A

Uniqueness of Normal Forms for Shallow Term Rewrite Systems

NICHOLAS R. RADCLIFFE, LUIS F. T. MORAES, AND RAKESH M. VERMA, University of Houston

Uniqueness of normal forms ($\mathrm{UN}^=$) is an important property of term rewrite systems. $\mathrm{UN}^=$ is decidable for ground (i.e., variable-free) systems and undecidable in general. Recently it was shown to be decidable for linear, shallow systems. We generalize this previous result and show that this property is decidable for shallow rewrite systems, in contrast to confluence, reachability and other properties, which are all undecidable for flat systems. Our result is also optimal in some sense, since we prove that the $\mathrm{UN}^=$ property is undecidable for two classes of linear rewrite systems: left-flat systems in which right-hand sides are of depth at most two and right-flat systems in which left-hand sides are of depth at most two.

CCS Concepts: •**Theory of computation** → **Equational logic and rewriting;** *Computability;*

Additional Key Words and Phrases: term rewrite systems, uniqueness of normal forms, decidability/undecidability, shallow rewrite systems, flat rewrite systems

ACM Reference Format:
Nicholas Radcliffe, Luis F. T. Moraes, and Rakesh Verma, 20xx. Decidability of Unicity for Shallow Term Rewrite Systems. *ACM Trans. Comput. Logic* V, N, Article A (January YYYY), 17 pages.
DOI: http://dx.doi.org/10.1145/0000000.0000000

1. INTRODUCTION

Term rewrite systems (TRSs), finite sets of rules, are useful in many computer science fields including theorem proving, rule-based programming, and symbolic computation. An important property of TRSs is confluence (also known as the Church-Rosser property), which implies unicity or uniqueness of normal forms ($\mathrm{UN}^=$). Normal forms are expressions to which no rule is applicable. A TRS has the $\mathrm{UN}^=$ property if there are *not* distinct normal forms n, m such that $n \overset{*}{\leftrightarrow}_R m$, where $\overset{*}{\leftrightarrow}_R$ is the symmetric closure of the rewrite relation induced by the TRS R. There is a related property called $UN^{\rightarrow}$, which is defined as: no term should have more than one normal form, i.e., if m and n are two normal forms reachable from the same term ($\overset{*}{\leftarrow} \circ \overset{*}{\rightarrow}$), then R does not have the $UN^{\rightarrow}$ property. This property is known to be undecidable for flat systems and also flat and right-linear systems [Godoy and Jacquemard 2009].

Uniqueness of normal forms is an interesting property in itself and well-studied [Terese 2003]. Confluence can be too strong a requirement for some applications such as lazy programming. Additionally, in the proof-by-consistency approach for inductive theorem proving, consistency is often ensured by requiring the $\mathrm{UN}^=$ property.

We study the decidability of uniqueness of normal forms. Uniqueness of normal forms is decidable for ground systems [Verma et al. 2001], but is undecidable in general [Verma et al. 2001]. Since the property is undecidable in general, we would like to know for which classes of rewrite systems, beyond ground systems, we can decide $\mathrm{UN}^=$. In [Zinn and Verma 2006; Zinn 2006] a polynomial time algorithm for this property was given for linear, shallow rewrite systems. A rewrite system is *linear* if vari-

A preliminary version of this paper appeared in IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2010).
Author's addresses: Luis F. T. Moraes and Rakesh Verma, Computer Science Department, University of Houston.

DOI: http://dx.doi.org/10.1145/0000000.0000000

ables occur at most once in each side of any rule. It is *shallow* if variables occur only at depth zero or depth one in each side of any rule. It is *flat* if both the left- and right-hand sides of all the rules have height zero or one. An example of a linear flat (in fact, ground) system that has $\mathrm{UN}^=$ but not confluence is $\{f(c) \to 1,\ c \to g(c)\}$. More sophisticated examples can be constructed using a sequential 'or' function in which the second argument gives rise to a nonterminating computation.

In this paper, we consider the class of shallow systems, i.e., we drop the linearity restriction of [Zinn and Verma 2006], and a subset of this class, the flat systems. For flat systems many properties are known to be undecidable including confluence, reachability, joinability, and existence of normal forms [Mitsuhashi et al. 2006; Verma 2008; Godoy and Hernández 2009]. On the other hand, the word problem is known to be decidable for shallow systems [Comon et al. 1994]. This paper shows that the uniqueness of normal forms problem is decidable for the class of shallow term rewrite systems, which is a significant generalization of [Zinn and Verma 2006] and also somewhat surprising since so many properties are undecidable for this class of systems. We also prove the undecidability of $\mathrm{UN}^=$ for two subclasses of linear systems: left-hand sides are flat and right-hand sides are of depth at most two and conversely right-flat and depth two left-hand sides, which improves the undecidability result of [Verma 2008] for the linear, depth-two subclass and shows that our result is optimal as far as linearity and depth restrictions are involved.

We would like to clarify the relationship between $UN^{\to}$ (see [Terese 2003] for a definition) and $UN^=$. It is well known in rewriting that $UN^=$ implies $UN^{\to}$ but not the other way around. For a simple example, well-known since [Klop 1980; Klop 1992], consider $a \to b$, $a \to c$, $c \to c$, $d \to c$, and $d \to e$. This example has $UN^{\to}$ since c is not a normal form but does not have $UN^=$ since normal forms b and e satisfy $b =_R e$, so $UN^{\to}$ does not imply $UN^=$. However, just because property A implies property B it does not automatically follow that if A is decidable for a class of inputs, then B is also decidable for the same class of inputs. For this we need the concept of a reduction and in fact the second author has shown [Verma 2009] that for variable-preserving rewrite systems $UN^=$ reduces to $UN^{\to}$.

Comparison with related work. Viewed at a very high level, the proof of decidability shows some flavor in common with that of some other decidability proofs of properties of rewrite systems such as [Godoy et al. 2003]. The basic insight is that, just as in algebra the terms that reduce to 0 are crucial in a sense, so in rewriting are the terms that reduce to (or are equivalent to) constants. We see a parallel between constants, which are height 0 terms in rewriting with the expression 0 in algebra. Of course, this observation is about as helpful in proofs of decidability as a compass is to someone lost in a maze. The details in both scenarios are vital and there are many twists and turns. The proof of undecidability shows some similarity with proofs in [Verma et al. 2001; Godoy and Tison 2007].

The structure of our decidability proof is as follows: in [Zinn and Verma 2006; Zinn 2006] it was shown that $\mathrm{UN}^=$ for shallow systems can be reduced to $\mathrm{UN}^=$ for flat systems, (ii) checking $\mathrm{UN}^=$ for flat systems can be reduced to searching for equational proofs between terms drawn from a finite set of terms, and (iii) existence of equational proofs between terms in part (ii) is done thanks to the decidability of the word problem by Comon et al. [Comon et al. 1994].

Our strategy for part (ii) above, assuming a flat TRS, R, is to show that a sufficiently small witness to non-$UN^=$ for R exists if, and only if, any witness at all exists. To see this, say $\langle M, N\rangle$ is a minimal witness to non-$UN^=$ (in that the sum of the sizes of M and N is minimal). We show that we can replace certain subterms of M and N that are not equivalent to constants with variables, obtaining a witness $\langle M', N'\rangle$. If the heights of M' and N' are both strictly less than $\max(1, C)$, where C is the number of constants in our rewrite system, then $\langle M', N' \rangle$ is sufficiently small. Otherwise, M' or N' must have a big subterm (i.e. a subterm whose height is greater than, or equal to, the number of constants), and this subterm is equivalent to a constant. However, in this case (when there is a constant that is equivalent to a big subterm of a component of a minimal witness), we can show that there is a small witness to non-$UN^=$. So, in all cases, we end up with a small witness.

This paper improves our previous work in [Radcliffe and Verma 2010] by strengthening the undecidability proof. In particular, the previous proofs work only for either left-nonlinear or right-nonlinear systems, whereas here the reductions give *linear* systems of the appropriate type.

1.1. Definitions

Terms. A *signature* is a set $\mathcal{F}$ along with a function *arity*: $\mathcal{F} \to \mathbb{N}$. Members of $\mathcal{F}$ are called *function symbols*, and $arity(f)$ is called the $arity$ of the function symbol f. Function symbols of arity zero are called $constants$. Let X be a countable set disjoint from $\mathcal{F}$ that we shall call the set of $variables$. The set $\mathcal{T}(\mathcal{F}, X)$ of $\mathcal{F}$*-terms over* X is defined to be the smallest set that contains X and has the property that $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, X)$ whenever $f \in \mathcal{F}$, $n = arity(f)$, and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, X)$. The set of function symbols with arity n is denoted by $\mathcal{F}_n$; in particular, the set of constants is denoted by $\mathcal{F}_0$. We use $root(t)$ to refer to the outermost function symbol of t.

The $size$, $|t|$, of a term t is the number of occurrences of constants, variables and function symbols in t. So, $|t| = 1$ if t is a constant or a variable, and $|t| = 1 + \Sigma_{i=1}^{n} |t_i|$ if $t = f(t_1, \ldots, t_n)$ for $n > 0$. The $height$ of a term t is 0 if t is a constant or a variable, and $1 + max\{height(t_1), \ldots, height(t_n)\}$ if $t = f(t_1, \ldots, t_n)$. If a term t has height zero or one, then it is called $flat$. A $position$ of a term t is a sequence of natural numbers that is used to identify the locations of subterms of t. The subterm of $t = f(t_0, \ldots, t_{n-1})$ at position p, denoted $t|_p$, is defined recursively: $t|_\lambda = t$, $t|_k = t_k$, for $0 \leq k \leq n-1$, and $t|_{k.p} = (t|_k)|_p$. If $t = f(t_0, \ldots, t_{n-1})$, then we call $t_0, \ldots, t_{n-1}$ the *depth*-1 subterms of t. If all variables appearing in t are either t itself or depth-1 subterms of t, then we say that t is *shallow*. The notation $g[a]$ focuses on (any) one occurrence of subterm a of term g, and $s\{u \mapsto v\}$ denotes the term obtained from term s by replacing all occurrences of the subterm u in s by term v.

