The Power of Greedy for Online Minimum Cost Matching on the Line

Eric Balkanski, Yuri Faenza, and Noemie Perivier

Columbia University

Abstract

We consider the online minimum cost matching problem on the line. In this problem, there are n servers and, at each of n time steps, a request arrives and must be irrevocably matched to a server that has not yet been matched to, with the goal of minimizing the sum of the distances between the matched pairs. Online minimum cost matching is a central problem in applications such as ride-hailing platforms and food delivery services. Despite achieving a worst-case competitive ratio that is exponential in n, the simple greedy algorithm, which matches each request to its nearest available free server, performs very well in practice. A major question is thus to explain greedy's strong empirical performance. In this paper, we aim to understand the performance of greedy over instances that are at least partially random.

When both the requests and the servers are drawn uniformly and independently from $[0, 1]$, we obtain a constant competitive ratio for greedy, which improves over the previously best-known $O(\sqrt{n})$ bound for greedy in this setting. We extend this constant competitive ratio to the excess supply setting where there is a linear excess of servers, which improves over the previously best-known $O(\log^3 n)$ bound for greedy in this setting.

We moreover show that in the semi-random model where the requests are still drawn uniformly and independently but where the servers are chosen adversarially, greedy achieves an $O(\log n)$ competitive ratio. Even though this one-sided randomness allows a large improvement in greedy's competitive ratio compared to the model where the requests are adversarial and arrive in a random order, we show that it is not sufficient to obtain a constant competitive ratio by giving a tight $\Omega(\log n)$ lower bound. These results invite further investigation about how much randomness is necessary and sufficient to obtain strong theoretical guarantees for the greedy algorithm for online minimum cost matching, on the line and beyond.

Contents

1 Introduction **2**
1.1 Technical overview . . . 4
1.2 Additional related work . . . 5

2 Preliminaries **5**

3 Greedy is Constant Competitive in the Fully Random Model **8**

4 Greedy is Logarithmic Competitive in the Random Requests Model **11**
4.1 Greedy is $O(\log n)$-competitive . . . 11
4.2 Overview of the lower bound . . . 12
4.3 Greedy is $\Omega(\log n)$-competitive . . . 15
4.3.1 Preliminaries . . . 15
4.3.2 Upper bound on the cost of the optimal offline matching . . . 17
4.3.3 Analysis of $(S_1, \ldots, S_n)$. . . . 18
4.3.4 Lower bound on $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)]$. . . . 22
4.3.5 Proof of Theorem 4 . . . 28

A Auxiliary Lemmas **34**

B Proof of the Hybrid Lemma (Lemma 5) **34**

C Missing Analysis from Section 3 **45**

D Missing Analysis from Section 4.1 **49**

E Hierarchical Greedy is $\Omega(n^{1/4})$ in the Random Requests Model **52**

F Missing Analysis from Section 4.3 **54**
F.1 Missing analysis from Section 4.3.1 . . . 54
F.2 Missing analysis from Section 4.3.3 . . . 56
F.3 Missing analysis from Section 4.3.4 . . . 60

1 Introduction

Matching problems form a core area of discrete optimization. In the 90s, a seminal paper by Karp, Vazirani, and Vazirani [KVV90] introduced online bipartite maximum matching problems and showed that, in the worst-case scenario, no deterministic algorithm can beat a simple greedy procedure, and no randomized algorithm can beat *ranking*, which is a greedy procedure preceded by a random shuffling of the order of the nodes. These elegant results and their natural application to online advertising spurred much research, especially from the late 2000s on (see, e.g., [M$^+$13] and the references therein for a survey). While more complex algorithms have been devised for models other than worst-case analysis, greedy techniques are often used as a competitive benchmark for comparisons, see, e.g., [FHK$^+$10, LFZ$^+$20, XSC$^+$19].

In the last few years, motivated by the surge of ride-sharing platforms, a second online matching paradigm has received much attention: online (bipartite) minimum cost matching. In this class of problems, one side of the market is composed of *servers* (sometimes called *drivers*) and is fully known at time 0. Nodes from the other side, often called *requests* or *customers*, arrive one at a time. When request i arrives, we must match it to one of the servers j, and incur a cost c_{ij}. Server j is then removed from the list of available servers, and the procedure continues. The goal is to minimize the total cost of the matching.

Given the motivating application to ride-sharing, it is natural to impose the condition that both servers and requests belong to some metric space (e.g., [GK19, KP93, Kan21, Rag16, TTC94]). Many algorithms in this area involve non-trivial procedures like randomized tree embeddings [MNP06, BBGN07], iterative segmentation of the space [Kan21] or primal-dual arguments [Rag18], or use randomization to bypass worst-case scenarios for deterministic algorithms [GL12].

Computational experiments suggest however that a simple greedy algorithm (sometimes also called nearest neighbor) that matches each incoming request to the closest available server works very well in practice. In particular, experiments have shown that greedy was more effective than other existing algorithms in most tests and has outstanding scalability [TSD$^+$16]. This performance substantiates the choice of many ride-sharing platforms to actually implement greedy procedures, in combination with other techniques [How, Lyf, Ubea, Ubeb]. In contrast, if we assume that n servers and n requests are adversarially placed on a line, the greedy algorithm only achieves a $2^n - 1$ competitive ratio [KP93, TTC94]. These results motivate the guiding question of this paper:

> Can we find a theoretical justification for the strong practical performance of the greedy algorithm for online minimum cost matching problems?

Understanding the strong practical performance of simple algorithms has motivated a lot of work on beyond the worst-case analysis of algorithms. Some examples include using properties such as curvature, stability, sharpness, and smoothness to obtain improved guarantees for greedy for submodular maximization [CC84, CRV17, PST20, RZ22, BQS21] and different semi-random models for analyzing local search for the traveling salesman problem [ERV14, KM15, ERV16, BFK22], k-means for clustering [AMR09, MR13], and greedy for online *maximum* matching [GM08, DJSW11, MJ13, Arn22]. In the context of online *minimum* cost matching, our understanding of the performance of greedy is limited. Despite its simplicity, greedy is hard to analyze because a greedy match at some time step can have complex consequences on the available servers in a different region at a much later time step. In other words, "the state of the system under the standard greedy algorithm is hard to keep track of analytically" [Kan21].

In this paper, we aim to understand the performance of greedy over instances that are at least partially random for online minimum cost matching on the line, which is a problem that has recently received significant attention [AALS21, GL12, PS21, GKS20, MN20, Rag18, KN04, NR17, FHK03]. We first consider the *fully random* model, where the n servers and n requests are all drawn uniformly and independently from $[0, 1]$. In this model, greedy achieves an $O(\sqrt{n})$ competitive ratio in the plane [TTC94] and there are more sophisticated algorithms such as hierarchical greedy [Kan21] and fair-bias [GGPW19] that are constant competitive in Euclidean spaces and on the line, respectively. Our first main result settles the asymptotic performance of greedy for matching on the line in the fully random model by showing that greedy achieves a constant competitive ratio in that setting.

Theorem 1. *For online matching on the line in the fully random model, the greedy algorithm achieves an $O(1)$-competitive ratio.*

We show that this constant competitiveness of greedy also holds in the *fully random $\epsilon-$excess model*, for every constant $\epsilon > 0$. This is the modification of the fully random model where there is a linear excess of servers, i.e., $(1 + \epsilon)n$ servers. This results improves over the previously best-known competitive ratio for greedy of $O(\log^3 n)$ in this setting [AALS21].

Theorem 2. *For any constant $\epsilon > 0$, greedy is $O(1)$-competitive in the fully random ϵ-excess model.*

It is widely acknowledged (see, e.g., [Fei21]) that i.i.d. instances often do not resemble "real" instances. We next therefore consider whether strong guarantees for greedy can also be obtained in a semi-random model. In particular, we consider a model that we call the *random requests model* where the n servers are adversarially chosen and the requests are, as in the fully random model, drawn uniformly and independently. In ride-sharing, this is motivated by the fact that there have been examples of drivers that behave adversarially to increase the prices of the rides, see, e.g., [Ubec]. Our next result shows that greedy is logarithmic competitive in the random requests model.

Theorem 3. *For online matching on the line in the random requests model, the greedy algorithm achieves an $O(\log n)$-competitive ratio.*

In the model where the servers and requests are chosen adversarially but where the arrival order is random, $O(n)$ and $\Omega(n^{0.22})$ upper and lower bounds are known for the competitive ratio of greedy [GK19]. Combined with this $\Omega(n^{0.22})$ lower bound, our result shows that the performance of greedy improves exponentially when the locations of the requests are also random. Our last main result shows that this competitive ratio of greedy in the random requests model is tight.

Theorem 4. *For online matching on the line in the random requests model, the greedy algorithm achieves an $\Omega(\log n)$-competitive ratio.*

Combined with the previous result, we obtain that greedy is $\Theta(\log n)$-competitive in the random requests model. Interestingly, this lower bound is obtained on an instance where the servers are very close to being uniformly spread.

These results improve our understanding of why and when greedy performs well for online minimum cost matching, but there remain many intriguing questions. In particular, we believe that it would be interesting to study semi-random requests and/or semi-random servers, for example, in a model where some fraction of the servers are adversarial and some fraction are random. Another interesting model, especially in the context of ride-sharing, would be one where the location of a small number of servers can be chosen (i.e., a mix of best-case and worst-case). Considering more general metric spaces beyond the line is of course also a direction for future work. Finally, it would be interesting to explore empirically which semi-random models exhibit a structure that most closely resembles the structure of real-world instances.

1.1 Technical overview

The main difficulty in analyzing the greedy algorithm is that there can be complex dependencies between a greedy match that occurred at some time step in some region of the line and the set of remaining servers that are available at a later time step in a completely different region of the line. In other words, a single greedy match at some time step can have a butterfly effect on the servers that will be available in the future in different regions. Algorithms such as hierarchical greedy that partition the interval in different regions have been designed to prevent matching decisions in one region from impacting the future available servers in another region, which does not necessarily lead to algorithms that are better than greedy but does give algorithms that are simpler to analyze.

The starting point of our analysis, both for the upper and lower bounds, is to consider a *hybrid algorithm* $\mathcal{H}^m_{\mathcal{A}}$ that matches the first m requests according to an algorithm $\mathcal{A}$ and then greedily matches each of the remaining requests to the closest available server. The algorithm $\mathcal{A}$ that is used is different for each of our results. We show a hybrid lemma that upper bounds, for any algorithm $\mathcal{A}$ that satisfies some fairly general properties, the difference $cost(\mathcal{H}^{m-1}_{\mathcal{A}}) - cost(\mathcal{H}^m_{\mathcal{A}})$ (i.e., between the total costs incurred by $\mathcal{H}^{m-1}_{\mathcal{A}}$ and $\mathcal{H}^m_{\mathcal{A}}$) as a function of the cost incurred by $\mathcal{A}$ to match the m^{th} request. This hybrid algorithm idea is similar to the path-coupling idea from [BD97] and was also used in [GL12] to analyze online matching algorithms, but with two important differences: in [GL12], their hybrid algorithm is used to analyze a randomized algorithm on a deterministic instance (instead of a deterministic algorithm on a randomized instance), and in their hybrid algorithm, $\mathcal{A}$ is an optimal offline algorithm (instead of an online algorithm).

In the fully random model, we consider the hybrid algorithm $\mathcal{H}^m_{\mathcal{A}}$ where $\mathcal{A}$ is the hierarchical greedy algorithm from [Kan21], which is known to be constant competitive in this model. We use 1) the hybrid lemma applied to this hybrid algorithm and 2) a bound on hierarchical greedy's cost for matching the m^{th} request to show that the difference $\mathbb{E}[cost(\mathcal{G}) - cost(\mathcal{A}^H)]$ (i.e., between the total cost of greedy and the total cost of hierarchical greedy) is $O(\sqrt{n})$. Since the expected optimal total cost is known to be $\Theta(\sqrt{n})$ in the fully random model and since hierarchical greedy is constant competitive, we get that greedy is also constant competitive. The analysis for the excess supply setting is different and relies on concentration arguments. In the random requests model, we use the hybrid lemma to show that the total cost of greedy is within an $O(\log n)$ factor of the total cost of a simple modification of the fair-bias algorithm from [GGPW19] that is constant competitive in the random requests model.

The most technical result of this paper is the $\Omega(\log n)$ lower bound in the random requests model. We consider an instance where there is a large number of servers at location 0, no servers in $(0, n^{-1/5}]$, and the remaining $1 - o(1)$ servers uniformly spread in $(n^{-1/5}, 1]$. We again analyze the difference $cost(\mathcal{H}^{m-1}_{\mathcal{A}}) - cost(\mathcal{H}^m_{\mathcal{A}})$, but with the hybrid algorithm $\mathcal{H}^m_{\mathcal{A}}$ where $\mathcal{A}$ is the algorithm that matches any request in $[0, n^{-1/5}]$ to a server at 0 and greedily matches any other request to the closest available server. We show that at any time step t, the set of available servers for $\mathcal{H}^{m-1}_{\mathcal{A}}$ and $\mathcal{H}^m_{\mathcal{A}}$ differ in at most one server. We then consider the distance δ_t at time t between these two different servers that are available to only one of the algorithms and we show that $cost(\mathcal{H}^{m-1}_{\mathcal{A}}) - cost(\mathcal{H}^m_{\mathcal{A}})$ can be lower bounded as a function of $\max_{t \geq m} \delta_t$. Due to the randomness of the requests, the main difficulty is to lower bound $\max_{t \geq m} \delta_t$ (e.g., the gap δ_t can either shrink or expand at each time step), which we do by giving a careful partial characterization of the remaining servers for $\mathcal{H}_{\mathcal{A}}^m$ at each time t.

