling for how intricate it can get to reason about weighted language equivalence, we show another example of two weighted automata over the singleton alphabet $A = \{a\}$ but with weight over the field of real numbers.

The leftmost states of these automata recognize the weighted language that assigns to the empty word weight $2$ and to any word $a^n$ ($n \geq 1$) the weight $1$. In the left-hand automaton, it is still relatively easy to convince oneself that this is the case, whereas for the right-hand automaton one needs some more ingenuity. We shall see that the algebraic proof is rather simple and instructive.

We start with the calculus for weighted bisimilarity obtained from the generic expression calculus of [Silva et al. 2011], and we extend this by adding three canonical equational axioms. More detailed, the syntactic expressions of our calculus are defined by the grammar[1]

$$E ::= x \mid \mathbf{0} \mid E \oplus E \mid \underline{r} \mid a.(r \bullet E) \mid \mu x.E,$$

where $x$ ranges over a finite set of syntactic variables, $a$ over a finite (input) alphabet, and $r$ over a Noetherian semiring. We show that each expression denotes a weighted language (cf. (4.1)): for example $\mathbf{0}$ denotes the empty weighted language, $\oplus$ is union of weighted languages, $\underline{r}$ denotes the language that assigns to the empty word the weight $r$ (and $0$ to all other words), $a.(r \bullet E)$ denotes a language that assigns to a word $aw$ the weight $r \cdot r_w$, where $r_w$ is the weight assigned to $w$ in the language denoted by the expression $E$, and $\mu$ is a fixpoint operator. From our abstract Kleene theorem (Theorem 4.2) we then obtain that for every state of a finite weighted automaton there exists an expression denoting the weighted language accepted by the given automaton starting from the given state $s$.

For our new axiomatisation of weighted language equivalence we consider first the following rules:

$$E_1 \equiv E_2[E_1/x] \implies E_1 \equiv \mu x.E_2 \qquad a.(0 \bullet E) \equiv \mathbf{0} \qquad \mathbf{0} \oplus E \equiv E$$
$$(E_1 \oplus E_2) \oplus E_3 \equiv E \oplus (E_2 \oplus E_3) \qquad E_1 \oplus E_2 \equiv E_2 \oplus E_1 \qquad \underline{r} \oplus \underline{s} \equiv \underline{r+s}$$
$$a.(r \bullet E) \oplus a.(s \bullet E) \equiv a.((r+s) \bullet E) \qquad \mu x.E \equiv E[\mu x.E/x] \qquad \underline{0} \equiv \mathbf{0}$$

As proved in [Silva et al. 2011] those axioms and rules together with $\alpha$-equivalence (i.e., renaming of variables bound by $\mu$ does not matter) and the replacement rule

$$E_1 \equiv E_2 \implies E[E_1/x] = E[E_2/x]$$

are sound and complete with respect to *weighted bisimilarity*.

Now we add the following three equational axioms to the above calculus:

$$a.(r \bullet (E_1 \oplus E_2)) \equiv a.(r \bullet E_1) \oplus a.(r \bullet E_2)$$
$$a.(r \bullet \underline{s}) \equiv a.(1 \bullet \underline{rs})$$
$$a.(r \bullet b.(s \bullet E)) \equiv a.((rs) \bullet b.(1 \bullet E))$$

Here $1$ is the multiplicative unit of the semiring. Our main result in Section 5 is that this augmented calculus is sound and complete with respect to *weighted language equivalence*.

In Section 6 we mention the special case of non-deterministic automata. In this case the syntactic expressions simplify to

$$E ::= x \mid \mathbf{0} \mid E \oplus E \mid \underline{1} \mid a.E \mid \mu x.E.$$

and the equational axioms and rules for bisimilarity of expressions derived from the work in [Silva et al. 2011] are

$$E_1 \equiv E_2[E_1/x] \implies E_1 \equiv \mu x.E_2 \qquad \mu x.E \equiv E[\mu x.E/x] \qquad \mathbf{0} \oplus E \equiv E$$
$$(E_1 \oplus E_2) \oplus E_3 \equiv E_1 \oplus (E_2 \oplus E_3) \qquad E_1 \oplus E_2 \equiv E_2 \oplus E_1 \qquad E \oplus E \equiv E$$

plus $\alpha$-equivalence and the replacement rule. Here we add the following two axioms

$$a.(E_1 \oplus E_2) \equiv a.E_1 \oplus a.E_2 \qquad \text{and} \qquad a.\mathbf{0} \equiv \mathbf{0}$$

to obtain a sound and complete calculus for *language equivalence* of non-deterministic automata.

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[1]Note that we shall require a guardedness condition: every variable bound by $\mu$ must be within the scope of a $a.(r \bullet (-))$ operator, see Section 5.

Notice that the latter calculus coincides with Rabinovich's result for trace equivalence of finite state labelled transition systems [Rabinovich 1994]. Two axiomatizations for weighted language equivalence were recently developed by Ésik and Kuich [2012] who build on axiomatizations of rational weighted languages over the semiring of natural numbers by Bloom and Ésik [2009]. These axiomatizations use a $*$-operations as in the axiomatizations of regular languages presented in [Krob 1991] and [Bloom and Ésik 1993a]. Ésik and Kuich's work provides one purely equational axiomatization, necessarily with an infinite set of equational axioms, and a simpler axiomatization which is not equational and in which $*$ is a unique fixpoint operator. Our calculus using a $\mu$-operator and the unique fixpoint rule is similar to the latter axiomatization. The idea to extend a sound and complete calculus for weighted bisimilarity with additional axioms as well as our proof method for soundness and completeness are new. Our result can also be seen as an extension of the second author's calculus for closed stream circuits [Milius 2010] to weighted automata over alphabets of arbitrary size and from weights in a field to weights in a semiring.

We restrict our application to the above two concrete calculi in the present paper. But our results on finitary coinduction can be applied to different combinations of monads $T$ and functors $F$. For example, for the monad of free semimodules for a semiring $\mathbb{S}$ used above, our method can be applied to calculi for an inductively defined class of functors $F$ in a uniform way. However, working out these details is non-trivial because the resulting generic calculus is syntactically more involved as it will be parametric in $F$. We therefore decided to treat this generic calculus in a subsequent paper.

## 2. PRELIMINARIES

We assume that readers are familiar with basic concepts and notions from category theory. Here we present some additional basic material needed throughout the paper. We denote by Set the category of sets and functions.

### 2.1. Semirings and semimodules

In our applications we will consider semimodules for a semiring. A *semiring* is a tuple $(\mathbb{S}, +, \cdot, 0, 1)$ where $(\mathbb{S}, +, 0)$ and $(\mathbb{S}, \cdot, 1)$ are monoids, the former of which is commutative, and multiplication distributes over finite sums (i. e., $r \cdot 0 = 0 = 0 \cdot r$, $r \cdot (s+t) = r \cdot s + r \cdot t$ and $(r+s) \cdot t = r \cdot t + s \cdot t$). We just write $\mathbb{S}$ to denote a semiring. An $\mathbb{S}$-*semimodule* is a commutative monoid $(M, +, 0)$ with an action $\mathbb{S} \times M \to M$ denoted by juxtaposition $rm$ for $r \in \mathbb{S}$ and $m \in M$, such that for every $r, s \in \mathbb{S}$ and every $m, n \in M$ the following laws hold:

$$\begin{aligned}
(r+s)m &= rm + sm & r(m+n) &= rm + rn \\
0m &= 0 & r0 &= 0 \\
1m &= m & r(sm) &= (r \cdot s)m
\end{aligned}$$

A $\mathbb{S}$-semimodule homomorphism is a monoid homomorphism $h \colon M_1 \to M_2$ such that $h(rm) = rh(m)$ for each $r \in \mathbb{S}$ and $m \in M_1$. We denote by

$$\mathbb{S}\text{-Mod}$$

the category of $\mathbb{S}$-semimodules and their homomorphisms.

A semimodule $M$ is *finitely generated* if there is a finite set $G \subseteq M$ such that every element of $M$ can be written as a finite linear combination of elements from $G$. Equivalently, there exists a surjective homomorphism $\mathbb{S}^n \to M$ for some natural number $n$. A semimodule $M$ is called *finitely presentable* if it can be presented by finitely many generators and relations. Equivalently, $M$ is a coequalizer of some parallel pair of semimodule homomorphisms $\mathbb{S}^m \rightrightarrows \mathbb{S}^n$, where $m$ and $n$ are natural numbers.

*Definition* 2.1 (*[Ésik and Maletti 2011]*).  A semiring $\mathbb{S}$ is called *Noetherian* if every subsemimodule of a finitely generated $\mathbb{S}$-semimodule is itself finitely generated.

Examples of Noetherian semirings are: every finite semiring, every field, every principal ideal domain such as the ring of integers and therefore every finitely generated commutative ring by Hilbert's Basis theorem. As recently proved by Ésik and Maletti [2010], the tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is not Noetherian. Also the semiring $(\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers is not Noetherian, as shown in the following example.

*Example* 2.2.  The $\mathbb{N}$-semimodule $\mathbb{N} \times \mathbb{N}$ (with the pointwise action) is finitely generated. But its subsemimodule generated by the infinitely many elements

$$(2, 1), (3, 2), (4, 3), \dots$$

is not finitely generated.

*Remark* 2.3.  In the literature (see e. g. [Golan 1999]) a semiring is sometimes called Noetherian if every of its ideals is finitely generated. This is the same notion that is considered in ordinary ring theory (see e. g. [Lang 1984, VI, Proposition 1.5]), and, in fact, for a ring $\mathbb{S}$ this notion is equivalent to the one in Definition 2.1. However, in general, this is not the case for semirings: while every ideal of the semiring $(\mathbb{N}, +, 0, \cdot, 1)$ is finitely generated, we have seen in the above example that this semiring is not Noetherian according to Definition 2.1.

LEMMA 2.4.  *For every semiring $\mathbb{S}$, finitely generated $\mathbb{S}$-semimodules are closed under finite products.*

PROOF. Clearly the terminal semimodule $\{0\}$ is finitely generated. Given two finitely generated semimodules $M$ and $N$ with the corresponding quotients $p \colon \mathbb{S}^m \to M$ and $q \colon \mathbb{S}^n \to N$ we have the quotient

$$\mathbb{S}^{m+n} = \mathbb{S}^m \times \mathbb{S}^n \xrightarrow{p \times q} M \times N.$$

□

The following proposition gives a slightly more easy criterion to verify Noetherianess of a semiring.

PROPOSITION 2.5.  *For a semiring $\mathbb{S}$ the following are equivalent:*

*(1) $\mathbb{S}$ is Noetherian,*

*(2) every subsemimodule of a free finitely generated semimodule $\mathbb{S}^n$ is finitely generated.*

PROOF.  (1) $\Rightarrow$ (2) trivially hold.

(2) $\Rightarrow$ (1). Suppose that $N$ is a subsemimodule of the finitely generated $\mathbb{S}$-semimodule $M$ via $m \colon N \to M$. Take a quotient $q \colon \mathbb{S}^n \to M$ and form the pullback of $m$ along $q$:

$$
\begin{array}{ccc}
N' & \xrightarrowtail{m'} & \mathbb{S}^n \\
{\scriptstyle q'}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
N & \xrightarrowtail[m]{} & M
\end{array}
$$

Since surjective and injective homomorphism are stable under pullback, we see that $N'$ is a submodule of $\mathbb{S}^n$ and $N$ is a quotient of $N'$. So $N'$ is finitely generated by assumption, and, hence, so is its quotient $N$.  □

We will use the following properties of Noetherian semirings.

PROPOSITION 2.6. *If $\mathbb{S}$ is a Noetherian semiring, then the following hold:*

*(1) every finitely generated semimodule is finitely presentable.*

*(2) finitely generated $\mathbb{S}$-semimodules are closed under finite limits.*

PROOF. Ad (1). Let $M$ be a finitely generated $\mathbb{S}$-semimodule, and take a surjective homomorphism $h\colon \mathbb{S}^n \to M$. Since $h$ is a regular epimorphism, it follows that $h$ is the coequalizer of its kernel pair. So we form the kernel pair $p, q\colon K \rightrightarrows \mathbb{S}^n$ of $h$. Then $K$ is a subsemimodule of the free finitely generated module $\mathbb{S}^{n+n}$. Hence, since $\mathbb{S}$ is Noetherian, $K$ is a finitely generated semimodule, too. So we have a surjective homomorphism $g\colon \mathbb{S}^m \to K$. This implies that $h$ is a coequalizer of the parallel pair $p \cdot g, q \cdot g\colon \mathbb{S}^m \to \mathbb{S}^n$, which shows that $M$ is finitely presentable.

Ad (2). It suffices to prove closedness under finite products and subsemimodules. The former was established in the Lemma 2.4 and the latter is by hypothesis. □

*Example* 2.7. For general (semi)rings finitely generated modules need not be finitely presentable. For a counterexample consider the ring $\mathbb{S} = (\mathbb{Z}_2)^{\mathbb{N}}$ and its ideal $I$ formed by all functions $f\colon \mathbb{N} \to \mathbb{Z}_2$ with finite support. Then the quotient $\mathbb{S}/I$ is clearly finitely generated as an $\mathbb{S}$-module (since there is a surjective homomorphism $q\colon \mathbb{S} \to \mathbb{S}/I$). But $\mathbb{S}/I$ is not finitely presented; it is easy to show that the kernel $I$ of $q$ is not finitely generated as an $\mathbb{S}$-module.

*Remark* 2.8. For a ring $\mathbb{S}$ the item (2) of Proposition 2.6 is actually equivalent to $\mathbb{S}$ being Noetherian. To see this recall that the ring $\mathbb{S}$ is Noetherian if and only if every of its ideals is finitely generated (see [Lang 1984, Chapter VI, Proposition 1.5]).

Now suppose that finitely generated $\mathbb{S}$-modules are closed under finite limits, and let $I$ be any ideal of $\mathbb{S}$. Form the quotient ring $\mathbb{S}/I$, i.e., the quotient homomorphism $c\colon \mathbb{S} \to \mathbb{S}/I$ is the coequalizer of the inclusion $i\colon I \hookrightarrow \mathbb{S}$ and the $0$-morphism $I \to \mathbb{S}$. Now notice that $I$ is a split quotient of the domain $K = \{ (x, y) \mid cx = cy \}$ of the kernel pair of $c$ via $q\colon K \to I$ with $q(x, y) = x - y$.

The quotient $\mathbb{S}/I$ is of course finitely generated (with one generator). Since the free $\mathbb{S}$-module $\mathbb{S}$ is also finitely presented, so are $K$ (by assumption) and $I$ (since finitely generated objects are closed under quotients).

Let us mention a few special cases of the category $\mathbb{S}$-Mod of $\mathbb{S}$-semimodules: for the Boolean semiring $\mathbb{S} = (\{0, 1\}, \vee, \wedge, 0, 1)$, $\mathbb{S}$-Mod is the category Jsl of (bounded) join-semilattices and join-preserving maps[2]. If $\mathbb{S}$ is a field, then $\mathbb{S}$-Mod is the category $\mathbb{S}$-Vec of vector spaces over $\mathbb{S}$ and linear maps; for $\mathbb{S}$ the ring of integers we get the category of Abelian groups and for $\mathbb{S}$ the natural numbers $\mathbb{S}$-Mod is the category of commutative monoids.

## 2.2. Coalgebras

Let $\mathcal{A}$ be a category, and let $F\colon \mathcal{A} \to \mathcal{A}$ be an endofunctor. A *coalgebra* for $F$ is a pair $(C, c)$ consisting of an object $C$ and a structure morphism $c\colon C \to FC$. For example, if $\mathcal{A} = \mathsf{Set}$, then we can understand coalgebras as systems, where the set $C$ consists of all states of the system and where the map $c$ provides the transitions whose type is described by the endofunctor $F$. Concrete examples of coalgebras for set endofunctors include various kinds of automata (deterministic, non-deterministic, Mealy, Moore), stream systems, probabilistic automata, weighted ones, labelled transition

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[2]We consider join-semilattices with a least element $0$. So a join-semilattice is, equivalently, a commutative idempotent monoid.

systems and many others. We now mention two leading examples that we will consider in our applications in Sections 5 and 6; for more examples see e. g. [Rutten 2000; Silva et al. 2011].

Firstly, non-deterministic automata are coalgebras for the set functor $FX = 2 \times (\mathcal{P}_f X)^A$, where $A$ is the finite input alphabet, and $\mathcal{P}_f$ is the finite powerset functor. A coalgebra $c\colon C \to 2 \times (\mathcal{P}_f C)^A$ is precisely the same as a set $C$ of states together with an image finite transition relation $\delta \subseteq C \times A \times C$ and a subset $C' \subseteq C$ of final states.

Our second leading example is weighted automata [Schützenberger 1961; Droste et al. 2009]. Let $\mathbb{S}$ be a semiring. We consider the functor $V_{\mathbb{S}}\colon \mathsf{Set} \to \mathsf{Set}$ defined on sets $X$ and maps $h\colon X \to Y$ as follows:

$$V_{\mathbb{S}}X = \{\, f\colon X \to \mathbb{S} \mid f \text{ has finite support}\,\}, \qquad V_{\mathbb{S}}h(f) = \big(y \mapsto \sum_{x \in h^{-1}(y)} f(x)\big), \quad (2.1)$$

where a function $f\colon X \to \mathbb{S}$ is said to have finite support if $f(x) \neq 0$ holds only for finitely many elements $x \in X$. In the sequel we will omit the subscript $\mathbb{S}$ from the above functor as we will always work with a fixed semiring $\mathbb{S}$. One can think of $VX$ as consisting of all formal linear combinations on elements of $X$; in other words, $VX$ is the free $\mathbb{S}$-semimodule on $X$. A weighted automaton with finite input alphabet $A$ is simply a coalgebra for the functor $FX = \mathbb{S} \times (VX)^A$. In more detail, a coalgebra $c\colon C \to \mathbb{S} \times (VX)^A$ is given by a set $C$ of states, a map $o\colon C \to \mathbb{S}$ associating an output weight with every state and a map $t\colon C \to (VX)^A$ encoding the transition relation in the following way: the state $s \in C$ can make a transition to $s' \in C$ with input $a \in A$ and weight $w \in \mathbb{S}$ if and only if $t(s)(a)(s') = w$.

Notice that taking $\mathbb{S}$ to be the Boolean semiring weighted automata are precisely the classical non-deterministic ones as $V$ and $\mathcal{P}_f$ are naturally isomorphic. So the first example is actually a special case of the second one.

For $F$-coalgebras to form a category we need morphisms: a *coalgebra homomorphism* from a coalgebra $(C, c)$ to a coalgebra $(D, d)$ is a morphism $h\colon C \to D$ preserving the transition structure, i. e., such that $d \cdot h = Fh \cdot c$. We write

$$\mathsf{Coalg}(F)$$

for the category of $F$-coalgebras and their homomorphisms.

An important concept in the theory of coalgebras is that of a final coalgebra. An $F$-coalgebra $(T, t)$ is said to be *final* if for every $F$-coalgebra $(C, c)$ there exists a unique coalgebra homomorphism $\dagger c$ from $(C, c)$ to $(T, t)$:

$$
\begin{array}{ccc}
C & \xrightarrow{\ c\ } & FC \\
{\scriptstyle \dagger c}\big\downarrow & & \big\downarrow{\scriptstyle F\dagger c} \\
T & \xrightarrow[\ t\ ]{} & FT
\end{array}
$$

We will write

$$\nu F$$

for the final coalgebra $T$, if it exists.[3] The final coalgebra is uniquely determined up to isomorphism. Moreover, the structure map $t\colon \nu F \to F(\nu F)$ of a final coalgebra is an isomorphism by Lambek's Lemma [Lambek 1968]. So $\nu F$ is a fixpoint of the endofunctor $F$. More generally, any coalgebra $(C, c)$ with $c$ an isomorphism is said to be a

---

[3]Existence of a final coalgebra can be guaranteed by mild assumptions on $F$, e. g., every bounded (or, equivalently, accessible) endofunctor on Set has a final coalgebra.

*fixpoint* of $F$. For an endofunctor on Set, the elements of the final coalgebra provide semantics for the behavior of the states of a system regarded as $F$-coalgebra $(C, c)$ via the unique coalgebra homomorphism $\dagger c$.

Let us note that finality also provides the basis for semantic equivalence. Let $(C, c)$ and $(D, d)$ be two coalgebras for an endofunctor $F$ on Set with the final coalgebra $(\nu F, t)$. Then two states $x \in C$ and $y \in D$ are called *behavioral equivalent* if $\dagger c(x) = \dagger d(y)$. If $F$ preserves weak pullbacks then behavioral equivalence coincides with the well-known notion of bisimilarity. The states $x$ and $y$ are called bisimilar if they are in a special relation called a *bisimulation* [Aczel and Mendler 1989]. We shall not define that concept here as it is not needed in the present paper; for details see [Rutten 2000]. Let us just remark that the coalgebraic notion of bisimulation generalises the concepts known, under the same name for concrete classes of systems, e. g., for labelled transition systems, where coalgebraic bisimulation coincides with Milner's strong bisimulation. The requirement that $F$ preserve weak pullbacks is not very restrictive; many functors of interest in coalgebra theory do indeed preserve weak pullbacks. We list some examples of interest in this paper.

*Examples* 2.9.