A $substitution$ is a mapping $\sigma : X \to \mathcal{T}(\mathcal{F}, X)$ that is the identity on all but finitely many elements of X. Substitutions are generally extended to a homomorphism on $\mathcal{T}(\mathcal{F}, X)$ in the following way: if $t = f(t_1, \ldots, t_k)$, then (abusing notation) $\sigma(t) = f(\sigma(t_1), \ldots, \sigma(t_k))$. Oftentimes, the application of a substitution to a term is written in postfix notation. A $unifier$ of two terms s and t is a substitution σ (if it exists) such that $s\sigma = t\sigma$. We assume familiarity with the concept of *most general unifier* [Terese 2003], which is unique up to variable renaming and denoted by mgu.

Term Rewrite Systems. A *rewrite rule* is a pair of terms, (l, r), usually written $l \to r$. For the rule $l \to r$, the *left-hand side* is $l \notin X$, and the *right-hand side* is r. Notice that l cannot be a variable. A rule, $l \to r$, can be applied to a term, t, if there exists a substitution, σ, such that $l\sigma = t'$, where t' is a subterm of t; in this case, t is rewritten by replacing the subterm $t' = l\sigma$ with $r\sigma$. The process of replacing the subterm $l\sigma$ with $r\sigma$ is called a *rewrite*. A *root rewrite* is a rewrite where $t' = t$. A rule $l \to r$ is *flat* (resp. shallow) if both l and r are flat (resp. shallow). The rule $l \to r$ is *collapsing* if r is a variable. A *term rewrite system* (or *TRS*) is a pair, $(\mathcal{T}, R)$, where R is a finite set of rules and $\mathcal{T}$ is the set of terms over some signature. A TRS, R, is *flat* (resp. shallow) if all of the rules in R are flat (resp. shallow). If we think of $\to$ as a relation, then $\xrightarrow{+}$ and $\xrightarrow{*}$ denote its transitive closure, and reflexive and transitive closure, respectively. Also, $\leftrightarrow$, $\xleftrightarrow{+}$, and $\xleftrightarrow{*}$ denote the symmetric closure, symmetric and transitive closure, and symmetric, transitive, and reflexive closure, respectively. We put an 'r' over arrows to denote a root rewrite, i.e., $\stackrel{r}{\leftrightarrow}$.

A *derivation* is a sequence of terms, $t_1, \ldots, t_n$, such that $t_i \to t_{i+1}$ for $i = 1, \ldots, n-1$; this sequence is often denoted by $t_1 \to t_2 \to \ldots \to t_n$. A *proof* is a sequence, $t_1, \ldots, t_n$, such that $t_i \leftrightarrow t_{i+1}$ for $i = 1, \ldots, n-1$; this sequence is generally denoted by $t_1 \leftrightarrow t_2 \leftrightarrow \ldots \leftrightarrow t_n$. If R is a rewrite system, then a proof is *over* R if it can be constructed using rules in R. If π is a proof, we say that $\pi \in s\stackrel{*}{\leftrightarrow}t$ if π is of the form $s \leftrightarrow \ldots \leftrightarrow t$ (it is possible for the proof sequence to consist of a single term, in which case $s = t$), s). We say that $\pi \in s\stackrel{+}{\leftrightarrow}t$ if $\pi \in s\stackrel{*}{\leftrightarrow}t$ and the proof sequence contains at least one step. We write $s\stackrel{*}{\leftrightarrow}t$ (resp. $s\stackrel{+}{\leftrightarrow}t$) to denote that there is a proof, π, with $\pi \in s\stackrel{*}{\leftrightarrow}t$ (resp. $\pi \in s\stackrel{+}{\leftrightarrow}t$).

A *normal form* is a term, $t \in \mathcal{T}(\mathcal{F}, X)$, such that no subterm of t can be rewritten. A term that is not a normal form, i.e., one with a subterm that *can* be rewritten, is called *reducible*. We denote the set of all normal forms for R by NF_R, or simply NF. A rewrite system R is $UN^=$ if it is *not* the case that R has two distinct normal forms, M and N, such that $M\stackrel{*}{\leftrightarrow}N$. If such a pair exists, then we say that the pair, $\langle M, N\rangle$, is a *witness* to non-$UN^=$. The *size of a witness*, denoted $|\langle M, N\rangle|$, is $|M| + |N|$. A *minimal witness* is a witness with minimal size. Finally, we define $SubMinWit_R$ to be set of all terms M' such that $\langle M, N\rangle$ is a minimal witness, and M' is a subterm of M.

2. PRELIMINARY RESULTS

We begin with a few simple results on when rules apply. They are used throughout the paper to show that normal forms are preserved under certain transformations. Before we begin, notice that it is relatively simpler to preserve normal forms when the relevant TRS is linear. For instance, imagine any *flat and linear* TRS such that $f(g(a), h(b))$ is a normal form. Since $g(a)$ is evidently a normal form, $f(g(a), g(a))$ would also be a normal form, when the TRS is linear. If the TRS is not linear, then there could be a rule of the form $f(x, x) \to t$, making $f(g(a), g(a))$ reducible. The results below handle such complications presented by non-linear rules.

DEFINITION 2.1.
Let R be a rewrite system, and let $l \to r = \rho \in R$ be a rule. The pattern of *ρ, denoted $Patt(\rho)$, is a set of equations $\{i = j \mid l|_i = l|_j, l|_i, l|_j \in X\}$.*

DEFINITION 2.2. *Let $t \in \mathcal{T}(\mathcal{F}, X)$ be a term with $root(l) = root(t)$. If $A = \{i_1, i_2, \ldots, i_k\}$ is the set of positions that appear in equations in $Patt(\rho)$, then the* pattern of t with respect to *ρ, denoted $Patt_\rho(t)$, is the set $\{i_a = i_b \mid t|_{i_a} = t|_{i_b}, i_a, i_b \in A\}$.*

Note that $Patt_\rho(t)$ is undefined if $root(l) \neq root(t)$.

LEMMA 2.3.
Let R be a flat TRS. Let $t \in \mathcal{T}(\mathcal{F}, X)$ be a term, and let $l \to r = \rho \in R$ be a rule. Then ρ can be applied to t at λ if, and only if, (i) $l|_i = t|_i$ whenever $l|_i$ is a constant, and (ii) $Patt_\rho(t)$ is defined and $Patt(\rho) \subseteq Patt_\rho(t)$.

PROOF. Assume that (i) and (ii) are satisfied. Since (i) is satisfied and $Patt_\rho(t)$ is defined, all we have to show is that there exists a substitution, σ, such that $l|_i\sigma = t|_i$ whenever $l|_i$ is a variable. We would like to define $x\sigma = t|_i$ whenever $l|_i = x$, but if $l|_i = l|_j = x$, then $t|_i = l|_i\sigma = l|_j\sigma = t|_j$, and hence it needs to be the case that $i = j \in Patt_\rho(t)$. But if $i = j \in Patt(\rho)$ and (ii) is satisfied, then we know that $i = j \in Patt_\rho(t)$. So, we can consistently define σ as above. Clearly, $l\sigma = t$, and thus ρ can be applied to t at λ.

Now assume that there exists a substitution, σ, with $l\sigma = t$. Obviously, $Patt_\rho(t)$ is defined and $l|_i = t|_i$ whenever $l|_i$ is a constant, and so we need to show that $Patt(\rho) \subseteq$

$Patt_\rho(t)$. Say $i = j \in Patt(\rho)$. Then $l|_i = l|_j$, and hence $t|_i = l|_i\sigma = l|_j\sigma = t|_j$. Therefore, $i = j \in Patt_\rho(t)$, and $Patt(\rho) \subseteq Patt_\rho(t)$. □

Consider the term $f(a, x, x, g(b))$. Let's assume that it is a normal form. We want to know if altering depth-1 subterms can make the term reducible. Clearly, replacing x with a constant could *potentially* make the term reducible, depending on the rules in the rule set. But what about replacing any of the depth-1 subterms with a normal form containing a fresh variable? Notice that such a replacement could not make condition (i) of the above lemma true if it had been false. But what if condition (i) is true and condition (ii) is false? Could replacing a depth-1 subterm, or even several depth-1 subterms, with terms containing fresh variables make condition (ii) true? This question is answered by the following proposition.

PROPOSITION 2.4.
Let R be a flat TRS, and let $M = f(s_1, \ldots, s_m)$ be a normal form for R. Let $S = \{t_{i_1}, \ldots, t_{i_n}\}$ be a set of normal forms, where $n \leq m$ and each term contains at least one fresh variable (relative to M). Further, say that $t_{i_j} \neq t_{i_k}$ whenever $s_{i_j} \neq s_{i_k}$ for all $i_j, i_k \in \{i_1, \ldots, i_n\}$. If M' is what one obtains from M by replacing each s_{i_j} with t_{i_j}, then $M' \in NF_R$.