1.2 Additional related work

In general metric spaces with adversarial requests and servers, [KP93] gave a $2n - 1$ deterministic competitive algorithm and proved that this competitive ratio is optimal for deterministic algorithms, On the line, [KP93] showed that the competitive ratio of greedy is at least $2^n - 1$. A deterministic algorithm with a sublinear competitive ratio was presented in [ABN$^+$14]. A few years later, [NR17] gave a $O(\log^2 n)$ competitive deterministic algorithm, which was then shown to be $O(\log n)$-competitive in [Rag18]. Regarding lower bounds, [FHK03] showed that no deterministic algorithm can achieve a competitive ratio strictly less than 9.001 on the line.

For randomized algorithms, still for adversarial requests and servers, [MNP06] and [CP07] obtained a $O(\log^3 n)$ competitive ratio in general metric spaces using randomized tree embeddings, which was later improved to $O(\log^2 n)$ by [BBGN07]. On the line, and for doubling metrics, [GL12] showed that a randomized greedy algorithm is $O(\log n)$ competitive. Recently, [PS21] improved the lower bound from [FHK03] to obtain an $\Omega(\sqrt{\log n})$ lower bound for the line that also holds for randomized algorithms. For general metrics, it was previously known that no randomized algorithm can achieve a competitive ratio better than $\Omega(\log n)$ [MNP06].

When the arrival order of the requests is random, [GK19] showed that greedy is $O(n)$ and $\Omega(n^{0.22})$ competitive. [Rag16] gave a deterministic algorithm that achieves a $O(\log n)$ competitive ratio, which is optimal even for randomized algorithms. When the requests are drawn i.i.d. from any distribution over the set of servers, [GGPW19] gave a $O((\log \log \log n)^2)$ competitive algorithm in general metric spaces that is also constant competitive on the line and for tree metrics. When the servers and requests are uniformly and independently distributed, [TTC94] showed that greedy achieves an $O(\sqrt{n})$ competitive ratio on the unit disk and [Kan21] showed that an algorithm called hierarchical greedy is constant competitive on the unit hypercube (and also analyzed the more challenging fully dynamic setting where the servers also arrive online).

Empirical evaluations of different algorithms on real spatial data have shown that greedy performs well in practice [TSD$^+$16]. The excess supply setting was studied in [AALS21], who showed that the total optimal cost is $O(1)$ and the total cost of greedy is $O(\log^3 n)$ when the number of excess servers is linear and when the requests and servers are random (but the arrival order can be adversarial). The results for hierarchical greedy from [Kan21] also extends to the excess supply setting. Recourse, i.e. allowing matching decisions to be revoked to some extent, has been considered in [MN20, GKS20]. In the offline non-bipartite version of the problem with $2n$ point drawn uniformly from $[0, 1]$, [FMR90] showed that greedy achieves a $\Theta(\log n)$ approximation.

2 Preliminaries

In the online matching on the line problem, there are n_s *servers* $S = \{s_1, \ldots, s_{n_s}\}$ and $n = n_r$ *requests* $R = (r_1, \ldots, r_n)$ such that $s_i, r_i \in [0, 1]$ for all i. Hence, an instance is given by a pair (S, R). The servers are known to the algorithm at time $t = 0$. At each time step $t \in [n]$, the algorithm observes request r_t and must irrevocably match it to a server that has not yet been matched. We denote by $s_{\mathcal{A}}(r_t)$ the server that gets matched to request r_t by (the current execution of) algorithm $\mathcal{A}$ and by $S_{\mathcal{A},0} \supseteq \cdots \supseteq S_{\mathcal{A},n}$ the sets of free servers obtained through the execution of $\mathcal{A}$, where $S_{\mathcal{A},0}$ is the initial set of servers, and for all $t \in [n]$, $S_{\mathcal{A},t}$ is the set of remaining free servers just after matching r_t. The cost incurred from matching r_t to $s_{\mathcal{A}}(r_t)$ is $\mathrm{cost}_t(\mathcal{A}, r_t) = |r_t - s_{\mathcal{A}}(r_t)|$ and the total cost of the matching produced by $\mathcal{A}$ on instance I is $\mathrm{cost}(\mathcal{A}, I) = \sum_{t=1}^{n} \mathrm{cost}_t(\mathcal{A}, r_t)$. We often abuse notation and write $\mathrm{cost}_t(\mathcal{A}), \mathrm{cost}(\mathcal{A})$, and S_t instead of $\mathrm{cost}_t(\mathcal{A}, r_t), \mathrm{cost}(\mathcal{A}, I)$, and $S_{\mathcal{A},t}$. We consider instances $I = (I_r, I_a)$ that have a random component $I_r \sim \mathcal{D}$ and an adversarial component $I_a \in \mathcal{I}_A$. The performance of an algorithm $\mathcal{A}$ is measured by its competitive ratio:

All models studied in the paper can be represented by a triple (n^u, n^d, n). Here, n^u (resp. n) is the cardinality of the set S^u of servers (resp. of the set R of requests) sampled independently from the uniform distribution $\mathcal{U}_{[0,1]}$. n^d is the number of adversarially placed servers (hence, $n^u + n^d = n_s$). The performance of an algorithm $\mathcal{A}$ is measured by its competitive ratio:

$$\max_{S^d \in [0,1]^{n^d}} \frac{\mathbb{E}_{S^u, R \sim \mathcal{U}_{[0,1]}, \mathcal{A}}[\mathrm{cost}(\mathcal{A}, (S^d \cup S^u, R))]}{\mathbb{E}_{S^u, R \sim \mathcal{U}_{[0,1]}}[\mathrm{cost}(OPT, (S^d \cup S^u, R))]}.$$

where OPT is the offline optimal matching when the requests are known at time $t = 0$. Although some papers in online optimization use a different notion of competitive ratio (see, e.g., survey [Meh13]), in the context of online matching on the line, most literature we are aware of use the same definition as ours. This is true, in particular, for papers over which we build [GGPW19, Kan21, GK19] or whose results we improve [AALS21, TTC94].

More precisely, we study the three following models.

- In the *fully random model*, $(n^u, n^d, n) = (n, 0, n)$, i.e., all servers S and requests R are drawn uniformly and independently from $[0, 1]$ and there is an equal number of servers and requests.

- For all constant $\epsilon > 0$, we define the *fully random $\epsilon-$excess model*, in which $(n^u, n^d, n) = ((1 + \epsilon)n, 0, n)$, i.e., all servers S and requests R are drawn uniformly and independently from $[0, 1]$ and there is a linear excess of ϵn servers.

- In the *random requests model*, $(n^u, n^d, n) = (0, n, n)$, i.e., the requests R are still drawn uniformly and independently from $[0, 1]$ but the servers are now chosen adversarially over all potential sequence of n requests in $[0, 1]$.

The greedy algorithm, denoted by $\mathcal{G}$, is the algorithm that matches each request r_t to the closest available server, i.e., $s_{\mathcal{G}}(r_t) = \arg\min_{s \in S_{\mathcal{G},t-1}} |s - r_t|$. We say that an algorithm $\mathcal{A}$ *makes neighboring matches* if it matches every request r_t either to the closest available server to its left or to its right. For any algorithm $\mathcal{A}$ and $m \in \{0, \ldots, n\}$, we define the hybrid algorithm $\mathcal{H}_{\mathcal{A}}^m$ that matches the first m requests according to $\mathcal{A}$ and then greedily matches the remaining requests to the closest available server. The following key lemma (proved in Appendix B) bounds $\mathbb{E}\big[cost(\mathcal{H}_{\mathcal{A}}^{m-1}) - cost(\mathcal{H}_{\mathcal{A}}^m)\big]$ as a function of $\mathrm{cost}_m(\mathcal{A})$ – that is, the cost for algorithm $\mathcal{A}$ to match the m^{th} request.

Lemma 5. ***(The Hybrid Lemma).*** *There exists a constant $C > 0$ such that for any online algorithm $\mathcal{A}$ that makes neighboring matches, for any instance with n servers $S = \{s_1, \ldots, s_n\}$ adversarially chosen, n requests $R = (r_1, \ldots, r_n)$ uniformly and independently drawn from $[0, 1]$, and for any event E_m that depends only on S_{m-1}, r_m, we have*

$$\mathbb{E}\big[cost(\mathcal{H}_{\mathcal{A}}^{m-1}) - cost(\mathcal{H}_{\mathcal{A}}^m) | E_m\big] \leq C \cdot \mathbb{E}\big[\big(1 + \log\big(\tfrac{1}{cost_m(\mathcal{A})}\big)\big) cost_m(\mathcal{A}) \big| E_m\big].$$

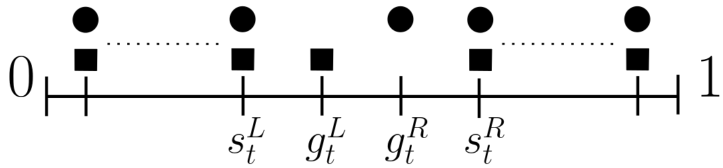

Figure 1: Set of servers S_t (free servers at time t for $\mathcal{H}_A^m$) and S'_t (free servers at time t for $\mathcal{H}_A^{m-1}$) in the case where $S_t \neq S'_t$, where the squares are the servers in S_t and the circles the servers in S'_t.

The idea of using hybrid algorithms for analyzing an online matching algorithm was used in [GL12], which has a similar hybrid lemma (see Section 1.1 for additional discussion). The proof of Lemma 5 relies on the following structural lemma (Lemma 6), in which, for a given realization R of the sequence of requests and a fixed value of $m \in [n]$, we consider a simultaneous execution of $\mathcal{H}_\mathcal{A}^m$ and $\mathcal{H}_\mathcal{A}^{m-1}$ on the sequence R. To ease the exposition, we drop the reference to the algorithms in the indices and write S_t and $s(r_t)$ instead of $S_{\mathcal{H}_\mathcal{A}^m,t}$ and $s_{\mathcal{H}_\mathcal{A}^m}(r_t)$ to denote, respectively, the set of free server for $\mathcal{H}_\mathcal{A}^m$ just after matching r_t and the server to which $\mathcal{H}_\mathcal{A}^m$ matches r_t. Similarly, we write S'_t and $s'(r_t)$ instead of $S_{\mathcal{H}_\mathcal{A}^{m-1},t}$ and $s_{\mathcal{H}_\mathcal{A}^{m-1}}(r_t)$ for the equivalent objects for $\mathcal{H}_\mathcal{A}^{m-1}$. Finally, to remove any ambiguity, we assume that all servers in S_0 are distinct (even if it means moving them an infinitesimal distance).

We now show that at all time steps t, the sets S_t and S'_t of free servers for the two algorithms are all identical except for at most one server in each of these sets and that, if they each have such a unique server, there is no server in $S_t \cup S'_t$ that is in between the two unique servers (see Figure 1 for an illustration).

We let $g_t^L < g_t^R$ denote these at most two servers in the symmetric difference of S_t and S'_t, and let $\delta_t := g_t^R - g_t^L$ be the distance between these two servers. If $S_t = S'_t$, then we write $g_t^L = g_t^R = \emptyset$ and $\delta_t = 0$. We also define $s_t^L = \max\{s \in S_t : s < g_t^L\}$ and $s_t^R = \min\{s \in S_t : s > g_t^R\}$ (with the convention that $s_t^L, s_t^R = \emptyset$ when there are no such servers).

Lemma 6. *Let $\mathcal{A}$ be any online algorithm, S_0 be n arbitrary servers and R be n arbitrary requests. Let $(S_0, \ldots, S_n)$ and $(S'_0, \ldots, S'_n)$ denote the set of free servers for $\mathcal{H}_\mathcal{A}^m$ and $\mathcal{H}_\mathcal{A}^{m-1}$ at each time steps. Then, the following propositions hold for all $t \in \{m, \ldots, n\}$:*

1. ***Differ in at most one server.*** $|S_t \setminus S'_t| = |S'_t \setminus S_t| \leq 1$.
2. ***Consecutiveness of the different servers.*** *If $g_t^L, g_t^R \neq \emptyset$, there is no server $s \in S_t \cup S'_t$ such that $g_t^L < s < g_t^R$.*
3. ***The values.*** *If $t < n$ and $S_t \neq S'_t$ (and assuming without loss of generality that $S_t = S'_t \cup \{g_t^L\} \setminus \{g_t^R\}$), then the values of $s(r_{t+1}), s'(r_{t+1}), \delta_{t+1}, g_{t+1}^L, g_{t+1}^R$ and an upper bound on $\Delta cost_{t+1} := |cost_{t+1}(\mathcal{H}^{m-1}) - cost_{t+1}(\mathcal{H}^m)|$ are given in Tables 2, 3 and 4 (in Appendix B):*
   - *if $s_t^L \neq \emptyset, s_t^R \neq \emptyset$, the values are given in Table 2, where $d_t^L = g_t^L - s_t^L$ and $d_t^R = s_t^R - g_t^R$,*
   - *if $s_t^L = \emptyset, s_t^R \neq \emptyset$, the values are given in Table 3, where $d_t^R = s_t^R - g_t^R$,*
   - *if $s_t^R = \emptyset, s_t^L \neq \emptyset$, the values are given in Table 4, where $d_t^L = g_t^L - s_t^L$,*
   - *if $s_t^L = \emptyset, s_t^R = \emptyset$, then $S_{t+1} = S'_{t+1} = \emptyset, \delta_{t+1} = 0$, and $|cost_{t+1}(\mathcal{H}^{m-1}) - cost_{t+1}(\mathcal{H}^m)| \leq \delta_t$.*

4. ***Gap remains zero after disappearing.*** *If $\delta_t = 0$, then $\delta_{t'} = 0$ for all $t' \geq t$.*

The proof is given in Appendix B.