(1) Let $\Sigma$ be a signature of operations symbols with prescribed finite arities, i. e. a sequence $(\Sigma_n)_{n<\omega}$ of sets. The associated polynomial functor $F_\Sigma$ is defined by the object assignment

$$F_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n.$$

All polynomial set functors preserve weak pullbacks.
(2) The finite powerset functor preserves weak pullbacks.
(3) Composites, products and coproducts of weak pullback preserving sets functors preserve weak pullbacks.
(4) It follows from (1)–(3) that the functors $X \mapsto 2 \times X^A$, $X \mapsto 2 \times (\mathcal{P}_f X)^A$ and $X \mapsto B \times X^A$ of deterministic, non-deterministic, and Moore automata, respectively, preserve weak pullbacks.
(5) The functor $V$ from (2.1) preserves weak pullbacks if and only if the monoid $(\mathbb{S}, +, 0)$ is
    (a) *positive*, i. e., $a + b = 0$ implies $a = 0 = b$ and
    (b) *refinable*, i. e., whenever $a_1 + a_2 = b_1 + b_2$ then there exists a $2 \times 2$-matrix with row sums $a_1$ and $a_2$ and column sums $b_1$ and $b_2$, respectively,
    see [Gumm and Schröder 2001] and the discussion in [Adámek et al. 2011]. So if $(\mathbb{S}, +, 0)$ is positive and refinable the type functor $X \mapsto \mathbb{S} \times (VX)^A$ of weighted automata preserves weak pullbacks.
(6) Giry's probability monad [Giry 1981] on the category of measurable spaces does not preserve weak pullbacks (see [Viglizzo 2005]).

We now mention some examples of final coalgebras in more detail.

*Examples* 2.10.

(1) For a polynomial set endofunctor $F_\Sigma$, the final coalgebra consists of all (finite and infinite) $\Sigma$-trees, i. e. rooted and ordered trees labelled in $\Sigma$ such that a node with $n$ children is labelled by an $n$-ary operation symbol. The coalgebra structure is given by the inverse of tree tupling.
(2) Classical deterministic automata with input alphabet $A$ are coalgebras for the functor $FX = 2 \times X^A$, where $2 = \{0, 1\}$, and the final $F$-coalgebra is carried by the set $\mathcal{P}(A^*)$ of all formal languages on $A$; its coalgebra structure is given by the two

maps $o\colon \mathcal{P}(A^*) \to 2$ and $t\colon \mathcal{P}(A^*) \to \mathcal{P}(A^*)^A$ where for a formal language $L \subseteq A^*$ we have

$$o(L) = 1 \iff \varepsilon \in L \quad \text{and} \quad t(L)(a) = L_a = \{w \mid aw \in L\}.$$

Moreover, for a deterministic automaton presented as an $F$-coalgebra $(C, c)$ the unique homomorphism $\dagger c\colon C \to \mathcal{P}(A^*)$ assigns to every state $s \in C$ the formal language it accepts.

(3) Deterministic Moore automata with input alphabet $A$ and outputs in the set $B$ are coalgebras for the functor $FX = B \times X^A$. The final $F$-coalgebra is carried by the set $B^{A^*}$. The coalgebra structure on $B^{A^*}$ is given by the two maps $o\colon B^{A^*} \to B$ and $t\colon B^{A^*} \to (B^{A^*})^A$ with

$$o(L) = L(\varepsilon) \quad \text{and} \quad t(L)(a) = \lambda w.L(aw).$$

We shall be interested in the case where $B = \mathbb{S}$ is a semiring, so $\mathbb{S}^{A^*}$ are weighted languages (or formal power series).

(4) In the example of non-deterministic automata as coalgebras for $FX = 2 \times (\mathcal{P}_f X)^A$ the elements of the final coalgebra can be thought of as representatives of all finitely branching processes with outputs in $2$ modulo strong bisimilarity. A more concrete description follows from the result on the final coalgebra for $\mathcal{P}_f$ given by Worrell [2005] (see also [Adámek et al. 2011]):
Consider all (rooted) finitely branching trees with edges labelled in $A$ and nodes labelled in $2$. Every such tree can be considered as an $F$-coalgebra in a canonical way (with the coalgebra structure assigning to a node $x$ of a tree the pair $(o, t)$, where $o$ is the node label of $x$ and $f$ is the function mapping an input symbol $a \in A$ to the finite set of child nodes of $x$ reachable by $a$-labelled edges). A *tree bisimulation* between a tree $t$ and a tree $s$ is a bisimulation $R$ between the corresponding coalgebras such that (a) the roots of $s$ and $t$ are in $R$, (b) whenever two nodes are related, then their parents are related and (c) only nodes of the same depth are related. A tree is said to be *strongly extensional* if there is no non-trivial tree bisimulation on the coalgebra induced by the tree. The final coalgebra consists of all finitely branching strongly extensional trees with nodes labelled in $2$ and edge labels from $A$ with the coalgebra structure given by the inverse of tree tupling.

(5) Finally, for weighted automata considered as coalgebras for the functor $FX = \mathbb{S} \times (VX)^A$ a final coalgebra exists since the functor is finitary. However, an explicit description of its elements does not seem to be known in general. In the following special case an explicit description easily follows from the recent work of Adámek et al. [2011]: let $(\mathbb{S}, +, 0)$ be a positive and refinable monoid (equivalently, $V$ preserves weak pullbacks, cf. Example 2.9(5)). Then the final coalgebra for $V$ is carried by the set of all strongly extensional, finitely branching, $\mathbb{S}$-labelled trees with the coalgebra structure given by the inverse of tree tupling. Similarly, it is not difficult to prove that the final coalgebra for $F$ is carried by the set of all strongly extensional, finitely branching, $(\mathbb{S}, A)$-labelled trees (i.e. each edge is labelled by a weight from $\mathbb{S}$ and an input symbol from $A$) with all nodes labelled in $\mathbb{S}$.
The latter trees are precisely the behaviors of weighted automata modulo weighted bisimilarity; in fact, weighted bisimilarity [Buchholz 2008] is precisely the behavioral equivalence for the above functor $F$ (see [Silva et al. 2011, Proposition 3.4]).

*Remark* 2.11.

(1) We shall need to work with quotients of coalgebras. In general, a (strong) *quotient* of an object $X$ in a category $\mathcal{A}$ is represented (up to isomorphism) by a strong epimorphism $q\colon X \twoheadrightarrow Y$; we shall simply call $Y$ a quotient of $X$. Similarly, a *quotient*

*coalgebra* is represented by a coalgebra homomorphism $q\colon (X, x) \twoheadrightarrow (Y, y)$ such that $q\colon X \twoheadrightarrow Y$ is a strong epimorphism in $\mathcal{A}$.

(2) Our choice of strong epimorphisms to represent quotients stems from the fact that in an Eilenberg-Moore category $\mathsf{Set}^T$ of a monad $T$ on Set the strong epimorphisms are precisely the surjective $T$-algebra homomorphisms. In general, epimorphisms may not be surjective in $\mathsf{Set}^T$, e.g. the embedding $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the categories of rings and semigroups (see [Adámek et al. 2009, Example 7.40(5)]), which are both isomorphic to $\mathsf{Set}^T$ for appropriate monads $T$.

### 2.3. Eilenberg-Moore-Algebras and the generalised powerset construction

The recent paper [Silva et al. 2010] provides a coalgebraic version of the powerset construction applicable to many different system types expressed as coalgebras for a set endofunctor. One considers an endofunctor $H$, giving the transition type of a class of systems, that is obtained as the composition of two functors $F$ and $T$ on Set, i.e. $H = FT$. Intuitively, $F$ gives the "behavior type" and $T$ the "branching behavior" of that class of systems. We already saw this in our two leading examples above: non-deterministic automata are $FT$-coalgebras where $FX = 2 \times X^A$ and $T = \mathcal{P}_\mathsf{f}$ is the finite powerset functor, and weighted automata are $FT$-coalgebras for $FX = \mathbb{S} \times X^A$ and $T = V$.

To apply the generalised powerset construction to a coalgebra $c\colon C \to FTC$ it is important that $T$ is the functor part of a monad and that $FTC$ is an Eilenberg-Moore algebra for $T$. We now briefly recall these concepts (see e.g. [MacLane 1998] for a detailed introduction).

A *monad* is a triple $(T, \eta, \mu)$, where $\eta\colon \mathrm{Id} \to T$ and $\mu\colon TT \to T$ are natural transformations such that $\mu \cdot \eta T = \mathrm{id}_T = \mu \cdot T\eta$ and $\mu \cdot T\mu = \mu \cdot \mu T$. An *Eilenberg-Moore* algebra for a monad $T$ (or $T$-*algebra*, for short) is a pair $(A, \alpha)$ consisting of an object $A$ and a structure morphism $\alpha\colon TA \to A$ such that $\alpha \cdot \eta_A = \mathrm{id}_A$ and $\alpha \cdot \mu_A = \alpha \cdot T\alpha$. A $T$-algebra homomorphism from $(A, \alpha)$ to $(B, \beta)$ is a morphism $h\colon A \to B$ such that $h \cdot \alpha = \beta \cdot Th$. Eilenberg-Moore algebras for a monad $T$ on Set form the category denoted by $\mathsf{Set}^T$. Clearly, for every set $X$, $(TX, \mu_X)$ is an Eilenberg-Moore algebra for $T$. Moreover, this $T$-algebra is *free* on $X$, i.e., for every $T$-algebra $(A, \alpha)$ and every map $f\colon X \to A$ there is a unique $T$-algebra homomorphism $f^\sharp\colon TX \to A$ such that $f^\sharp \cdot \eta_X = f$:

$$
\begin{array}{ccccc}
TTX & \xrightarrow{\ \mu_X\ } & TX & \xleftarrow{\ \eta_X\ } & X \\
{\scriptstyle Tf^\sharp}\big\downarrow & & \big\downarrow{\scriptstyle f^\sharp} & \swarrow{\scriptstyle f} & \\
TA & \xrightarrow[\ \alpha\ ]{} & A & &
\end{array}
\tag{2.2}
$$

Notice also that we have $f^\sharp = \alpha \cdot Tf$.

Now we are ready to recall the generalised powerset construction from [Silva et al. 2010]. Let $F$ be an endofunctor on Set with the final coalgebra $\nu F$ and let $T$ be a monad. Suppose we are given an $FT$-coalgebra $(C, c)$ such that $FTC$ carries some $T$-algebra structure. Then we can form the $F$-coalgebra $c^\sharp\colon TC \to FTC$ and consider the unique $F$-coalgebra homomorphism $\dagger(c^\sharp)$ into the final coalgebra $\nu F$

as summarised by the following diagram:

$$
\begin{array}{ccc}
C & \xrightarrow{\;c\;} & FTC \\
{\scriptstyle\eta_C}\big\downarrow & \nearrow{\scriptstyle c^\sharp} & \big\downarrow{\scriptstyle F\dagger(c^\sharp)} \\
TC & & \\
{\scriptstyle\dagger(c^\sharp)}\big\downarrow & & \\
\nu F & \xrightarrow{\;t\;} & F(\nu F).
\end{array}
\tag{2.3}
$$

*Notation* 2.12.  For every $FT$-coalgebra $(C, c)$ we denote the map $\dagger(c^\sharp) \cdot \eta_C$ arising from the generalized powerset construction by

$$\ddagger c\colon C \to \nu F.$$

and we call $\ddagger c$ the *coalgebraic language map* of $(C, c)$.

*Definition* 2.13.  Let $(C, c)$ and $(D, d)$ be $FT$-coalgebras and let $x \in C$ and $y \in D$. The states $x$ and $y$ are called *(coalgebraically) language equivalent* if $\ddagger c(x) = \ddagger d(y)$ holds.

In concrete instances, the construction of the coalgebra $(TC, c^\sharp)$ is determinisation and the map $\ddagger c\colon C \to \nu F$ assigns to states of the coalgebra $C$ their language or set of traces.

For example, as we saw previously, non-deterministic automata are $FT$-coalgebras where the functor is $FX = 2 \times X^A$ and the monad is $T = \mathcal{P}_\mathsf{f}$. The construction extending the coalgebra structure $c\colon C \to 2 \times (\mathcal{P}_\mathsf{f}X)^A$ to $c^\sharp\colon \mathcal{P}_\mathsf{f}C \to 2 \times (\mathcal{P}_\mathsf{f}C)^A$ is precisely the usual powerset construction determinising the given non-deterministic automaton. Moreover, the final coalgebra for $F$ consists of all formal languages, and the map $\ddagger c$ provides the usual language semantics of a non-deterministic automaton. In contrast, as we saw in Example 2.10(4), the final coalgebra for $FT$ provides the bisimilarity semantics taking into account the non-deterministic branching of automata (thus, for example, a non-deterministic automaton and its determinisation are in general not equivalent in this semantics).

In our second leading example of weighted automata we consider $FT$-coalgebras for the functor $FX = \mathbb{S} \times X^A$ and the monad $T = V$. The construction extending a coalgebra $c\colon C \to \mathbb{S} \times (VX)^A$ to $c^\sharp$ can be understood as determinisation of the given weighted automaton again. Moreover, we saw in Example 2.10(2) that the final coalgebra for $F$ is carried by the set $\mathbb{S}^{A^*}$ of weighted languages, and so the map $\ddagger c\colon C \to \mathbb{S}^{A^*}$ assigns to a state of a weighted automaton the weighted language it accepts. To summarise: behavioral equivalence of $FT$-coalgebras coincides with weighted bisimilarity, while behavioral equivalence of $F$-coalgebras yields weighted language equivalence.

## 2.4. Liftings of functors to algebras

We have seen that the category of Eilenberg-Moore algebras for a set monad $T$ plays an important rôle for the generalised powerset construction presented in the previous section. For our work in the present paper we make use of functors $F$ that lift to the category $\mathsf{Set}^T$ and we shall study fixpoints of $F$ and its lifting. We now briefly recall the necessary background material.

Let $F\colon \mathsf{Set} \to \mathsf{Set}$ be a functor and let $(T, \eta, \mu)$ be a monad on $\mathsf{Set}$. We denote by $U\colon \mathsf{Set}^T \to \mathsf{Set}$ the forgetful functor mapping a $T$-algebra to its underlying set. A *lifting*

of $F$ to $\mathsf{Set}^T$ is a functor $\bar{F}\colon \mathsf{Set}^T \to \mathsf{Set}^T$ such that the square below commutes:

$$
\begin{array}{ccc}
\mathsf{Set}^T & \xrightarrow{\;\bar{F}\;} & \mathsf{Set}^T \\
{\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle U} \\
\mathsf{Set} & \xrightarrow[\;F\;]{} & \mathsf{Set}
\end{array}
$$

In general, a lifting of $F$ need not be unique. It is well-known that to have a lifting of $F$ to $\mathsf{Set}^T$ is the same as to have a distributive law of the monad $T$ over the functor $F$ (see [Applegate 1965; Johnstone 1975]). Recall from *loc. cit.* that a *distributive law* of $T$ over $F$ is a natural transformation $\lambda\colon TF \to FT$ such that the following two laws hold:

$$\lambda \cdot \eta F = F\eta \qquad \text{and} \qquad \lambda \cdot \mu F = F\mu \cdot \lambda T \cdot T\lambda. \tag{2.4}$$

*Remark* 2.14. Suppose that $F$ has a lifting to $\mathsf{Set}^T$. Then $FTC$ carries a $T$-algebra structure for every object $C$: apply the lifting $\bar{F}$ to the free $T$-algebra $(TC, \mu_C)$. Thus, the generalised powerset construction described in (2.3) can be applied to every coalgebra $c\colon C \to FTC$.

The functors in our leading examples have liftings to the respective Eilenberg-Moore categories. For the case of non-deterministic automata recall that we have $FX = 2 \times X^A$ and $T = \mathcal{P}_{\mathsf{f}}$ and notice that $\mathsf{Set}^{\mathcal{P}_{\mathsf{f}}}$ is (equivalent to) the category $\mathsf{Jsl}$ of join-semilattices. The lifting $\bar{F}$ maps a join-semilattice $X$ to $2 \times X^A$, where $2$ carries the join-semilattice structure with $0 \le 1$, and on the product and the power to the set $A$ one takes the join-semilattice structure componentwise. More generally, every non-deterministic functor as defined in [Silva et al. 010a] canonically lifts to $\mathsf{Jsl}$.

For the case of weighted automata we have $FX = \mathbb{S} \times X^A$ and $T = V$. Then $\mathsf{Set}^V$ is (equivalent to) the category $\mathbb{S}$-$\mathsf{Mod}$ of $\mathbb{S}$-semimodules. The lifting $\bar{F}$ maps a semimodule $X$ to $\mathbb{S} \times X^A$, again with the obvious componentwise structure.

We leave it to the reader to work out the distributive laws corresponding to the liftings.

## 3. COALGEBRAS OVER ALGEBRAS

For the results in the current paper we will study the move from coalgebras for a functor $F$ to coalgebras for the lifted functor $\bar{F}$ more thoroughly. In this section we develop the necessary mathematical theory of finitary coinduction that we later use to obtain desired general soundness and completeness theorems. The main contributions of this section are: in subsection 3.1, the proof that locally finitely presentable coalgebras are closed under quotients (Lemma 3.16); in subsection 3.3, the proof that the final $FT$-coalgebra also carries a $T$-algebra structure (Lemma 3.24) and the relation between the final $FT$-coalgebra and the final $\bar{F}$-coalgebra (Proposition 3.26); and in subsection 3.4, the relation between the rational fixpoints of $FT$ and $\bar{F}$ (recall (1.1) and see Theorem 3.41).

### 3.1. Locally finitely presentable coalgebras

For the soundness and completeness proofs of the expression calculi presented in [Silva et al. 010a], locally finite coalgebras play an important rôle, and for the sound and complete calculus for linear systems given in [Milius 2010] one uses locally finite dimensional coalgebras. More precisely, expressions modulo the equations and rules of the calculus form a final locally finite (or, locally finite dimensional, respectively) coalgebra. In [Milius 2010], *locally finitely presentable coalgebras* were introduced as a

common generalization of locally finite and locally finite dimensional coalgebras. Next, we recall the necessary material and further extend the theory so as to be able to relate the final locally finitely presentable coalgebras for $FT$ and $\bar{F}$.

For a general category, local finiteness of coalgebras is based on a notion of finiteness of objects of the category, and the latter is captured by locally finitely presentable categories; we now briefly recall the basics from [Adámek and Rosický 1994]. A functor is *finitary* if it preserves filtered colimits, and an object $X$ of a category $\mathcal{A}$ is called *finitely presentable* if its hom-functor $\mathcal{A}(X, -)$ is finitary. A category $\mathcal{A}$ is called *locally finitely presentable* (*lfp*, for short) if

(1) it is cocomplete and
(2) has a set of finitely presentable objects such that every object of $\mathcal{A}$ is a filtered colimit of objects from that set.

We write $\mathcal{A}_{\mathrm{fp}}$ for the full subcategory of $\mathcal{A}$ given by all finitely presentable objects.

Our categories of interest, Set and $\mathbb{S}$-Mod, are locally finitely presentable with the expected notion of finitely presentable objects: finite sets, and finitely presentable $\mathbb{S}$-semimodules, respectively. In the special instances of Jsl and vector spaces over a field $\mathbb{S}$ the finitely presentable objects are finite join-semilattices and finite dimensional vector spaces, respectively. Other examples of lfp categories are the categories of posets, graphs, groups and, in fact, every finitary variety of algebras is lfp. The corresponding notions of finitely presentable objects are: finite posets or graphs and those groups or algebras presented by finitely many generators and relations. Notice that finitary varieties are precisely the Eilenberg-Moore categories for finitary set monads, so $\mathsf{Set}^T$ is lfp for every finitary monad $T$ on Set (here we call a monad finitary if its the underlying functor is finitary). In contrast, the category of complete partial orders (cpo's) and continuous maps is not lfp; there are no non-trivial finitely presentable objects.

*Assumption* 3.1. For the rest of this section we assume that $\mathcal{A}$ is an lfp category and that $F\colon \mathcal{A} \to \mathcal{A}$ is a finitary functor on $\mathcal{A}$.

*Examples* 3.2. There are many examples of finitary functors on lfp categories. We mention only those two of interest in the current paper.
(1) Every non-deterministic functor on Set as defined in [Silva et al. 010a] is finitary. All these functors lift to finitary functors on Jsl (e. g. the functor $FX = 2 \times X^A$).
(2) The functor $FX = \mathbb{S} \times X^A$ is finitary on Set and it lifts to a finitary functor of $\mathbb{S}$-Mod.

*Remark* 3.3. (1) We shall need the following property of lfp categories, and we recall this from [Adámek and Rosický 1994]: Every morphism $f$ in an lfp category $\mathcal{A}$ can be factorized as a strong epi $e$ followed by a monomorphism $m$, i.e. $f = m \cdot e$. This factorisation has the following diagonalisation property: for every commutative square

$$
\begin{array}{ccc}
A & \xrightarrow{\ e\ } & B \\
{\scriptstyle f}\big\downarrow & {\scriptstyle d}\ \diagup & \big\downarrow{\scriptstyle g} \\
C & \xrightarrowtail[\ m\ ]{} & D
\end{array}
$$

with $m$ a monomorphism and $e$ a strong epimorphism there exists a unique morphism $d\colon B \to C$ such that $m \cdot d = g$ and $d \cdot e = f$.

(2) It follows that $\mathsf{Coalg}(F)$ also has factorisations whenever $F$ preserves monomorphisms. Given the coalgebra homomorphism $f\colon (C, c) \to (D, d)$ we take its strong epi-mono factorisation $f = m \cdot e$ in $\mathcal{A}$. By diagonalisation, we obtain a unique $F$-coalgebra

structure on the codomain of $e$ such that $e$ and $m$ are coalgebra homomorphisms:

$$
\begin{array}{ccc}
C & \xrightarrow{\;c\;} & FC \\
{\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle Fe} \\
E & \dashrightarrow & FE \\
{\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle Fm} \\
D & \xrightarrow[\;d\;]{} & FD
\end{array}
$$

Notice that we do not claim that $e$ is a strong epimorphism in $\mathsf{Coalg}(F)$ (and, in general, this claim is false). Also observe that for $\mathcal{A} = \mathsf{Set}$ the above argument works for all endofunctors since set endofunctors preserve all non-empty monomorphisms $m$ and the case of $m\colon \emptyset \to A$ is trivial.

*Notation* 3.4. We denote by $\mathsf{Coalg_f}(F)$ the category of all coalgebras $p\colon P \to FP$ with a finitely presentable carrier $P$.

In the current setting, local finiteness of coalgebras is captured by the following notion introduced in [Milius 2010].