PROOF. We say that $M' = f(s'_1, \ldots, s'_m)$, where $s'_q = \begin{cases} t_q & \text{if } q \in \{i_1, \ldots, i_n\} \\ s_q & \text{otherwise} \end{cases}$. By Lemma 2.3 and the above observations, we simply need to demonstrate, for an arbitrary rule $\rho \in R$, that if $Patt(\rho) \not\subseteq Patt_\rho(M)$, then $Patt(\rho) \not\subseteq Patt_\rho(M')$ (i.e. if ρ cannot be applied to M, then it cannot be applied to M', making M' a normal form).

So, assume that $Patt(\rho) \not\subseteq Patt_\rho(M)$. We need to show that $s'_j \neq s'_k$ whenever $s_j \neq s_k$. We consider three cases: (i) $s'_j, s'_k \notin S$, (ii) $s'_j \in S$, $s'_k \notin S$, and (iii) $s'_j, s'_k \in S$. In case (i), $s_j = s'_j$ and $s_k = s'_k$, so clearly $s'_j \neq s'_k$ whenever $s_j \neq s_k$. In case (ii), s'_j contains a fresh variable, whereas $s'_k = s_k$ does not, so $s'_j \neq s'_k$. Hence, it is (vacuously) the case that $s'_j \neq s'_k$ whenever $s_j \neq s_k$. Since case (iii) is an hypothesis, we see that, in all cases, $s'_j \neq s'_k$ whenever $s_j \neq s_k$, and hence $Patt_\rho(M') \subseteq Patt_\rho(M)$. Therefore, $Patt(\rho) \not\subseteq Patt_\rho(M')$, and $M' \in NF_R$. □

LEMMA 2.5.
If R is any TRS such that $f(t_1, \ldots, t_m) \in SubMinWit_R$, then $t_i \overset{*}{\leftrightarrow}_R t_j$ is impossible for $t_i \neq t_j$. This is equivalent to saying that there is no term s that is equivalent to both t_i and t_j via R.

PROOF. Let $\langle M, N \rangle$ be a minimal witness to non-$UN^=$ for R, and say that $f(t_1, \ldots, t_m)$ is a subterm of N. Assume that the lemma is false, i.e., there is a term, s, such that $s \overset{*}{\leftrightarrow}_R t_i$ and $s \overset{*}{\leftrightarrow}_R t_j$ with $t_i \neq t_j$. Then we would have $t_i \overset{*}{\leftrightarrow} t_j$. Since $|t_i| + |t_j| < |f(t_1, \ldots, t_m)| < |M| + |N|$, we see that $\langle t_i, t_j \rangle$ violates the minimality of $\langle M, N \rangle$, and hence the lemma must be true. □

2.1. Normal Forms Equivalent to Constants

Let E be a finite set of equations. Following the authors of [Comon et al. 1994], we extend E to $\widehat{E}$ by closing under the following inference rules:

(1) $\dfrac{g = d,\ l = r}{d\sigma = r\sigma}$ if $l, g \notin X$ and $\sigma = mgu(l, g)$

(2) $\dfrac{x = d,\ y = r}{d = r\{y \mapsto x\}}$ if $y \in X$ and $x \in \mathcal{F}_0 \cup X$

(3) $\dfrac{g[a] = d,\ a = b}{g[b] = d}$ if $a, b \in \mathcal{F}_0$

Notice that if E is flat, then $\widehat{E}$ is flat, as well.

We can think of a rewrite system as a set of equations: if $s \to t$ is a rule in R, then $s \leftrightarrow t$ is its corresponding equation. We write E_R for the set of equations obtained in this way from a rewrite system R. Clearly, if s and t are terms in $\mathcal{T}(\mathcal{F}, X)$, then they are R-equivalent if and only if they are E_R equivalent. Also, from [Comon et al. 1994] we know that terms are E_R equivalent if, and only if, they are $\widehat{E_R}$-equivalent. In [Comon et al. 1994], the authors show that, if R is a shallow TRS and $s, t \in \mathcal{T}(\mathcal{F}, X)$, then there is a procedure that produces, for any proof, $\pi \in s \overset{*}{\leftrightarrow}_R t$, over R, a new proof, which is denoted by $\pi_{1rr} \in s \overset{*}{\leftrightarrow}_{\widehat{E_R}} t$, over $\widehat{E_R}$, such that there is at most one root rewrite step in π_{1rr}.

Consider the following example: $R = \{f(x,x) \to c, f(x,x) \to g(a,x), g(a,x) \to g(a,x), a \to h(b), b \to h(c)\}$. It is easy to check that $\widehat{E_R} = E_R \cup \{c \leftrightarrow g(a,x)\}$. We use $\widehat{E_R}$ to search for a minimal witness to non-$UN^=$ for R; in particular, we will use the fact that for every proof $s \overset{*}{\leftrightarrow}_R t$, there is a proof $s \overset{*}{\leftrightarrow}_{\widehat{E_R}} t$ with at most one root rewrite.

Clearly, c is an R-normal form, so if we are looking for a minimal witness to non-$UN^=$ for R, $\langle c, ?\rangle$ might be a good first guess. We know that $c \leftrightarrow_{\widehat{E_R}} f(x,x)$, so maybe $\langle c, f(u,v)\rangle$ is a minimal witness, for some normal forms u and v. This is not possible. First, notice that $f(x,x)$ appears on the LHS of a rule, so $f(t,t)$ cannot be a normal form, for arbitrary term t. Second, notice that if $f(t,t)$ is equivalent to another normal form, then we can assume it is of the form $f(u,v)$, because we have already "used up" our only root rewrite by using $c \leftrightarrow_{\widehat{E_R}} f(x,x)$. So, maybe we can plug some term, t, into x, and then rewrite one instance of it to a normal form u, and another instance of it to a normal form v, obtaining a minimal witness of the form $\langle c, f(u,v)\rangle$? This cannot be the case, because if $\langle c, f(u,v)\rangle$ is a minimal witness, then (by Lemma 2.5 and the fact that $u \overset{*}{\leftrightarrow} v$) $\langle u, v\rangle$ would violate the minimality of $\langle c, f(u,v)\rangle$. So, we should consider $c \leftrightarrow_{\widehat{E_R}} g(a,x)$ as *the* (one and only) rewrite step in our proof. We know that a is not a normal form, and must, therefore, be rewritten to one - $h(h(c))$. But what about x? Should we plug anything into it? Say we were to plug t into x, and then rewrite t to some normal form, u. This would be unnecessary, because non-linearity is not an issue here, and so we can leave x as it is. So, $\langle c, g(h(h(c)), x)\rangle$ is a minimal witness, and the relevant proof looks like: $c \leftrightarrow_{\widehat{E_R}} g(a,x) \leftrightarrow_{\widehat{E_R}} g(h(b),x) \leftrightarrow_{\widehat{E_R}} g(h(h(c)),x)$.

Now, here is the interesting part. Notice that we have *four* R-normal forms equivalent to constants, but only *three* constants in R, i.e, $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} c$, $h(c) \overset{*}{\leftrightarrow}_{\widehat{E_R}} b$, $h(h(c)) \overset{*}{\leftrightarrow}_{\widehat{E_R}} a$, and $g(h(h(c)),x) \overset{*}{\leftrightarrow}_{\widehat{E_R}} c$. From the Pigeonhole Principle, we can conclude that there must be some constant in R that is equivalent to two distinct normal forms (of course, we already knew this, but in general this technique will be useful). We generalize the lessons learned from this example in the following results.

LEMMA 2.6. *Let R be a flat TRS. Let $\langle M_0, M_1\rangle$ be a minimal witness to non-$UN^=$ for R, and say $M = f(t_1, \ldots, t_m)$ is a subterm of M_0. Let c be a constant, and let $c \overset{r}{\leftrightarrow}_{\widehat{E_R}} f(s_1, \ldots, s_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t_1, \ldots, t_m) = M$ be a proof with a single root rewrite. If s_i is not a constant, then $height(t_i) = 0$.*

PROOF. Let S_{const} be the set of positive integers, i, such that $s_i \in \mathcal{F}_0$. If none of the s_i's is a variable, then there is nothing to show; so, assume at least one of the s_i's is a variable. Now, let

$$s'_j = \begin{cases} s_j & \text{if } j \in S_{const} \\ x_{s_j} & \text{otherwise} \end{cases} \quad \text{and} \quad t'_j = \begin{cases} t_j & \text{if } j \in S_{const} \\ x_{s_j} & \text{otherwise} \end{cases}$$

where x_{s_j} is a fresh variable not appearing in M_0 or M_1, and $x_{s_i} = x_{s_j}$ if and only if $s_i = s_j$. We show that (i) $f(s'_1, \ldots, s'_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t'_1, \ldots, t'_m)$, (ii) $f(t'_1, \ldots, t'_m) \in NF_R$, and (iii) for $i \notin S_{const}$, $height(t_i) = 0$.