3 Greedy is Constant Competitive in the Fully Random Model

In this section, we show that greedy achieves a constant competitive ratio in the fully random model where both the servers and requests are drawn uniformly and independently from $[0, 1]$. In addition, we show that this result also holds when there is a linear excess supply of servers.

The setting with n servers. We recall that in this setting, the competitive ratio of any algorithm $\mathcal{A}$ is given by:

$$\frac{\mathbb{E}_{(R,S)\sim\mathcal{U}(0,1)^n\times\mathcal{U}(0,1)^n,\mathcal{A}}[\text{cost}(\mathcal{A}, (S, R))]}{\mathbb{E}_{(R,S)\sim\mathcal{U}(0,1)^n\times\mathcal{U}(0,1)^n}[\text{cost}(OPT, (S, R))]}.$$

The main idea of the analysis is to consider a hybrid algorithm that first runs the hierarchical greedy algorithm from [Kan21], and then greedily matches the remaining requests to the closest available server.

We first present the hierarchical greedy algorithm introduced in [Kan21], which we denote by $\mathcal{A}^H$. To describe it, we need to define the sequence $\mathcal{I}_{\ell_0}, ..., \mathcal{I}_0$, where $\ell_0 = \log(n)$, which are increasingly refined partitions of $[0, 1]$. More precisely, $\mathcal{I}_{\ell_0} = \{[0, 1]\}$ and $\mathcal{I}_\ell$ is the partition obtained by dividing each interval in $\mathcal{I}_{\ell+1}$ into two intervals of equal length, i.e., $\mathcal{I}_\ell = \left(\cup_{]x,y]\in\mathcal{I}_{\ell+1}}\{]x,(x+y)/2],](x+y)/2, y]\}\right) \cup \left(\cup_{[0,y]\in\mathcal{I}_{\ell+1}}\{[0, y/2],]y/2, y]\}\right)$. The partitions obtained through this process can be organized in a binary tree, where the nodes at level ℓ are the intervals of $\mathcal{I}_\ell$ and the leafs are the intervals of $\mathcal{I}_0$.

Given a request r_t, let $I(r_t)$ be the leaf interval to which r_t belongs and $J(r_t)$ be the lowest-level ancestor interval of $I(r_t)$ in the tree such that $J(r_t) \cap S_{t-1} \neq \emptyset$, i.e., such that $J(r_t)$ contains some free servers when request r_t arrives. The hierarchical greedy algorithm matches r_t to a free server in $J(r_t)$. For our purposes, we assume that it matches r_t to the closest free server in $J(r_t)$. A request r_t is said to be matched at level ℓ if $J(r_t) \in \mathcal{I}_\ell$. There are two known results about hierarchical greedy that are important for our analysis. The first one upper bounds the number of requests matched at each level and the second one is its constant competitiveness.

Lemma 7 ([Kan21])**.** *There is a constant $C' > 0$ such that, for all $\ell \in \{0, \ldots, \ell_0\}$, $\mathbb{E}[|\{r_t : J(r_t) \in I_\ell\}|] \leq C'\sqrt{n2^{\ell-\ell_0}}2^{\ell_0-\ell}$.*

Theorem 8 ([Kan21])**.** *In the fully random model, we have that $\mathbb{E}[cost(\mathcal{A}^H)] = O(\sqrt{n})$.*

Next, we show the following bound on the cost incurred by hierarchical greedy when matching a request at level ℓ (proof deferred to Appendix C).

Lemma 9. *For all $t \in [n]$, if r_t is matched at level ℓ, then we have $cost_t(\mathcal{A}^H) \log(1/cost_t(\mathcal{A}^H)) \leq 2^{\ell-\ell_0}(\log(2)(\ell_0 - \ell) + 1)$.*

The next lemma is the main lemma of this section and shows that the difference between the total cost of greedy and hierarchical greedy is $O(\sqrt{n})$.

Lemma 10. *In the fully random model, we have that $\mathbb{E}[cost(\mathcal{G}) - cost(\mathcal{A}^H)] = O(\sqrt{n})$.*

Proof. We first note that since the hierarchical greedy algorithm matches every request r_t to the closest free server in $J(r_t)$, and since $r_t \in J(r_t)$ by definition of $J(r_t)$, hierarchical greedy makes neighboring matches, which is the condition needed to apply the hybrid lemma to the hybrid algorithm $\mathcal{H}^m$. We get that

$$
\begin{aligned}
&\mathbb{E}[cost(\mathcal{G}) - cost(\mathcal{A}^H)] \\
&= \sum_{m=1}^{n} \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)] && \mathcal{H}^n = \mathcal{A}^H, \mathcal{H}^0 = \mathcal{G} \\
&\leq C \sum_{m=1}^{n} \mathbb{E}[\big(1 + \log\big(\tfrac{1}{cost_m(\mathcal{A}^H)}\big)\big)\mathrm{cost}_m(\mathcal{A}^H)] && \text{Hybrid lemma} \\
&\leq C \sum_{m=1}^{n} \mathbb{E}[\log\big(\tfrac{1}{cost_m(\mathcal{A}^H)}\big)\mathrm{cost}_m(\mathcal{A}^H)] + C\mathbb{E}[cost(\mathcal{A}^H)] \\
&\leq C \sum_{m=1}^{n} \mathbb{E}[\log\big(\tfrac{1}{cost_m(\mathcal{A}^H)}\big)\mathrm{cost}_m(\mathcal{A}^H)] + O(\sqrt{n}) && \text{Theorem 8} \\
&= C \sum_{m=1}^{n} \sum_{\ell=0}^{\ell_0} \mathbb{P}(J(r_m) \in \mathcal{I}_\ell) \cdot \mathbb{E}[\log\big(\tfrac{1}{cost_m(\mathcal{A}^H)}\big)\mathrm{cost}_m(\mathcal{A}^H) | J(r_m) \in \mathcal{I}_\ell] + O(\sqrt{n}) \\
&\leq C \sum_{m=1}^{n} \sum_{\ell=0}^{\ell_0} \mathbb{P}(J(r_m) \in \mathcal{I}_\ell) \cdot 2^{\ell-\ell_0}(\log(2)(\ell_0 - \ell) + 1) + O(\sqrt{n}) && \text{Lemma 9} \\
&= C \sum_{\ell=0}^{\ell_0} 2^{\ell-\ell_0}(\log(2)(\ell_0 - \ell) + 1) \cdot \sum_{m=1}^{n} \mathbb{P}(J(r_m) \in \mathcal{I}_\ell) + O(\sqrt{n}) \\
&= C \sum_{\ell=0}^{\ell_0} 2^{\ell-\ell_0}(\log(2)(\ell_0 - \ell) + 1) \cdot \mathbb{E}[|\{r_t : J(r_t) \in I_\ell\}|] + O(\sqrt{n}) \\
&\leq CC'\sqrt{n} \sum_{\ell=0}^{\ell_0} 2^{(\ell-\ell_0)/2}(\log(2)(\ell_0 - \ell) + 1) + O(\sqrt{n}) && \text{Lemma 7} \\
&= CC'\sqrt{n} \sum_{j=0}^{\ell_0} 2^{-j/2}(\log(2)j + 1) + O(\sqrt{n}) \\
&= CC'\sqrt{n} \left(\log(2) \sum_{j=0}^{\ell_0} j \left(\frac{1}{\sqrt{2}}\right)^j + \sum_{j=0}^{\ell_0} \left(\frac{1}{\sqrt{2}}\right)^j \right) + O(\sqrt{n}) \\
&= O(\sqrt{n}).
\end{aligned}
$$

□

The last result needed is that the optimal cost in the fully random model is known to be $\Theta(\sqrt{n})$.

Lemma 11 ([Kan21])**.** *In the fully random model, we have that* $\mathbb{E}[\mathtt{OPT}] = \Theta(\sqrt{n})$.

By combining Theorem 8, Lemma 10, and Lemma 11, we obtain the main result of this section.

Theorem 1. *For online matching on the line in the fully random model, the greedy algorithm achieves an* $O(1)$*-competitive ratio.*

The excess supply setting. We consider here an extension of the previous model where there is a linear excess of servers. For any constant $\epsilon > 0$, we define the *fully random ϵ-excess model*, where an instance consist of n requests and $n(1+\epsilon)$ servers all drawn uniformly and independently from $[0,1]$. The competitive ratio of any algorithm $\mathcal{A}$ is given by:

$$\frac{\mathbb{E}_{(R,S)\sim\mathcal{U}(0,1)^n\times\mathcal{U}(0,1)^{n(1+\epsilon)},\mathcal{A}}[\text{cost}(\mathcal{A},(S,R))]}{\mathbb{E}_{(R,S)\sim\mathcal{U}(0,1)^n\times\mathcal{U}(0,1)^{n(1+\epsilon)}}[\text{cost}(OPT,(S,R))]}.$$

In this setting, the hybrid approach with hierarchical greedy used above does not give a constant competitive ratio. However, we are still able to prove that greedy is constant competitive with a different argument. Unlike the model with n servers, the analysis for the excess supply setting does not rely on the hybrid lemma but on concentration arguments. All proofs are in Appendix C.

The main technical contribution here lies in showing that, thanks to the excess of servers, there is an exponentially small probability that there is a large area around the n-th request that contains no available servers. More formally, for $\ell, m \in [0,1]$, we let $x_{(\ell,m)} = |\{t \in [n-1] : r_t \in (\ell,m)\}|$ be the number of requests out of the $n-1$ first requests that arrived in the interval (ℓ,m), and we let $y_{(\ell,m)} = |\{t \in [n(1+\epsilon)] : s_t \in (\ell,m)\}|$ be the total number of servers that lie in the interval (ℓ,m).

Lemma 12. *Let $\epsilon > 0$ be a constant. There are constants C_ϵ, C'_ϵ such that, in the fully random ϵ-excess model, we have that for all $z \in [\frac{4+\epsilon}{\epsilon n}, 1]$,*

$$\mathbb{P}(\exists \ell, m \in [0,1] : x_{(\ell,m)} = y_{(\ell,m)}, (r_n - \ell \geq z \text{ or } \ell = 0), (m - r_n \geq z \text{ or } m = 1) \mid r_n) \leq C'_\epsilon e^{-nzC_\epsilon}.$$

Using Lemma 12, we then upper bound the expected cost incurred by greedy at the last step.

Lemma 13. *Let $\epsilon > 0$ be a constant. There is a constant C''_ϵ such that, in the fully random ϵ-excess model, we have $\mathbb{E}[cost_n(\mathcal{G})] \leq \frac{C''_\epsilon}{n}$.*

Last, we observe that, because of servers getting less and less dense as requests arrive, the expected cost at each step of the greedy algorithm increases.

Lemma 14. *Let $\epsilon > 0$ be a constant. Then, in the fully random ϵ-excess model, we have that for all $i \in [n-1]$, $\mathbb{E}[cost_i(\mathcal{G})] \leq \mathbb{E}[cost_{i+1}(\mathcal{G})]$.*

Using Lemma 13 and Lemma 14, we conclude that $\mathbb{E}[cost(\mathcal{G})] = \sum_{i=1}^n \mathbb{E}[cost_i(\mathcal{G})] \leq n \cdot \mathbb{E}[cost_n(\mathcal{G})] \leq C''_\epsilon$. We have thus shown the following.

Lemma 15. *Let $\epsilon > 0$ be a constant. There exists a constant $C''_\epsilon > 0$ such that in the fully random ϵ-excess model, we have $\mathbb{E}[cost(\mathcal{G})] \leq C''_\epsilon$.*

In order to conclude the proof of Theorem 2, it suffices to lower bound the cost of the optimal solution in the *fully random ϵ-excess model.*

Lemma 16 ([Kan21]). *For any constant $\epsilon > 0$, we have that in the fully random ϵ-excess model, $\mathbb{E}[OPT] = \Theta(\frac{1}{\epsilon})$.*

We can then conclude the following result on the performance of the greedy algorithm.

Theorem 2. *For any constant $\epsilon > 0$, greedy is $O(1)$-competitive in the fully random ϵ-excess model.*

4 Greedy is Logarithmic Competitive in the Random Requests Model

In this section, we show that greedy achieves an $\Theta(\log n)$ competitive ratio in the random requests model where the servers are chosen adversarially and the requests are drawn uniformly and independently from $[0, 1]$. Thus, unlike in the fully random model, servers and requests can be distributed in a significantly different manner in this model.

4.1 Greedy is $O(\log n)$-competitive

We first show the $O(\log n)$ upper bound. We note that, even though hierarchical greedy and greedy are both constant-competitive in the fully random model, hierarchical greedy is only $\Omega(n^{1/4})$-competitive in the random requests model (see Appendix E). The main lemma (Lemma 18) shows that greedy is at most a logarithmic factor away from any online algorithm that makes neighboring matches. To prove that lemma, we first need to lower bound the probability that the cost incurred by any online algorithm at any time step is small. The proof is in Appendix D.