*Definition* 3.5. An $F$-coalgebra $(S,s)$ is called *locally finitely presentable* if the canonical forgetful functor $\mathsf{Coalg_f}(F)/(S,s) \to \mathcal{A}_{\mathrm{fp}}/S$ is cofinal.

*Remark* 3.6. More explicitly $(S,s)$ is locally finitely presentable if and only if the following two conditions are satisfied:

(1) for every $f\colon X \to S$ where $X$ is a finitely presentable object of $\mathcal{A}$ there exists a coalgebra $(P,p)$ from $\mathsf{Coalg_f}(F)$, a coalgebra homomorphism $h\colon (P,p) \to (S,s)$ and a morphism $f'\colon X \to P$ such that $h \cdot f' = f$.
(2) The factorisation in (1) is essentially unique in the sense that for every $f''\colon X \to P$ with $h \cdot f'' = f$ there exists a homomorphism $\ell\colon (P,p) \to (Q,q)$ in $\mathsf{Coalg_f}(F)$ and a coalgebra homomorphism $h'\colon (Q,q) \to (S,s)$ such that $\ell \cdot f' = \ell \cdot f''$.

*Example* 3.7. (1) For $\mathcal{A} = \mathsf{Set}$ an $F$-coalgebra is locally finitely presentable if and only if every finite subset of its carrier is contained in a finite subcoalgebra. As discussed in [Milius 2010], if $F$ preserves weak pullbacks the above notion coincides with that of local finiteness considered in [Silva et al. 010a].

(2) Analogously for $\mathcal{A} = \mathsf{Jsl}$, an $F$-coalgebra is locally finitely presentable if and only if every finite sub-join-semilattice of its carrier is contained in a finite subcoalgebra.

(3) For $\mathcal{A} = \mathbb{F}\text{-}\mathsf{Vec}$, the category of vector spaces over a field $\mathbb{F}$, an $F$-coalgebra is locally finitely presentable if and only if every finite dimensional subspace of its carrier is contained in a finite dimensional subcoalgebra, i. e., the given coalgebra is locally finite dimensional.

The following theorem gives an easier characterisation of locally finitely presentable coalgebras, and in particular of the *final* locally finitely presentable coalgebra, which can be described by considering *only* those coalgebras with a finitely presentable carrier.

THEOREM 3.8 ([MILIUS 2010]). *(1) A coalgebra is locally finitely presentable if and only if it is a filtered colimit of a diagram of coalgebras from* $\mathsf{Coalg_f}(F)$.

*(2) A locally finitely presentable coalgebra $(R,r)$ is final in the category of all locally finitely presentable coalgebras if and only if for every coalgebra $(P,p)$ from $\mathsf{Coalg_f}(F)$ there exists a unique homomorphism from $(P,p)$ to $(R,r)$.*

An immediate consequence of point (1) in the previous theorem is that the final locally finitely presentable coalgebra for a finitary functor $F$ always exists.

COROLLARY 3.9. *(1) The final locally finitely presentable $F$-coalgebra $\varrho F$ of a finitary functor $F$ exists and is constructed as the colimit of $\mathsf{Coalg_f}(F)$; in symbols:*

$$\varrho F = \mathrm{colim}(\mathsf{Coalg_f}(F) \hookrightarrow \mathsf{Coalg}(F)).$$

*(2) Furthermore, $\varrho F$ is a fixpoint of $F$.*

For the proof of point (2) in the above Corollary 3.9, see [Adámek et al. 2006, Theorem 3.3]. The colimit construction in point (1) is exactly the construction given in *loc. cit.* of the initial iterative algebra for $F$. We shall not recall the notion of iterative algebras here as this plays no rôle in the present paper, but just mention the following result to make an explicit connection of the work here and in [Silva et al. 010a; Silva et al. 2011; Milius 2010] to iterative theories of [Elgot 1975].

COROLLARY 3.10. *The final locally finitely presentable coalgebra for $F$ is equivalently characterised as the initial iterative algebra for $F$.*

Continuing on the above connection, in [Adámek et al. 2006] it was proved that the monad of free iterative algebras for $F$ is the free iterative monad $\mathcal{R}$ on $F$. Thus, our Corollary 5.15 below (page 47) and the corresponding theorem in [Silva et al. 010a; Silva et al. 2011] provide a new syntactic characterisation of the closed terms in the free iterative theory (i. e., $\mathcal{R}0$, where $0$ denotes the initial object).

Next we return to our study of locally finitely presentable coalgebras. We shall continue to use the notation

$$\varrho F$$

for the final locally finitely presentable coalgebra for $F$ in analogy to the notation $\nu F$ for the final $F$-coalgebra, and we will call $\varrho F$ the *rational fixpoint* of $F$.

*Example* 3.11. We mention a number of examples of rational fixpoints $\varrho F$ to illustrate that they capture finite system behavior; further examples can be found in [Adámek et al. 2006; Adámek et al. 2009].

(1) For a polynomial endofunctor $F = F_\Sigma$ on Set (see Example 2.9(1)), recall that the final coalgebra is carried by the set of all $\Sigma$-trees, and $\varrho F$ consists of all *rational* $\Sigma$-trees, i. e., $\Sigma$-trees having, up to isomorphism, only finitely many subtrees (see [Ginali 1979]).

(2) For the special case $FX = 2 \times X^A$ on Set, recall that a coalgebra is a deterministic automaton, and the final coalgebra is carried by the set $\mathcal{P}(A^*)$ of all formal languages on $A$. Here $\varrho F$ is the subcoalgebra given by all regular languages.

(3) Let $\mathbb{F}$ be a field. For the functor $FX = \mathbb{F} \times X$ on Set, $\varrho F$ consists of all streams $\sigma$ that are eventually periodic, i. e., $\sigma = uvvv\cdots$ where $u$ and $v$ are finite words on $\mathbb{F}$. However, for the lifting $\bar{F}$ to $\mathbb{F}$-Vec, $\varrho\bar{F}$ is the subcoalgebra of $\mathbb{F}^\omega$ given by all rational streams (see [Milius 2010] for details).

(4) Similarly, for the lifted functor $\bar{F}X = \mathbb{S} \times X^A$ on the category of $\mathbb{S}$-Mod for a semiring $\mathbb{S}$, the final coalgebra is carried by the set $\mathbb{S}^{A^*}$ of formal power series (or weighted languages) on $\mathbb{S}$. We will see later in this section that, whenever $\mathbb{S}$ is Noetherian, $\varrho\bar{F}$ can be characterised by those coalgebras with a carrier

freely generated by a finite set $X$; equivalently, a weighted automaton with the finite state set $X$. By the Kleene-Schützenberger theorem [Schützenberger 1961] (see also [Berstel and Reutenauer 1988]) it follows that $\varrho \bar{F}$ is the subcoalgebra of all rational formal power series. Our sound and complete calculus for language equivalence of weighted automata in Section 5 is based on this example. Note that in the special case that $\mathbb{S}$ is a field and $A$ is the singleton set, one can use a different (but equivalent as it coincides with $\varrho \bar{F}$) definition of rational formal power series [Rutten 2003].

In all the examples above, the rational fixpoint $\varrho F$ always occurs as a subcoalgebra of $\nu F$. This is no coincidence as we will now prove.

Recall from [Adámek and Rosický 1994] that a *finitely generated* object is an object $X$ such that its covariant hom-functor $\mathcal{A}(X, -)$ preserves directed unions (i.e., colimits of directed diagrams of monomorphisms). Clearly, every finitely presentable object is finitely generated, but in general the converse does not hold. In fact, finitely generated objects are closed under quotients (whereas finitely presentable objects are not, in general), and an object is finitely generated if and only if it is a quotient of a finitely presentable object. Therefore, to say that finitely generated and finitely presentable objects coincide (cf. Proposition 3.12 below) is equivalent to the statement that finitely presentable objects are closed under quotients. The following proposition follows from [Adámek et al. 2003, Proposition 4.6 and Remark 4.3]. We include a proof for the convenience of the reader.

PROPOSITION 3.12. *Suppose that in an lfp category $\mathcal{A}$ finitely generated objects are finitely presentable, and that $F$ preserves monomorphisms. Then $\varrho F$ is the subcoalgebra of $\nu F$ given by the union of images of all coalgebra homomorphisms $(P, p) \to (\nu F, t)$ where $(P, p)$ ranges over $\mathsf{Coalg_f}(F)$.*

PROOF. Recall that for every coalgebra $p \colon P \to FP$, $\dagger p \colon P \to \nu F$ denotes the unique coalgebra homomorphism. Let $R$ be the union from the statement of the proposition:

$$R = \bigcup \mathsf{im}(\dagger p) \qquad \text{where } p \colon P \to FP \text{ ranges over } \mathsf{Coalg_f}(F).$$

More precisely, for every $(P, p)$ in $\mathsf{Coalg_f}(F)$, let $I = \mathsf{im}(\dagger p)$ be the subobject of $\nu F$ given by factorizing $\dagger p$ as a strong epimorphism $e \colon P \to I$ followed by a monomorphism $m \colon I \to \nu F$. Since $\mathsf{Coalg_f}(F)$ is a filtered category it follows that the subobjects $\mathsf{im}(\dagger p)$ and their inclusions form a directed diagram $\mathcal{D}$, and $R$ is the colimit of this diagram. In addition, from Remark 3.3(2) we see, since $F$ preserves monomorphisms, that $I$ carries a coalgebra $i \colon I \to FI$ such that $(I, i)$ is a quotient coalgebra of $(P, p)$ via $e$ and a subcoalgebra of $\nu F$ via $m$. Thus, the union $R$ is a subcoalgebra of $\nu F$: indeed, being a colimit of a diagram of coalgebras, $R$ carries a canonical coalgebra structure, and, in addition, the cocone given by all monomorphisms $m \colon I \to \nu F$ factors through a monomorphism $R \to \nu F$ (see [Adámek and Rosický 1994]).

Furthermore, by assumption, we have that the quotient $I$ of the finitely presentable object $P$ is finitely presentable, too. So $\mathcal{D}$ is actually a full subcategory of $\mathsf{Coalg_f}(F)$. Since we have the morphism $e \colon (P, p) \to (I, i)$, we see that the inclusion of $\mathcal{D}$ into $\mathsf{Coalg_f}(F)$ is cofinal. It follows that the colimits of $\mathcal{D}$ and $\mathsf{Coalg_f}(F)$ are the same, in symbols: $R \cong \varrho F$, which completes the proof. $\square$

*Example* 3.13. Let us list some examples of categories in which our first assumption of Proposition 3.12 holds, i.e., finitely generated and finitely presentable objects coincide.

(1) The categories of sets, of posets and of graphs obviously have the desired property since finitely presentable objects are just finite sets (or posets or graphs, respectively).

(2) The categories Jsl of join-semilattices, of vector spaces over a field and of Abelian groups satisfy the property. More generally, the category $\mathbb{S}$-Mod satisfies this assumption whenever $\mathbb{S}$ is a Noetherian semiring (see Proposition 2.6).

(3) A *locally finite variety* is a finitary variety in which free algebras on finite sets are themselves finite (e.g., Boolean algebras, distributive lattices or join-semilattices). It is not difficult to see that in such a category finitely presentable and finitely generated objects coincide and are precisely the finite algebras.

(4) In the categories of commutative monoids and commutative semigroups finitely presentable and finitely generated objects coincide as proved by Rèdei [1965] (see also [Rosales and Garcia-Sánchez 1999] and, for a rather short proof, [Freyd 1968]). Notice that commutative monoids are $\mathbb{S}$-Mod for $\mathbb{S}$ the natural numbers, which we have already seen do *not* form a Noetherian semiring. So the proof is different than what we saw in Proposition 2.6.

(5) The category of presheaves on finite sets (equivalently, finitary endofunctors of Set) (see [Adámek et al. 2009]).

We have seen that for many interesting categories it holds that finitely generated and finitely presentable objects coincide. However there are many other relevant categories in which this fails:

*Example* 3.14.

(1) In the category of groups, finitely generated objects are precisely those groups having a presentation by finitely many generators, and finitely presentable groups are precisely those groups with a presentation by finitely many generators and finitely many relations. It is well-known that there exist finitely generated groups that are not finitely presented.
(2) Similarly, in the category of all (not necessarily commutative) monoids finitely presentable and finitely generated objects do not coincide (see e.g. [Ruškuc 1999] for a finitely generated monoid that is not finitely presentable)
(3) In the category of $\mathbb{S}$-modules for the ring $\mathbb{S} = (\mathbb{Z}_2)^{\mathbb{N}}$, finitely presentable and finitely generated objects do not coincide as shown in Example 2.7.

With the next example we show that Proposition 3.12 does not hold without the assumption that finitely presentable and finitely generated objects coincide.

*Example* 3.15. We take as $\mathcal{A}$ the category of algebras for the signature $\Sigma$ with a unary and a binary operation symbol. Then the natural numbers $\mathbb{N}$ with the operations of addition and $n \mapsto 2 \cdot n$ is an object of $\mathcal{A}$. Thus, we have an endofunctor $FX = \mathbb{N} \times X$ on $\mathcal{A}$, and its final coalgebra is carried by the set $\mathbb{N}^\omega$ of all streams of natural numbers with the obvious componentwise algebra structure. Now consider the $F$-coalgebra $\alpha \colon A \to FA$, where $A$ is the free (term) algebra on one generator $x$ and $\alpha$ is uniquely determined by the assignment $\alpha(x) = (1, 2 \cdot x)$. The unique $F$-coalgebra homomorphism $h \colon A \to \nu F$ maps $x$ to the stream $(1, 2, 4, 8, \cdots)$ of powers of 2, and we have

$$h(2 \cdot x) = h(x + x) = (2, 4, 8, 16, \ldots).$$

Now notice that $(A, \alpha)$ lies in $\mathsf{Coalg_f}(F)$, and so there is also a unique $F$-coalgebra homomorphism $h_0 \colon A \to \varrho F$. However, we will now prove that

$$h_0(2 \cdot x) \neq h_0(x + x), \tag{3.1}$$

and this implies that $\varrho F$ is not a subcoalgebra of $\nu F$.

We prove (3.1) by contradiction. Suppose that $h_0(2 \cdot x) = h_0(x + x)$. By the construction of $\varrho F$ (see Corollary 3.9), we know that there is a coalgebra $\beta : B \to FB$ in $\mathsf{Coalg_f}(F)$

and an $F$-coalgebra homomorphism $g : A \to B$ in $\mathcal{A}$ with $g(2 \cdot x) = g(x + x)$. Since $B$ is finitely presented it is the quotient in $\mathcal{A}$ of a free algebra $A'$ on a finite set $Y$ of generators modulo finitely many relations, via a quotient homomorphism $q : A' \to B$, say. Next, observe that there is an $F$-coalgebra structure $\alpha' : A' \to FA'$ such that $q$ is an $F$-coalgebra morphism: indeed, choose some map $s : B \to A'$ with $q \cdot s = \mathrm{id}$; then extending $Fs \cdot \beta \cdot q \cdot \eta_Y$, where $\eta_Y : Y \to A'$ is the embedding of generators yields the desired $F$-coalgebra $A' \to FA'$.

Now choose a term $t_x$ in $A'$ with $q(t_x) = g(x)$. This implies that $q(2 \cdot t_x) = q(t_x + t_x)$ since $q$ is a coalgebra homomorphism. Since $g$ is a coalgebra homomorphism it merges the right-hand component of $\alpha(2 \cdot x)$ and $\alpha(x+x)$, in symbols: $g(2 \cdot (2 \cdot x)) = g((2 \cdot x) + (2 \cdot x))$. It follows that $q$ satisfies: $q(2 \cdot (2 \cdot t_x)) = q((2 \cdot t_x) + 2 \cdot t_x))$.

Continuing to use that $g$ and $q$ are homomorphisms, we obtain the following infinite list of elements (terms) of $A'$ that are merged by $q$ (we write these pairs as equations):
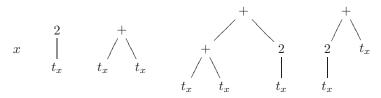
$$
\begin{aligned}
2 \cdot t_x &= t_x + t_x \\
2 \cdot (2 \cdot t_x) &= (2 \cdot t_x) + (2 \cdot t_x) \\
2 \cdot (2 \cdot (2 \cdot t_x)) &= (2 \cdot (2 \cdot t_x)) + (2 \cdot (2 \cdot t_x)) \\
&\vdots
\end{aligned}
\tag{3.2}
$$

We need to prove that there exists no finite set of relations $E \subseteq A' \times A'$ generating the above congruence $q \colon A' \to B$.

Suppose the contrary, let $A'_0$ be the $\Sigma$-subalgebra of $A'$ generated by $\{t_x\}$, and let $T \subseteq A'_0$ be the subset of those terms (or finite $\Sigma$-trees on $Y$) such that every path in $t$ from the root to the generator $t_x$ has the same length. For example, of the following finite $\Sigma$-trees
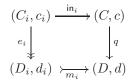
$$
\begin{array}{ccccc}
x & 
\begin{array}{c} 2 \\ | \\ t_x \end{array} &
\begin{array}{c} + \\ / \ \backslash \\ t_x \quad t_x \end{array} &
\begin{array}{c} + \\ / \quad \backslash \\ \begin{array}{c} + \\ / \backslash \\ t_x \ t_x \end{array} \quad \begin{array}{c} 2 \\ | \\ t_x \end{array} \end{array} &
\begin{array}{c} + \\ / \ \backslash \\ \begin{array}{c} 2 \\ | \\ t_x \end{array} \quad t_x \end{array}
\end{array}
$$

the first four are in $T$ but not the fifth one. For every $\Sigma$-tree in $t$ we call the maximal length of a path from the root to the generator $t_x$ the *height* of that tree. Now let $t$ and $s$ be $\Sigma$-trees of different height in $T$. Then we clearly have $k(t) \neq k(s)$, where $k : B \to \nu F$ is the unique $F$-coalgebra homomorphism; this follows from the fact that for a tree $t$ of height $n$ in $T$ we have

$$
k(t) = (2^n, 2^{n+1}, 2^{n+2}, \dots).
$$

Thus, the equation $t = s$ is not in the congruence generated by $E$ (otherwise, we would have $q(t) = q(s)$ which implies $k(t) = k(s)$). Now let $\ell$ be the height of the tallest $\Sigma$-tree that occurs in a relation from $E$. Then the $\ell + 1$-st equation in (3.2) with $x$ replaced by $t_x$ is not generated by $E$ as this equation is of the form $2 \cdot t = t' + t''$ with $t$, $t'$ and $t''$ of height $\ell + 1$. If $s$ and $s'$ are terms of height greater that $k$ related by the smallest congruence generated by $E$, then $s$ and $s'$ must have the same head symbol. So $2 \cdot t$ and $t' + t''$ are not related. Thus, we arrive at the desired contradiction.

We already mentioned that one way to say that finitely generated and finitely presentable objects coincide is to say that finitely presentable objects are closed under quotients. The following lemma extends the latter property to locally finitely presentable coalgebras.

LEMMA 3.16. *Under the assumptions of Proposition 3.12 every quotient coalgebra of a locally finitely presentable coalgebra is itself locally finitely presentable.*

PROOF. Let $q\colon (C,c) \to (D,d)$ be a quotient coalgebra, where $(C,c)$ is a locally finitely presentable $F$-coalgebra. So $q\colon C \to D$ is a strong epimorphism in $\mathcal{A}$. By Theorem 3.8(1), $(C,c)$ is a filtered colimit of a diagram of coalgebras $(C_i, c_i)$ from $\mathsf{Coalg_f}(F)$ with the colimit injections $\mathsf{in}_i\colon (C_i, c_i) \to (C,c)$. For every $i$ factorize $q \cdot \mathsf{in}_i$ as a strong epi- followed by a monomorphism in $\mathsf{Coalg}(F)$ (see Remark 3.3(2)):

$$
\begin{array}{ccc}
(C_i, c_i) & \xrightarrow{\;\mathsf{in}_i\;} & (C,c) \\
\Big\downarrow{\scriptstyle e_i} & & \Big\downarrow{\scriptstyle q} \\
(D_i, d_i) & \underset{m_i}{\rightarrowtail} & (D,d)
\end{array}
$$

By assumption, each $(D_i, d_i)$ lies in $\mathsf{Coalg_f}(F)$. Moreover, each connecting morphism $c_{ij}\colon (C_i, c_i) \to (C_j, c_j)$ induces a coalgebra homomorphism $d_{ij}\colon (D_i, d_i) \to (D_j, d_j)$ turning the $D_i$ into a filtered diagram (with the same diagram scheme as for the $C_i$). To conclude our proof it suffices to show that $D$ is a colimit of this new diagram. We shall now prove that $D$ is the *union* of its subobjects $m_i\colon D_i \to D$, i.e., $D$ has no proper subobject containing every $m_i$. It then follows that $D = \mathrm{colim}\, D_i$ (see [Adámek and Rosický 1994], 1.63). So let $m\colon M \to D$ be a subobject containing all $m_i$, i.e., for every $i$ we have monomorphisms $n_i\colon D_i \to M$ such that $m \cdot n_i = m_i$. Now the outside of the following square commutes:

$$
\begin{array}{ccccc}
\coprod_i C_i & \xrightarrow{\;[\mathsf{in}_i]_i\;} & C & \xrightarrow{\;q\;} & D \\
\Big\downarrow{\scriptstyle [n_i \cdot e_i]_i} & & {\scriptstyle s}\nearrow & & \Big\| \\
M & & \underset{m}{\longrightarrow} & & D
\end{array}
$$

For every $i$ we have

$$q \cdot \mathsf{in}_i = m_i \cdot e_i = m \cdot n_i \cdot e_i.$$

Moreover, notice that the copairing $[\mathsf{in}_i]_i$ is a strong epimorphism since it is the copairing of all the injections of the colimit $C$. Since strong epimorphisms compose, we see that the upper edge of the above diagram is a strong epimorphism. Hence, we get, by diagonalisation, the morphism $s\colon D \to M$ such that $m \cdot s = \mathrm{id}$ showing $m$ to be a split epimorphism, whence an isomorphism. This completes the proof. □

## 3.2. The overall setting – algebraic categories

For our soundness and completeness proofs in Section 4 we need to consider coalgebras for a lifted endofunctor on categories of Eilenberg-Moore algebras. So we will now focus our attention on *algebraic categories* $\mathcal{A}$, i.e., $\mathcal{A} = \mathsf{Set}^T$ for a finitary monad $T$ on Set. Recall that a kernel pair of a morphism $f\colon X \to Y$ is a pair $k_1, k_2\colon R \to X$ forming a pullback of $f$ with itself, and that in every algebraic category kernel pairs exist and are formed in Set, i.e.

$$R = \{\, (x,y) \mid x,y \in X, fx = fy \,\}$$

where $k_1, k_2$ are the projections.