Part (i). If $j \notin S_{const}$, then $s'_j = t'_j = x_{s_j}$. So, say $j \in S_{const}$. In this case, $s'_j = s_j \overset{*}{\leftrightarrow}_{\widehat{E_R}} t_j = t'_j$. So, $f(s'_1, \ldots, s'_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t'_1, \ldots, t'_m)$. ***Part (ii)***. Let $j, j' \notin S_{const}$, and say $t_j \neq t_{j'}$. In order to apply Proposition 2.4, we need to show that $t'_j \neq t'_{j'}$. From Lemma 2.5, we know that $t_j \overset{*}{\leftrightarrow}_{\widehat{E_R}} s_j \neq s_{j'} \overset{*}{\leftrightarrow}_{\widehat{E_R}} t_{j'}$, and hence $t'_j = x_{s_j} \neq x_{s_{j'}} = t'_{j'}$. Therefore, we can apply Proposition 2.4 to obtain that $f(t'_1, \ldots, t'_m) \in NF_R$. ***Part (iii)***. Notice that, by (i) and $f(s'_1, \ldots, s'_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} c$, we have $f(t_1, \ldots, t_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} c \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(t'_1, \ldots, t'_m) = N$. Also, since N contains at least one fresh variable not appearing in M_0 or M_1, we know that $M \neq N$ and $C[N] \neq M_0$ or M_1, where $C[]$ is a context and $M_0 = C[M]$. Hence $\langle C[N], M_1 \rangle$ is a witness to non-$UN^=$, with $|C[N]| \leq |M_0|$. But $\langle M_0, M_1 \rangle$ is a minimal witness, so $|C[N]| = |C[M]|$ and $|N| = |M|$. Since $|t'_i| = 1$ for all $i \notin S_{const}$, it must be the case that $|t_i| = 1$. Thus, we have that $height(t_i) = height(t'_i) = 0$ for all $i \notin S_{const}$. □

COROLLARY 2.7. *Under the same assumptions as Lemma 2.6 plus the assumption that at least one of the s_i's is a constant, there is a j such that $s_j \in \mathcal{F}_0$ and $height(t_j) = height(f(t_1, \ldots, t_m)) - 1$ with $1 \leq j \leq m$.*

PROOF. Since $height(t_i) = 0$ whenever $s_i \notin \mathcal{F}_0$, we know that $height(t_i) \leq height(t_j)$ whenever $s_i \notin \mathcal{F}_0$ and $s_j \in \mathcal{F}_0$. So, amongst the direct subterms of $f(t_1, \ldots, t_m)$ with maximal height, there must be one, t_j, such that $s_j \in \mathcal{F}_0$. □

PROPOSITION 2.8. *Let R be a flat TRS, and let $c \in \mathcal{F}_0$. Let $\langle M, N \rangle$ be a minimal witness, and let N' be a subterm of N such that $height(N') = k$. Further, let $\pi \in c \overset{*}{\leftrightarrow} N'$ be a proof over R. Then we can find either (i) $1 + k$ distinct normal forms equivalent to constants, the normal forms having heights $0, 1, \ldots, k$, or (ii) a witness, $\langle N_0, N_1 \rangle$, to non-$UN^=$, such that N_0 and N_1 are flat.*

PROOF. We proceed by induction on $height(N')$. For the base case we assume that $height(N') = 0$. If the proof is trivial, i.e., if $c = N'$, then we have $1 = 1 + height(N')$ normal form (with height zero) equivalent to a constant. So, assume that π has at least one step.

We know that there is a proof, $\pi_{1rr} \in c \overset{+}{\leftrightarrow}_{\widehat{E_R}} N'$, such that there is only one root rewrite step in π_{1rr}. Since the first step in π_{1rr} is necessarily a root rewrite, π_{1rr} must have the form $c \overset{r}{\leftrightarrow} w\sigma = N'$, where the rule applied is $c \to w$ or $w \to c$, and $height(w) = 0$ (notice that if $c \overset{r}{\leftrightarrow} u \overset{*}{\leftrightarrow} N'$ for some term u with $height(u) > 0$, then we would need a second root rewrite to get back to N'). If $w \in X$, then $x \leftrightarrow c \leftrightarrow y$, where x, y are distinct variables. Therefore, $\langle x, y \rangle$ is a witness to non-$UN^=$ with x and y flat. If $w \in \mathcal{F}_0$, then we have found $1 = 1 + height(N')$ normal form (with height zero) equivalent to a constant.

For the inductive step, assume that $height(N') > 0$, and that the proposition holds for any height strictly less than $height(N')$. Now, π_{1rr} has the form

$$c \overset{r}{\leftrightarrow}_{\widehat{E_R}} f(t_1, \ldots, t_m) \overset{*}{\leftrightarrow}_{\widehat{E_R}} f(u_1, \ldots, u_m) = N'$$

and $t_i \overset{*}{\leftrightarrow}_{\widehat{E_R}} u_i$ for $1 \leq i \leq m$. We have two cases: (i) there is an i such that $t_i \in \mathcal{F}_0$, and (ii) there is no such i. For (i), by Corollary 2.7, there exists an i such that t_i is a constant and $height(u_i) = k - 1$. So, we can apply the inductive hypothesis to conclude that we have either (i) $1 + (1 + (height(N') - 1)) = 1 + height(N')$ distinct normal forms, with heights $0, 1, \ldots, height(N')$, equivalent to constants (the first $height(N') - 1$ normal forms come from the inductive hypothesis, and the final normal form is N' itself, which is equivalent to c), or (ii) a witness, $\langle N_0, N_1 \rangle$, to non-$UN^=$, such that N_0 and N_1 are flat.

In case (ii), if $c \leftrightarrow_{\widehat{E_R}} f(s_1, \ldots, s_m)$ is the rule used for $c \leftrightarrow_{\widehat{E_R}} f(t_1, \ldots, t_m)$, then s_i is a variable for $1 \leq i \leq m$. We need to show that $f(s_1, \ldots, s_m) \in NF_R$. From Lemma 2.5, we know that $t_i \neq t_j$ whenever $u_i \neq u_j$ for $1 \leq i, j \leq m$. Since $t_i \neq t_j$ implies that $s_i \neq s_j$, we see that $s_i \neq s_j$ whenever $u_i \neq u_j$. We can assume that the variables $s_1, \ldots, s_m$ are fresh relative to $f(u_1, \ldots, u_m)$, and so we can replace u_i with s_i in $f(u_1, \ldots, u_m)$, obtaining $f(s_1, \ldots, s_m) \in NF_R$ by Proposition 2.4. Since $f(s_1, \ldots, s_m)$ is a normal form, we can replace the variables appearing in $f(s_1, \ldots, s_m)$ with fresh variables to produce a new normal form, $f(s'_1, \ldots, s'_m)$, such that $f(s_1, \ldots, s_m) \leftrightarrow_{\widehat{E_R}} c \leftrightarrow_{\widehat{E_R}} f(s'_1, \ldots, s'_m)$. So, $\langle f(s_1, \ldots, s_m), f(s'_1, \ldots, s'_m)\rangle$ is our witness with $f(s_1, \ldots, s_m)$ and $f(s'_1, \ldots, s'_m)$ flat. □

COROLLARY 2.9. *Let R be a flat TRS, and let $c \in \mathcal{F}_0$. Let $\langle M, N\rangle$ be a minimal witness, and let N' be a subterm of N, with $height(N') \geq |\mathcal{F}_0|$. Further, let $\pi \in c \overset{*}{\leftrightarrow}_R N'$ be a proof over R. Then we can find either (i) a witness, $\langle M_0, M_1\rangle$, to non-$UN^=$, such that M_0 and M_1 are flat, or (ii) a witness, $\langle N_0, N_1\rangle$, to non-$UN^=$, such that $height(N_0), height(N_1) \leq |\mathcal{F}_0|$.*

PROOF. By Proposition 2.8, we know that we can find either (a) a witness, $\langle M_0, M_1\rangle$, to non-$UN^=$, such that M_0 and M_1 are flat, or (b) $1 + height(N')$ distinct normal forms equivalent to constants. If (a) is the case, then we are done. So assume that (b) is true. Since there are $1 + height(N') > |\mathcal{F}_0|$ normal forms equivalent to, at most, $|\mathcal{F}_0|$ constants, we know, by the Pigeonhole Principle, that a single constant is equivalent to two distinct normal forms. From the above observation, we know that the normal forms have heights 0, 1, 2, …, $height(N')$. The smallest (height-wise) $1 + |\mathcal{F}_0|$ normal forms each have height no more than $|\mathcal{F}_0|$. So, we know that we can find a witness, $\langle N_0, N_1\rangle$, to non-$UN^=$, such that $height(N_0), height(N_1) \leq |\mathcal{F}_0|$. □

PROPOSITION 2.10. *Let R be a flat TRS. Then, either (i) there does not exist a constant $c \in \mathcal{F}_0$ and normal form $N \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N$ and $height(N) \geq |\mathcal{F}_0|$, or (ii) there exists a witness, $\langle N_0, N_1\rangle$ to non-$UN^=$ for R such that $height(N_0), height(N_1) \leq k = max\{1, |\mathcal{F}_0|\}$. Further, there is an effective procedure to decide whether (i) or (ii) is the case.*

PROOF. Consider all ground[1] normal forms over the signature of the rewrite system, i.e., consisting of constants and function symbols appearing in the finitely many rules of R, with height less than, or equal to, k; we use $NF_{\leq k}$ to denote this set. Notice that if there is a constant, $c \in \mathcal{F}_0$, and an element of $SubMinWit_R$, N, with $height(N) \geq |\mathcal{F}_0|$, such that $c \overset{*}{\leftrightarrow} N$, then by Corollary 2.9 there is a witness, $\langle N_0, N_1\rangle$, to non-$UN^=$ for R with $height(N_0), height(N_1) \leq k$. By a result in [Comon et al. 1994], the word problem is decidable for flat systems. So, we can construct the set of all pairs, (s, t), such that $s, t \in NF_{\leq k}$ and $s \overset{*}{\leftrightarrow}_R t$. If we do not find a witness to non-$UN^=$ in $NF_{\leq k}$, then we know that there is no $c \in \mathcal{F}_0$ and $N \in SubMinWit_R$ such that $height(N) \geq |\mathcal{F}_0|$ and $c \overset{*}{\leftrightarrow}_R N$. Otherwise, we have found the witness $\langle N_0, N_1\rangle$ with $height(N_0), height(N_1) \leq k$. □

2.2. Shrinking Witnesses

Say $\langle f(a, g(b, f(c, x))), h(y, y, h(a, b, c))\rangle$ is a witness to non-$UN^=$ for some TRS. Can we replace big subterms of a component of the witness, without changing the fact that it is a witness, i.e., if we replace $g(b, f(c, x))$ with a variable, z, will $\langle f(a, z), h(y, y, h(a, b, c))\rangle$ still be a witness? We show that we can replace depth-1 subterms that are *not* equiv-

[1]As in [Zinn and Verma 2006; Zinn 2006], for nonlinear rewrite systems also we can expand the signature of the rewrite system with 3α new constants, where α is the maximum arity of a function symbol in the rules, and focus on ground normal forms.

alent to a constant with a variable. This shrinks the size of the witness; in particular, only depth-1 subterms of such a shrunk witness that are equivalent to a constant can have height greater than, or equal to, the number of constants in the TRS. So, a shrunk minimal witness either has small components, or there is a large subterm of a component of a minimal witness that is equivalent to a constant. If the latter is the case, then we know, by Corollary 2.9, that there is a small witness.