Lemma 17. *In the random requests model, for any online algorithm $\mathcal{A}$ and any time step $t \in [n]$, we have that $\mathbb{P}(cost_t(\mathcal{A}) \geq 1/n^4) \geq 1 - 2/n^3$ and $\mathbb{E}[cost_t(\mathcal{A})] \geq \frac{1}{2(n+1)}$.*

Next, to show that Lemma 18 holds for any online algorithm $\mathcal{A}$ that makes neighboring matches, we use the hybrid lemma on the hybrid algorithm $\mathcal{H}^m_{\mathcal{A}}$, and we abuse notation with $\mathcal{H}^m$.

Lemma 18. *In the random requests model, there exists a constant $C > 0$ such that, for any instance and for any online algorithm $\mathcal{A}$ that makes neighboring matches, $\mathbb{E}[cost(\mathcal{G})] \leq C \log(n)\mathbb{E}[cost(\mathcal{A})]$.*

Proof. Note that $\{\text{cost}_m(\mathcal{A}) \geq 1/n^4\}$ is an event that depends only on S_{m-1} and r_m and that $\mathcal{A}$ makes neighboring matches; hence we can use the hybrid lemma to get

$$
\begin{aligned}
&\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)] \\
&= \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) \mid \text{cost}_m(\mathcal{A}) \geq 1/n^4] \cdot \mathbb{P}(\text{cost}_m(\mathcal{A}) \geq 1/n^4) \\
&\qquad + \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) \mid \text{cost}_m(\mathcal{A}) < 1/n^4] \cdot \mathbb{P}(\text{cost}_m(\mathcal{A}) < 1/n^4) \\
&\leq \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) \mid \text{cost}_m(\mathcal{A}) \geq 1/n^4] \cdot \mathbb{P}(\text{cost}_m(\mathcal{A}) \geq 1/n^4) + n \cdot 2/n^3 && \text{Lemma 17} \\
&\leq C\mathbb{E}[\big(1 + \log\big(\tfrac{1}{cost_m(\mathcal{A})}\big)\big)\text{cost}_m(\mathcal{A}) \mid \text{cost}_m(\mathcal{A}) \geq 1/n^4] \cdot \mathbb{P}(\text{cost}_m(\mathcal{A}) \geq 1/n^4) + 2n^{-2} && \text{Hybrid lemma} \\
&\leq C(1 + 4\log(n))\mathbb{E}[\text{cost}_m(\mathcal{A}) \mid \text{cost}_m(\mathcal{A}) \geq 1/n^4] \cdot \mathbb{P}(\text{cost}_m(\mathcal{A}) \geq 1/n^4) + 2n^{-2} \\
&\leq C(1 + 4\log(n))\mathbb{E}[\text{cost}_m(\mathcal{A})] + 2n^{-2} \\
&= C' \log(n)\mathbb{E}[\text{cost}_m(\mathcal{A})].
\end{aligned}
$$

Since $\mathcal{H}^n = \mathcal{A}$ and $\mathcal{H}^0 = \mathcal{G}$, we conclude that

$$
\begin{aligned}
\mathbb{E}[cost(\mathcal{G}) - cost(\mathcal{A})] = \sum_{m=1}^{n} \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)] &\leq C' \log(n) \sum_{m=1}^{n} \mathbb{E}[cost_m(\mathcal{A})] \\
&= C' \log(n)\mathbb{E}[cost(\mathcal{A})].
\end{aligned}
$$

□

It remains to show the existence of a $O(1)$-competitive online algorithm that makes neighboring matches in the random requests model, which is the case for a simple modification of the algorithm fair-bias from [GGPW19]. The proof is deferred to Appendix D.

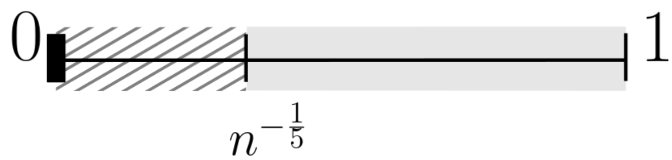

Figure 2: Lower bound instance. There are $n^{4/5} + 4\log^2(n)\sqrt{n}$ servers at 0, no server in the dashed area, and $n - (n^{\frac{4}{5}} + 4\log^2(n)\sqrt{n})$ servers uniformly distributed in the gray area.

Lemma 19. *In the random requests model, there exists a $O(1)$-competitive algorithm that makes neighboring matches.*

We are now ready to prove the main result of Section 4.1.

Theorem 3. *For online matching on the line in the random requests model, the greedy algorithm achieves an $O(\log n)$-competitive ratio.*

Proof. By Lemma 19, there exists $\mathcal{A}$ a $O(1)$-competitive algorithm in the random requests model that makes neighboring matches. For such an algorithm $\mathcal{A}$, we have, by Lemma 18, that $\mathbb{E}[cost(\mathcal{G})] \leq C\log(n)\mathbb{E}[cost(\mathcal{A})]$. We conclude that greedy is $O(\log n)$-competitive. □

4.2 Overview of the lower bound

The $\Omega(\log n)$ lower bound is the main technical proof of this paper. It is obtained by analyzing another hybrid algorithm to show that, on some instance, greedy makes mistakes that have an intricate cascading effect on the cost of future requests. In this section, we give an overview of the proof of the lower bound. A reader interested in the complete analysis can directly skip to Section 4.3.

Description of the instance.

There are $n^{4/5} + 4\log(n)^2\sqrt{n}$ servers located at point 0, there are no servers in the interval $(0, n^{-1/5}]$ and the remaining $n - (n^{4/5} + 4\log(n)^2\sqrt{n})$ servers are uniformly spread in the interval $(n^{-1/5}, 1]$. More precisely, for all $j \in [n^{4/5} + 4\log(n)^2\sqrt{n}]$, we set $s_j = 0$. Then, we let $\tilde{n} := n - 4\log(n)^2\sqrt{n}/(1-n^{-1/5})$, and for all $j \in \{1, \ldots, n-n^{4/5}-4\log(n)^2\sqrt{n})\}$, we set $s_{(n^{4/5}+4\log(n)^2\sqrt{n})+j} = n^{-1/5} + \frac{j}{\tilde{n}}$ (see Figure 2 for an illustration of the instance). We note that, interestingly, the servers are almost uniform since a $1 - o(1)$ fraction of the servers are uniformly spread in an interval $(o(1), 1]$.

Analysis of the instance. We compare the greedy algorithm to the algorithm $\mathcal{A}$ that, for all $t \in [n]$, matches r_t to a free server at location 0 if $r_t \in [0, n^{-1/5}]$ and $S_{\mathcal{A},t-1} \cap \{0\} \neq \emptyset$, and, otherwise, matches r_t greedily. Note that for the instance defined above, $\mathcal{A}$ is a better algorithm than greedy since the expected total number of requests in $[0, n^{-1/5}]$ is $n^{-1/5} \cdot n = n^{4/5}$, which is less than the number of servers at position 0. The main part of the proof is to lower bound $\mathbb{E}[cost(\mathcal{H}_{\mathcal{A}}^{m-1}) - cost(\mathcal{H}_{\mathcal{A}}^{m})]$, i.e., the increase in cost from switching from algorithm $\mathcal{A}$ to the greedy algorithm $\mathcal{G}$ one step earlier in hybrid algorithm $\mathcal{H}_{\mathcal{A}}^{m-1}$ compared to $\mathcal{H}_{\mathcal{A}}^{m}$. As we will show, matching a request in $[0, n^{-1/5}]$ greedily at time $t = m$ instead of matching it to a server at location 0 causes a cascading increase in costs at future time steps for $\mathcal{H}_{\mathcal{A}}^{m-1}$ compared to $\mathcal{H}_{\mathcal{A}}^{m}$ due to the different available servers, even though these two algorithms both match requests greedily at time steps $t > m$.

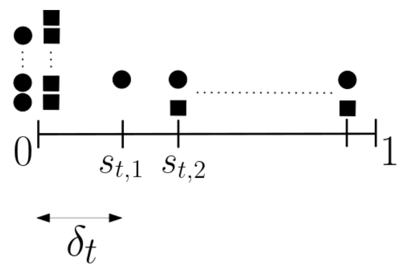

Figure 3: Sets of free servers for $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ at all time steps (with the circles for S_t and the squares for S'_t).

The first lemma shows that at every time step t, there are at most two servers in the symmetric difference between the sets of free servers $S_{\mathcal{H}^m_{\mathcal{A}},t}$ and $S_{\mathcal{H}^{m-1}_{\mathcal{A}},t}$, and that the potential extra free server in $S_{\mathcal{H}^{m-1}_{\mathcal{A}},t}$ is always located at 0 whereas the potential extra free server in $S_{\mathcal{H}^m_{\mathcal{A}},t}$ is the leftmost free server that is not at location 0 (see Figure 3). To ease notation, we write $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ instead of $\mathcal{H}^m_{\mathcal{A}}$ and $\mathcal{H}^{m-1}_{\mathcal{A}}$ and S_t and S'_t instead of $S_{\mathcal{H}^m,t}$ and $S_{\mathcal{H}^{m-1},t}$.

Lemma 20. *Let R be n arbitrary requests and S_0 be n arbitrary servers. Then, for all $t \in \{0, \dots, m-1\}$, we have $S_t = S'_t$, and for all $t \geq m$, either $S_t = S'_t$ or $S'_t = S_t \cup \{0\} \setminus \{\min\{s \in S_t : s > 0\}\}$ (and $\{s \in S_t : s > 0\} \neq \emptyset$).*

To bound $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)]$, we analyze the gap $\delta_t := \min\{s \in S_t : s > 0\}$ between the unique available server in $S'_t \setminus S_t = \{0\}$ and the unique available server in $S_t \setminus S'_t = \{\min\{s \in S_t : s > 0\}\}$. If $S_t = S'_t$, then there is no gap and we define $\delta_t = 0$. The next lemma formally bounds $\mathbb{E}[\sum_{t=m+1}^{n}(cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)|\delta_m, S_m]$ as a function of the gap δ_t.

Lemma 21. *For all $m \in [n]$, we have that*

$$\mathbb{E}\Big[\sum_{t=m+1}^{n}(cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)|\delta_m, S_m\Big] \geq \frac{1}{2}\mathbb{E}\Big[\max_{t\in\{0,\dots,\min(t_{\{0\}},t_w)-m\}} \delta_{t+m} - \delta_m|\delta_m, S_m\Big] - \mathbb{P}(t^d > t_{\{0\}}|\delta_m, S_m),$$

where $s_{t,1} = \min\{s > 0 : s \in S_t\}$ and $s_{t,2} = \min\{s > s_{t,1} : s \in S_t\}$; $t_w := \min\{t \geq m : s_{t,2} - s_{t,1} > s_{t,1},\ or\ s_{t,2} = \emptyset\}$, $t^d = \min\{t \geq m : \delta_t = 0\}$ and $t_{\{0\}} := \min\{t \geq m|\ S_t \cap \{0\} = \emptyset\}$.

To prove Lemma 21, we first show some structural properties of the process $\{(\delta_t, S_t)\}_{t\geq 0}$. In particular, we partially characterize the transitions from (δ_t, S_t) to (δ_{t+1}, S_{t+1}) (Lemma 41), and show that if at some time step t, there remains servers at 0 and the gap $s_{2,t} - s_{1,t}$ between the two first servers with positive location in S_t is smaller than the gap δ_t, then the expected difference in cost $\mathbb{E}[cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)]$ at step t is lower bounded by $(\delta_t - \delta_{t-1})/2$.

By Lemma 21, it remains to lower bound the maximum gap δ_t, for $t \geq m$. To analyze this gap, we first need to introduce some additional notation and terminology. We consider a partition $I_0, I_1, \dots$ of $(0, 1]$ into intervals of geometrically increasing size, where $I_i = (y_{i-1}, y_i]$ and $y_i = (3/2)^i n^{-1/5}$ (with the convention $y_{-1} = 0$). In addition, we say that a sequence of requests is regular if, for any $i \in [n]$, the number of requests between any time steps t and t' that are in the interval $[(i-1)/n, i/n]$ sufficiently concentrates (Definition 29). By concentration bounds, a sequence of requests is regular with high probability.

Lemma 22. *With probability at least $1 - n^{-\Omega(\log(n))}$, the sequence of requests is regular.*

When a sequence of requests is regular, we can bound, for algorithm $\mathcal{H}^m$, the gap $s_{t,j+1} - s_{t,j}$ between the j^{th} and $j+1^{th}$ free servers $s_{t,j}$ and $s_{t,j+1}$ with positive location at time $t \in [(1-o(1))n]$ (Lemma 30).

The main technical lemma of the proof is to lower bound the maximum gap δ_t over all $t \geq m$, which we do in the next lemma, where c_1, d_1, c_3 are positive constants.

Lemma 23. *For all $i \in [d_1 \log(n)]$ and $m \leq c_1 n$,*

$$\mathbb{P}\Big(\max_{t\in\{m,\dots,\min(n-n^{c_3},t_{\{0\}})\}} \delta_t \geq y_{i-1} | R \text{ is regular}, \delta_m, S_m\Big) \geq \frac{\delta_m}{y_i} - n^{-\Omega(\log(n))}.$$

Challenges to prove Lemma 23. The main difficulty in proving Lemma 23 is that the value of δ_t at each time step t is dependent on the value of S_t. However, S_t lies in an exponentially-sized state space and it is difficult to compute the exact distribution of S_t at all time steps. The key idea is to separate the analysis of $(\delta_1, \dots, \delta_n)$ and $(S_1, \dots, S_n)$. We first show that with high probability, the servers in $(S_1, \dots, S_n)$ become globally unavailable from left to right (see below an overview of the proof for a more precise statement). Then, we lower bound the probability that for any y and any arbitrary sequence of sets $(S_1 \supseteq \dots \supseteq S_n)$, $\delta = 0$ before all servers in the interval $(0, y]$ have become unavailable. Combining these two properties leads to the desired result.