*Assumption* 3.17. For the rest of the paper we assume that $\mathcal{A} = \mathsf{Set}^T$ for the finitary monad $(T, \eta, \mu)$, and we also assume that in $\mathsf{Set}^T$ finitely generated algebras are

closed under taking kernel pairs. In addition we require that $F\colon \mathsf{Set} \to \mathsf{Set}$ is a finitary endofunctor weakly preserving pullbacks and having a lifting $\bar{F}\colon \mathsf{Set}^T \to \mathsf{Set}^T$.

From the above assumptions that $F$ and $T$ are finitary endofunctors on $\mathsf{Set}$ we know that the final coalgebras $\nu F$ and $\nu(FT)$ exist, see e. g. [Barr 1993]. We also know that the rational fixpoint $\varrho(FT)$ of $FT$ exists since $FT$ is a finitary functor of $\mathsf{Set}$. Recall that $\mathsf{Set}^T$ is an lfp category, and notice that the lifting $\bar{F}$ is finitary because $F$ is finitary and filtered colimits in $\mathsf{Set}^T$ are formed on the level of $\mathsf{Set}$. Thus, the final $\bar{F}$-coalgebra and the rational fixpoint $\varrho\bar{F}$ also exists. Notice that, in general, $\varrho\bar{F}$ is different from the rational fixpoint of $F\colon \mathsf{Set} \to \mathsf{Set}$ as demonstrated by Example 3.11(3).

For some results in the previous section we assumed finitely presentable objects to be closed under quotients (or, equivalently, finitely generated objects to be finitely presentable). In this section, we restrict our attention to algebras for the finitary monad $T$ and we assume that finitely generated $T$-algebras are closed under kernel pairs. The kernel pair of a morphism gives always a congruence, and, conversely every congruence relation $\sim$ on an algebra $A$ is the kernel of the corresponding quotient homomorphism $A \to A/\sim$. Our assumption above is thus requiring finitely generated algebras to be *closed under congruences*; more precisely, every congruence of a finitely generated algebra $A$ is itself finitely generated (as a subalgebra of $A \times A$). After a few examples below we will see that this assumption implies finitely presentable algebras to be closed under quotients.

*Example* 3.18.  Let us come back to the categories in Example 3.13 and see whether they satisfy our assumptions.

(a) We have seen that finitely generated commutative monoids and semigroups are also finitely presentable (cf. Example 3.13(4)). However congruences of finitely generated commutative monoids need not be finitely generated as a monoid. Consider the following example from [Chapman et al. 2006]: let $R$ be the congruence on $\mathbb{N}$, the free commutative monoid on one generator, defined by

$$(x, y) \in R \ \text{iff} \ (x \geq 1 \text{ and } y \geq 1) \text{ or } x = y.$$

It is easy to see that $\{(x, 1) \mid x \geq 1\}$ is contained in $R$. But the elements of this set cannot be expressed as a sum of two other nontrivial elements of $R$. Therefore $R$ cannot be finitely generated as a monoid.

(b) All other categories from Example 3.13 satisfy the condition that finitely generated objects are closed under taking kernel pairs: indeed, for sets, posets, graphs (cf. 3.13(1)) and locally finite varieties (cf. 3.13(3)) this clearly holds, and for semimodules of a Noetherian semiring (cf. 3.13(2)) see Proposition 2.6.

(c) The following categories from Example 3.13 are categories of algebras for a monad on $\mathsf{Set}$ (in each case we list the monad):

| **category** | **is $\mathsf{Set}^T$ for ...** |
|---|---|
| Set | $T = \mathrm{Id}$ |
| Jsl | $T = \mathcal{P}_{\mathsf{f}}$ |
| $\mathbb{F}$-Vec | $T = V$, where $\mathbb{S} = \mathbb{F}$ is a field |
| abelian groups | $T = V$, $\mathbb{S} = \mathbb{Z}$ the ring of integers |
| $\mathbb{S}$-Mod | $T = V$, $\mathbb{S}$ is a (Noetherian) semiring |
| commutative semigroups | $TX = $ non-empty bags on $X$[4] |
| commutative monoids | $TX = $ bags on $X$ |

---

[4]A *bag* is a finite multiset.

The categories of posets, graphs and finitary endofunctors of Set are not (equivalent to) $\mathsf{Set}^T$ for any finitary monad $T$ on Set.

(d) None of the categories from Example 3.14 has finitely generated objects closed under kernel pairs; this follows from the next lemma.

From the counterexample in Example 3.18(a) and the lemma below, it follows that our Assumption 3.17 is strictly stronger than the one used in Proposition 3.12.

LEMMA 3.19. *In an algebraic category, if finitely generated algebras are closed under kernel pairs then they are finitely presentable.*

PROOF. Let $A$ be a finitely generated algebra. So $A$ is the quotient of some finitely presentable algebra $B$ via the surjective homomorphism $q\colon B \to A$. Then $q$ is the coequalizer of its kernel pair $f, g\colon K \rightrightarrows B$. Since $A$ and $B$ are finitely generated so is $K$. Hence, $K$ is a quotient of the finitely presentable algebra $L$ via $p\colon L \to K$. As $p$ is an epimorphism it follows that $q$ is the coequalizer of $f \cdot p$ and $g \cdot p$. Since $L$ and $B$ are finitely presentable, and finitely presentable objects are closed under finite colimits, also $A$ is finitely presentable. □

*Remark* 3.20.

(1) The lifted functor $\bar{F}$ on $\mathsf{Set}^T$ preserves monomorphisms since monomorphisms are just injective $T$-algebra homomorphisms and since $F$ preserves all injective maps (it even preserves weak pullbacks by assumption). Thus, the previous lemma ensures that $\varrho\bar{F}$ is a subcoalgebra of $\nu\bar{F}$ (see Proposition 3.12) and that locally finitely presentable $\bar{F}$-coalgebras are closed under quotients (see Lemma 3.16).
(2) Actually, the previous lemma holds more generally: in any lfp category, where strong epimorphisms are regular and finitely generated objects are closed under kernel pairs, we have that finitely generated objects are finitely presentable.

### 3.3. Final coalgebras over algebras

In this subsection we show that the final coalgebra $\nu F$ lifts to a final coalgebra of $\bar{F}$, and we prove that $\nu(FT)$ also carries the structure map of a $\bar{F}$-coalgebra and that $\nu F$ is a quotient coalgebra of $\nu(FT)$.

*Remark* 3.21. For the results in this subsection rational fixpoints are not necessary. Our results here hold, more generally, for any monad $T$ and any endofunctor $F$ on an arbitrary category such that $\nu(FT)$ and $\nu F$ exist and $F$ has a lifting to the category of $T$-algebras.

*Notation* 3.22. From now on we write

$$\mathfrak{t}\colon \nu(FT) \to FT(\nu(FT)) \qquad \text{and} \qquad \widetilde{\mathfrak{t}}\colon \nu F \to F(\nu F),$$

for the structure maps of the final $FT$ and $F$-coalgebra, respectively.
We also write

$$\lambda\colon TF \to FT$$
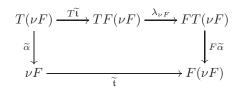
for the distributive law that (uniquely) corresponds to the lifting $\bar{F}\colon \mathsf{Set}^T \to \mathsf{Set}^T$.

First, let us recall that in our setting the final coalgebra for $F$ lifts to a final coalgebra for $\bar{F}$. This result essentially follows from the work in [Bartels 2004] (see Theorem 3.2.3) and also cf. [Plotkin and Turi 1997]. More explicitly, one obtains the unique

coalgebra homomorphism $\widetilde{\alpha}\colon T(\nu F) \to \nu F$ as displayed below:

$$
\begin{array}{ccc}
T(\nu F) \xrightarrow{\ T\widetilde{\mathfrak{t}}\ } TF(\nu F) \xrightarrow{\ \lambda_{\nu F}\ } FT(\nu F) \\
\widetilde{\alpha} \downarrow \hspace{6cm} \downarrow F\widetilde{\alpha} \\
\nu F \xrightarrow{\hspace{5cm}\widetilde{\mathfrak{t}}\hspace{1cm}} F(\nu F)
\end{array}
$$

It is then easy to prove that $(\nu F, \widetilde{\alpha})$ is an Eilenberg-Moore algebra for $T$ such that $\widetilde{\mathfrak{t}}\colon \nu F \to \bar{F}(\nu F)$ is a $T$-algebra homomorphism, and, moreover, $(\nu F, \widetilde{\mathfrak{t}})$ is a final $\bar{F}$-coalgebra. So we shall write $\nu F$ for both the final coalgebras for $F$ and its lifting $\bar{F}$.

*Example* 3.23. We only mention how the terminal coalgebras lifts for the two concrete examples for which we discuss a sound and complete expression calculus in Sections 5 and 6. Some further examples of the setting of this subsection may be found in [Silva et al. 2010].

(1) In the case of non-deterministic automata we saw that the functor $FX = 2 \times X^A$ lifts to $\mathsf{Set}^{\mathcal{P}_f}$, and so, the final coalgebra for the lifting $\bar{F}$ is carried by the set of formal languages with the join-semilattice structure given by union of formal languages.
(2) For the case of weighted automata we saw that the functor $FX = \mathbb{S} \times X^A$ lifts to the category $\mathsf{Set}^V$ of $\mathbb{S}$-semimodules. Hence, the final coalgebra for the lifting $\bar{F}$ is carried by the set $\mathbb{S}^{A^*}$ of weighted languages with the canonical (pointwise) structure of a semimodule.

Next, we want to relate the final coalgebras for $F$ and $FT$. As a first step we show in the following lemma that the isomorphism of every fixpoint of $FT$ (and whence the structural map of the final $FT$ coalgebra), say $c\colon C \xrightarrow{\cong} FTC$, is a $T$-algebra homomorphism. In other words, $(C, c)$ can be regarded as an $\bar{F}$-coalgebra. Recall that the the generalized powerset constructions turns $(C, c)$ into the $\bar{F}$-coalgebra $(TC, c^\sharp)$ (see diagram (2.3)).

LEMMA 3.24. *Every fixpoint $(C, c)$ of $FT$ has a unique $T$-algebra structure $\gamma\colon TC \to C$ such that $c\colon C \to FTC$ a $T$-algebra homomorphism. Furthermore,*

$$\gamma\colon (TC, c^\sharp) \to (C, F\gamma \cdot c)$$

*is an $\bar{F}$-coalgebra homomorphism.*

PROOF. On $FTC$ we have the $T$-algebra structure

$$\bar{F}(TC, \mu_C) = (TFTC \xrightarrow{\lambda_{TC}} FTTC \xrightarrow{F\mu_C} FTC)$$

Since the forgetful functor $U\colon \mathsf{Set}^T \to \mathsf{Set}$ creates isomorphisms we have that

$$\gamma = (TC \xrightarrow{Tc} TFTC \xrightarrow{(F\mu \cdot \lambda T)_C} FTC \xrightarrow{c^{-1}} C),$$

is the unique $T$-algebra structure on $C$ such that $c$ is a $T$-algebra homomorphism. To see that $\gamma$ is an coalgebra homomorphism for the set functor $F$ consider the commuta-

tive diagram below:

$$
\begin{array}{ccccccc}
TC & \xrightarrow{\ Tc\ } & TFTC & \xrightarrow{\ \lambda_{TC}\ } & FTTC & \xrightarrow{\ F\mu_C\ } & FTC \\
\gamma \downarrow & & & & F\mu_C \downarrow & & \downarrow F\gamma \\
C & & \xrightarrow{\hspace{4cm} c} & & FTC & \xrightarrow[F\gamma]{\hspace{1.5cm}} & FC
\end{array}
\qquad (3.3)
$$

Now we will show that the coalgebras in the upper and lower rows are actually $\bar{F}$-coalgebras. Firstly, as just proved, $c$ is a $T$-algebra homomorphism and so is the $T$-algebra structure $\gamma$ (by one of the axioms of Eilenberg-Moore algebras) whence $F\gamma = \bar{F}\gamma$ is a $T$-algebra homomorphism. This proves $(C, F\gamma \cdot c)$ to be an $\bar{F}$-coalgebra. Secondly, $Tc$ is clearly a $T$-algebra homomorphism and $F\mu_C \cdot \lambda_{TC}$ is the structure of the $T$-algebra $\bar{F}(TC, \mu_C)$ whence a $T$-algebra homomorphism. This proves that the coalgebra in the upper row is an $\bar{F}$-coalgebra. So $\gamma$ is an $\bar{F}$-coalgebra homomorphism.

We still need to show that the coalgebra structure in the top row of (3.3) is $c^\sharp$. This follows from the universal property of the free algebra $TC$ by showing that the $T$-algebra homomorphism $F\mu_C \cdot \lambda_{TC} \cdot Tc$ extends $c$:

$$
\begin{aligned}
F\mu_C \cdot \lambda_{TC} \cdot Tc \cdot \eta_C &= F\mu_C \cdot \lambda_{TC} \cdot \eta_{FTC} \cdot c && \text{naturality of } \eta, \\
&= F\mu_C \cdot F\eta_{TC} \cdot c && \lambda \text{ a distributive law,} \\
&= c && \text{since } \mu \cdot \eta T = \mathrm{id}.
\end{aligned}
$$

□

From the above lemma we have $T$-algebra structures on the final coalgebra and on the rational fixpoint for $FT$, and hence, both can be given structures of $\bar{F}$-coalgebras. We now fix notation for these structures for the rest of paper.

*Notation* 3.25. We already fixed the notation $\mathfrak{t}$ and $\widetilde{\mathfrak{t}}$ for the structures of the final coalgebras $\nu(FT)$ and $\nu F$, respectively. We will henceforth write

$$
\mathfrak{r} \colon \varrho(FT) \to FT(\varrho(FT))
$$

for the structure map of the rational fixpoint of $FT$. We also denote the $T$-algebra structures on $\nu(FT)$ and $\varrho(FT)$ obtained from the previous lemma by

$$
\alpha \colon T(\nu(FT)) \to \nu(FT) \qquad \text{and} \qquad \beta \colon T(\varrho(FT)) \to \varrho(FT).
$$

Thus, both $\nu(FT)$ and $\varrho(FT)$ are $\bar{F}$-coalgebras with the structure maps

$$
\nu(FT) \xrightarrow{\ \mathfrak{t}\ } FT(\nu(FT)) \xrightarrow{\ F\alpha\ } F(\nu(FT)) \quad \text{and} \quad \varrho(FT) \xrightarrow{\ \mathfrak{r}\ } FT(\varrho(FT)) \xrightarrow{\ F\beta\ } F(\varrho(FT))
$$

Notice also that $\alpha$ and $\beta$ are $\bar{F}$-coalgebra homomorphisms:

$$
\alpha \colon (T\nu(FT), \mathfrak{t}^\sharp) \to (\nu(FT), F\alpha \cdot \mathfrak{t}) \quad \text{and} \quad \beta \colon (T(\varrho(FT)), \mathfrak{r}^\sharp) \to (\varrho(FT), F\beta \cdot \mathfrak{r}).
$$

Taking a step further into deepening the understanding of the relation between final $FT$-coalgebra and the final $\bar{F}$-coalgebra, we now prove that the latter is actually a quotient of the former, meaning it is the codomain of a surjective coalgebra homomorphism.
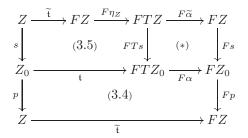
PROPOSITION 3.26. *The final $\bar{F}$-coalgebra is a quotient coalgebra of the final $FT$-coalgebra.*

PROOF. Consider the following $\bar{F}$-coalgebra homomorphism obtained by using the universal property of $\nu F$ (just within this proof we abuse notation and write $F$ instead
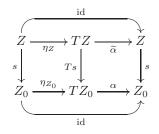
of $\bar{F}$, and we also write $Z$ for $\nu F$ and $Z_0$ for $\nu FT$):

$$
\begin{array}{ccc}
Z_0 \xrightarrow{\ t\ } FTZ_0 \xrightarrow{\ F\alpha\ } FZ_0 \\
\ \downarrow p \qquad\qquad\qquad\qquad \downarrow Fp \\
Z \xrightarrow{\qquad\ \widetilde{t}\ \qquad} FZ
\end{array}
\tag{3.4}
$$

Since all horizontal morphisms are $T$-algebra homomorphisms, then so is $p\colon Z_0 \to Z$. To see that $p$ is surjective we show it has a splitting $s\colon Z \to Z_0$ in Set. To obtain $s$ we use the universal property of $Z_0$; there is a unique $FT$-coalgebra homomorphism $s$ such that the diagram below commutes:

$$
\begin{array}{ccc}
Z \xrightarrow{\ \widetilde{t}\ } FZ \xrightarrow{\ F\eta_Z\ } FTZ \\
\ \downarrow s \qquad\qquad\qquad\qquad \downarrow FTs \\
Z_0 \xrightarrow{\qquad\ t\ \qquad} FTZ_0
\end{array}
\tag{3.5}
$$

To see that $p \cdot s = \mathrm{id}$ holds, we verify that the following diagram commutes:

$$
\begin{array}{ccccc}
Z \xrightarrow{\ \widetilde{t}\ } FZ \xrightarrow{\ F\eta_Z\ } FTZ \xrightarrow{\ F\widetilde{\alpha}\ } FZ \\
\ \downarrow s \quad (3.5) \quad FTs\downarrow \quad (*) \quad \downarrow Fs \\
Z_0 \xrightarrow{\qquad\ t\ \qquad} FTZ_0 \xrightarrow{\ F\alpha\ } FZ_0 \\
\ \downarrow p \qquad\qquad (3.4) \qquad\qquad \downarrow Fp \\
Z \xrightarrow{\qquad\qquad \widetilde{t}\ \qquad\qquad} FZ
\end{array}
$$

Indeed, the upper left-hand and lower parts commute as indicated, but we do not claim that part $(*)$ commutes. This part commutes when precomposed with $F\eta_Z$; to see this remove $F$ and consider



where the left-hand square commutes by the naturality of $\eta$ and the upper and lower triangle by the unit law of $T$-algebras. $\quad\square$

From the previous theorem we obtain an alternative way to define, for a given $FT$-coalgebra $(C, c)$, a coalgebraic language map $C \to \nu F$, namely as

$$
C \xrightarrow{\ \dagger c\ } \nu(FT) \xrightarrow{\ p\ } \!\!\!\!\twoheadrightarrow \nu F ,
$$

where $\dagger c$ is the unique $FT$-coalgebra homomorphism. We shall now prove that this map coincides with the coalgebraic language map $\ddagger c\colon C \to \nu F$ from Notation 2.12. We first prove that $FT$-coalgebra homomorphisms preserve coalgebraic language equivalence:
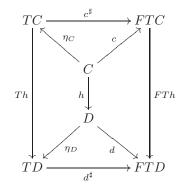
LEMMA 3.27.  *Let* $h\colon (C,c) \to (D,d)$ *be an* $FT$-*coalgebra homomorphism. Then we have*

$$\ddagger c = (C \xrightarrow{\ h\ } D \xrightarrow{\ \ddagger d\ } \nu F).$$

PROOF.  Given $h\colon (C,c) \to (D,d)$ we prove that $Th\colon TC \to TD$ is an $F$-coalgebra homomorphism. To see this consider the following diagram:

$$
\begin{array}{ccc}
TC & \xrightarrow{\ \ c^{\sharp}\ \ } & FTC \\
\end{array}
$$

The outside of the diagram consists of $T$-algebra homomorphisms. By the freeness of $TC$ it suffices to show that it commutes when extended by $\eta_C$, and this follows from the commutativity of the inner parts; to see this, use naturality of $\eta$ and the fact that $h$ is an $FT$-coalgebra homomorphism.