DEFINITION 2.11.
Let R be a rewrite system. Say X contains, for each term (up to renaming of variables), t, a variable $x_{\overline{t}}$, where $x_{\overline{s}} = x_{\overline{t}}$ if, and only if, $s \overset{*}{\leftrightarrow}_R t$. Let $t = f(t_1, \ldots, t_n)$ be a term in $\mathcal{T}(\mathcal{F}, X)$. Then, we define $\phi(t)$ as:

$$\phi(t) = \begin{cases} x_{\overline{t}} & \textit{if } t \textit{ is not equivalent to a constant} \\ t & \textit{otherwise} \end{cases}$$

Let $u = f(u_1, \ldots, u_m)$ for $m > 0$ and $v \in X$. We define the function α that maps terms to terms as follows: $\alpha(u) = f(\phi(u_1), \ldots, \phi(u_m))$ and $\alpha(v) = v$.

Notice that $\alpha(c) = c$ for $c \in \mathcal{F}_0$, since α only affects depth-1 subterms.

LEMMA 2.12. *Let R be a flat TRS, and let $u \leftrightarrow_R v$ be a proof over R, where $u \leftrightarrow_R v$ is not a root rewrite. Then, there is a proof $\alpha(u) \overset{*}{\leftrightarrow}_R \alpha(v)$.*

PROOF. Say $u = f(u_1, \ldots, u_m)$ and $v = f(v_1, \ldots, v_m)$ (notice that if $u \leftrightarrow_R v$ is not a root rewrite, then neither u nor v can have height zero). Since the rewrite is not a root rewrite, we know that there are u_i and v_i such that $u_i \leftrightarrow_R v_i$, and $u_j = v_j$ for all $j \neq i$. If u_i, v_i are equivalent to a constant, then $\phi(u_i) = u_i$ and $\phi(v_i) = v_i$, and hence $\alpha(u) \leftrightarrow_R \alpha(v)$. If u_i, v_i are not equivalent to a constant, then $\phi(u_i) = x_{\overline{u_i}} = x_{\overline{v_i}} = \phi(v_i)$, and hence $\alpha(u) = \alpha(v)$. □

LEMMA 2.13. *Let R be a flat TRS, and let $u \leftrightarrow_R v$ be a proof over R, where $u \leftrightarrow_R v$ is a root rewrite. If the rewrite has the form $u = w\sigma \to x\sigma = v$ (i.e. it uses a collapsing rule $w \to x$), then $\alpha(u) \leftrightarrow_R \phi(v)$; otherwise $\alpha(u) \leftrightarrow_R \alpha(v)$.*

PROOF. In case of a collapsing rule, any instantiations of x appearing as depth-1 subterms of u are equal to v, and so they are replaced by $\phi(v)$ in $\alpha(u)$. Since constants in w are never replaced, $\alpha(u) \leftrightarrow_R \phi(v)$. Otherwise, if s is a depth-1 subterm of u or v that is an instantiation of a shared variable, then every depth-1 instance of s is replaced by $\phi(s)$ in $\alpha(u)$ and $\alpha(v)$. So, $\alpha(u) \leftrightarrow_R \alpha(v)$. □

PROPOSITION 2.14. *Let R be a flat TRS. Let s and t be terms not equivalent to a constant and $\pi \in s \overset{*}{\leftrightarrow} t$ be a proof over R. Then, either there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$ for some variable y, or there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(t)$.*

PROOF. We know that there is a proof, π_{1rr}, over $\widehat{E_R}$ with at most one root rewrite. If π_{1rr} has zero steps, then $\alpha(s) = \alpha(t)$, and so $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(t)$. Assume that π_{1rr} has at least one step, and say that it has the form $s = s_0 \leftrightarrow_{\widehat{E_R}} \ldots \leftrightarrow_{\widehat{E_R}} s_k = t$ for some $k \geq 1$. We consider three cases: (i) π_{1rr} has no root rewrite; (ii) the only root rewrite in π_{1rr} uses a collapsing rule; and (iii) the only root rewrite in π_{1rr} does not use a collapsing rule.

In cases (i) and (iii), we know, by lemmas 2.12 and 2.13, that there is a proof $\alpha(s_i) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(s_{i+1})$ for $0 \leq i \leq k-1$. Therefore, there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(t)$.

In case (ii), let $w\sigma = s_j \leftrightarrow_{\widehat{E_R}} s_{j+1} = x\sigma$ be the instance of the collapsing rule, $w \to x$, for some $0 \leq j \leq k-1$. For $i < j$, we know that there is a proof $\alpha(s_i) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(s_{i+1})$. By Lemma 2.13, we know that $\alpha(s_j) \leftrightarrow_{\widehat{E_R}} \phi(s_{j+1})$, and so there is a proof $\alpha(s) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \phi(s_{j+1})$.

Since the terms in π_{1rr} cannot be equivalent to a constant (since s, t are not equivalent to a constant), we know that $\phi(s_{j+1}) = x_{\overline{s_{j+1}}}$, and so the proof is complete $\square$

Remark 2.15. As mentioned above, for any term v not equivalent to a constant, $\phi(v)$ can be chosen so that it does not appear as a subterm of any finite number of terms. Therefore, $\phi(s_{j+1})$ can be chosen so that it does not appear as a subterm of $s_0, s_1, \ldots, s_k$. We can always choose a fresh variable that does not appear in a finite set of terms.

PROPOSITION 2.16. *Let R be a flat TRS, and let $\langle M, N\rangle$ be a minimal witness to non-$UN^=$ for R, with M, N not equivalent to a constant. Then either $\langle \alpha(M), y\rangle$ or $\langle \alpha(M), \alpha(N)\rangle$ is a witness for some variable, y.*

PROOF. We know from Proposition 2.14 that either there is a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$ for some variable y, or there is a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(N)$. So, we need to show that (i) $\alpha(M)$, $\alpha(N)$, and y are normal forms, and that (ii) $\alpha(M) \neq y$ (whenever $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$) and $\alpha(M) \neq \alpha(N)$.

For (i), we need to show that if s and t are depth-1 subterms of M (or N) that are not equivalent to constants, then $\phi(s) \neq \phi(t)$ whenever $s \neq t$. So, say that $s \neq t$. If $s \overset{*}{\leftrightarrow}_{\widehat{E_R}} t$, then $\langle s, t\rangle$ would violate the minimality of $\langle M, N\rangle$, since $|s| + |t| < |M| \leq |M| + |N|$. So, we know that s and t are not equivalent, and hence $\phi(s) \neq \phi(t)$. We know by Proposition 2.4 that $\alpha(M)$ and $\alpha(N)$ are normal forms, because the variables replacing subterms of M and N can be chosen so that they are fresh. Since variables are always normal forms, we know that $\alpha(M)$, $\alpha(N)$, and y are normal forms.

For (ii), if M is not a variable, then $\alpha(M)$ is not a variable, and hence $\alpha(M) \neq y$. If M is a variable, then, by Remark 2.15, we can choose y so that it does not appear as a subterm of M. So, $\alpha(M) = M \neq y$.

To see that $\alpha(M) \neq \alpha(N)$, we need to consider two cases. If $root(M) \neq root(N)$, then clearly $\alpha(M) \neq \alpha(N)$, since α does not affect the outermost function symbol. If $root(M) = root(N)$, then it must be the case that $M|_i \neq N|_i$ for some integer, i. In order for $\alpha(M) = \alpha(N)$ to be true, $M|_i$ and $N|_i$ must be replaced by the same variable. But this only happens when $M|_i$ and $N|_i$ are equivalent, and if $M|_i$ and $N|_i$ were equivalent, then (setting $M' = M|_i$ and $N' = N|_i$) $\langle M', N'\rangle$ would be a witness with $|M'| < |M|$ and $|N'| < |N|$. This would violate the minimality of $\langle M, N\rangle$, so $M|_i$ and $N|_i$ cannot be equivalent, and hence $M|_i$ and $N|_i$ must be replaced by distinct variables. Therefore, $\alpha(M) \neq \alpha(N)$. $\square$

3. DECIDABILITY FOR FLAT AND SHALLOW REWRITE SYSTEMS

LEMMA 3.1. *Let R be a flat TRS, and say that there is no constant $c \in \mathcal{F}_0$ and normal form $N' \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N'$ and $height(N') \geq |\mathcal{F}_0|$. Let $\langle M, N\rangle$ be a minimal witness to non-$UN^=$ for R. Then $height(\alpha(M)), height(\alpha(N)) \leq k = max\{1, |\mathcal{F}_0|\}$.*

PROOF. We know that (i) all depth-1 subterms of $\alpha(M)$ and $\alpha(N)$ that are not equivalent to a constant are necessarily variables, and (ii) there is no constant $c \in \mathcal{F}_0$ and normal form $N' \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N'$ and $height(N') \geq |\mathcal{F}_0|$. Hence, the depth-1 subterms of $\alpha(M)$ and $\alpha(N)$ are either (i) variables or (ii) elements of $SubMinWit_R$ with height strictly less than $|\mathcal{F}_0|$. This means that the heights of $\alpha(M)$ and $\alpha(N)$ are at most $max\{1, |\mathcal{F}_0|\}$. $\square$

THEOREM 3.2. *Let R be a flat TRS. If there is a witness to non-$UN^=$ for R, then there exists a witness, $\langle N_0, N_1\rangle$, with $height(N_0), height(N_1) \leq k = max\{1, |\mathcal{F}_0|\}$. Hence $UN^=$ is decidable for R.*

PROOF. By Proposition 2.10, we know that there is either (i) *no* constant $c \in \mathcal{F}_0$ and normal form $N' \in SubMinWit_R$ such that $c \overset{*}{\leftrightarrow}_{\widehat{E_R}} N'$ and $height(N') \geq |\mathcal{F}_0|$, or (ii) a witness, $\langle N_0, N_1 \rangle$ to non-$UN^=$ for R such that $height(N_0), height(N_1) \leq k$. Further, there is an effective procedure to decide if (i) or (ii) is the case.