Overview of the proof of Lemma 23. The first part of the proof analyzes the sets of free servers $S_0 \supseteq \dots \supseteq S_n$ obtained with algorithm $\mathcal{H}^m$. We say that an interval I is depleted at time t if $S_t \cap I = \emptyset$. We let $t_I := \min\{t \geq 0 | S_t \cap I = \emptyset\}$, i.e., t_I is the time at which I is depleted. For simplicity, we write t_i instead of t_{I_i}. We first show in Lemma 34 that for some constant $c_2 \in (1/2, 1)$, if $t_{i-1} \leq n - (1-c_2)^{i-1}n$, then, $t_{i-1} < t_i$. Then, we show in Lemma 37 that if $t_0 < \dots < t_{i-1} \leq n - (1-c_2)^{i-1}n$ and $t_{i-1} < t_i$, then, $t_i \leq n - (1-c_2)^i n$. To show this last result, we lower bound the number of requests matched in I_i until time $\overline{t_i} = \min(t_i, t_{i-1} + c_2(n - t_{i-1}))$. We first show that

$$\begin{aligned}|\{j \in [\overline{t_i}] \,| s_{\mathcal{H}^m}(r_j) \in I_i\}| \geq &\Big[|\{j \in [\overline{t_i}] : r_j \in I_i\}| - |\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \notin I_i\}|\Big] \\ &+ |\{j \in \{t_{i-1} + 1 + c_1(n - t_{i-1}), \dots, \overline{t_i}\} : r_j \in [\tfrac{3}{4}y_{i-1}, y_{i-1}], s_{\mathcal{H}^m}(r_j) \in I_i\}|.\end{aligned}$$

We then lower bound each of these terms separately using Lemma 35, Lemma 36, Lemma 30, and the regularity of the requests sequence. We deduce from this lower bound that if $t_i > n-(1-c_2)^i n$, then the number of requests matched in I_i exceeds the initial number of free servers in I_i, which is a contradiction. Hence the bound $t_i \leq n - (1-c_2)^i n$. Finally, by combining the above inequalities shown in Lemma 34 and Lemma 37, we show inductively that there is a constant $d_1 > 0$ such that the intervals $\{I_i\}_{i\in[d_1 \log(n)]}$ are depleted in increasing order, i.e. that $m < t_1 < \dots < t_{d_1 \log(n)} \leq n - n^{c_3}$ and that $m < t_{\{0\}}$, (Lemma 32), which is the main lemma of this first part.

In the second part of the proof of Lemma 23, we use the characterization from Lemma 41 to prove the following in Lemma 43: conditioning on the gap δ_m and available servers S_m, and for all $y \in [\delta_m, 1]$, we have

$$\mathbb{P}\Big(\min(t_{(0,y]}, t_{\{0\}}) \leq \min(t^d, t_{\{0\}}) \Big| \delta_m, S_m\Big) \geq \frac{\delta_m}{y}.$$

In other words, starting from a gap δ_m, the probability that the gap has not yet disappeared at the time all the servers in $(0, y]$ have been depleted, or that all the servers at location 0 are depleted before either of these events occurs, is lower bounded by $\frac{\delta_m}{y}$.

The third and last part combines the first two parts to obtain Lemma 23. Since we have shown in the first part that the intervals $\{I_j\}$ are depleted in increasing order of j, we have that just before the time t_{y_i} where $(0, y_i] = \cup_{j\leq i} I_j$ is depleted, none of the intervals I_j for $j < i$ have free servers left, hence $\min\{s > 0 : s \in S_{t_{y_i}-1}\} \in I_i$. Hence, if $\delta_{t_{y_i}-1} \neq 0$, we have by the definition of δ_t that $\delta_{t_{y_i}-1} = \min\{s > 0 : s \in S_{t_{y_i}-1}\} \in I_i = (y_{i-1}, y_i]$, which, in particular, implies $\delta_{t_{y_i}-1} \geq y_{i-1}$. Thus, to prove the desired result, it suffices to lower bound the probability that $\delta_{t_{y_i}-1} \neq 0$ and that $t_{y_i} \leq t_{\{0\}}$ and $t_{y_i} \leq n - n^{c_3}$. By using the second part, we show that it is lower bounded by $\frac{\delta_m}{y_i} - n^{-\Omega(\log(n))}$.

Concluding the proof. By combining the main lemma (Lemma 23) with Lemma 21, we can show the following bounds on $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)]$.

Lemma 24.

1. *For any $m > c_1 n$, we have:* $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in [0, y_0]] = -O(n^{-1/5})$.
2. *for any $m \leq c_1 n$, we have:* $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in [0, y_0]] = \Omega(\log(n) n^{-1/5})$.
3. *For any $m \in [n]$, we have:* $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in (y_0, 1]] = 0$.

The last lemma needed is the following bound on `OPT`.

Lemma 25. *For any $n \in \mathbb{N}$, the expected cost `OPT` of the optimal offline matching for our lower bound instance satisfies:* $E[\texttt{OPT}] = O(n^{3/5})$.

By doing a telescoping sum over all $m \in [n]$ and using that $\mathcal{H}^n = \mathcal{A}$ and $\mathcal{H}^0 = \mathcal{G}$, we obtain from Lemma 24 and 25 the lower bound.

Theorem 4. *For online matching on the line in the random requests model, the greedy algorithm achieves an $\Omega(\log n)$-competitive ratio.*

4.3 Greedy is $\Omega(\log n)$-competitive

In this section, we give a more detailed proof of our lower bound result. All omitted proofs can be found in Appendix E.

4.3.1 Preliminaries

Description of the instance.

There are $n^{4/5} + 4\log(n)^2\sqrt{n}$ servers located at point 0, there are no servers in the interval $(0, n^{-1/5}]$ and the remaining $n - (n^{4/5} + 4\log(n)^2\sqrt{n})$ servers are uniformly spread in the interval $(n^{-1/5}, 1]$. More precisely, for all $j \in [n^{4/5} + 4\log(n)^2\sqrt{n}]$, we set $s_j = 0$. Then, we let $\tilde{n} := n - 4\log(n)^2\sqrt{n}/(1-n^{-1/5})$, and for all $j \in \{1, \ldots, n - n^{4/5} - 4\log(n)^2\sqrt{n})\}$, we set $s_{(n^{4/5}+4\log(n)^2\sqrt{n})+j} = n^{-1/5} + \frac{j}{\tilde{n}}$ (see Figure 2 for an illustration of the instance). We note that, interestingly, the servers are almost uniform since a $1 - o(1)$ fraction of the servers are uniformly spread in an interval $(o(1), 1]$.

We now give bounds on the number of servers contained in each subinterval of $[n^{-1/5}, 1]$.

Fact 26. *Let* $\tilde{n} := n - 4\log(n)^2\sqrt{n}/(1-n^{-1/5})$. *For any* $I \subseteq [n^{-1/5}, 1]$, *we have* $|S_0 \cap I| \in [\tilde{n}|I| - 1, \tilde{n}|I| + 3]$

Basic definitions and notations. We first introduce some notation and terminology.

Definition 27. *We consider a partition* $I_0, I_1, \ldots$ *of* $(0,1]$ *into intervals of geometrically increasing size, where* $I_i = (y_{i-1}, y_i]$ *and* $y_i = (3/2)^i n^{-1/5}$ *(with the convention* $y_{-1} = 0$*).*

We now define an algorithm $\mathcal{A}$ to which we will compare the greedy algorithm.

Definition 28. *We let* $\mathcal{A}$ *be the algorithm that, for all* $t \in [n]$, *matches* r_t *to a free server at location* 0 *if* $r_t \in [0, n^{-1/5}]$ *and* $S_{\mathcal{A},t-1} \cap \{0\} \neq \emptyset$, *and, otherwise, matches* r_t *greedily. For all* $m \geq 0$, *we recall that* $\mathcal{H}^m$ *denotes the hybrid algorithm that matches the first* m *requests according to* $\mathcal{A}$, *then, matches greedily the remaining requests to the remaining free servers.*

A useful tool: regularity of the requests sequence. Informally, we define a sequence of requests R regular if in every time interval, its realized density is not much different from its expected density. We now give some intuition about why we define such a notion. Throughout the proof, many random events can be shown to occur with high probability by successive applications of simple Chernoff bounds. Once the sequence of requests is assumed to be regular, these events become deterministic events, which greatly simplifies the analysis.

More formally, we start by discretizing the interval $[0,1]$ as $\mathcal{D} = \{\frac{i}{n} : i \in \{0, \ldots, n\}\}$. For any interval $I = [i_L, i_R] \subseteq [0,1]$, we also consider $d^+(I)$, the smallest interval with end points in $\mathcal{D}$ that contains I, and $d^-(I)$, the largest interval with end points in $\mathcal{D}$ contained in I.

1. $d^+(I) := [d_L^+, d_R^+]$, with $d_L^+ = \max\{x \in \mathcal{D} | x \leq i_L\}$ and $d_R^+ = \min\{x \in \mathcal{D} | x \geq i_R\}$
2. $d^-(I) := [d_L^-, d_R^-]$, with $d_L^- = \min\{x \in \mathcal{D} | x \geq i_L\}$ and $d_R^- = \max\{x \in \mathcal{D} | x \leq i_R\}$.

Definition 29. *We say that a realization* R *of the sequence of requests is regular if for all* $d, d' \in \mathcal{D}$ *such that* $d < d'$, *and for all* $t, t' \in [n]$ *such that* $t < t'$,

1. $|\{j \in \{t, \ldots, t'\}|\, r_j \in [d, d']\}| \geq (d'-d)(t'-t) - \log(n)^2\sqrt{(d'-d)(t'-t)}$,
2. *and if* $(d'-d)(t'-t) = \Omega(1)$, *then*
$$|\{j \in \{t, \ldots, t'\}|\, r_j \in [d, d']\}| \leq (d'-d)(t'-t) + \log(n)^2\sqrt{(d'-d)(t'-t)}.$$

We now show that R is regular with high probability.

Lemma 22. *With probability at least* $1 - n^{-\Omega(\log(n))}$, *the sequence of requests is regular.*

Proof. Note that for all $d, d' \in D$ such that $d < d'$ and $t, t' \in [n]$ such that $t < t'$, $|\{j \in \{t, \ldots, t'\}|\, r_j \in [d, d']\}|$ follows a binomial distribution $\mathcal{B}(t'-t, d'-d)$. Hence the lemma results from a direct application of Chernoff Bounds (Lemma 48) and a union bound over all $d, d' \in D$ and $t, t' \in [n]$. □

We now show a property that is implied by the regularity of a sequence R of requests. We define $m_t = |S_t \cap (0,1]|$, and we denote by $0 < s_{t,1} < \ldots < s_{t,m_t} \leq 1$ the locations of the m_t free servers with positive location in S_t. For some small $\epsilon > 0$, we define $c_3 = \frac{4}{5} + \epsilon$. The following lemma upper bounds the distance between two consecutive free servers with positive location in S_t at time $t \in [n - o(n)]$ for algorithm $\mathcal{H}^m$ assuming that R is regular.

Lemma 30. *Assume that the sequence of requests is regular. Then, for n large enough and for all $t \in [n - n^{c_3}]$ and $j \in [m_t - 1]$, we have $s_{t,j+1} - s_{t,j} \leq 2\log(n)^4 n^{1-2c_3}$.*

4.3.2 Upper bound on the cost of the optimal offline matching

We first introduce a useful lemma.

Lemma 31. *Let $m \geq 0$ and $R = \{r_1, \ldots, r_{|R|}\}$ be a set of at most m requests uniformly drawn from the interval $(0, 1]$ and $Z = \{z_1, \ldots, z_m\}$ be a set of m servers such that for all $i \in \{1, \ldots, m\}$, $z_i = \frac{i}{m}$. Then, the optimal matching M^* between Z and R satisfies $\mathbb{E}(cost(M^*)) = O(\sqrt{m})$.*

Proof. We assume without loss of generality that R contains exactly m requests and we let $r_{(1)} < \ldots < r_{(m)}$ denote the ordered statistics of R. In this case, we claim that an optimal matching M^* between R and Z is to match each $r_{(i)}$ to z_i for all $i \in \{1, \ldots, m\}$ (see the proof of Theorem 2.5 in [AALS21] for a proof of this fact).