Now by the uniqueness of coalgebra homomorphisms into $\nu F$ we have $\dagger(d^{\sharp}) \cdot Th = \dagger(c^{\sharp})$, and so we conclude

$$
\begin{aligned}
\ddagger d \cdot h \ &= \ \dagger(d^{\sharp}) \cdot \eta_D \cdot h && \text{definition of } \ddagger d \\
&= \ \dagger(d^{\sharp}) \cdot Th \cdot \eta_C && \text{naturality of } \eta \\
&= \ \dagger(c^{\sharp}) \cdot \eta_C && \text{from above} \\
&= \ \ddagger c && \text{definition of } \ddagger c.
\end{aligned}
$$

□

PROPOSITION 3.28.  *Let* $(C,c)$ *be an* $FT$-*coalgebra. Then we have*

$$
\begin{array}{ccc}
C & \xrightarrow{\ \dagger c\ } & \nu(FT) \\
 & {}_{\ddagger c} \searrow & \downarrow {}^{p} \\
 & & \nu F
\end{array}
$$

*where* $\dagger c$ *is the unique* $FT$-*coalgebra homomorphism*

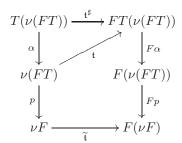PROOF.  (1) We first prove that for the final $FT$-coalgebra $(\nu(FT), \mathsf{t})$ we have

$$\ddagger \mathsf{t} = p\colon \nu(FT) \to \nu F.$$

The desired equation follows from the following one by precomposing with $\eta_{\nu(FT)}$:

$$\dagger(\mathsf{t}^{\sharp}) = (T(\nu(FT)) \xrightarrow{\ \alpha\ } \nu(FT) \xrightarrow{\ p\ } \nu F),$$

where $\alpha$ is the $T$-algebra structure of $\nu(FT)$ and $\dagger(\mathsf{t}^{\sharp})\colon (\nu(FT), \mathsf{t}^{\sharp}) \to (\nu F, \widetilde{\mathsf{t}})$ the unique $F$-coalgebra homomorphism (cf. Diagram (2.3)). Now to establish the latter equation

one proves that $p \cdot \alpha$ is an $F$-coalgebra homomorphism:

$$
\begin{array}{ccc}
T(\nu(FT)) & \xrightarrow{\;\;t^{\sharp}\;\;} & FT(\nu(FT)) \\
{\scriptstyle\alpha}\downarrow & \nearrow{\scriptstyle t} & \downarrow{\scriptstyle F\alpha} \\
\nu(FT) & & F(\nu(FT)) \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle Fp} \\
\nu F & \xrightarrow[\;\;\widetilde{t}\;\;]{} & F(\nu F)
\end{array}
$$

The lower part commutes by the definition of $p$ in (3.4), and for the upper triangle one uses that both $t^{\sharp}$ and $t \cdot \alpha$ are $T$-algebra homomorphisms that are equal when composed with $\eta_{\nu(FT)}$, in symbols:

$$
t^{\sharp} \cdot \eta_{\nu(FT)} = t = t \cdot \alpha \cdot \eta_{\nu(FT)}.
$$

It follows that the outside of the above diagram commutes as desired.

(2) We are now ready to prove the statement of the proposition. Given the $FT$-coalgebra $(C, c)$ we use item (1) above and Lemma 3.27 to conclude that

$$
\ddagger c = \ddagger t \cdot \dagger c = p \cdot \dagger c,
$$

which completes the proof. $\square$

As a corollary we get the main result of [Silva et al. 2010] that behavioral equivalence implies coalgebraic language equivalence.

COROLLARY 3.29. *Let $(C, c)$ and $(D, d)$ be $FT$-coalgebras. Then for every $x \in C$ and $y \in D$ we have*

$$
\dagger c(x) = \dagger d(y) \quad \Longrightarrow \quad \ddagger c(x) = \ddagger d(y).
$$

We have now a formal relation between the final coalgebras for $FT$ and for $\bar{F}$. These final coalgebras contain, respectively, canonical representatives for bisimilarity and language equivalence. As demonstrated by the previous corollary, the abstract result that the final coalgebra for $\bar{F}$ is a quotient of the final coalgebra for $FT$ instantiates, for non-deterministic automata and labelled transition systems, to the well-known fact that language (or trace) equivalence is coarser than bisimilarity. Similarly, in the case of weighted automata, we have that weighted bisimilarity implies weighted language equivalence.

### 3.4. Locally Finitely Presentable Coalgebras over Algebras

The aim of this subsection is to establish our main result concerning final coalgebras and rational fixpoints as explained in the introduction (see (1.1)). In fact, we will show that the rational fixpoint $\varrho\bar{F}$ is a quotient of the rational fixpoint $\varrho(FT)$, thus these rational fixpoints share the same relationship that we saw for the corresponding final coalgebras in Proposition 3.26. In addition, we use the fact that we work with algebras to improve on the finality criterium for $\varrho\bar{F}$ from Theorem 3.8(2), and we show that $\varrho\bar{F}$ can be constructed from only those coalgebras with a *free* finitely presentable carrier.

These results lay down the foundation for the work in the subsequent sections: establishing that soundness and completeness proofs amount to proving that expressions modulo axioms of a calculus are isomorphic to $\varrho\bar{F}$ (Section 4) and the application of this to our calculus for weighted language equivalence in Section 5.

*Remark* 3.30. Let us collect some facts which are true for every finitary monad $T$ on Set and every functor $F$ having a lifting to $\mathsf{Set}^T$ and that we will subsequently need in our proofs.

(1) Every free algebra $TX$ is *projective*: for every (strong) epimorphism $q\colon A \to B$ in $\mathsf{Set}^T$ (i.e., $q$ is a surjective homomorphism) and every $T$-algebra homomorphism $f\colon TX \to B$ there exists a homomorphism $g\colon TX \to A$ such that $q \cdot g = f$:

$$TX \xrightarrow{\ g\ } A$$
$$f \searrow \quad \downarrow q$$
$$B$$

Since $q$ is surjective we have a (not necessarily homomorphic) map $s\colon B \to A$ with $q \cdot s = \mathrm{id}$. Then we use the freeness of $TX$ to extend the map $s \cdot f \cdot \eta_X\colon X \to A$ to the homomorphism $g\colon TX \to A$, which has the desired property.

(2) As we mentioned already, finitely presentable algebras are precisely those algebras that are presentable by finitely many generators and relations. In category theoretic terms, an algebra $A$ is finitely presentable if and only if it is the (reflexive) coequalizer of a parallel pair $f, g\colon TX \to TY$ of homomorphism between *free* finitely presentable algebras, i.e., free algebras on the finite sets $X$ and $Y$ (cf. [Adámek et al. 011a, Proposition 5.17]).

(3) The monad $T$ yields a functor $T'\colon \mathsf{Coalg}(FT) \to \mathsf{Coalg}(\bar{F})$; it assigns to every $FT$-coalgebra $c\colon X \to FTX$ the coalgebra $c^{\sharp}\colon TX \to FTX$ obtained by the generalized powerset construction, and on morphisms $T'$ acts like $T$. It is easy to see that $T'$ is finitary; this follows essentially from the fact that the filtered colimits in $\mathsf{Coalg}(FT)$ and $\mathsf{Coalg}(\bar{F})$ are formed on the level of Set (since the forgetful functors of $\mathsf{Coalg}(FT)$, $\mathsf{Coalg}(\bar{F})$ and $\mathsf{Set}^T$ create filtered colimits).

*Notation* 3.31. We denote by

$$\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$$

the full subcategory of $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$ given by coalgebras with a *free* finitely presentable carrier. That means that the objects of $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ are of the form $TX \to FTX$ with $X$ a finite set.

*Remark* 3.32. Observe that the objects of $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ are precisely the results of applying the generalized powerset construction in Section 2.3 to every finite coalgebra $c\colon X \to FTX$. Indeed, $T'(X, c) = (TX, c^{\sharp})$ lies in $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$, and, conversely, for every $d\colon TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ it is easy to see that $d = (d \cdot \eta_X)^{\sharp}$.

LEMMA 3.33. *The category* $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ *is closed in* $\mathsf{Coalg}(F)$ *under finite coproducts.*

PROOF. The empty $FT$-coalgebra $0 \to FT0$ extends uniquely to an $\bar{F}$-coalgebra $T0 \to FT0$, and this is the initial object of $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$.

Let $c^{\sharp}\colon TX \to FTX$ and $d^{\sharp}\colon TY \to FTY$ be objects of $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ with the corresponding $FT$-coalgebras $c\colon X \to FTX$ and $d\colon Y \to FTY$. Now form

$$k = (\, X + Y \xrightarrow{c+d} FTX + FTY \xrightarrow{\ \mathsf{can}\ } FT(X+Y)\,),$$

where $\mathsf{can} = [FT\mathsf{inl}, FT\mathsf{inr}]$, and extend $k$ to the $T$-algebra homomorphism

$$k^{\sharp}\colon T(X+Y) \to FT(X+Y).$$

It is not difficult to verify that this $\bar{F}$-coalgebra is the coproduct of $(TX, c^\sharp)$ and $(TY, d^\sharp)$ in $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$. To see this, first verify that $T\mathsf{inl}\colon TX \to T(X + Y)$ and $T\mathsf{inr}\colon TY \to T(X + Y)$ are $\bar{F}$-coalgebra homomorphisms. Next we show that they serve as the coproduct injections. Suppose we have two $\bar{F}$-coalgebra homomorphisms $f\colon (TX, c^\sharp) \to (A, a)$ and $g\colon (TY, d^\sharp) \to (A, a)$. Let $f_0 = f \cdot \eta_X$ and $g_0 = g \cdot \eta_Y$. Now extend the morphism $h_0 = [f_0, g_0]\colon X + Y \to A$ to a $T$-algebra homomorphism $h\colon T(X+Y) \to A$. Then one readily verifies using the universal properties of free $T$-algebras that $h$ is the unique $\bar{F}$-coalgebra homomorphism from $(T(X + Y), k^\sharp)$ to $(A, a)$ such that $h \cdot T\mathsf{inl} = f$ and $h \cdot T\mathsf{inr} = g$.  □

The next proposition is the key to the main results of this section. It uses the full strength of our Assumption 3.17, in particular that finitely generated algebras are closed under taking kernel pairs.

PROPOSITION 3.34.    *Every coalgebra in $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$ is the coequalizer of a pair of morphisms in $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$.*

PROOF.    Let $a\colon A \to \bar{F}A$ be a coalgebra from $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$, so $A$ is a finitely presentable $T$-algebra. From Remark 3.30(2) we recall that $A$ is the coequalizer of some pair $TX' \rightrightarrows TX$ of $T$-algebra homomorphisms with $X'$ and $X$ finite sets via some $q\colon TX \to A$. Being a functor on Set, $F$ preserves epimorphisms. Thus, $\bar{F}q$ is a strong epimorphism in $\mathsf{Set}^T$. Now we use that $TX$ is projective to obtain a coalgebra structure $c\colon TX \to FTX$ as displayed below:

$$
\begin{array}{ccc}
TX & \overset{c}{-\!\!-\!\!\to} & FTX \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Fq} \\
A & \underset{a}{\longrightarrow} & FA
\end{array}
\qquad (3.6)
$$

Now since $\mathsf{Set}^T$ is a category with pullbacks we know that every coequalizer in that category is the coequalizer of its kernel pair. So let $f, g\colon K \to TX$ be the kernel pair of $q$ in $\mathsf{Set}^T$. Notice that since $TX$ and $A$ are finitely presentable $T$-algebras, so is $K$ because finitely presentable (equivalently, finitely generated) $T$-algebras are closed under taking kernel pairs by Assumption 3.17. Since the forgetful functor $\mathsf{Set}^T \to \mathsf{Set}$ preserves limits we have a pullback in Set, and since $F$ weakly preserves pullbacks $Ff, Fg$ form a weak pullback of $Fq$ along itself in Set. Thus, we have a map $k\colon K \to FK$ such that the diagram below commutes:

$$
\begin{array}{ccc}
K & \overset{k}{-\!\!-\!\!\to} & FK \\
{\scriptstyle f}\downdownarrows{\scriptstyle g} & & {\scriptstyle Ff}\downdownarrows{\scriptstyle Fg} \\
TX & \overset{c}{\longrightarrow} & FTX \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle Fq} \\
A & \underset{a}{\longrightarrow} & FA
\end{array}
\qquad (3.7)
$$

Notice that we do not claim that $k$ is a $T$-algebra homomorphism. However, since $K$ is a finitely presentable $T$-algebra it is the coequalizer of some pair $TY' \rightrightarrows TY$ of $T$-algebra homomorphisms, $Y'$ and $Y$ finite, via $p\colon TY \to K$. Now we choose some splitting $s\colon K \to TY$ of $p$ in Set, i. e., $s$ is a map such that $p \cdot s = \mathsf{id}$. Next we extend the map

$d_0 = Fs \cdot k \cdot p \cdot \eta_Y$ to a $T$-algebra homomorphism $d \colon TY \to FTY$:

$$
\begin{array}{ccc}
Y & & \\
{\scriptstyle \eta_Y}\downarrow & \searrow {\scriptstyle d_0} & \\
TY & \xrightarrow{\;d\;} & FTY \\
{\scriptstyle p}\downarrow & & \downarrow {\scriptstyle Fp} \\
K & \xrightarrow[k]{} & FK
\end{array}
\qquad (3.8)
$$

(Notice that to obtain $d$ we cannot simply use projectivity of $TY$ similarly as in (3.6) since $k$ is not necessarily a $T$-algebra homomorphism.)

We do not claim that this makes $p$ a coalgebra homomorphism (i.e., we do not claim the lower square in (3.8) commutes). However, $f \cdot p$ and $g \cdot p$ are $\bar{F}$-coalgebra homomorphisms from $(TY, d)$ to $(TX, c)$. To see that

$$
c \cdot (f \cdot p) = F(f \cdot p) \cdot d
$$

it suffices that this equation of $T$-algebra homomorphisms holds when both sides are precomposed with $\eta_Y$. To see this we compute

$$
\begin{array}{rll}
c \cdot f \cdot p \cdot \eta_Y & = \; Ff \cdot k \cdot p \cdot \eta_Y & \text{see (3.7),} \\
& = \; Ff \cdot Fp \cdot d_0 & \text{outside of (3.8),} \\
& = \; Ff \cdot Fp \cdot d \cdot \eta_Y & \text{definition of } d.
\end{array}
$$

Similarly, $g \cdot p$ is a coalgebra homomorphism. Since $p$ is an epimorphism in $\mathsf{Set}^T$ it follows that $q$ is a coequalizer of $f \cdot p$ and $g \cdot p$. Thus $f \cdot p$ and $g \cdot p$ form the desired pair of morphisms in $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ such that $(A, a)$ is a coequalizer of them, which completes the proof. □

As a consequence of the previous proposition we obtain that the rational fixpoint $\varrho\bar{F}$ can be constructed just using those coalgebras obtained by applying the generalized powerset construction to finite $FT$-coalgebras.

COROLLARY 3.35. *The rational fixpoint of $\bar{F}$ is the colimit of all coalgebras in* $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$*; in symbols:*

$$
\varrho\bar{F} = \operatorname{colim}(\mathsf{Coalg}_{\mathsf{free}}(\bar{F}) \hookrightarrow \mathsf{Coalg}(\bar{F})).
$$

PROOF. We first show that $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$ is the closure of $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ under coequalizers in the category $\mathsf{Coalg}(\bar{F})$. Since finitely presentable algebras are closed under finite colimits and finite colimits in $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$ are formed on the level of $\mathsf{Set}^T$ we clearly see that the closure of $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ under coequalizers is a subcategory of $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$. But, by the previous proposition, each object of $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$ is a coequalizer of some parallel pair of morphisms from $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$, which establishes the desired statement.

It is easy to prove that the colimit of $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ and the filtered colimit of its closure under coequalizers coincide. But the latter is $\varrho\bar{F}$ by Corollary 3.9. □

Furthermore, and playing a crucial rôle in simplifying our proof burden for completeness later, we have that a locally finitely presentable coalgebra $r \colon R \to \bar{F}R$ is final for *all* locally finitely presentable coalgebras if there is a unique homomorphism from those coalgebras whose carrier is *free on a finite set* to $(R, r)$. This means that when proving finality of $(R, r)$ one does not need to show the existence of a unique homomorphism for all coalgebras but only for the much smaller class of coalgebras from $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$.

COROLLARY 3.36. *A locally finitely presentable $\bar{F}$-coalgebra $(R, r)$ is final in the category of all locally finitely presentable $\bar{F}$-coalgebras if and only if for every coalgebra $(TX, c^\sharp)$ from $\mathsf{Coalg_{free}}(\bar{F})$ there exists a unique coalgebra homomorphism from $(TX, c^\sharp)$ to $(R, r)$.*

PROOF. Necessity of a unique coalgebra homomorphism from each $(TX, c^\sharp)$ to $(R, r)$ is clear. For sufficiency let $a\colon A \to FA$ be a coalgebra in $\mathsf{Coalg_f}(\bar{F})$. By Proposition 3.34, we have a coequalizer diagram

$$(TX, c^\sharp) \underset{g}{\overset{f}{\rightrightarrows}} (TY, d^\sharp) \overset{q}{\longrightarrow} (A, a),$$

with $(TX, c^\sharp)$ and $(TY, d^\sharp)$ in $\mathsf{Coalg_{free}}(\bar{F})$. The unique coalgebra homomorphism $h\colon (TY, d^\sharp) \to (R, r)$ satisfies $h \cdot f = h \cdot g$ since both of these are coalgebra homomorphisms from $(TX, c^\sharp)$ to $(R, r)$. So by the universal property of the coequalizer we get a unique coalgebra homomorphism $k\colon (A, a) \to (R, r)$. The desired result now follows from Theorem 3.8(2). $\square$

We are now ready to relate the rational fixpoints of $FT$ and $\bar{F}$. Recall the congruence quotient $p\colon \nu(FT) \to \nu F$ from Proposition 3.26 and notice that the rational fixpoint $\varrho(FT)$ is a subcoalgebra of $\nu(FT)$ (see Proposition 3.12). From our assumptions we also know that $\varrho\bar{F}$ is a subcoalgebra of $\nu F$ (recall from Section 3.3 that $\nu F$ denotes the final $\bar{F}$-coalgebra).

*Notation* 3.37. We denote the corresponding inclusion homomorphisms by

$$i\colon \varrho(FT) \to \nu(FT) \qquad \text{and} \qquad j\colon \varrho\bar{F} \to \nu F.$$

Furthermore, recall from Notation 3.25 that $\varrho(FT)$ is an $\bar{F}$-coalgebra with the structure $F\beta \cdot \mathfrak{r}$, where $\mathfrak{r}$ is the coalgebra map of the rational fixpoint $\varrho(FT)$, and $\beta$ its $T$-algebra structure.

LEMMA 3.38. *The coalgebra*

$$\varrho(FT) \overset{\mathfrak{r}}{\longrightarrow} FT(\varrho(FT)) \overset{F\beta}{\longrightarrow} F(\varrho(FT))$$

*is a locally finitely presentable $\bar{F}$-coalgebra.*

PROOF. By Theorem 3.8(1) the coalgebra $(\varrho(FT), \mathfrak{r})$ is the filtered colimit of the inclusion functor $I\colon \mathsf{Coalg_f}(FT) \hookrightarrow \mathsf{Coalg}(FT)$. The finitary functor $T'\colon \mathsf{Coalg}(FT) \to \mathsf{Coalg}(\bar{F})$ from Remark 3.30(3) preserves this colimit, and so the coalgebra $T'(\varrho(FT), \mathfrak{r}) = (T(\varrho(FT), \mathfrak{r}^\sharp)$ is the filtered colimit of the diagram of all $\bar{F}$-coalgebras $T'(C, c) = (TC, c^\sharp)$. (Notice that the corresponding diagram scheme contains the same objects but fewer connecting morphisms than $\mathsf{Coalg_{free}}(\bar{F})$ from Notation 3.31—here we consider only the morphisms $Th$ for $h$ an $FT$-coalgebra homomorphism.)

Thus, since the carrier of every object in this diagram is a finitely presentable algebra, we can apply Theorem 3.8 to conclude that $(T(\varrho(FT)), \mathfrak{r}^\sharp)$ is a locally finitely presentable coalgebra. We also know that

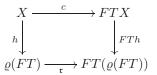$$\beta\colon (T(\varrho(FT)), \mathfrak{r}^\sharp) \to (\varrho(FT), F\beta \cdot \mathfrak{r})$$

is a homomorphism of $\bar{F}$-coalgebras (see Notation 3.25). This is a strong epimorphism in $\mathsf{Set}^T$ (because $\beta \cdot \eta_{\varrho(FT)} = \mathrm{id}$). Hence, being a quotient of a coalgebra that is locally finitely presentable, $(\varrho(FT), F\beta \cdot \mathfrak{r})$ also has that property (see Lemma 3.16). $\square$

Next, we show that the coalgebra $(\varrho(FT), F\beta \cdot \mathfrak{r})$ is weakly final[5] among the locally finitely presentable $\bar{F}$-coalgebras.

LEMMA 3.39. *For every locally finitely presentable $\bar{F}$-coalgebra there exists a canonical homomorphism into the coalgebra $(\varrho(FT), F\beta \cdot \mathfrak{r})$.*

PROOF. It suffices to show the statement for every coalgebra from $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$. It then follows that every coalgebra from $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$ (being a coequalizer of a pair of morphisms in $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$) admits a homomorphism into $\varrho(FT)$. Hence, every filtered colimit of coalgebras from $\mathsf{Coalg}_{\mathsf{f}}(\bar{F})$ admits a homomorphism into $\varrho(FT)$.

Now suppose we are given $c^\sharp \colon TX \to FTX$ from $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$. Consider the corresponding $FT$-coalgebra $c \colon X \to FTX$. Since $X$ is a finite set we obtain a unique $FT$-coalgebra homomorphism $h$ from $(X, c)$ to the final locally finite coalgebra $\varrho(FT)$:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ c\ \ } & FTX \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle FTh} \\
\varrho(FT) & \xrightarrow[\ \ \mathfrak{r}\ \ ]{} & FT(\varrho(FT))
\end{array}
$$

We apply the functor $T' \colon \mathsf{Coalg}(FT) \to \mathsf{Coalg}(\bar{F})$ to obtain an $\bar{F}$-coalgebra homomorphism $Th$ from $(TX, c^\sharp)$ to $(TR, \mathfrak{r}^\sharp)$. Then compose with the $\bar{F}$-coalgebra homomorphisms $\beta \colon T(\varrho(FT) \to \varrho(FT)$ to obtain the desired coalgebra homomorphism from $(TX, c^\sharp)$ to $(\varrho(FT), \beta \cdot \mathfrak{r})$. ☐

As a consequence of the previous lemma, also every quotient of $(\varrho(FT), F\beta \cdot \mathfrak{r})$ is weakly final among the locally finitely presentable $\bar{F}$-coalgebras:

COROLLARY 3.40. *Every quotient coalgebra of $(\varrho(FT), F\beta \cdot \mathfrak{r})$ admits a homomorphism from every locally finitely presentable coalgebra for $\bar{F}$.*

At last, we can state the formal relation between the rational fixpoints of $FT$ and $\bar{F}$:

THEOREM 3.41. *The rational fixpoint of $\bar{F}$ is the image of $\varrho(FT)$ under the quotient $p \colon \nu(FT) \to \nu F$ from Proposition 3.26, that is, there is a surjective $\bar{F}$-coalgebra homomorphism $q \colon \varrho(FT) \to \varrho\bar{F}$ such that the following square commutes (using Notation 3.37):*

$$
\begin{array}{ccc}
\varrho(FT) & \rightarrowtail^{\ i\ } & \nu(FT) \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\
\varrho\bar{F} & \rightarrowtail[\ j\ ] & \nu F
\end{array}
$$

PROOF. We first need to verify that $i \colon \varrho(FT) \to \nu(FT)$ is a homomorphism of $\bar{F}$-coalgebras. By definition we have $\mathfrak{t} \cdot i = FTi \cdot \mathfrak{r}$, and since $\mathfrak{t}$ is invertible we get:

$$
i = \mathfrak{t}^{-1} \cdot FTi \cdot \mathfrak{r}.
$$

---

[5]A *weakly final* object in a category is an object $W$ such that for every object $X$ there exists a (not necessarily unique) morphism $X \to W$.