If (ii) is the case, then we have our witness. So, assume that (i) is the case, and let $\langle M, N \rangle$ be a minimal witness to non-$UN^=$ for R. If M and N are equivalent to a constant, c, and $height(M), height(N) < |\mathcal{F}_0|$, then we are done. So, we assume (without loss of generality) that M, N are not equivalent to a constant, and thus we can apply Proposition 2.14. Hence there is either a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} y$ for some variable y, or a proof $\alpha(M) \overset{*}{\leftrightarrow}_{\widehat{E_R}} \alpha(N)$. By Lemma 3.1, we know that $height(\alpha(M)), height(\alpha(N)) \leq k$. Hence, by Proposition 2.16, either $\langle \alpha(M), y \rangle$ or $\langle \alpha(M), \alpha(N) \rangle$ is a witness to non-$UN^=$ with $height(\alpha(M)), height(\alpha(N)), |y| \leq k$.

So, if there is a witness to non-$UN^=$ for R, then there is a witness, $\langle N_0, N_1 \rangle$, with $height(N_0), height(N_1) \leq k$. The following algorithm, on input R, determines if R is $UN^=$: Enumerate all ground normal forms over the signature of the rewrite system, i.e., consisting of constants and function symbols appearing in the finitely many rules of R, with height less than, or equal to, k; say they are $N_0, \ldots, N_n$. In [Comon et al. 1994], the authors show that the word problem is decidable for shallow TRS. So, for $0 \leq i < j \leq n$, check if $N_i \overset{*}{\leftrightarrow}_{\widehat{E_R}} N_j$. If $N_i \overset{*}{\leftrightarrow}_{\widehat{E_R}} N_j$ for some $0 \leq i < j \leq n$, then R is not $UN^=$; otherwise, R is $UN^=$. □

Now that we have shown that $UN^=$ is decidable for flat rewrite systems, we extend this result to shallow rewrite systems. We do this by *flattening* a shallow rewrite system, i.e., transforming a shallow rewrite system into a flat one in a way that preserves $UN^=$.

THEOREM 3.3. *Let R be a shallow TRS. Then $UN^=$ is decidable for R.*

4. UNDECIDABILITY FOR LINEAR AND LEFT/RIGHT-FLAT SYSTEMS

We begin by introducing a problem known to be undecidable.

4.1. Post Correspondence Problem

An instance P of the Post Correspondence Problem (PCP) is defined as follows:

Definition 4.1. Given a finite set of tiles $\{\langle u_i, v_i \rangle \mid 1 \leq i \leq n\}$ where u_i, v_i are words under some finite alphabet Γ, we must decide whether a sequence of indices $i_1 \cdots i_k$ exists such that $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$.

Given a PCP instance P we consider $|P|$ to be the number of tiles defined for that instance. If a sequence of indices is meant as a candidate solution to the PCP instance, we call it a *tile sequence*. We use the convention that Γ^* refers to the words generated by the alphabet.

4.2. Linear and Right-Flat Construction

We will construct a linear and right-flat TRS $\mathcal{R}$ that reduces PCP to the $UN^=$ problem between two normal forms: 0 and 1. Thus, if $0 \overset{*}{\longleftrightarrow} 1$ we violate $UN^=$ and there is a solution to P; otherwise, P has no solution and $UN^=$ is preserved. A correct reduction implies $UN^=$ must be undecidable for this class of TRS.

Our construction will be composed of two parts. Part one will convert an arbitrary string into a pair of identical strings. The only normal form found in part one is 0. Part two will convert an arbitrary tile sequence into a pair of strings generated by the tiles. The only normal form found in part two is 1. Both parts can reach a solution to P. Thus, if a solution exists, then $0 \overset{*}{\longleftrightarrow} 1$.

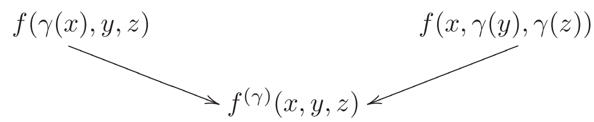
Since strings are central to our construction we will work with a few conventions. The terms representing strings are sequences of unary symbols ended by $\emptyset$. Furthermore, strings and the terms that represent them are used interchangeably; we may refer to $a(b(\emptyset))$ as ab. For a string s we denote its reversal s^R. Note that for $s = s_1 s_2$, we have $s^R = s_2^R s_1^R$. We liberally use γ as a placeholder for the appropriate symbol in the alphabet Γ.

Our initial set of rules corresponds to part one:

$$\mathcal{R}_0 \;:=\; \left\{ \begin{array}{r} f(\gamma(x), \emptyset, \emptyset) \to 0 \\ f(\emptyset, x, y) \to g(x, y) \end{array} \;\middle|\; \gamma \in \Gamma \right\}$$

$$\mathcal{R}_S \;:=\; \left\{ \begin{array}{r} f(\gamma(x), y, z) \to f^{(\gamma)}(x, y, z) \\ f(x, \gamma(y), \gamma(z)) \to f^{(\gamma)}(x, y, z) \end{array} \;\middle|\; \gamma \in \Gamma \right\}$$

Since we are working with equivalences, the orientation of a rule has no bearing on reachability. We use this to our advantage by simulating the rule $f(\gamma(x), y, z) \longleftrightarrow f(x, \gamma(y), \gamma(z))$. Notice the following structure:



In our construction, the superscripted version of a function symbol will have a reduced set of applicable rewrites. By making sure only two rewrites apply, these two rewrites can be considered a single rewrite. In a derivation between non-superscripted terms, rewriting to $f^{(\gamma)}$ fixes the next rewrite we perform. Therefore, we should view $\mathcal{R}_S$ as the set of rules $f(\gamma(x), y, z) \longleftrightarrow f(x, \gamma(y), \gamma(z))$.

The following lemmas concern $\mathcal{R}_0 \cup \mathcal{R}_S$ unless otherwise specified.

LEMMA 4.2. *$f(x, \emptyset, \emptyset) \xleftrightarrow[\mathcal{R}_S]{*} f(\emptyset, y, z)$ iff $x, y, z \in \Gamma^*$.*

PROOF. Clearly the rules in $\mathcal{R}_S$ only allow the removal of symbols $\gamma \in \Gamma$. □

LEMMA 4.3. *$f(s, y, z) \xleftrightarrow[\mathcal{R}_S]{*} f(\emptyset, s^R(y), s^R(z))$ where $s \in \Gamma^*$.*

PROOF. We proceed by induction on the length of s. For $|s| = 1$, the rules in $\mathcal{R}_S$ suffice. Suppose our lemma holds for $|s| = n - 1$. Given s of length n, we can write $s = \gamma(s')$ for some $\gamma \in \Gamma$. The rules in $\mathcal{R}_S$ allow $f(\gamma(s'), y, z) \xleftrightarrow{*} f(s', \gamma(y), \gamma(z))$. If we consider $y' = \gamma(y)$ and $z' = \gamma(z)$ our induction hypothesis applies and we are done. □

LEMMA 4.4. *Let $p = f(x, \emptyset, \emptyset)$ for some x. Let $q = f(\emptyset, y, z)$ for some y, z. For a pair of terms (p, q) where $p \xleftrightarrow[\mathcal{R}_S]{*} q$ then:*

— *$\nexists\, p' = f(x', \emptyset, \emptyset)$ for some $x' \neq x$ such that $p' \xleftrightarrow[\mathcal{R}_S]{*} q$*
— *$\nexists\, q' = f(\emptyset, y', z')$ for some $(y', z') \neq (y, z)$ such that $p \xleftrightarrow[\mathcal{R}_S]{*} q'$*

PROOF. We can consider $\mathcal{R}_S$ to be $\{f(\gamma(x), y, z) \longleftrightarrow f(x, \gamma(y), \gamma(z))\}$ since we are only interested in non-superscripted terms. Let π be a mapping from terms of the form $f(s_1, s_2, s_3)$ to $s_1^R s_2 s_1^R s_3$. Suppose $p' \xleftrightarrow[\mathcal{R}_S]{*} q$. By Lemma 4.2, π is well defined for p, p', and q. Clearly there is no $p' \neq p$ such that $\pi(p') = \pi(p)$. However, if $p' \xleftrightarrow[\mathcal{R}_S]{*} q$ then $\pi(p') = \pi(q) = \pi(p)$ since the value is conserved under $\mathcal{R}_S$. A similar argument applies to q'. $\square$

Informally, we can show Lemma 4.4 holds by observing there is no choice of rewrite if $f|_2 = f|_3 = \emptyset$. Once we apply that rewrite we are presented with a series of meaningless choices: either backtrack or perform the only other rewrite. This is the case until we reach a term where $f|_1 = \emptyset$ or we get stuck on the way. The situation is the same if we start at $f|_1 = \emptyset$ and work our way toward $f|_2 = f|_3 = \emptyset$.