Now, it is a known fact that for all $i \in \{1, \ldots, m\}$, $r_{(i)}$ follows a Beta distribution $B(i, m+1-i)$ (see [Sin10]). In particular, we have that $\mathbb{E}[r_{(i)}] = \frac{i}{m+1}$ and $\text{std}(r_{(i)}) = \sqrt{\frac{i(m-i+1)}{(m+1)^2(m+2)}} \leq \frac{1}{\sqrt{m}}$. We thus obtain

$$\begin{aligned}
\mathbb{E}(\text{cost}(M^*)) &= \sum_{i=1}^{m} \mathbb{E}(|r_{(i)} - z_i|) \\
&\leq \sum_{i=1}^{m} \mathbb{E}(|r_{(i)} - \mathbb{E}(r_{(i)})|) + \mathbb{E}(|\mathbb{E}(r_{(i)}) - z_i|) \\
&\leq \sum_{i=1}^{m} \text{std}(r_{(i)}) + \sum_{i=1}^{m} \mathbb{E}(|\mathbb{E}(r_{(i)}) - z_i|) && \text{(lemma 49)} \\
&\leq \sum_{i=1}^{m} \frac{1}{\sqrt{m}} + \sum_{i=1}^{m} |\tfrac{i}{m+1} - \tfrac{i}{m}| \\
&= O(\sqrt{m}).
\end{aligned}$$

□

We now give an upper bound on the cost of the optimal offline matching for our lower bound instance.

Lemma 25. *For any $n \in \mathbb{N}$, the expected cost $\texttt{OPT}$ of the optimal offline matching for our lower bound instance satisfies: $E[\texttt{OPT}] = O(n^{3/5})$.*

Proof. For a given realization R of the requests sequence, we partition the requests into $R_1 = \{r \in R : r \in [0, n^{-1/5}]\}$ and $R_2 = \{r \in R : r \in (n^{-1/5}, 1]\}$. We also let $\overline{R}_1$ be the $n^{4/5}$ requests of R_1 that arrived first, or $\overline{R}_1 = R_1$ if $|R_1| < n^{4/5}$, and let $\overline{R}_2$ be the $n - (n^{4/5} + 4\log(n)^2\sqrt{n})$ requests of R_2 that arrived first, or $\overline{R}_2 = R_2$ if $|R_2| < n - (n^{4/5} + 4\log(n)^2\sqrt{n})$.

We now define the following matching M, where for all $r \in R$, $s_M(r)$ denotes the server to which r is matched, and for any subset $\tilde{R}$ of the requests, $M|_{\tilde{R}}$ denote the restriction of M to $\tilde{R}$:

- For all $r \in \overline{R}_1$, $s_M(r) = 0$.

- $M|_{\overline{R}_2}$ is an optimal matching between $\overline{R}_2$ and $S_0 \cap (n^{-1/5}, 1]$.
- The remaining requests are matched arbitrarily to the remaining free servers.

Note that M is well defined since $|\overline{R}_1| \leq n^{4/5} \leq |S_0 \cap \{0\}|$ and $|\overline{R}_2| \leq n - (n^{4/5} + 4\log(n)^2\sqrt{n}) \leq |S_0 \cap (n^{-1/5}, 1]|$.

Now, for all $r \in R_1$, since $r \in [0, n^{-1/5}]$, we have $|r - s_M(r)| = |r - 0| \leq n^{-1/5}$, hence

$$\mathbb{E}[\text{cost}(M|_{\overline{R}_1})] = \mathbb{E}[\sum_{r \in \overline{R}_1} |s_M(r) - r|] \leq \mathbb{E}[|\overline{R}_1|] \cdot n^{-1/5} \leq n^{4/5} \cdot n^{-1/5} = n^{3/5}. \tag{1}$$

Next, note that the requests in $\overline{R}_2$ are uniform i.i.d. in $(n^{-1/5}, 1]$ and the servers in $S_0 \cap (n^{-1/5}, 1]$ are uniformly spread in $(y_0, 1]$, hence by using Lemma 31 and a simple scaling argument, we get

$$\mathbb{E}[\text{cost}(M|_{\overline{R}_2})] = O(\sqrt{|\overline{R}_2|}) = O(\sqrt{n}). \tag{2}$$

Now, note that $|R_1| = |\{r \in R : r \in [0, n^{-1/5}]\}|$ follows a binomial distribution $\mathcal{B}(n, n^{-1/5})$ with mean $n^{4/5}$ and standard deviation $\sqrt{n^{4/5}(1 - n^{-1/5})}$, thus by Lemma 49, we have $\mathbb{E}[\max(0, |R_1| - n^{4/5})] \leq \mathbb{E}[||R_1| - n^{4/5}|] \leq \sqrt{n^{4/5}(1 - n^{-1/5})} \leq \sqrt{n}$. Since by definition, $R_1 \setminus \overline{R}_1$ contains $\max(0, |R_1| - n^{4/5})$ elements, we thus have

$$\mathbb{E}[|R_1 \setminus \overline{R}_1|] = \mathbb{E}[\max(0, |R_1| - n^{4/5})] \leq \sqrt{n}.$$

We also have that $|R_2| = |\{r \in R : r \in (n^{-1/5}, 1]\}|$ follows a binomial distribution $\mathcal{B}(n, 1 - n^{-1/5})$ with mean $n - n^{4/5}$ and standard deviation $\sqrt{(n - n^{4/5})n^{-1/5}}$. Hence, by Lemma 49, we have $\mathbb{E}[\max(0, |R_2| - (n - n^{4/5}))] \leq \mathbb{E}[||R_2| - (n - n^{4/5})|] \leq \sqrt{(n - n^{4/5})n^{-1/5}} \leq \sqrt{n}$. Since by definition, $R_2 \setminus \overline{R}_2$ contains $\max(0, |R_2| - (n - n^{4/5} - 4\log(n)^2\sqrt{n}))$ elements, we thus have

$$\begin{aligned}
\mathbb{E}[|R_2 \setminus \overline{R}_2|] &= \mathbb{E}[\max(0, |R_2| - (n - n^{4/5} - 4\log(n)^2\sqrt{n}))] \\
&\leq \mathbb{E}[\max(0, |R_2| - (n - n^{4/5}))] + 4\log(n)^2\sqrt{n} \\
&= \tilde{O}(\sqrt{n}).
\end{aligned}$$

Since for all $r \in R$, we have $|s_M(r) - r| \leq 1$, we get

$$\mathbb{E}[\text{cost}(M|_{(R_1 \setminus \overline{R}_1) \cup (R_2 \setminus \overline{R}_2)})] \leq \mathbb{E}[|R_1 \setminus \overline{R}_1|] + \mathbb{E}[|R_2 \setminus \overline{R}_2|] = \tilde{O}(\sqrt{n}). \tag{3}$$

Combining (1), (2) and (3), we finally get

$$\mathbb{E}[\texttt{OPT}] \leq \mathbb{E}[\text{cost}(M)] = \mathbb{E}[\text{cost}(M|_{(R_1 \setminus \overline{R}_1) \cup (R_2 \setminus \overline{R}_2)})] + \mathbb{E}[\text{cost}(M|_{\overline{R}_1})] + \mathbb{E}[\text{cost}(M|_{\overline{R}_2})] = O(n^{3/5}). \qquad \square$$

4.3.3 Analysis of $(S_1, \ldots, S_n)$.

We first introduce a few constants that will be used thoughout the proof. We recall that $c_3 = \frac{4}{5} + \epsilon$ for some small $\epsilon > 0$. We also define the following constants: $c_1 = \frac{2}{9}(1 - \epsilon), c_2 = \frac{2}{3}\left((1 + \epsilon) + \frac{1}{9}(1 - \epsilon)\right)$. Note that in particular, we have $1 > c_2 > 1/2 > c_1 > 0$. In addition, we define $d_1 := (1 - c_3)/\log(1/(1 - c_2))$.

In this section, we consider a fixed value $m \leq c_1 n$ and we give some global property of the sequence $(S_1, \ldots, S_n)$ of sets of free servers for $\mathcal{H}^m$. More precisely, we first define for all interval I

the time $t_I := \min\{t \geq 0 | S_t \cap I = \emptyset\}$ at which the last free server of I is matched to some request (we say that I is **depleted** at time t_I). The objective is to show that during the execution of $\mathcal{H}^m$, and for all $i < j$, the interval I_i is depleted at an earlier time step than I_j, and that all intervals $\{I_i\}_{i\in[d_1 \log(n)]}$ are depleted between times m and $n - n^{c_3}$ (which is formally stated in the next lemma, whose proof is given at the end of the section).

Lemma 32. *Let $m \leq c_1 n$ and consider algorithm $\mathcal{H}^m$. Then, assuming that the sequence of requests R is regular, we have that $c_1 n < t_1 < \ldots < t_{d_1 \log(n)} \leq n - n^{c_3}$. In addition, we have $c_1 n \leq t_{\{0\}}$.*

Before presenting the proof of Lemma 32, we introduce a few technical properties. We first show a simple but useful lemma.

Lemma 33. *For all $i \in \{0, \ldots, d_1 \log(n)\}$, we have $(1 - c_2)^i n \geq n^{c_3}$.*

Proof. Let $i \in \{0, \ldots, d_1 \log(n)\}$. Then,

$$(1 - c_2)^i n = e^{-i \log(1/(1-c_2)) + \log(n)} \geq e^{-\frac{(1-c_3)\log(n)}{log(1/(1-c_2))} \log(1/(1-c_2)) + \log(n)} = e^{-(1-c_3)\log(n) + log(n)} = n^{c_3},$$

where the inequality is by definition of d_1 and since $i \leq d_1 \log(n)$. ☐

We now show a number of properties that are satisfied for all $i \in [d_1 \log(n)]$ under the assumption that the sequence of requests R is regular.

We first show that if the depletion time t_{i-1} of interval I_{i-1} is small enough, then I_{i-1} is depleted before I_i.

Lemma 34. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular and that $t_{i-1} \leq n - (1 - c_2)^{i-1} n$. Then, $t_{i-1} < t_i$.*

Next, we show that if the intervals $I_1, \ldots, I_{i-1}$ are depleted in increasing order of i and that t_{i-1} is small enough, then t_i is also small enough. To this end, we first introduce a couple lemmas. The first one upper bounds the number of requests that arrived in I_i and were matched outside of I_i until time $\min(t_i, t_{i-1} + c_2(n - t_{i-1}))$.

Lemma 35. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular, that $t_0 < \ldots < t_{i-1} \leq n - (1 - c_2)^{i-1} n$ and that $t_{i-1} < t_i$. Let $\overline{t_i} := \min(t_i, t_{i-1} + c_2(n - t_{i-1}))$. Then,*

$$|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \notin I_i\}| = \tilde{O}(\sqrt{n}).$$

The next lemma lower bounds the number of requests that arrived in the interval $[\frac{3}{4} y_{i-1}, y_{i-1}]$ and were matched inside I_i from time $t_{i-1} + 1 + c_1(n - t_{i-1})$ to time $\min(t_i, t_{i-1} + c_2(n - t_{i-1}))$.

Lemma 36. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular, that $t_0 < \ldots < t_{i-1}$ and that $t_{i-1} < t_i$. Let $\overline{t_i} := \min(t_i, t_{i-1} + c_2(n - t_{i-1}))$. Then,*

$$\begin{aligned} &|\{j \in \{t_{i-1} + 1 + c_1(n - t_{i-1}), \ldots, \overline{t_i}\} : r_j \in [\tfrac{3}{4} y_{i-1}, y_{i-1}], s_{\mathcal{H}^m}(r_j) \in I_i\}| \\ &\geq \frac{1}{2}(\overline{t_i} - t_{i-1} - c_1(n - t_{i-1}))|I_i| - \tilde{O}(\sqrt{n}). \end{aligned}$$

Using the two above lemmas, we show that if the intervals $I_1, \ldots, I_{i-1}$ are depleted in increasing order of i and t_{i-1} is small enough, then $t_i \leq n - (1 - c_2)^i n$.

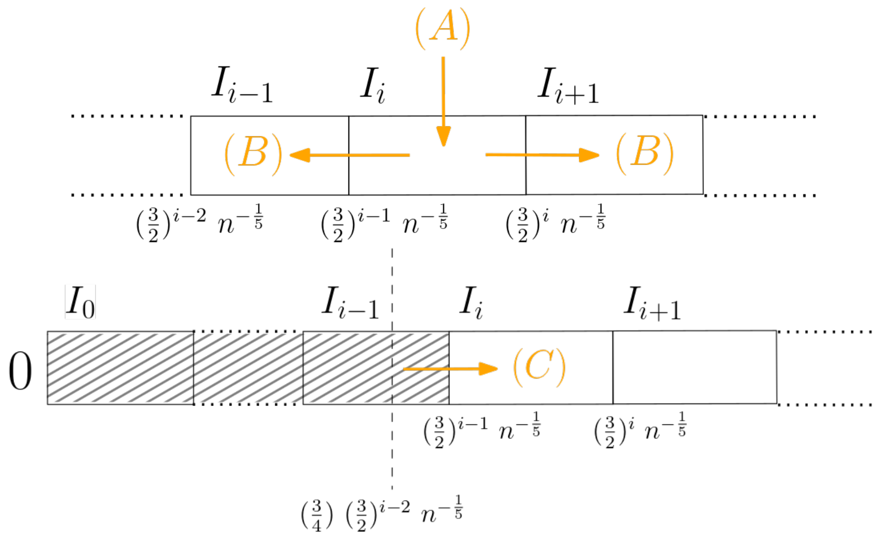

Figure 4: Requests in and out of I_i up to time $\overline{t}_i = \min(t_i, t_{i-1} + c_2(n - t_{i-1}))$, with (A) the total number of requests that arrived in I_i from time 0 to $\overline{t}_i$, (B) the total number of requests that arrived in I_i and were matched outside I_i from time 0 to $\overline{t}_i$, and (C) the total number of requests that arrived in $[\frac{3}{4}y_{i-1}, y_{i-1}]$ and were matched inside I_i from time $t_{i-1} + 1 + c_1(n - t_{i-1})$ to time $\overline{t}_i$ (note that there are no free servers in the dashed area for times $t \geq t_{i-1}$).