Thus, $i$ is a $T$-algebra homomorphism since all three morphisms on the right-hand side of the above equation are. Now the following diagram commutes as desired:
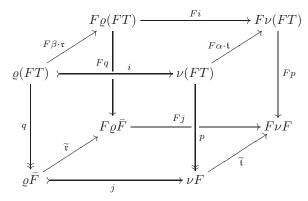
$$
\begin{array}{ccccc}
\varrho(FT) & \xrightarrow{\ \mathfrak{r}\ } & FT(\varrho(FT)) & \xrightarrow{\ F\beta\ } & F(\varrho(FT)) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle FTi} & & \downarrow{\scriptstyle Fi} \\
\nu(FT) & \xrightarrow{\ \mathfrak{t}\ } & FT(\nu(FT)) & \xrightarrow{\ F\alpha\ } & F(\nu(FT))
\end{array}
$$

Let $I$ be the image in $\nu F$ of $\varrho(FT)$ under $p$, i. e., we take the image factorisation $m \cdot e$ of $p \cdot i$. Then $I$ is a sub-$T$-algebra of $\nu F$. Since $\bar{F}$ preserves monomorphisms (cf. Remark 3.20), it follows that $I$ carries the structure $z \colon I \to \bar{F}I$ of an $\bar{F}$-coalgebra making it a subcoalgebra of $\nu F$ (see Remark 3.3(2)). We will prove that $I$ is the final locally finitely presentable $\bar{F}$-coalgebra and a quotient coalgebra of $(\varrho(FT), F\beta \cdot \mathfrak{r})$.

Firstly, by an application of Lemma 3.16 we see that the quotient $(I, z)$ is locally finitely presentable since the coalgebra $(\varrho(FT), F\beta \cdot \mathfrak{r})$ also has this property (see Lemma 3.38). Thus, by Corollary 3.36, we only need to prove that for every $\bar{F}$-coalgebra $c^{\sharp} \colon TX \to FTX$ from the category $\mathsf{Coalg}_{\mathsf{free}}(\bar{F})$ there exists a unique coalgebra homomorphism from $(TX, c^{\sharp})$ to $(I, z)$. Since $(I, z)$ is a subcoalgebra of the final $\bar{F}$-coalgebra $\nu F$ the uniqueness of a homomorphism is clear, and the existence of a homomorphism is clear since $(I, z)$ is a quotient coalgebra of $(\varrho(FT), F\beta \cdot \mathfrak{r})$ (use Corollary 3.40).

This proves that $I \cong \varrho\bar{F}$ and composing this isomorphism with $m$ and $e$ yields $q$ and $j$ as displayed in the square above.  $\square$

Let us summarize the four fixpoints from the previous theorem and their coalgebra structures in one picture for future reference:



From Proposition 3.28 we also see that the quotient map $p$ is the coalgebraic language map $\ddagger\mathfrak{t}$ for the final $FT$-coalgebra $\mathfrak{t} \colon \nu(FT) \to FT(\nu(FT))$ and the diagonal of the front square is $\ddagger\mathfrak{r}$ for the final locally finite $FT$-coalgebra $\mathfrak{r} \colon \varrho(FT) \to FT(\varrho(FT))$:

$$
\begin{array}{ccc}
\varrho(FT) & \xrightarrow{\ i\ } & \nu(FT) \\
\downarrow{\scriptstyle q} & {\scriptstyle \ddagger\mathfrak{r}} \searrow & \downarrow{\scriptstyle p} \\
\varrho\bar{F} & \xrightarrow{\ j\ } & \nu F
\end{array}
$$

In this section, we have developed the theory of locally finitely presentable coalgebras (over algebras). All the abstract work and results in this section will play a prominent rôle in the rest of the paper; they enable stating and proving a Kleene like

theorem and soundness and completeness of axiomatisation results for coalgebraic language equivalence, for a large class of systems, uniformly. We will demonstrate this with our calculus for weighted automata in Section 5. The first pay-off of this abstract work appears immediately in the next section, where we will narrow down what proof obligations one has after extending a sound and complete calculus for bisimilarity with extra axioms in order to guarantee that the resulting calculus is sound and complete with respect to (coalgebraic) language equivalence.

## 4. SOUNDNESS, COMPLETENESS AND KLEENE'S THEOREM IN GENERAL

In this section we obtain a generalisation of Kleene's classical theorem from automata theory [Kleene 1956] to the setting of $FT$-coalgebras as presented in Section 3. We also present generic coalgebraic formulations of soundness and completeness of an expression calculus that we will then instantiate in the concrete example of weighted automata in the next section. The goal is to push as much work as possible to the present abstract setting and only do the minimal necessary amount of work in concrete instances.

We still work in the setting as described in Assumption 3.17. Thus we consider a finitary endofunctor $F\colon \mathsf{Set} \to \mathsf{Set}$ that weakly preserves pullbacks and has a lifting $\bar{F}\colon \mathsf{Set}^T \to \mathsf{Set}^T$, for a finitary monad $(T, \eta, \mu)$ such that in $\mathsf{Set}^T$ finitely generated algebras are closed under taking kernel pairs.

Let us first consider our two leading examples. For the functor $FX = 2 \times X^A$ and the monad $T = \mathcal{P}_{\mathsf{f}}$ consider the expression calculus obtained from (the structure of) the functor $FT$; we recalled the syntax in the introduction. Let $\mathsf{Exp}$ denote the *closed* syntactic expressions, i. e., those expressions in which every variable is bound by a $\mu$-operator, and let $\equiv$ be the least equivalence on $\mathsf{Exp}$ generated by the proof rules of the calculus. Then, as proved in [Silva et al. 010a], $\mathsf{Exp}/{\equiv}$ is isomorphic to $\varrho(F\mathcal{P}_{\mathsf{f}})$.

Similarly, for the semiring $\mathbb{S}$, $FX = \mathbb{S} \times X^A$ and $T = V$ one can define an expression calculus with closed syntactic expressions $\mathsf{Exp}$, and proof rules such that $\mathsf{Exp}/{\equiv}$ is isomorphic to $\varrho(FV)$ (see [Silva et al. 2011]).

In each case we write $q_0\colon \mathsf{Exp} \to \mathsf{Exp}/{\equiv}$ for the canonical quotient map. This motivates the following definition.

*Definition* 4.1.   We call a set $\mathsf{Exp}$ with a surjective map $q_0\colon \mathsf{Exp} \to \varrho(FT)$ an (abstract) *expression calculus* (for $FT$). The elements of $\mathsf{Exp}$ are referred to as *expressions*.

Besides the $FT$-bisimilarity semantics from [Silva et al. 010a; Silva et al. 2011] for which the calculi from the introduction are sound and complete, there is a different semantics that we now introduce.

Let us fix an expression calculus $q_0\colon \mathsf{Exp} \to \varrho(FT)$ for the rest of this section. Then we see that every expression $E$ in $\mathsf{Exp}$ denotes an element $[\![E]\!]$ of the final coalgebra $\nu F$. More precisely, the semantics function $[\![-]\!]\colon \mathsf{Exp} \to \nu F$ is defined by

$$[\![-]\!] = (\,\mathsf{Exp} \xrightarrow{\;q_0\;} \!\!\!\!\!\to \varrho(FT) \xrightarrow{\;\ddagger\mathfrak{r}\;} \nu F\,), \tag{4.1}$$

where $\ddagger\mathfrak{r}$ is the coalgebraic language map of $\varrho(FT)$.

In our leading examples this semantics is the usual language semantics; for non-deterministic automata $[\![E]\!]$ is the formal language the expression $E$ denotes, and, similarly, in the example of weighted automata $[\![E]\!]$ is the weighted language denoted by $E$.

We now prove a Kleene like theorem that establishes a one-to-one correspondence between expressions and states of finite $FT$-coalgebras.

THEOREM 4.2. *Every state of a finite coalgebra for $FT$ can equivalently be presented by an expression and vice versa. More precisely, we have:*
*(1) Let $E$ be an expression in* Exp*, then there exists a finite $FT$-coalgebra $(S, g)$ and a state $s \in S$ having the behavior $[\![E]\!]$, i. e., $\ddagger g(s) = [\![E]\!]$.*
*(2) Conversely, let $(S, g)$ be a finite $FT$-coalgebra and let $s \in S$ be a state. Then there exists an expression $E$ such that the behavior of $s$ is $[\![E]\!]$; in symbols: $\ddagger g(s) = [\![E]\!]$.*

PROOF. Ad (1). Given the expression $E$ we have $q_0(E) \in \varrho(FT)$. Since $\varrho(FT)$ is locally finitely presentable there exists a finite $FT$-coalgebra $(S, g)$, a state $s \in S$ and a coalgebra homomorphism $h \colon (S, g) \to (\varrho(FT), \mathfrak{r})$ with $h(s) = q_0(E)$. We compose this with the coalgebraic language map $\ddagger \mathfrak{r}$ to obtain:

$$[\![E]\!] = \ddagger \mathfrak{r} \cdot q_0(E) = \ddagger \mathfrak{r} \cdot h(s) = \ddagger g(s),$$

where the last equation uses Lemma 3.27.

Ad (2). Given the $FT$-coalgebra $(S, g)$ and $s \in S$ form the $\bar{F}$-coalgebra $(TS, g^\sharp)$ and take the unique $\bar{F}$-coalgebra homomorphism $f$ into the final locally finitely presentable coalgebra $\varrho\bar{F}$. Let $E$ be such that $q \cdot q_0(E) = f \cdot \eta_S(s)$, where $q \colon \varrho(FT) \to \varrho\bar{F}$ is the quotient homomorphism from Theorem 3.41. Now composing with $j \colon \varrho\bar{F} \to \nu F$ yields $[\![E]\!] = \ddagger g(s)$ as before. □

Next, we will show that, for $F$ and $T$ satisfying our assumptions, it is always possible to "add proof rules" to an existing expression calculus in order to arrive at a sound and complete calculus w. r. t. the language semantics given by $[\![-]\!]$ in (4.1).

*Definition* 4.3. Let $(\mathsf{E}, e)$ be an $\bar{F}$-coalgebra and let $f \colon$ Exp $\to \mathsf{E}$ be a map. We call $(\mathsf{E}, e, f)$ *sound* if for two expressions $E$ and $F$ in Exp, $f(E) = f(F)$ implies that $[\![E]\!] = [\![F]\!]$, and $(\mathsf{E}, e, f)$ is called *complete* if $[\![E]\!] = [\![F]\!]$ implies $f(E) = f(F)$.

One should think of $\mathsf{E}$ in the above definition as a quotient coalgebra of $(\mathsf{Exp}/\equiv) = \varrho(FT)$ obtained by adding proof rules so as to obtain a coarser equivalence $\equiv_D$ with $\mathsf{E} = (\mathsf{Exp}/\equiv_D)$. In fact, we have the following

THEOREM 4.4 (SOUNDNESS). *Every quotient coalgebra of the $\bar{F}$-coalgebra $(\varrho(FT), F\beta \cdot \mathfrak{r})$ is sound.*

PROOF. Let $\mathsf{E}$ be a quotient coalgebra of $\varrho(FT)$ via $q \colon \varrho(FT) \to \mathsf{E}$ and let $j \colon \mathsf{E} \to \nu F$ be the unique coalgebra homomorphism. We consider the map $q \cdot q_0 \colon$ Exp $\to \mathsf{E}$ and verify the soundness by proving that the diagram below commutes:

$$
\begin{array}{ccccc}
\mathsf{Exp} & \xrightarrow{\ q_0\ } & \varrho(FT) & \xrightarrow{\ q\ } & \mathsf{E} \\
 & & \downarrow{\ddagger\mathfrak{r}} & & \downarrow{j} \\
 & & & \nu F &
\end{array}
$$
$$[\![-]\!] \qquad\qquad\qquad (4.2)$$

The left-hand part commutes by the definition of the semantic map $[\![-]\!]$ (see (4.1)), and the right-hand part commutes since all its arrows are $\bar{F}$-coalgebra homomorphisms and using finality of $\nu F$.

Now whenever for two expressions $E$ and $F$ in Exp we have $q \cdot q_0(E) = q \cdot q_0(F)$ we clearly have $[\![E]\!] = [\![F]\!]$, and this is the desired soundness. □

In particular, we see that $(\varrho(FT), F\beta \cdot \mathfrak{r}, q_0)$ itself is sound. Now recall that the final locally finitely presentable coalgebra $\varrho\bar{F}$ is the (greatest) quotient of $(\varrho(FT), F\beta \cdot \mathfrak{r})$ via the homomorphism $q \colon \varrho(FT) \to \varrho\bar{F}$ (see Theorem 3.41). So, in addition we have

THEOREM 4.5 (COMPLETENESS). *The final locally finitely presentable coalgebra $\varrho\bar{F}$ together with the map $q \cdot q_0 \colon \mathsf{Exp} \to \varrho\bar{F}$ is complete.*

PROOF. Recall the four $\bar{F}$-coalgebra homomorphisms from the statement of Theorem 3.41. Now consider diagram (4.2) where $\mathsf{E} = \varrho\bar{F}$. If for two expression $E$ and $F$ in $\mathsf{Exp}$ we have $[\![E]\!] = [\![F]\!]$ then $q \cdot q_0(E) = q \cdot q_0(F)$ since $j \colon \varrho\bar{F} \to \nu F$ is injective. Therefore we obtain the desired completeness. □

Intuitively, this theorem states that, under our assumptions, it is always possible to obtain a complete calculus for (coalgebraic) language equivalence as a quotient of a calculus for bisimilarity (hence by adding new sound rules). However, the theorem does not give any indication how the added rules should look like in concrete instances and not even whether it suffices to add finitely many new rules.

One may wonder at this point about the relevance of the theorems in this section because we did not introduce any concrete syntax and proof rules. But we shall see in the next sections that from the above abstract results we automatically obtain soundness, completeness and Kleene theorems for concrete syntactic calculi once we have established that the quotient formed by concrete syntactic expressions modulo proof rules forms the rational fixpoint $\varrho\bar{F}$.

## 5. EXPRESSION CALCULUS FOR WEIGHTED AUTOMATA

In [Milius 2010] the second author has presented a sound and complete expression calculus for linear systems presented in the form of closed stream circuits, which are equivalent to weighted automata with unary input alphabet $A = \{*\}$ and weights in a field. In this section we are going to use the ideas from *loc. cit.* and apply the results from Section 3.1 to provide a sound and complete expression calculus for the language equivalence of weighted automata. This extends the previous work to weighted systems with several different inputs and from weights in a field to weights in a semiring.

As discussed in the introduction, an axiomatization for weighted language equivalence also follows from [Ésik and Kuich 2012]. Their result holds for so-called proper commutative semirings, a class of semirings containing all Noetherian semirings but also the semiring of natural numbers. Our work here is independent.

*Assumption* 5.1. In this section we work with the category $\mathbb{S}\text{-Mod}$ for a semiring $\mathbb{S}$ such that finitely generated semimodules are closed under kernel pairs. We consider the free-semimodule monad $T = V$ (see (2.1)) and the functor $F = \mathbb{S} \times (-)^A$.

The above assumption on $\mathbb{S}$ holds whenever $\mathbb{S}$ is Noetherian (see Proposition 2.6). Notice that the functor $F$ has a canonical lifting $\bar{F}$ to $\mathbb{S}\text{-Mod} = \mathsf{Set}^V$. So our assumptions in 3.17 clearly hold.

As we saw in Example 2.10, coalgebras for the composite $FV$ are weighted automata with weights in the semiring $\mathbb{S}$, and the final coalgebra for $F$ and its lifting is carried by the set $\mathbb{S}^{A^*}$ of all weighted languages.

The expression calculus one obtains in this particular instance from the work in [Silva et al. 2011] allows one to reason about the equivalence of weighted automata w. r. t. weighted bisimilarity. We will now recall the syntax and proof rules of this calculus. The syntactic expressions are defined by the following grammar

$$E \ ::= \ x \mid \mathbf{0} \mid E \oplus E \mid \underline{r} \mid a.(r \bullet E) \mid \mu x.E^g,$$
$$E^g \ ::= \ \mathbf{0} \mid E^g \oplus E^g \mid \underline{r} \mid a.(r \bullet E) \mid \mu x.E^g.$$

Notice that the variable binding operator $\mu x.-$ is only applied to *guarded* expressions, i. e., expressions $E^g$ where each occurrence of $x$ is within the scope of an operator $a.(r \bullet -)$.

We write Exp for the set of all *closed expressions* defined by the above grammar. One puts on these expressions certain rules and equations stating that $\mu$ is a unique fixpoint operator, that $\oplus$ is a commutative and associative binary operation with the neutral element $\mathbf{0}$, etc; here is the list of rules:

$$\underline{0} \equiv \mathbf{0} \qquad \underline{r} \oplus \underline{s} \equiv \underline{r+s}$$
$$\mathbf{0} \oplus E \equiv E \qquad E_1 \oplus E_2 \equiv E_2 \oplus E_1 \qquad (E_1 \oplus E_2) \oplus E_3 \equiv E_1 \oplus (E_2 \oplus E_3)$$
$$a.(0 \bullet E) \equiv \mathbf{0} \qquad a.(r \bullet E) \oplus a.(s \bullet E) \equiv a.((r+s) \bullet E)$$
$$\mu x.E \equiv E[\mu x.E/x] \qquad E_1 \equiv E_2[E_1/x] \implies E_1 \equiv \mu x.E_2$$

We call the last two rules pertaining to $\mu$ the *fixpoint axiom* (FP, for short) and the *uniqueness rule*, respectively. In addition the rules contain $\alpha$-equivalence, i. e., renaming of bound variables does not matter, and the *replacement rule* (also called congruence rule):

$$\frac{E_1 \equiv E_2}{E[E_1/x] \equiv E[E_2/x]}, \qquad (5.1)$$

where $E_1$, $E_2$ and $E$ are expressions and $x$ is a free variable in $E$. We write $\equiv$ for the least equivalence on Exp generated by the above rules.

The main result of [Silva et al. 2011] is that this calculus is sound and complete for bisimilarity equivalence of weighted automata. As previously mentioned, the key fact used in order to prove soundness and completeness is that the set $\mathsf{E} = \mathsf{Exp}/\equiv$ of closed syntactic expressions modulo the proof rules above is (isomorphic to) the final locally finite coalgebra $\varrho(FV)$.

Now we will turn to a different semantics of the expressions in Exp, the weighted languages described by them. The canonical quotient map $q_0 \colon \mathsf{Exp} \to \mathsf{E} = \varrho(FV)$ gives us an expression calculus in the sense of Definition 4.1, and we obtain the corresponding semantics map from (4.1):

$$[\![-]\!] \colon \mathsf{Exp} \xrightarrow{q_0} \mathsf{E} = \varrho(FV) \xrightarrow{\ddagger \mathfrak{r}} \nu F = \mathbb{S}^{A^*};$$

it assigns to every expression the weighted language it denotes.

*Remark* 5.2. Before we proceed we gather a number of facts that we will need for the subsequent technical development.

(1) In [Silva et al. 2011] a measure of complexity $N(E)$ for guarded expressions is defined. For the special instance of the calculus we are considering here, $N(E)$ is defined as:

$$N(\mathbf{0}) = N(\underline{r}) = N(a.(r \bullet E)) = 0$$
$$N(E_1 \oplus E_2) = 1 + \max\{N(E_1), N(E_2)\}$$
$$N(\mu x.E) = 1 + N(E).$$

Notice that for every guarded expression we clearly have $N(E_1) = N(E_1[E_2/x])$ for every expression $E_2$.

(2) For every set $X$, every element of $VX$ can be written as a formal linear combination

$$\sum_{i=1}^{n} r_i x_i, \quad \text{with } x_i \in X, r_i \in \mathbb{S} \text{ for } i = 1, \ldots, n.$$

This formal linear combination corresponds to $f \colon X \to \mathbb{S}$ with $f(x_i) = r_i$ for $i = 1, \ldots, n$, and $f(y) = 0$ else.

(3) As usual we denote by $[E]$ the equivalence classes in $\mathsf{E} = \varrho(FV)$. By Lemma 3.38, we see that $\mathsf{E}$ has a canonical structure of a $V$-algebra, i. e., $\mathsf{E}$ is an $\mathbb{S}$-semimodule.

It is straightforward to work out that the semimodule addition is

$$[E_1] + [E_2] = [E_1 \oplus E_2]$$

with the neutral element $[\mathbf{0}]$ and that the action of the semiring $\mathbb{S}$ satisfies the following laws:

$$
\begin{aligned}
r[\mathbf{0}] &= [\mathbf{0}] \\
r[E_1 \oplus E_2] &= r[E_1] + r[E_2] \\
r[\mu x.E] &= r[E[\mu x.E/x]] \\
r[\underline{s}] &= [\underline{rs}] \\
r[a.(s \bullet E)] &= [a.((rs) \bullet E)]
\end{aligned}
\tag{5.2}
$$

From now on we will omit the square brackets indicating equivalence classes w.r.t. $\equiv$ and simply write $E$ for elements of $\mathsf{E}$. We shall also write $rE$ for any expression in $r[E]$.