LEMMA 4.5. $0 \xleftrightarrow{*} g(y,z)$ *iff* $(y,z) = (s^R, s^R)$ *for some* $s \in \Gamma^* \backslash \{\varepsilon\}$.

PROOF. There is only one rewrite applicable at each end term: $f(x,\emptyset,\emptyset) \to 0$ and $f(\emptyset,y,z) \to g(y,z)$. Thus, our proof will have the form $0 \leftarrow f(x,\emptyset,\emptyset) \xleftrightarrow{*} f(\emptyset,y,z) \to g(y,z)$. We know by Lemma 4.2 that $x = s$ for some $s \in \Gamma^*$. Furthermore, s cannot be the empty string due to how we constructed the rules in $\mathcal{R}_0$. By Lemma 4.3 we know $(y,z) = (s^R, s^R)$. By Lemma 4.4 we know $\Longleftarrow$. $\square$

The first part of our construction is concluded. The second part of our construction uses many of the same techniques.

For each tile $\langle u_i, v_i \rangle$ in P let it be represented by the function symbol $t_i : 1$. Let $n = \max(|u_i|, |v_i|)$. We create n rules for each tile. Note that $\gamma^n_{u_i}$ and $\gamma^n_{v_i}$ refer to the nth symbol in u_i and v_i, respectively. If $k > |u_i|$ then $\gamma^k_{u_i}$ leaves the variable unchanged (the concrete instantiation of the rule has only the variable in that position). Same for $k > |v_i|$. Here are the rules:

$$\mathcal{R}_1 \quad := \quad \left\{ \begin{array}{r} h(t_i(x),\emptyset,\emptyset) \to 1 \\ h(\emptyset,x,y) \to g(x,y) \end{array} \right\}$$

$$\mathcal{R}_T \quad := \quad \left\{ \begin{array}{r} h(t_i(x),y,z) \to h^{(i,0)}(x,y,z) \\ h^{(i,k)}(x,\gamma^k_{u_i}(y),\gamma^k_{v_i}(z)) \to h^{(i,k-1)}(x,y,z) \\ h(x,\gamma^n_{u_i}(y),\gamma^n_{v_i}(z)) \to h^{(i,n-1)}(x,y,z) \end{array} \right\}$$

The rules in $\mathcal{R}_T$ were constructed to simulate rules, much like $\mathcal{R}_S$. However, in $\mathcal{R}_T$ we fix a longer chain of rewrites so we can simulate $h(t_i(x),y,z) \xleftrightarrow{*} h(x,u_i^R(y),v_i^R(z))$ for non-superscripted terms:

$$h(T_i(x),y,z) \longrightarrow h^{(i,0)}(x,y,z) \xleftarrow{\;*\;} h(x,u_i^R(y),v_i^R(z))$$

For example, if $t_1 = \langle aab, bb \rangle$ then we would have a sequence of rules:

$$\begin{array}{r} h(t_1(x),y,z) \to h^{(1,0)}(x,y,z) \\ h^{(1,1)}(x,a(y),b(z)) \to h^{(1,0)}(x,y,z) \\ h^{(1,2)}(x,a(y),b(z)) \to h^{(1,1)}(x,y,z) \\ h(x,b(y),z) \to h^{(1,2)}(x,y,z) \end{array}$$

The following lemmas concern $\mathcal{R}_1 \cup \mathcal{R}_T$ unless otherwise specified.

LEMMA 4.6. $h(x,\emptyset,\emptyset) \xleftrightarrow[\mathcal{R}_T]{*} h(\emptyset,y,z)$ *iff* $x = t_{i_1} \cdots t_{i_n}$ *and* $y, z \in \Gamma^*$.

PROOF. Clearly, the rules in $\mathcal{R}_T$ only allow the removal of symbols t_i from $h|_1$ and the removal of symbols $\gamma \in \Gamma$ from $h|_2$ and $h|_3$. $\square$

LEMMA 4.7. *$h(t, y, z) \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} h(\emptyset, s_a^R(y), s_b^R(z))$ where $t = t_{i_1} \cdots t_{i_n}$, $s_a = u_{i_1} \cdots u_{i_n}$ and $s_b = v_{i_1} \cdots v_{i_n}$.*

PROOF. We proceed by induction on the length of t. For $|t| = 1$, the rules in $\mathcal{R}_T$ suffice. Suppose our lemma holds for $|t| = n - 1$. Given t of length n, we can write $t = t_i(t')$ for some t_i. The rules in $\mathcal{R}_T$ allow $h(t_i(t'), y, z) \overset{*}{\longleftrightarrow} h(t', u_i^R(y), v_i^R(z))$. If we consider $y' = u_i^R(y)$ and $z' = v_i^R(z)$ our induction hypothesis applies and we are done. $\square$

LEMMA 4.8. *Let $p = h(x, \emptyset, \emptyset)$ for some x. Let $q = h(\emptyset, y, z)$ for some y, z. For a pair of terms (p, q) where $p \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} q$ then:*

— *$\nexists\, p' = h(x', \emptyset, \emptyset)$ for some $x' \neq x$ such that $p' \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} q$*
— *$\nexists\, q' = h(\emptyset, y', z')$ for some $(y', z') \neq (y, z)$ such that $p \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} q'$*

PROOF. We can consider $\mathcal{R}_T$ to be $\{h(t_i(x), y, z) \overset{*}{\longleftrightarrow} h(x, u_i^R(y), v_i^R(z))\}$ since we are only interested in non-superscripted terms. Let π_a, π_b be mappings from tile sequences $t_{i_1} \cdots t_{i_n}$ to $u_{i_1} \cdots u_{i_n}$ and $v_{i_1} \cdots v_{i_n}$, respectively. Let π be a mapping from terms of the form $h(t_1, s_2, s_3)$ to $(\pi_a(t_1))^R s_2 (\pi_b(t_1))^R s_3$. Suppose $p' \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} q$. By Lemma 4.6, π is well defined for p, p', and q. Clearly there is no $p' \neq p$ such that $\pi(p') = \pi(p)$ (holds as long as $t_i \neq t_j$ for $i \neq j$). However, if $p' \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} q$ then $\pi(p') = \pi(q) = \pi(p)$ since the value is conserved under $\mathcal{R}_T$. A similar argument applies to q'. $\square$

The informal argument used in part one unfortunately does not apply. Let $t_1 = \langle abb, ba\rangle$ and $t_2 = \langle bb, ba\rangle$. We have the following equivalence with our rules: $h(t_1, \emptyset, \emptyset) \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} h(\emptyset, bba, ab) \overset{*}{\underset{\mathcal{R}_T}{\longleftrightarrow}} h(t_2, a, \emptyset)$. Although it may seem like an error, the values of all three terms under π are indeed the same.

LEMMA 4.9. *$1 \overset{*}{\longleftrightarrow} g(y, z)$ iff $(y, z) = (s_a^R, s_b^R)$ where $s_a = u_{i_1} \cdots u_{i_n}$ and $s_b = v_{i_1} \cdots v_{i_n}$ for some nonempty tile sequence $i_1 \cdots i_n$.*

PROOF. There is only one rewrite applicable at each end term: $h(x, \emptyset, \emptyset) \to 1$ and $h(\emptyset, y, z) \to g(y, z)$. Thus, our proof will have the form $1 \leftarrow h(x, \emptyset, \emptyset) \overset{*}{\longleftrightarrow} h(\emptyset, y, z) \to g(y, z)$. We know by Lemma 4.6 that $x = t$ for some $t = t_{i_1} \cdots t_{i_n}$. Furthermore, t cannot be an empty sequence due to how we constructed the rules in $\mathcal{R}_1$. By Lemma 4.7 we know $(y, z) = (s_a^R, s_b^R)$. By Lemma 4.8 we know $\Longleftarrow$. $\square$