Lemma 37. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular, that $t_0 < \ldots < t_{i-1} \leq n - (1 - c_2)^{i-1} n$, and that $t_{i-1} < t_i$. Then, $t_i \leq n - (1 - c_2)^i n$.*

Proof. Fix $i \in [d_1 \log(n)]$. We start by lower bounding the number of requests that were matched to servers inside I_i until time $\overline{t_i} := \min(t_i, t_{i-1} + c_2(n - t_{i-1}))$ included. First, we have (see Figure 4):

$$
\begin{aligned}
&|\{j \in [\overline{t_i}]\ | s_{\mathcal{H}^m}(r_j) \in I_i\}| \\
&= |\{j \in [\overline{t_i}]\ | r_j \in I_i, s_{\mathcal{H}^m}(r_j) \in I_i\}| + |\{j \in [\overline{t_i}]\ | r_j \notin I_i, s_{\mathcal{H}^m}(r_j) \in I_i\}| \\
&\geq \Big[|\{j \in [\overline{t_i}] : r_j \in I_i\}| && \text{(A)} \\
&- |\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \notin I_i\}| \Big] && \text{(B)} \\
&+ |\{j \in \{t_{i-1} + 1 + c_1(n - t_{i-1}), \ldots, \overline{t_i}\} : r_j \in [\tfrac{3}{4} y_{i-1}, y_{i-1}], s_{\mathcal{H}^m}(r_j) \in I_i\}| && \text{(C)}
\end{aligned}
$$

where the lower bound in (C) is since $I_i = (y_{i-1}, y_i]$; hence $[\frac{3}{4} y_{i-1}, y_{i-1}] \subseteq [0, 1] \setminus I_i$.

We now bound each of these three terms separately. Since we assumed that the sequence of requests is regular, by applying the first regularity condition with $t = 0$, $t' = \overline{t_i}$, $[d, d'] = d^-(I_i)$), we have that

$$
|\{j \in [\overline{t_i}] : r_j \in I_i\}| \geq |\{j \in [\overline{t_i}] : r_j \in d^-(I_i)\}| \geq d^-(I_i)\overline{t_i} - \log(n)^2 \sqrt{d^-(I_i)\overline{t_i}} = |I_i|\overline{t_i} - \tilde{O}(\sqrt{n}).
$$

By Lemma 35, we have that

$$
|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \notin I_i\}| = \tilde{O}(\sqrt{n}),
$$

and by Lemma 36, we have that

$$
\begin{aligned}
&|\{j \in \{t_{i-1}+1+c_1(n-t_{i-1}),\ldots,\overline{t_i}\} : r_j \in [\tfrac{3}{4}y_{i-1}, y_{i-1}], s_{\mathcal{H}^m}(r_j) \in I_i\}| \\
&\geq \frac{1}{2}(\overline{t_i} - t_{i-1} - c_1(n-t_{i-1})|I_i| - \tilde{O}(\sqrt{n}).
\end{aligned}
$$

Combining the four previous inequalities gives

$$
|\{j \in [\overline{t_i}] : s_{\mathcal{H}^m}(r_j) \in I_i\}| \geq |I_i|\Big[\overline{t_i} + \frac{1}{2}(\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}))\Big] - \tilde{O}(\sqrt{n}).
$$

Now, $|\{j \in [\overline{t_i}] : s_{\mathcal{H}^m}(r_j) \in I_i\}|$ is trivially upper bounded by the initial number of servers available in I_i, which, by Fact 26, is at most $|I_i|\tilde{n}+1 < |I_i|(n+1)$. By combining this upper bound with the above lower bound and by simplifying the $|I_i|$ on both sides, we obtain

$$
n+1 > \overline{t_i} + \frac{1}{2}(\overline{t_i} - t_{i-1} - c_1(n-t_{i-1})) - \tilde{O}(\sqrt{n}/|I_i|) = \overline{t_i} + \frac{1}{2}(\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}) - \tilde{O}(n^{7/10}), \quad (4)
$$

where the equality is since $|I_i| = \Omega(n^{-1/5})$ for all $i \geq 0$.

Next, we show that the previous inequality implies that $\overline{t_i} = t_i$. Assume by contradiction that $\overline{t_i} = t_{i-1} + c_2(n-t_{i-1})$. We get

$$
\begin{aligned}
&t_{i-1} + c_2(n-t_{i-1}) + \frac{1}{2}(t_{i-1} + c_2(n-t_{i-1}) - t_{i-1} - c_1(n-t_{i-1})) - \tilde{O}(n^{7/10}) \\
&= t_{i-1} + (n-t_{i-1})(c_2(1+1/2) - c_1/2) - \tilde{O}(n^{7/10}) \\
&= t_{i-1} + (n-t_{i-1})(1+\epsilon) - \tilde{O}(n^{7/10}) \\
&= n + \epsilon(n-t_{i-1}) - \tilde{O}(n^{7/10}) \\
&\geq n + \epsilon n^{c_3} - \tilde{O}(n^{7/10}) \\
&> n+1,
\end{aligned}
$$

where the second equality is since $c_2(1+1/2) - c_1/2 = \frac{3}{2}\cdot\frac{2}{3}\big((1+\epsilon) + \frac{1}{9}(1-\epsilon)\big) - \frac{1}{2}\cdot\frac{2}{9}(1-\epsilon) = (1+\epsilon)$. The first inequality is since $t_{i-1} \leq n - (1-c_2)^{i-1}n \leq n - n^{c_3}$ (by using the assumption of the lemma and from Lemma 30), and the last inequality is since we set $c_3 > 3/4$ and assumed n large enough.

Hence, by (4), we cannot have $\overline{t_i} = t_{i-1} + c_2(n-t_{i-1})$, thus $\overline{t_i} = \min(t_{i-1} + c_2(n-t_{i-1}), t_i) = t_i$.

Using the assumption that $t_{i-1} \leq n - (1-c_2)^{i-1}n$, we conclude that

$$
t_i \leq t_{i-1} + c_2(n-t_{i-1}) = c_2 n + (1-c_2)t_{i-1} \leq c_2 n + (1-c_2)(n - (1-c_2)^{i-1}n) = n - (1-c_2)^i n. \quad \square
$$

Finally, we show in the two following lemmas that I_1 is not yet depleted at time $c_1 n$, and that if it is the case, we also have that $\{0\}$ is not yet depleted at time $c_1 n$.

Lemma 38. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular and that $t_1 < t_2$. Then, $c_1 n < t_1$.*

Lemma 39. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular and that $c_1 n < t_1$, then $c_1 n < t_{\{0\}}$.*

We are now ready to present the proof of Lemma 32, that we restate below for convenience.

Lemma 32. *Let $m \leq c_1 n$ and consider algorithm $\mathcal{H}^m$. Then, assuming that the sequence of requests R is regular, we have that $c_1 n < t_1 < \ldots < t_{d_1 \log(n)} \leq n - n^{c_3}$. In addition, we have $c_1 n \leq t_{\{0\}}$.*

Proof. Fix $m \in [c_1 n]$ and assume that the sequence of requests is regular. We first show by induction on i that, for n sufficiently large, we have $t_0 < \ldots < t_i \leq n - (1-c_2)^i n$ for all $i \in \{0, \ldots, d_1 \log(n)\}$.

The base case is immediate since by construction of the instance, $I_0 \cap S_0 = (0, n^{-1/5}] \cap S_0 = \emptyset$, which implies that $t_0 = 0 = n - (1-c_2)^0 n$.

Now, for n large enough, let $i \in [d_1 \log(n)]$ and assume that $t_0 < \ldots < t_{i-1} \leq n - (1-c_2)^{i-1} n$. Then, in particular, we have that $t_{i-1} \leq n - (1-c_2)^{i-1} n$, hence $t_{i-1} < t_i$ by Lemma 34. By combining this with the assumption that $t_0 < \ldots < t_{i-1} \leq n - (1-c_2)^{i-1} n$, we obtain that $t_i \leq n - (1-c_2)^i n$ by Lemma 37. Hence, we get $t_0 < \ldots < t_i \leq n - (1-c_2)^i n$, which concludes the inductive case.

By applying the previous inequalities with $i = d_1 \log(n)$, and by Lemma 33, we thus have $t_1 < \ldots < t_{d_1 \log(n)} \leq n - (1-c_2)^{d_1 \log(n)} n \leq n - n^{c_3}$. In addition, since $t_1 < t_2$ and since we assumed $m \leq c_1 n$, we have that $m \leq c_1 n < t_1$ by Lemma 38, which also implies that $m \leq c_1 n < t_{\{0\}}$ by Lemma 39. We conclude that $m < t_1 < \ldots < t_{d_1 \log(n)} \leq n - n^{c_3}$ and that $m < t_{\{0\}}$. □

4.3.4 Lower bound on $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)]$.

The objective of this section is to use the characterization of the remaining servers $(S_1, \ldots, S_n)$ from Lemma 32 in Section 4.3.3 to prove the following lemma, in which we lower bound the total difference of cost between algorithms $\mathcal{H}^{m-1}$ and $\mathcal{H}^m$ conditioned on the location of request r_m. The proof is given at the end of the section.

Lemma 24.

1. *For any $m > c_1 n$, we have: $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in [0, y_0]] = -O(n^{-1/5})$.*
2. *for any $m \leq c_1 n$, we have: $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in [0, y_0]] = \Omega(\log(n) n^{-1/5})$.*
3. *For any $m \in [n]$, we have: $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in (y_0, 1]] = 0$.*

Structural properties. In order to prove Lemma 24, we first introduce a few structural properties about the sets $(S_0, \ldots, S_t)$ and $(S'_0, \ldots, S'_t)$ of free servers for $\mathcal{H}^{m-1}$ and $\mathcal{H}^m$. We first show that at every time step t, there are at most two servers in the symmetric difference between S_t and S'_t, and that the potential extra free server in S'_t is always located at 0 whereas the potential extra free server in S_t is the leftmost free server in S_t that is not at location 0 (see Figure 3).

Lemma 20. *Let R be n arbitrary requests and S_0 be n arbitrary servers. Then, for all $t \in \{0, \ldots, m-1\}$, we have $S_t = S'_t$, and for all $t \geq m$, either $S_t = S'_t$ or $S'_t = S_t \cup \{0\} \setminus \{\min\{s \in S_t : s > 0\}\}$ (and $\{s \in S_t : s > 0\} \neq \emptyset$).*

Armed with the previous lemma, we define the gap $\delta_t := \min\{s \in S_t : s > 0\}$ between the unique available server in $S'_t \setminus S_t = \{0\}$ and the unique available server in $S_t \setminus S'_t = \{\min\{s \in S_t : s > 0\}\}$. In the following, we let $s_{t,1} = \min\{s > 0 : s \in S_t\}$ and $s_{t,2} = \min\{s > s_{t,1} : s \in S_t\}$ denote the first two servers with positive location for $\mathcal{H}^m$ just after matching r_t.

$r_{t+1} \in \dots$	$[0, \frac{\delta_t}{2}]$	$[\frac{\delta_t}{2}, \frac{\delta_t+w_t}{2}]$	$[\frac{\delta_t+w_t}{2}, \delta_t + \frac{w_t}{2}]$	$[\delta_t + \frac{w_t}{2}, \delta_t + w_t]$	$[\delta_t + w_t, 1]$
S_{t+1}	$S_t \setminus \{0\}$	$S_t \setminus \{\delta_t\}$	$S_t \setminus \{\delta_t\}$	$S_t \setminus \{\delta_t + w_t\}$	$\exists s \in [\delta_t + w_t, 1] \cap S_t :$ $S_t \setminus \{s\}$
δ_{t+1}	δ_t	0	$\delta_t + w_t$	δ_t	δ_t
$\mathbb{E}[\Delta \text{cost}_{t+1} \| \dots]$	≥ 0	≥ 0	$\geq \begin{cases} \frac{w_t}{2} & \text{if } w_t \leq \delta_t \\ 0 & \text{otherwise.} \end{cases}$	≥ 0	≥ 0

Table 1: Values of (δ_{t+1}, S_{t+1}) and expected value of Δcost_{t+1} conditioning on (δ_t, S_t) and on r_{t+1}, assuming that $S_t \cap \{0\} \neq \emptyset$, $\delta_t \neq 0$ and $|S_t \cap (\delta_t, 1]| \geq 1$, and where $w_t := s_{t,2} - s_{t,1}$.

Definition 40. *For all $t \in [n]$, we let* $\delta_t := \begin{cases} 0 & \text{if } S_t = S'_t. \\ s_{t,1} & \text{otherwise.} \end{cases}$

We now present a few properties satisfied by $\{(\delta_t, S_t)\}_{t \geq m}$. We start by a partial characterization of the value of (δ_t, S_t) and of the difference of cost $\Delta\text{cost}_{t+1} := \text{cost}_{t+1}(\mathcal{H}^{m-1}) - \text{cost}_{t+1}(\mathcal{H}^m)$ between the costs incurred by $\mathcal{H}^{m-1}$ and $\mathcal{H}^m$ at time step t as a function of δ_t and S_t.