(4) Furthermore, since $\mathsf{E} = \varrho(FV)$ we have the coalgebra structure $\mathfrak{r} \colon \mathsf{E} \to FV(\mathsf{E})$ and we have the Eilenberg-Moore algebra structure $\beta \colon V(\mathsf{E}) \to \mathsf{E}$ which gives us an $\bar{F}$-coalgebra structure $F\beta \cdot \mathfrak{r}$ on $\mathsf{E}$ (see Notation 3.25). For further reference we note that the coalgebra structure $\mathfrak{r} \colon \mathsf{E} \to \mathbb{S} \times (V\mathsf{E})^A$ acts, for example, as follows:

$$
\begin{aligned}
\mathfrak{r}(a.(s \bullet E)) &= \left(0, \lambda b. \left\{ \begin{array}{ll} sE & \text{if } b = a \\ 0 & \text{else} \end{array} \right\} \right), \\
\mathfrak{r}(\underline{s}) &= (s, \lambda b.\mathbf{0}),
\end{aligned}
\tag{5.3}
$$

(since we omit equivalence classes here, we do have the formal linear combination $sE \in V(\mathsf{E})$).

From the generic Kleene theorem 4.2 we obtain immediately a Kleene like theorem stating that that every state of a weighted automaton can equivalently be specified by an expression of our calculus.

COROLLARY 5.3. *(1) For every expression $E$ in* Exp *there exists a finite weighted automaton $S$ and a state $s \in S$ such that the weighted language accepted by $s$ is $[\![E]\!]$.*
*(2) For every state $s$ of a finite weighted automaton there exists an expression that denotes the same weighted language as the one accepted by the state $s$.*

Indeed, this is just a restatement of Theorem 4.2 noting that finite weighted automata are precisely finite $FV$-coalgebras.

In classical automata theory one obtains, of course, an algorithmic construction of an expression for a given state of an automaton. The above theorem does not provide such a construction. However, in our theory the respective construction does occur, namely later in the proof of Theorem 5.14.

## 5.1. Axiomatization of weighted language equivalence

We are now going to add the following three additional equational laws to the calculus from the previous section:

$$a.(r \bullet (E_1 \oplus E_2)) \equiv_D a.(r \bullet E_1) \oplus a.(r \bullet E_2) \tag{5.4}$$

$$a.(r \bullet b.(s \bullet E)) \equiv_D a.((rs) \bullet b.(1 \bullet E)) \tag{5.5}$$

$$a.(r \bullet \underline{s}) \equiv_D a.(1 \bullet \underline{rs}) \tag{5.6}$$

Notice that we write $\equiv_D$ for the least equivalence generated by all the above rules (i.e., all the rules from the previous section and the three last ones).

We denote by $\mathsf{E}_D = \mathsf{Exp}/{\equiv_D}$ the closed expression modulo all these proof rules. Notice that $\mathsf{E}_D$ is a quotient of $\mathsf{E}$ via $q \colon \mathsf{E} \to \mathsf{E}_D$, say.

*Remark* 5.4. Observe that the following equational law

$$a.(r \bullet \mathbf{0}) \equiv_D \mathbf{0} \tag{5.7}$$

is provable from the other laws. Using (5.6) we have

$$a.(r \bullet \mathbf{0}) \equiv_D a.(r \bullet \underline{0}) \equiv_D a.(1 \bullet \underline{r0}) \equiv_D a.(1 \bullet \underline{0r}) \equiv_D a.(0 \bullet \underline{r}) \equiv_D \mathbf{0}.$$

LEMMA 5.5. *The quotient* $\mathsf{E}_D$ *is an* $\mathbb{S}$*-semimodule and* $q\colon \mathsf{E} \to \mathsf{E}_D$ *is a homomorphism of semimodules.*

PROOF. We only need to prove that the three additional equational laws in (5.4)–(5.6) respect the semimodule structure of $\mathsf{E}$, i. e., the semimodule operations on $\mathsf{E}_D$ are well-defined on equivalence classes.

For the addition this follows from the replacement rule (5.1). We verify well-definedness for the action of the semiring $\mathbb{S}$ for each of the three equational laws:

Ad (5.4) we have

$$\begin{aligned}
s(a.(r \bullet E_1) \oplus a.(r \bullet E_2)) &\equiv_D s(a.(r \bullet E_1)) \oplus s(a.(r \bullet E_2)) &&\text{see (5.2)}\\
&\equiv_D a.((sr) \bullet E_1) \oplus a.((sr) \bullet E_2) &&\text{see (5.2)}\\
&\equiv_D a.((sr) \bullet (E_1 \oplus E_2)) &&\text{see (5.4)}\\
&\equiv_D s(a.(r \bullet (E_1 \oplus E_2))) &&\text{see (5.2)}.
\end{aligned}$$

Ad (5.5) we have

$$\begin{aligned}
c(a.(r \bullet b.(s \bullet E))) &\equiv_D a.((cr) \bullet b.(s \bullet E)) &&\text{see (5.2)}\\
&\equiv_D a.((crs) \bullet b.(1 \bullet E)) &&\text{by (5.5)}\\
&\equiv_D c(a.((rs) \bullet b.(1 \bullet E))) &&\text{see (5.2)}.
\end{aligned}$$

Ad (5.6) we have

$$\begin{aligned}
c(a.(r \bullet \underline{s})) &\equiv_D a.((cr) \bullet \underline{s}) &&\text{see (5.2)}\\
&\equiv_D a.(1 \bullet \underline{crs}) &&\text{by (5.6)}\\
&\equiv_D a.(c \bullet \underline{rs}) &&\text{by (5.6)}\\
&\equiv_D c(a.(1 \bullet \underline{rs})) &&\text{see (5.2)}.
\end{aligned}$$

This completes the proof. □

LEMMA 5.6. *For the action of the semiring* $\mathbb{S}$ *in* $\mathsf{E}_D$ *we have the following provable identity:*

$$r(a.(s \bullet E)) \equiv_D a.(r \bullet sE).$$

PROOF. Recall from (5.2) that $r(a.(s \bullet E)) = a.((rs) \bullet E)$. Now the proof proceeds by induction on the complexity $N(E)$ of expressions. Here are the different cases (we drop the subscript in $\equiv_D$):

(1) For $E = \mathbf{0}$ we apply (5.7) and get

$$a.((rs) \bullet \mathbf{0}) \equiv \mathbf{0} \equiv a.(r \bullet \mathbf{0}) \equiv a.(r \bullet (s\mathbf{0})).$$

(2) For $E = \underline{t}$ we use (5.6) and (5.2) to obtain

$$a.((rs) \bullet \underline{t}) \equiv a.(1 \bullet \underline{rst}) \equiv a.(1 \bullet (r\underline{st})) \equiv a.(r \bullet \underline{st}) \equiv a.(r \bullet (s\underline{t})).$$

(3) For a sum $E = E_1 \oplus E_2$ we compute

$$\begin{aligned}
a.((rs) \bullet (E_1 \oplus E_2)) &\equiv a.((rs) \bullet E_1) \oplus a.((rs) \bullet E_2) &&\text{by (5.4)}\\
&\equiv a.(r \bullet sE_1) \oplus a.(r \bullet sE_2) &&\text{by induction hypothesis}\\
&\equiv a.(r \bullet (sE_1 \oplus sE_2)) &&\text{by (5.4)}\\
&\equiv a.(r \bullet s(E_1 \oplus E_2)) &&\text{by (5.2)}.
\end{aligned}$$

(4) For $E = b.(t \bullet E')$ we use (5.5) and obtain

$$
\begin{aligned}
a.((rs) \bullet b.(t \bullet E')) &\equiv a.((rst) \bullet b.(1 \bullet E')) \\
&\equiv a.(r \bullet b.((st) \bullet E')) \\
&\equiv a.(r \bullet s(b.(t \bullet E'))).
\end{aligned}
$$

(5) Finally, for a $\mu$-term $E = \mu x.E'$ one simply uses the induction hypothesis on $E'[\mu x.E'/x]$ to obtain

$$
\begin{aligned}
a.((rs) \bullet (\mu x.E')) &\equiv a.((rs) \bullet E'[\mu x.E'/x]) \\
&\equiv a.(r \bullet (sE'[\mu x.E'/x])) \\
&\equiv a.(r \bullet s(\mu x.E')).
\end{aligned}
$$

This completes the proof. $\quad\square$

*Remark* 5.7. Note that for a commutative semiring $\mathbb{S}$ we have

$$
r(a.(s \bullet E)) \equiv_D a.(s \bullet rE); \tag{5.8}
$$

indeed, one computes

$$
\begin{aligned}
r(a.(s \bullet E)) &\equiv_D a.((rs) \bullet E) && \text{by (5.2)} \\
&\equiv_D a.((sr) \bullet E) && \text{by commutativity} \\
&\equiv_D s(a.(r \bullet E)) && \text{by (5.2)} \\
&\equiv_D a.(s \bullet rE) && \text{by Lemma 5.6.}
\end{aligned}
$$

Before we proceed to prove that the axiomatisation above is sound and complete, let us revisit the examples from the introduction.

*Example* 5.8. (1) We start by considering the following two automata:



Using the Kleene theorem for weighted automata from [Silva et al. 2011], one can compute expressions $FV$-equivalent (and thus also $\bar{F}$-equivalent) to the leftmost states of the automata above, which we denote by $E_1$ and $E_2$, respectively.

$E_1 = a.(2 \bullet E) \qquad E = \mu x.b.(1 \bullet c.(6 \bullet x)) \oplus d.(2 \bullet \underline{2}) \oplus \underline{1}$

$E_2 = a.(2 \bullet E') \qquad E' = \mu y.b.(2 \bullet c.(3 \bullet (b.(6 \bullet c.(1 \bullet y)) \oplus d.(1 \bullet \underline{4}) \oplus \underline{1}))) \oplus d.(4 \bullet \underline{1}) \oplus \underline{1}$

The two expressions are not bisimilar, but they denote the same weighted language, therefore our goal is to show $E_1 \equiv_D E_2$. Using the congruence rule, we only need to show $E \equiv_D E'$. We will show that $E \equiv_D b.(2 \bullet c.(3 \bullet (b.(6 \bullet c.(1 \bullet E)) \oplus d.(1 \bullet \underline{4}) \oplus \underline{1}))) \oplus d.(4 \bullet \underline{1}) \oplus \underline{1}$, which then, using the uniqueness of fixpoints, yields the desired result.
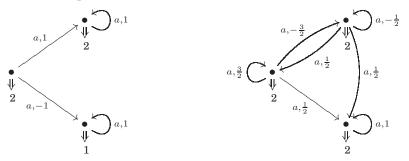
Using the fixpoint rule twice we obtain:

$$E \overset{\text{(FP)}}{\equiv}_D b.(1 \bullet c.(6 \bullet E)) \oplus d.(2 \bullet \underline{2}) \oplus \underline{1}$$

$$\overset{\text{(FP)}}{\equiv}_D b.(1 \bullet c.(6 \bullet (b.(1 \bullet c.(6 \bullet E)) \oplus d.(2 \bullet \underline{2}) \oplus \underline{1}))) \oplus d.(2 \bullet \underline{2}) \oplus \underline{1}$$

$$\overset{\text{(5.5)}}{\equiv}_D b.(2 \bullet c.(3 \bullet (b.(6 \bullet c.(1 \bullet E)) \oplus d.(2 \bullet \underline{2}) \oplus \underline{1}))) \oplus d.(2 \bullet \underline{2}) \oplus \underline{1}$$

$$\overset{\text{(5.6)}}{\equiv}_D b.(2 \bullet c.(3 \bullet (b.(6 \bullet c.(1 \bullet E)) \oplus d.(1 \bullet \underline{4}) \oplus \underline{1}))) \oplus d.(4 \bullet \underline{1}) \oplus \underline{1}$$

(2) For another example, consider the automata



As before, one can compute expressions equivalent to the leftmost states of the automata above, which we denote by $E_1$ and $E_2$, respectively.

$$E_1 = a.(1 \bullet E'') \oplus a.(-1 \bullet E) \oplus \underline{2} \qquad\qquad E = \mu y.a.(1 \bullet y) \oplus \underline{1}$$

$$E_2 = \mu x.a.(\tfrac{3}{2} \bullet x) \oplus a.(-\tfrac{3}{2} \bullet E') \oplus a.(\tfrac{1}{2} \bullet E'') \oplus \underline{2} \qquad E'' = \mu y.a.(1 \bullet y) \oplus \underline{2}$$

$$E' = \mu z.a.(-\tfrac{1}{2} \bullet z) \oplus a.(\tfrac{1}{2} \bullet x) \oplus a.(\tfrac{1}{2} \bullet E'') \oplus \underline{2}$$

The goal is now to show that $E_1 \equiv_D E_2$. Using the fixpoint axiom we first observe that

$$E_2 \overset{\text{(FP)}}{\equiv}_D a.(\tfrac{3}{2} \bullet E_2) \oplus a.(-\tfrac{3}{2} \bullet E'[E_2/x]) \oplus a.(\tfrac{1}{2} \bullet E'') \oplus \underline{2}$$

$$\overset{\text{(5.4)}}{\equiv}_D a.(1 \bullet (\tfrac{3}{2}E_2 \oplus -\tfrac{3}{2}E'[E_2/x] \oplus \tfrac{1}{2}E'')) \oplus \underline{2} \qquad\qquad \text{using (5.8)},$$

and

$$E_1 \overset{\text{(5.4)}}{\equiv}_D a.(1 \bullet (E'' \oplus -E)) \oplus \underline{2} \qquad\qquad\qquad \text{using (5.8)}.$$

Using the replacement rule, it suffices to prove that $\tfrac{3}{2}E_2 \oplus -\tfrac{3}{2}E'[E_2/x] \oplus \tfrac{1}{2}E'' \equiv E'' \oplus -E$. Using the fixpoint axiom we obtain:

$$\tfrac{3}{2}E_2 \oplus -\tfrac{3}{2}E'[E_2/x] \oplus \tfrac{1}{2}E''$$

$$\overset{\text{(FP)}}{\equiv}_D a.(\tfrac{9}{4} \bullet E_2) \oplus a.(-\tfrac{9}{4} \bullet E'[E_2/x]) \oplus a.(\tfrac{3}{4} \bullet E'') \oplus \underline{3}$$

$$\oplus a.(\tfrac{3}{4} \bullet E'[E_2/x]) \oplus a.(-\tfrac{3}{4} \bullet E_2) \oplus a.(-\tfrac{3}{4} \bullet E'') \oplus \underline{-3}$$

$$\oplus a.(\tfrac{1}{2} \bullet E'') \oplus \underline{1}$$

$$\overset{\text{(5.4)}}{\equiv}_D a.(1 \bullet (\tfrac{3}{2}E_2 \oplus -\tfrac{3}{2}E'[E_2/x] \oplus \tfrac{1}{2}E'')) \oplus \underline{1} \qquad\qquad \text{using (5.8)}$$

By the unique fixpoint rule we can now conclude that

$$\frac{3}{2}E_2 \oplus -\frac{3}{2}E'[E_2/x] \oplus \frac{1}{2}E'' \equiv_D \mu x.a(1 \bullet x) \oplus \underline{1}.$$

Now, we use a unique fixpoint argument to verify that $E'' \oplus -E$ is equivalent to $\mu x.a(1 \bullet x) \oplus \underline{1}$, too:

$$E'' \oplus -E$$
$$\overset{(\mathrm{FP})}{\equiv_D} \ a.(1 \bullet E'') \oplus \underline{2} \oplus \ a.(-1 \bullet E') \oplus \underline{-1}$$

$$\overset{(5.4)}{\equiv_D} \ a.(1 \bullet (E'' \oplus -E')) \oplus \underline{1} \qquad\qquad \text{using (5.8) and } \underline{r} \oplus \underline{s} = \underline{r+s}$$

So by the unique fixpoint rule we obtain $E'' \oplus -E \equiv_D \mu x.a(1 \bullet x) \oplus \underline{1}$.

### 5.2. Soundness of the calculus

Next we show that our calculus is sound for reasoning about weighted language equivalence.

In order to achieve our goal we will show that $\mathsf{E}_D$ is a coalgebra for the lifting $\bar{F} \colon \mathbb{S}\text{-Mod} \to \mathbb{S}\text{-Mod}$, and it is a quotient coalgebra of

$$\mathsf{E} = \varrho(FV) \xrightarrow{\ \mathfrak{r}\ } FV(\varrho(FV)) \xrightarrow{\ F\beta\ } F(\varrho(FV)) = F(\mathsf{E}).$$

Then we apply the general soundness theorem from Section 4.

LEMMA 5.9. *The map $Fq \cdot F\beta \cdot \mathfrak{r} \colon \mathsf{E} \to F(\mathsf{E}_D)$ is well-defined on the equivalence classes of $\equiv_D$.*

PROOF. It is sufficient to show that $F\beta \cdot \mathfrak{r}$ merges both sides of the three equations (5.4)–(5.6). We shall use the following notation for (certain) elements of $M^A$, where $M$ is some semimodule: for $s \in M$ we write $a \mapsto s$ for the function $f \colon A \to M$ with $f(a) = s$ and $f(b) = 0$ for $b \neq a$. We also make use of the fact that $\beta \colon V(\mathsf{E}) \to \mathsf{E}$ is a $V$-algebra structure in the following form: for $E_1 + E_2$ and $rE$ a formal sum and formal scalar product, respectively, in $V(\mathsf{E})$ we have $\beta(E_1 + E_2) = E_1 + E_2$ and $\beta(rE) = rE$, where the operations on the right-hand side are the semimodule operations of $\mathsf{E}$.

Ad (5.4) we compute

$$
\begin{aligned}
F\beta \cdot \mathfrak{r}(a.(r \bullet E_1) \oplus a.(r \bullet E_2)) &= F\beta(0, (a \mapsto rE_1) + (a \mapsto rE_2)) && \text{see (5.3)} \\
&= F\beta(0, a \mapsto (rE_1 + rE_2)) && \text{semimodule struc-} \\
& && \text{ture on } V(\mathsf{E})^A \\
&= (0, a \mapsto (rE_1 \oplus rE_2)) && \beta \text{ is a } V\text{-algebra} \\
&= (0, a \mapsto r(E_1 \oplus E_2)) && \mathsf{E} \text{ is a semimodule} \\
&= F\beta(0, a \mapsto r(E_1 \oplus E_2)) && \beta \text{ is a } V\text{-algebra} \\
&= F\beta \cdot \mathfrak{r}(a.(r \bullet (E_1 \oplus E_2))) && \text{see (5.3).}
\end{aligned}
$$

Ad (5.5) we compute

$$
\begin{aligned}
F\beta \cdot \mathfrak{r}(a.(r \bullet b.(s \bullet E))) &= F\beta(0, a \mapsto r(b.(s \bullet E))) && \text{see (5.3)} \\
&= (0, a \mapsto r(b.(s \bullet E))) && \beta \text{ is a } V\text{-algebra} \\
&= (0, a \mapsto b.((rs) \bullet E)) && \text{see (5.2)} \\
&= (0, a \mapsto (rs)(b.(1 \bullet E)) && \text{see (5.2)} \\
&= F\beta(0, a \mapsto (rs)(b.(1 \bullet E))) && \beta \text{ is a } V\text{-algebra} \\
&= F\beta \cdot \mathfrak{r}(a.((rs) \bullet b.(1 \bullet E))) && \text{see (5.3).}
\end{aligned}
$$

Ad (5.6) we compute

$$
\begin{aligned}
F\beta \cdot \mathfrak{r}(a.(r \bullet \underline{s})) \;&=\; F\beta(0, a \mapsto r\underline{s}) && \text{see (5.3)} \\
&=\; (0, a \mapsto r\underline{s}) && \beta \text{ is a } V\text{-algebra} \\
&=\; (0, a \mapsto \underline{rs}) && \text{see (5.2)} \\
&=\; (0, a \mapsto 1\underline{rs}) && \mathsf{E} \text{ is a semimodule} \\
&=\; F\beta(0, a \mapsto 1\underline{rs}) && \beta \text{ is a } V\text{-algebra} \\
&=\; F\beta \cdot \mathfrak{r}(a.(1 \bullet \underline{rs})) && \text{see (5.3).}
\end{aligned}
$$

This completes the proof. $\square$

COROLLARY 5.10. *There is a coalgebra structure* $c\colon \mathsf{E}_D \to F(\mathsf{E}_D)$ *such that* $q$ *is a* $\bar{F}$*-coalgebra homomorphism from the coalgebra* $(\mathsf{E}, F\beta \cdot \mathfrak{r})$ *to* $(\mathsf{E}_D, c)$.

PROOF. Define $c([E]) = Fq \cdot F\beta \cdot \mathfrak{r}(E)$. Then $c$ is well-defined by Lemma 5.9, a semimodule homomorphism since $q$, $\beta$ and $\mathfrak{r}$ are so, and $c \cdot q = Fq \cdot (F\beta \cdot \mathfrak{r})$ clearly holds. $\square$

THEOREM 5.11 (SOUNDNESS). *The calculus is sound: whenever we have* $E_1 \equiv_D E_2$ *for two expressions, then also* $[\![E_1]\!] = [\![E_2]\!]$.

This is just an application of Theorem 4.4 to the quotient coalgebra $q\colon (\mathsf{E}, F\beta \cdot \mathfrak{r}) \to (\mathsf{E}_D, c)$ for $\bar{F}$.

## 5.3. Completeness

We are ready to prove the completeness of our calculus w. r. t. weighted language equivalence of expressions. The key ingredient for our completeness result is the fact that $\mathsf{E}_D$ is the final locally finitely presentable coalgebra for $\bar{F}\colon \mathbb{S}\text{-Mod} \to \mathbb{S}\text{-Mod}$.