LEMMA 4.10. *$0 \overset{*}{\longleftrightarrow} 1$ iff P has a solution.*

PROOF. Any proof of $0 \overset{*}{\longleftrightarrow} 1$ must go through some term $g(x, y)$. Due to Lemma 4.5 and Lemma 4.9 the only term $g(x, y)$ that both 0 and 1 can reach must have x and y as a pair of identical strings generated by the tiles in P. Thus, P must have a solution. $\square$

$$
\begin{aligned}
0 &\xleftrightarrow{*} f(bbaabbbaa, \emptyset, \emptyset)\\
&\xleftrightarrow{*} f^{(b)}(baabbbaa, \emptyset, \emptyset)\\
&\xleftrightarrow{*} f(baabbbaa, b, b)\\
&\xleftrightarrow{*} f^{(b)}(aabbbaa, b, b)\\
&\xleftrightarrow{*} f(aabbbaa, bb, bb)\\
&\xleftrightarrow{*} f^{(a)}(abbbaa, bb, bb)\\
&\xleftrightarrow{*} f(abbbaa, abb, abb)\\
&\xleftrightarrow{*} f^{(a)}(bbbaa, abb, abb)\\
&\xleftrightarrow{*} f(bbbaa, aabb, aabb)\\
&\xleftrightarrow{*} f^{(b)}(bbaa, aabb, aabb)\\
&\xleftrightarrow{*} f(bbaa, baabb, baabb)\\
&\xleftrightarrow{*} f^{(b)}(baa, baabb, baabb)\\
&\xleftrightarrow{*} f(baa, bbaabb, bbaabb)\\
&\xleftrightarrow{*} f^{(b)}(aa, bbaabb, bbaabb)\\
&\xleftrightarrow{*} f(aa, bbbaabb, bbbaabb)\\
&\xleftrightarrow{*} f^{(a)}(a, bbbaabb, bbbaabb)\\
&\xleftrightarrow{*} f(a, abbbaabb, abbbaabb)\\
&\xleftrightarrow{*} f^{(a)}(\emptyset, abbbaabb, abbbaabb)\\
&\xleftrightarrow{*} f(\emptyset, aabbbaabb, aabbbaabb)\\
&\xleftrightarrow{*} g(aabbbaabb, aabbbaabb)
\end{aligned}
$$

$$
\begin{aligned}
&\xleftrightarrow{*} h(\emptyset, aabbbaabb, aabbbaabb)\\
&\xleftrightarrow{*} h^{(1,2)}(\emptyset, aabbbaabb, abbbaabb)\\
&\xleftrightarrow{*} h^{(1,1)}(\emptyset, aabbbaabb, bbbaabb)\\
&\xleftrightarrow{*} h^{(1,0)}(\emptyset, abbbaabb, bbaabb)\\
&\xleftrightarrow{*} h(t_1, abbbaabb, bbaabb)\\
&\xleftrightarrow{*} h^{(3,2)}(t_1, bbbaabb, bbaabb)\\
&\xleftrightarrow{*} h^{(3,1)}(t_1, bbaabb, baabb)\\
&\xleftrightarrow{*} h^{(3,0)}(t_1, baabb, aabb)\\
&\xleftrightarrow{*} h(t_3t_1, baabb, aabb)\\
&\xleftrightarrow{*} h^{(2,1)}(t_3t_1, aabb, abb)\\
&\xleftrightarrow{*} h^{(2,0)}(t_3t_1, abb, bb)\\
&\xleftrightarrow{*} h(t_2t_3t_1, abb, bb)\\
&\xleftrightarrow{*} h^{(3,2)}(t_2t_3t_1, bb, bb)\\
&\xleftrightarrow{*} h^{(3,1)}(t_2t_3t_1, b, b)\\
&\xleftrightarrow{*} h^{(3,0)}(t_2t_3t_1, \emptyset, \emptyset)\\
&\xleftrightarrow{*} h(t_3t_2t_3t_1, \emptyset, \emptyset) \xleftrightarrow{*} 1
\end{aligned}
$$

Fig. 1. An example for $P = \{\langle a, baa\rangle, \langle ab, aa\rangle, \langle bba, bb\rangle\}$.

Finally, we add the set of rules that guarantee 0 and 1 are the only normal forms. These rules do not disturb any of the results above.

$$
\mathcal{R}_{nf} \;:=\; \left\{ \begin{array}{cc} f(x,y,z) \to f(x,y,z) & h(x,y,z) \to h(x,y,z) \\ f^{(\gamma)}(x,y,z) \to f^{(\gamma)}(x,y,z) & h^{(i,j)}(x,y,z) \to h^{(i,j)}(x,y,z) \\ g(x,y) \to g(x,y) & \gamma(x) \to \gamma(x) \\ \emptyset \to \emptyset & t_i(x) \to t_i(x) \end{array} \right\}
$$

Now that 0 and 1 are the only normal forms, their equivalence implies a violation of $UN^=$. Thus, our complete set of rules is: $\mathcal{R} := \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_S \cup \mathcal{R}_T \cup \mathcal{R}_{nf}$.

THEOREM 4.11. *$UN^=$ is undecidable for linear TRS that are right-flat and have left-hand sides of depth two.*

PROOF. Direct consequence of Lemma 4.10, which proves our construction reduces $UN^=$ to solving PCP. □

4.3. Linear and Left-Flat Construction

If we reverse the orientation of all rules in $\mathcal{R}$ we run into a small problem: the $\mathcal{R}$ normal forms 0 and 1 are no longer normal forms after reorientation. To remedy this, we replace the rules $\{f(\gamma(x), \emptyset, \emptyset) \to 0,\ h(T_i(x), \emptyset, \emptyset) \to 1\}$ with the following modifications:

$$\mathcal{R}_j := \left\{\begin{array}{l} j_0(x) \to 0 \\ j_0(x) \to f(x, \emptyset, \emptyset) \\ j_1(x) \to h(\emptyset, \emptyset, x) \\ j_1(x) \to 1 \end{array}\right\}$$

Thus, 0 and 1 remain normal forms after reorientation and $Var(r) \subset Var(l)$. However, we must now disallow the empty string as a solution somewhere else in the construction. To that end, we replace $\{f(\emptyset, x, y) \to g(x, y),\ h(\emptyset, x, y) \to g(x, y)\}$ with the following:

$$\mathcal{R}_g := \left\{\begin{array}{c} g^{(\gamma,\gamma)}(x, y) \to f(\emptyset, \gamma(x), \gamma(y)) \\ g^{(\gamma_i,\gamma_j)}(x, y) \to h(\emptyset, \gamma_i(x), \gamma_j(y)) \end{array}\right\}$$

Any rules that have not been replaced are simply reoriented. Thus, our final rule set is:

$$\begin{aligned} \mathcal{R} \ := \ & \mathcal{R}_j \cup \mathcal{R}_g \cup \mathcal{R}_S^{-1} \cup \mathcal{R}_T^{-1} \\ & \cup \mathcal{R}_{nf} \backslash \{g(x, y) \to g(x, y)\} \\ & \cup \{g^{(\gamma_i,\gamma_j)}(x, y) \to g^{(\gamma_i,\gamma_j)}(x, y)\} \end{aligned}$$

THEOREM 4.12. *$UN^=$ is undecidable for linear TRS that are left-flat and have right-hand sides of depth two.*

PROOF. All proofs in Section 4.2 can be easily adapted for this modified TRS. □

5. CONCLUSION

The $\mathrm{UN}^=$ property of TRSs is shown to be decidable for the shallow class and undecidable for the class of linear TRS in which one side of the rule is allowed to be at most depth-two and the other side is flat. Among the fundamental properties of TRSs only the word problem and the $\mathrm{UN}^=$ property are now known to be decidable for the shallow class. An important direction for future research is to give a complete classification of the basic properties for all subclasses of linear, depth-two TRSs (see also [Verma 2008] in this regard).

Acknowledgments.

We thank Ross Greenwood and the reviewers of FSTTCS 2010 for careful readings and constructive comments.

REFERENCES

H. Comon, M. Haberstrau, and J. Jouannaud. 1994. Syntacticness, cycle-syntacticness, and shallow theories. *Inf. Comput.* 111, 1 (1994), 154–191.

Guillem Godoy and Hugo Hernández. 2009. Undecidable properties of flat term rewrite systems. *Appl. Algebra Eng. Commun. Comput.* 20, 2 (2009), 187–205.

Guillem Godoy and Florent Jacquemard. 2009. Unique Normalization for Shallow TRS. In *Rewriting Techniques and Applications, 20th International Conference, RTA 2009, Brasília, Brazil, June 29 - July 1, 2009, Proceedings*. 63–77.

Guillem Godoy and Sophie Tison. 2007. On the Normalization and Unique Normalization Properties of Term Rewrite Systems. In *Proc. Conf. on Automated Deduction*. 247–262.

G. Godoy, A. Tiwari, and R. Verma. 2003. On the Confluence of Linear Shallow Rewrite Systems. *Proceedings of the Symposium on Theoretical Aspects of Computer Science, Lecture Notes in Computer Science* 2607 (2003), 85–96.

J.W. Klop. 1980. *Combinatory Reduction Systems*. Ph.D. Dissertation. Mathematisch Centrum, Amsterdam.

J.W. Klop. 1992. Rewrite Systems. In *Handbook of Logic in Computer Science*. Oxford.

Ichiro Mitsuhashi, Michio Oyamaguchi, and Florent Jacquemard. 2006. The Confluence Problem for Flat TRSs. In *8th Artificial Intelligence and Symbolic Computation Conference*. 68–81.

N. Radcliffe and Rakesh M. Verma. 2010. Uniqueness of Normal Forms is Decidable for Shallow Term Rewrite Systems. In *Proc. Conf. on Foundations of Software Technology & Theoretical Comp. Sci.* 284–295.

Terese. 2003. *Term Rewriting Systems*. Cambridge University Press, Cambridge.

Rakesh Verma. 2009. Complexity of Normal Form Problems and Reductions for Term Rewriting Problems. *Fundamenta Informaticae* 92, 1-2 (2009), 145–168.

R.M. Verma, M. Rusinowitch, and D. Lugiez. 2001. Algorithms and Reductions for Rewriting Problems. *Fundamenta Informaticae* 46, 3 (2001), 257–276. Also in Proc. of Int'l Conf. on Rewriting Techniques and Applications 1998.

Rakesh M. Verma. 2008. New Undecidability Results for Properties of Term Rewrite Systems. In *Proc. (elec.) of 9th Workshop on Rule-based Programming (RULE)*.

Julian Zinn. 2006. *A Polynomial Algorithm for Uniqueness of Normal Forms of Linear, Shallow Term Rewrite Systems*. Master's thesis. University of Houston.

J. Zinn and R. Verma. 2006. A Polynomial-time algorithm for Uniqueness of Normal Forms of linear, shallow rewrite systems. In *Proc. IEEE Conf. on Logic in Computer Science*. short presentation.