LEMMA 5.12. *The map* $c\colon \mathsf{E}_D \to F(\mathsf{E}_D)$ *is a semimodule isomorphism.*

PROOF. We first define the map $d\colon F(\mathsf{E}_D) \to \mathsf{E}_D$ by

$$
d(r, \langle [E^a] \rangle_{a \in A}) = [\underline{r} \oplus \bigoplus_{a \in A} a.(1 \bullet E^a)].
$$

By the replacement rule (5.1), $d$ is well-defined. We first prove that $d$ preserves sums:

$$
\begin{aligned}
d(\langle r, \langle [E_1^a] \rangle_{a \in A} \rangle &+ \langle s, \langle [E_2^a] \rangle_{a \in A} \rangle) \\
&= d(r + s, \langle [E_1^a \oplus E_2^a] \rangle_{a \in A}) && \text{addition in } \bar{F}(\mathsf{E}) \\
&= (r \oplus s) \oplus \bigoplus_{a \in A} a.(1 \bullet (E_1^a \oplus E_2^a)) && \text{definition of } d \\
&= \left( r \oplus \bigoplus_{a \in A} a.(1 \bullet E_1^a) \right) \oplus \left( s \oplus \bigoplus_{a \in A} a.(1 \bullet E_2^a) \right) && \text{by (5.4)} \\
&= d(r, \langle [E_1^a] \rangle_{a \in A}) \oplus d(s, \langle [E_2^a] \rangle_{a \in A}) && \text{definition of } d.
\end{aligned}
$$

We now prove that $c$ and $d$ are mutually inverse. It then follows that $d$ is a semimodule homomorphism since the forgetful functor $\mathbb{S}\text{-Mod} \to \mathsf{Set}$ creates isomorphisms. To see that $c \cdot d = \mathrm{id}$ we compute:

$$
\begin{aligned}
c \cdot d(r, \langle [E^a] \rangle_{a \in A}) \;&=\; c \left( [\underline{r} \oplus \bigoplus_{a \in A} 1 \bullet E^a] \right) && \text{definition of } d \\
&=\; c([\underline{r}]) + c \left( [\bigoplus_{a \in A} 1 \bullet E^a] \right) && c \text{ semimodule homomorphism} \\
&=\; (r, \lambda b.[\mathbf{0}]) + \sum_{a \in A} (0, a \mapsto [E^a]) && \text{see 5.9, 5.10 and (5.3)} \\
&=\; (r, \langle [E^a] \rangle_{a \in A}) && \text{semimodule structure} \\
& && \text{on } \mathbb{S} \times (\mathsf{E}_D)^A.
\end{aligned}
$$

Finally, we verify that $d \cdot c = \mathrm{id}$, and we show this by induction on the complexity $N(E)$ of expressions $E$:

For $E = \mathbf{0}$ we have

$$d \cdot c([\mathbf{0}]) = d(0, \langle[\mathbf{0}]\rangle_{a \in A}) = \left[\underline{0} \oplus \bigoplus_{a \in A} a.(1 \bullet 0)\right] = [\mathbf{0} \oplus \mathbf{0}] = [\mathbf{0}],$$

by the definitions of $c$ and $d$ and using (5.7).

For $E = \underline{r}$ we obtain

$$d \cdot c([\underline{r}]) = d(r, \langle[\mathbf{0}]\rangle_{a \in A}) = \left[\underline{r} \oplus \bigoplus_{a \in A} a.(1 \bullet \mathbf{0})\right] = [\underline{r}],$$

where the last step uses the semimodule structure on $\mathsf{E}_D$ and (5.7).

Next, for $E = E_1 \oplus E_2$ we simply use that $c$ and $d$ preserve sums and the induction hypothesis to obtain

$$d \cdot c([E_1 \oplus E_2]) = d(c(E_1) + c(E_2)) = d(c(E_1)) \oplus d(c(E_2)) = [E_1] \oplus [E_2] = [E_1 \oplus E_2].$$

For $E = a.(r \bullet E')$ we compute

$$
\begin{aligned}
d \cdot c([a.(r \bullet E')]) &= d(0, a \mapsto r[E']) && \text{see (5.3)} \\
&= [\underline{0} \oplus a.(1 \bullet (rE'))] && \text{definition of } d \text{ and semi-} \\
&&& \text{module structure of } \mathsf{E}_D \\
&= [a.(1 \bullet (rE'))] \\
&= [r(a.(1 \bullet E'))] && \text{by Lemma 5.6} \\
&= [a.(r \bullet E')] && \text{see (5.2).}
\end{aligned}
$$

Finally, for a $\mu$-expression $E = \mu x.E'$ we simply use the fixpoint axiom and the induction hypothesis to obtain

$$d \cdot c([\mu x.E']) = d \cdot c([E'[\mu x.E'/x]]) = [E'[\mu x.E'/x]] = [\mu x.E'].$$

This completes the proof.  □

*Notation* 5.13. For expressions $E_1$ and $E_2$ we denote by

$$E_1\{E_2/x\}$$

the *syntactic replacement* of $x$ by $E_2$ in $E_1$, i.e., one substitutes $E_2$ without first renaming its free variables that are bound in $E_1$.

For example, for $E_1 = \mu x.(a.(3 \bullet x))$ and $E_2 = b.(2 \bullet x)$ we have

$$E_1\{E_2/x\} = \mu x.(a.(3 \bullet b.(2 \bullet x))).$$

THEOREM 5.14. *For every $\bar{F}$-coalgebra $(VS, g)$ with $S$ a finite set there exists a unique coalgebra homomorphism from $(VS, g)$ to $(\mathsf{E}_D, c)$.*

PROOF. Since the coalgebra $(\mathsf{E}_D, c)$ is a quotient of the coalgebra $(\mathsf{E}, F\beta \cdot \mathfrak{r})$ we obtain the existence of a homomorphism from Corollary 3.40. It remains to verify its uniqueness.

So let $m: (VS, g) \to (\mathsf{E}_D, c)$ by any $\bar{F}$-coalgebra homomorphism. Let us assume that $S = \{s_1, \ldots, s_n\}$. It suffices to prove that the $m(s_i)$ are uniquely determined.

In order to prove this we will first define closed expressions $\langle\!\langle s_i \rangle\!\rangle$ and then show that these are provably equivalent to $m(s_i)$.

The expressions $\langle\!\langle s_i \rangle\!\rangle$ are defined by an $n$-step process. Recall Remark 5.2(2), and let $g(s_i) \in \mathbb{S} \times (VS)^A$ be

$$g(s_i) = \left( r_i, \left\langle \sum_{j=1}^{n} r_{ij}^a s_j \right\rangle_{a \in A} \right), \qquad i = 1, \dots, n. \tag{5.9}$$

Our expressions will involve the scalars $r_i$, the coefficients $r_{ij}^a$ and $n$ variables $x_1, \dots, x_n$. For every $i = 1, \dots, n$ let

$$E_i^0 = \mu x_i. \left( \underline{r_i} \oplus \bigoplus_{a \in A} (a.(r_{i1}^a \bullet x_1) \oplus \cdots \oplus a.(r_{in}^a \bullet x_n)) \right).$$

Now define for $k = 0, \dots, n-1$

$$E_i^{k+1} = \begin{cases} E_i^k \{ E_{k+1}^k / x_{k+1} \} & \text{if } i \neq k+1 \\ E_i^k & \text{if } i = k+1. \end{cases}$$

It is easy to see that the set of free variables of $E_i^k$ is $\{ x_{k+1}, \dots x_n \} \setminus \{ x_i \}$, and moreover, every occurrence of those variables is free.

We also see that for every $i$,

$$\begin{aligned} E_i^n &= E_i^0 \{ E_1^0 / x_1 \} \{ E_2^1 / x_2 \} \cdots \{ E_{i-1}^{i-2} / x_{i-1} \} \{ E_{i+1}^i / x_{i+1} \} \cdots \{ E_n^{n-1} / x_n \} \\ &= E_i^{i-1} \{ E_{i+1}^i / x_{i+1} \} \cdots \{ E_n^{n-1} / x_n \}. \end{aligned}$$

Observe that $E_i^n$ is a closed term. Moreover, the variable $x_i$ from $E_i^0$ is never syntactically replaced and it is bound by the outermost $\mu x_i$. All other occurrences of $x_i$ in $E_i^n$ are not bound by this $\mu$-operator (but by $\mu$-operators further inside the term). We define

$$\langle\!\langle s_i \rangle\!\rangle = E_i^n.$$

From now on we will denote equivalence classes $[E]$ of expressions in $\mathsf{E}_D$ simply by expressions $E$ representing them.

It is our goal to prove that $m(s_i) \equiv_D \langle\!\langle s_i \rangle\!\rangle$. Let us write $m_i$ for (some representative of) $m(s_i)$, for short. We use the fact that $m$ is a coalgebra homomorphism, Lemma 5.12 and equation (5.9) to obtain

$$\begin{aligned} m_i &= c^{-1} \cdot Fm \cdot g(s_i) \\ &= c^{-1} \cdot Fm \left( r_i, \left\langle \sum_{j=1}^{n} r_{ij}^a s_j \right\rangle_{a \in A} \right) \\ &= c^{-1} \left( r_i, \left\langle \sum_{j=1}^{n} r_{ij}^a m_j \right\rangle_{a \in A} \right) \\ &= \underline{r_i} \oplus \bigoplus_{a \in A} a. \left( 1 \bullet \sum_{j=1}^{n} r_{ij}^a m_j \right). \end{aligned} \tag{5.10}$$

For the proof of $m_i \equiv_D \langle\!\langle s_i \rangle\!\rangle$, we show the case $n = 3$ in detail; the general case is completely analogous and is left to the reader.

We start by proving that $m_1 \equiv_D E_1^0[m_2/x_2][m_3/x_3]$ by an application of the uniqueness rule: from (5.10) we get

$$\begin{aligned} m_1 &\equiv_D \underline{r_1} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{11}^a m_1 + r_{12}^a m_2 + r_{13}^a m_3)) \\ &= (\underline{r_1} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{11}^a x_1 + r_{12}^a x_2 + r_{13}^a x_3))[m_2/x_2][m_3/x_3])[m_1/x_1]. \end{aligned}$$

Next, we prove that $m_2 \equiv_D E_2^1[m_3/x_3]$. Notice that

$$E_1^0[m_2/x_2][m_3/x_3] = E_1^0[m_3/x_3][m_2/x_2]$$

since $m_2$ and $m_3$ are closed. Then, applying (5.10), we have

$$
\begin{aligned}
m_2 \ \equiv_D \ & \underline{r_2} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{21}^a m_1 + r_{22}^a m_2 + r_{23}^a m_3)) \\
\equiv_D \ & \underline{r_2} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{21}^a E_1^0[m_2/x_2][m_3/x_3] + r_{22}^a m_2 + r_{23}^a m_3)) \\
= \ & \left( \underline{r_2} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{21}^a E_1^0[m_3/x_3] + r_{22}^a x_2 + r_{23}^a m_3)) \right) [m_2/x_2],
\end{aligned}
$$

and so we can apply the uniqueness rule to obtain the desired equation.

Now we are able to prove that $m_1 \equiv_D E_1^0\{E_2^1/x_2\}[m_3/x_3]$. Notice first that we have $E_1^0\{E_2^1/x_2\} = E_1^0[E_2^1/x_2]$ since $x_1$ (which is bound in $E_1^0$) is not free in $E_2^1$. Now we obtain

$$
\begin{aligned}
E_1^0[E_2^1/x_2][m_3/x_3] \ & \equiv_D \ E_1^0[m_3/x_3][E_2^1[m_3/x_3]/x_2] \\
& \equiv_D \ E_1^0[m_3/x_3][m_2/x_2] \\
& \equiv_D \ m_1.
\end{aligned}
$$

Finally, we show that $m_3 \equiv_D E_3^2$ by another application of the uniqueness rule: we have

$$
\begin{aligned}
m_3 \ \equiv_D \ & \underline{r_3} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{31}^a m_1 + r_{32}^a m_2 + r_{33}^a m_3)) \\
\equiv_D \ & \underline{r_3} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{31}^a E_1^0\{E_2^1/x_2\}[m_3/x_3] + r_{32}^a E_2^1[m_3/x_3] + r_{33}^a m_3)) \\
= \ & \left( \underline{r_3} \oplus \bigoplus_{a \in A} a.(1 \bullet (r_{31}^a E_1^0\{E_2^1/x_2\} + r_{32}^a E_2^1 + r_{33}^a x_3)) \right) [m_3/x_3].
\end{aligned}
$$

So we have proved $m_3 \equiv_D E_3^2 = E_3^3 = \langle\!\langle s_3 \rangle\!\rangle$. This implies that

$$m_2 \equiv_D E_2^1[m_3/x_3] \equiv_D E_2^1[E_3^2/x_3] = E_2^1\{E_3^2/x_3\} = E_2^3 = \langle\!\langle s_2 \rangle\!\rangle,$$

where the third step holds since the bound variables $x_1$ and $x_2$ of $E_2^1$ are also bound in $E_3^2$. Similarly, we have

$$m_1 \equiv_D E_1^0\{E_2^1/x_2\}[m_3/x_3] \equiv_D E_1^0\{E_2^1/x_2\}[E_3^2/x_3] = E_1^0\{E_2^1/x_2\}\{E_3^2/x_3\} = E_1^3 = \langle\!\langle s_1 \rangle\!\rangle.$$

This completes the proof. $\square$

COROLLARY 5.15. *The coalgebra* $(\mathsf{E}_D, c)$ *is (isomorphic to) the rational fixpoint* $\varrho \bar{F}$.

PROOF. We prove, equivalently, that $(\mathsf{E}_D, c)$ is the final locally finitely presentable coalgebra for $\bar{F}$. To see that $(\mathsf{E}_D, c)$ is a locally finitely presentable coalgebra we use that the coalgebra $(\mathsf{E}, F\beta \cdot \mathfrak{r})$ from Remark 5.2(4) is locally finitely presentable (see Lemma 3.38). Since $\mathsf{E}_D$ is a quotient coalgebra of $\mathsf{E}$ by Corollary 5.10, we see that $\mathsf{E}_D$ is locally finitely presentable, too (apply Lemma 3.16). The finality of $(\mathsf{E}_D, c)$ now follows from the previous theorem and Corollary 3.36. $\square$

THEOREM 5.16 (COMPLETENESS). *Whenever we have* $[\![E_1]\!] = [\![E_2]\!]$ *for two expressions, then they are provably equivalent, in symbols:* $E_1 \equiv_D E_2$.

This is just an application of Theorem 4.5 to $\mathsf{E}_D = \varrho \bar{F}$ with the map $q \cdot q_0 \colon \mathsf{Exp} \to \mathsf{E}_D$.

## 6. EXPRESSION CALCULUS FOR NON-DETERMINISTIC AUTOMATA

In this section we present some details of an interesting special case of the work in the previous section—the case of non-deterministic automata. The calculus becomes

somewhat simpler in this case but all results are just consequences of the more general results of Section 5.

Here $\mathbb{S}$ is the Boolean semiring, and so the category $\mathbb{S}$-Mod is the category Jsl, of join-semilattices and join-preserving maps, which is isomorphic to the category of Eilenberg-Moore algebras for the finite powerset monad $\mathcal{P}_f$.

Once again, we work with the functor $FX = 2 \times X^A$, where $A$ is a fixed finite input alphabet and $2$ the two element join-semilattice.

In [Silva et al. 010a], one considers the language Exp of closed and guarded expressions defined by the following grammar

$$
\begin{aligned}
E &::= x \mid \mathbf{0} \mid E \oplus E \mid \underline{1} \mid a.E \mid \mu x.E^g, \\
E^g &::= \mathbf{0} \mid E^g \oplus E^g \mid \underline{1} \mid a.E \mid \mu x.E^g.
\end{aligned}
$$

Notice that this is just a simplification of the syntax of the calculus from Section 5. Indeed, $a.E$ corresponds to $a.(1 \bullet E)$, and we do not need the expressions $\underline{0}$ and $a.(0 \bullet E)$ as they are provably equivalent to $\mathbf{0}$. These syntactic expressions describe precisely the behaviors of finite non-deterministic automata. Our set of axioms from Section 5 now states that (1) $\mu$ is a unique fixpoint operator, (2) $\oplus$ is an associative, commutative and idempotent[6] binary operation with the neutral element $\mathbf{0}$ and that (3) the $\alpha$-equivalence (i. e., renaming of bound variables does not matter) and the replacement rules are valid. In fact, those are exactly the axioms and rules considered in [Silva et al. 010a], where they were proven sound and complete with respect to bisimilarity.

To obtain a sound and complete axiomatisation for language equivalence we only need to add the following two axioms to the above axiomatisation:

$$a.(E_1 \oplus E_2) \equiv a.E_1 \oplus a.E_2 \qquad \text{and} \qquad a.\mathbf{0} = \mathbf{0}. \tag{6.1}$$

These new axioms correspond to (5.4) and (5.6), and the other axiom (5.5) of the previous section already trivially holds in the current special case. In this way we recover the result of [Rabinovich 1994] for labelled transition systems (which are just non-deterministic automata where every state is considered final). Also note that the result of [Silva et al. 010a] coincides precisely with the results in [Milner 1989] for labelled transition systems and bisimilarity, which constituted the base of Rabinovich's work.

From Section 5, we get: (1) a Kleene Theorem: every state of a non-deterministic automaton is language equivalent to an expression in the calculus and vice-versa; (2) soundness of the calculus from Theorem 5.11 and (3) completeness of the calculus from Theorem 5.16, which uses that the coalgebra $\mathsf{E}_D$ of expressions from Exp modulo *all* the axioms is final among all locally finite coalgebras for $\bar{F}$ (cf. Corollary 5.15).

## 7. CONCLUSIONS AND FUTURE WORK

In this paper, we have presented a general methodology to extend sound and complete calculi with respect to behavioral equivalence to sound and complete calculi with respect to coalgebraic language equivalence. To achieve this goal we have developed a mathematical theory of finitary coinduction for functors having a lifting to a category of algebras (satisfying certain finiteness conditions). We illustrated our general framework by applying it to two concrete instances, non-deterministic automata and weighted automata. For the former, we recovered the results of [Rabinovich 1994], whereas for the latter we presented a new sound and complete axiomatisation of weighted language equivalence for automata with weights over a Noetherian semiring.

---

[6]That $\oplus$ is idempotent can be derived using the semiring action: $E \oplus E = 1E \oplus 1E = (1 + 1)E = 1E = E$.

A key fact to be established in our soundness and completeness proofs is that expressions modulo proof rules form the final locally finitely presentable coalgebra. The development of the mathematical theory of these coalgebras was started in [Milius 2010], and we continue this in the current paper.

Even though we did not present the details, our method is generic. For non-deterministic systems it applies to all coalgebras for $F\mathcal{P}_f$ and for weighted systems we can deal with coalgebras of type $FV$, where $F$ is from an inductively defined class of functors. However, working out these details is non-trivial; for example, the generic calculus is syntactically more involved as it is parametric in the functor $F$. We therefore decided to treat this generic calculus in a subsequent paper.

In [Myers 2011] sound and complete expression calculi for a finitary endofunctor $F$ over an arbitrary variety $\mathcal{A}$ have been derived from a presentation of the functor $F$ directly in the variety $\mathcal{A}$. The semantics of the expressions is given by considering the image of the rational fixpoint of $F$ inside the final $F$-coalgebra. However, in order to prove completeness, a full understanding of the interplay between the operators of the functor presentation and those of the variety is assumed. This is different from our approach to extend sound and complete calculi with respect to $FT$-equivalence to sound and complete calculi with respect to $F$-equivalence in the category of $T$-algebras. As a consequence our approach is more concrete, and it requires to explicitly understand the relationship between the final $FT$-coalgebra and the final $F$-coalgebra and the corresponding rational fixpoints, respectively.

It is known that language equivalence for automata with weights in a field is decidable [Schützenberger 1961; Flouret and Laugerotte 1997]. This method can be easily generalised to weights in a skew field. Language equivalence is also decidable for weights in $\mathbb{N}$ [Harju and Karhumäki 1991], but it becomes undecidable if one uses weights in a tropical semiring [Krob 1994]. More recently, a decidability result for automata with weights in a proper and effectively presentable semiring has been obtained in [Ésik and Maletti 2010]. Using an argument similar to [Ésik and Maletti 2010] from our completeness result it follows that coalgebraic language equivalence for the calculi we have developed is decidable when weights are from an effectively presentable semiring[7]. On the one hand, equivalence is semi-decidable: if two expressions are equivalent one can simply enumerate all strings over some finite alphabet, and check which of these strings correspond to valid proof chains: because of the completeness, if such a proof chain exists, it will be found using this enumeration. On the other hand, non-equivalence is also semi-decidable: if two expressions are nonequivalent, there must be a word witnessing this. Thus we can simply enumerate all words and for each word compute the weight associated with the languages of the two expressions.

We presented the main results of the theory for the base category Set. In the future we plan to extend this to more general base categories in order to deal with systems whose state spaces have extra structure, e.g., they form posets, graphs or presheaves.

Unfortunately, our main result on final locally finitely presentable coalgebras (Theorem 3.41) uses the assumption that finitely generated objects are closed under kernel pairs. This assumption is somewhat restrictive, and we intend to study whether this can be relaxed. This would allow to consider other monads $T$, i.e., other branching types like, for instance, various kinds of probabilistic systems.

As we saw in our work, the generalised powerset construction lets us move from systems of type $FT$ to systems of type $F$ (in the category of $T$-algebras) and hence

---

[7]A semiring for which its elements have a finitary presentation, and such that sums and products can be effectively computed from this presentation (see [Ésik and Maletti 2010]).

from bisimilarity to language equivalence. On the other hand, in coalgebraic trace semantics [Hasuo et al. 2007] one considers functors of the form $TF$ and works with coalgebras for (the lifting of $F$) to the Kleisli category of $T$ (e. g. $TF = \mathcal{P}(1 + A \times -)$ for non-deterministic automata). This allows to deal with some monads $T$ that are not finitary, e. g. the full powerset monad or the subdistribution monad. However, that approach does not allow to consider the monad $V$ of free semimodules. It would therefore be desirable to find a common framework that accommodates both these approaches.

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