Lemma 41. *All following properties hold at any time $t \in \{m, \dots, n-1\}$:*

1. *if $\delta_t = 0$, then for all $t' \geq t$, we have $\delta_{t'} = 0$ and $\Delta cost_{t+1} = 0$,*
2. *if $S_t \cap \{0\} \neq \emptyset$, then $\Delta cost_{t+1} \geq 0$.*
3. *if $S_t \cap \{0\} \neq \emptyset$, $\delta_t \neq 0$ and $|S_t \cap (\delta_t, 1]| \geq 1$, then the values of (δ_{t+1}, S_{t+1}) and the expected value of $\Delta cost_{t+1}$ conditioning on (δ_t, S_t) and on r_{t+1} are as given in Table 1, where $w_t := s_{t,2} - s_{t,1}$ and where we write $\mathbb{E}[\Delta cost_{t+1}|...]$ instead of $\mathbb{E}[\Delta cost_{t+1}|(\delta_t, S_t), S_t \cap \{0\} \neq \emptyset, \delta_t \neq 0, |S_t \cap (\delta_t, 1]| \geq 1, r_{t+1} \in \dots]$.*
4. *if $\delta_{t+1} \neq \delta_t$, then $S_{t+1} = S_t \setminus \{\delta_t\}$.*
5. $\mathbb{1}_{S_t \cap \{0\} = \emptyset, \delta_t \neq 0} \cdot \mathbb{E}[\Delta cost_{t+1}|(\delta_t, S_t)] \geq -\mathbb{1}_{S_t \cap \{0\} = \emptyset, \delta_t \neq 0} \cdot \mathbb{P}(\delta_{t+1} = 0|(\delta_t, S_t))$.

In Lemma 43, we use the properties given in Lemma 41 to lower bound the probability that the gap δ has not yet disappeared at the time all servers in $(0, y]$ have been depleted, or that all the servers at location 0 are depleted before either of these events occurs. We first recall that for any interval $I \subseteq [0, 1]$, $t_I := \min\{t \geq m | \; S_t \cap I = \emptyset\}$ is the time at which I is depleted. We also define a couple additional stopping times for $\{(\delta_t, S_t)\}$.

Definition 42.

- ***Distance between** $s_{t,2}$ **and** $s_{t,1}$ **becomes large or** $s_{t,2} = \emptyset$.* *Let $t^w := \min\{t \geq m : s_{t,2} - s_{t,1} > s_{t,1}, \text{ or } s_{t,2} = \emptyset\}$.*
- *δ **disappears.** Let $t^d := \min\{t \geq m : \delta_t = 0\}$.*

Lemma 43. *Conditioning on the gap δ_m and available servers S_m, and for all $y \in [\delta_m, 1]$, we have*

$$\mathbb{P}\Big(\min(t_{(0,y]}, t_{\{0\}}) \leq \min(t^d, t_{\{0\}})\Big|\delta_m, S_m\Big) \geq \frac{\delta_m}{y},$$

We conclude this part by two simple properties. The first is about the initial gap δ_m just after matching request r_m.

Lemma 44.

1. *If $\delta_m > 0$, then $r_m \in [0, y_0]$.*
2. *For all $m \in [n]$, $\delta_m \in [0, 2n^{-1/5}]$.*
3. *For all $m \in [c_1 n]$, $\mathbb{E}[\delta_m | r_m \in [0, y_0]] \geq \frac{n^{-1/5}}{4} - n^{-\Omega(\log(n))}$.*

Finally, we show that if R is regular, then for all $i \in [d_1 \log(n)]$, the interval $(0, y_i]$ is depleted before all servers at location 0 are depleted, and we upper bound the probability that all servers at location 0 are depleted before δ disappears.

Lemma 45. *For all $m \in [n]$ and $i \in [d_1 \log n]$,*

1. *if R is regular, then $t_{(0,y_i]} \leq t_{\{0\}}$.*
2. *$\mathbb{P}(t^d > t_{\{0\}} | r_m \in [0, y_0]) = O(n^{-1/5})$.*

Lower bound on $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)]$ as a function of the gap δ. Using the structural properties stated above, we lower bound the expected difference of cost for matching requests $r_{m+1}, \ldots, r_n$.

Lemma 21. *For all $m \in [n]$, we have that*

$$\mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)|\delta_m, S_m\Big] \geq \frac{1}{2}\mathbb{E}\Big[\max_{t \in \{0,\ldots,\min(t_{\{0\}}, t_w)-m\}} \delta_{t+m} - \delta_m | \delta_m, S_m\Big] - \mathbb{P}(t^d > t_{\{0\}} | \delta_m, S_m),$$

where $s_{t,1} = \min\{s > 0 : s \in S_t\}$ and $s_{t,2} = \min\{s > s_{t,1} : s \in S_t\}$; $t_w := \min\{t \geq m : s_{t,2} - s_{t,1} > s_{t,1},\ \text{or } s_{t,2} = \emptyset\}$, $t^d = \min\{t \geq m : \delta_t = 0\}$ and $t_{\{0\}} := \min\{t \geq m |\ S_t \cap \{0\} = \emptyset\}$.

The full proof is in Appendix 4.3 and we only present here the main steps: by the second property of Lemma 41, we have that while there still are some free servers at location 0, the difference of cost Δcost_{t+1} is always nonnegative. Moreover, we also have, by the third property of Lemma 41 (and the values given in Table 1) that as long as $\delta_t \neq 0$, $|S_t \cap (\delta_t, 1]| \geq 1$, $|S_t \cap \{0\}| \neq \emptyset$ and $w_t \geq \delta_t$, the expected value of Δcost_{t+1} is at least the increase in δ. A telescoping sum over all time steps yields the result.

We also give a simple lower bound on the expected difference of cost for matching requests $r_1, \ldots, r_m$.

Lemma 46. *For all $m \in [n]$, $\mathbb{E}\Big[\sum_{t=1}^{m} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)) | r_m \in [0, y_0]\Big] \geq -n^{-1/5}$.*

Main technical lemma. We are now ready to present the main technical lemma of this part, which is a lower bound on the probability that the gap δ ever exceeds y_{i-1} for all i sufficiently small.

Lemma 23. *For all $i \in [d_1 \log(n)]$ and $m \leq c_1 n$,*

$$\mathbb{P}\Big(\max_{t\in\{m,\ldots,\min(n-n^{c_3},t_{\{0\}})\}} \delta_t \geq y_{i-1} | R \textit{ is regular}, \delta_m, S_m\Big) \geq \frac{\delta_m}{y_i} - n^{-\Omega(\log(n))}.$$

Proof. Fix $m \in \{1, \ldots, c_1 n\}$ and $i \in [d_1 \log(n)]$. For simplicity, we write t_i to denote t_{I_i}, the time at which I_i is depleted during the execution of $\mathcal{H}^m$, and we write t_{y_i} to denote $t_{(0,y_i]}$, the time at which $(0, y_i]$ is depleted.

In the remainder of the proof, we condition on the fact that the sequence of requests is regular. In particular, by Lemma 32, we have that

$$m < t_1 < \ldots < t_{d_1 \log(n)} \leq n - n^{c_3}. \tag{5}$$

We start by lower bounding the probability that $\delta_{t_{y_i}-1} > 0$ conditioning on the variables δ_m, S_m. First, note that if $m < t_{y_i} \leq t^d$, then by definition of t^d, we have that $\delta_{t_{y_i}-1} > 0$. In addition, by definition of t_i, t_{y_i}, and since $I_i = (y_{i-1}, y_i] \subseteq (0, y_i]$, we have $t_i \leq t_{y_i}$. Since by (5), we have $m < t_i$, we get that $m < t_{y_i}$. Finally, since R is regular, we also have, by Lemma 45, that $\min(t_{y_i}, t_{\{0\}}) = t_{y_i}$. Hence, if $\min(t_{y_i}, t_{\{0\}}) \leq \min(t^d, t_{\{0\}})$, then $t_{y_i} = \min(t_{y_i}, t_{\{0\}}) \leq \min(t^d, t_{\{0\}}) \leq t^d$. Therefore, we have

$$\begin{aligned}
&\mathbb{P}(\delta_{t_{y_i}-1} > 0 | \text{R is regular}, \delta_m, S_m)\\
&\geq \mathbb{P}(m < t_{y_i} \leq t^d | \text{R is regular}, \delta_m, S_m)\\
&= \mathbb{P}(t_{y_i} \leq t^d | \text{R is regular}, \delta_m, S_m)\\
&\geq \mathbb{P}\Big(\min(t_{y_i}, t_{\{0\}}) \leq \min(t^d, t_{\{0\}}) | \text{R is regular}, \delta_m, S_m\Big)\\
&\geq \mathbb{P}\Big(\min(t_{y_i}, t_{\{0\}}) \leq \min(t^d, t_{\{0\}}) | \delta_m, S_m\Big) - n^{-\Omega(\log(n))} && \text{R is regular w.h.p. by Lemma 22}\\
&\geq \frac{\delta_m}{y_i} - n^{-\Omega(\log(n))}. && \text{Lemma 43} \quad (6)
\end{aligned}$$

Next, we assume that $\delta_{t_{y_i}-1} > 0$ and we lower bound $\max_{t\in\{m,\ldots,\min(n-n^{c_3},t_{\{0\}})\}} \delta_t$. By definition of δ, we have that for all $t \geq 0$, either $\delta_t = 0$ or $\delta_t = \min\{x > 0 | x \in S_t\}$. Since we assumed $\delta_{t_{y_i}-1} > 0$, we thus have

$$\delta_{t_{y_i}-1} = \min\{x > 0 | x \in S_{t_{y_i}-1}\}. \tag{7}$$

Now, by (5), we have that for all $j \leq i-1$, $t_j < t_i$ (i.e., $(y_{j-1}, y_j]$ is depleted before $(y_{i-1}, y_i]$). Recalling that $t_i = \min\{t \geq m : S_t \cap (y_{i-1}, y_i] = \emptyset\}$ and that $t_{y_i} = \min\{t \geq m : S_t \cap (0, y_i] = \emptyset\}$, we get that $t_i = t_{y_i}$ and that $(0, y_{i-1}] \cap S_{t_{y_i}-1} = \Big(\bigsqcup_{j=0}^{i-1}(y_{j-1}, y_j]\Big) \cap S_{t_{y_i}-1} = \emptyset$. Hence,

$$\min\{x > 0 | x \in S_{t_{y_i}-1}\} \geq y_{i-1}. \tag{8}$$

Combining (8) and (7), we get that $\delta_{t_{y_i}-1} \geq y_{i-1}$. In addition, since $m < t_i \leq n - n^{c_3}$ by (5) and $t_{y_i} = t_i$ as argued above, we have that $m < t_{y_i} \leq n - n^{c_3}$. Since R is regular, we also have, by Lemma 45, that $t_{y_i} \leq t_{\{0\}}$. We deduce that $\max_{t \in \{m,\dots,\min(n-n^{c_3}, t_{\{0\}})\}} \delta_t \geq \delta_{t_{y_i}-1} \geq y_{i-1}$. As a result,

$$\mathbb{P}\Big(\max_{t \in \{m,\dots,\min(n-n^{c_3}, t_{\{0\}})\}} \delta_t \geq y_{i-1} | \text{R is regular}, \delta_m, S_m\Big) \geq \mathbb{P}(\delta_{t_{y_i}-1} > 0 | \text{R is regular}, \delta_m, S_m).$$

Combining this with (6), we finally obtain

$$\mathbb{P}\Big(\max_{t \in \{m,\dots,\min(n-n^{c_3}, t_{\{0\}})\}} \delta_t \geq y_{i-1} | \text{R is regular}, \delta_m, S_m\Big) \geq \frac{\delta_m}{y_i} - n^{-\Omega(\log(n))}. \qquad \square$$

Concluding the proof. We now present the proof of Lemma 24, that we restate below for convenience.

Lemma 24.

1. *For any $m > c_1 n$, we have:* $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in [0, y_0]] = -O(n^{-1/5})$.
2. *for any $m \leq c_1 n$, we have:* $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in [0, y_0]] = \Omega(\log(n) n^{-1/5})$.
3. *For any $m \in [n]$, we have:* $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in (y_0, 1]] = 0$.

Proof. Let $m \in [n]$. Since $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ make the same decisions at all time steps when $r_m \in (y_0, 1]$, it is immediate that $\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in (y_0, 1]] = 0$, which shows the third point of the lemma.

We now show the first two points. By Lemma 21, we have that

$$\mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)) | \delta_m, S_m\Big] \geq \frac{1}{2}\mathbb{E}\Big[\max_{t \in \{0,\dots,\min(t_{\{0\}}, t_w)-m\}} \delta_{t+m} - \delta_m | \delta_m, S_m\Big] - \mathbb{P}(t^d > t_{\{0\}} | \delta_m, S_m). \tag{9}$$

Thus, we first get

$$\begin{aligned}
&\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m) | r_m \in [0, y_0]] \\
&= \mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)) | r_m \in [0, y_0]\Big] + \mathbb{E}\Big[\sum_{t=1}^{m} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)) | r_m \in [0, y_0]\Big] \\
&\geq \mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)) | r_m \in [0, y_0]\Big] - n^{-1/5} \\
&= \int_{(x,S) \in \mathcal{X}} \mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)) | (\delta_m, S_m) = (x, S), r_m \in [0, y_0]\Big] \\
&\qquad \cdot \mathrm{d}\mathbb{P}((x, S) | r_m \in [0, y_0]) - n^{-1/5} \\
&= \int_{(x,S) \in \mathcal{X}} \mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m)) | (\delta_m, S_m) = (x, S)\Big] \cdot \mathrm{d}\mathbb{P}((x, S) | r_m \in [0, y_0]) - n^{-1/5}
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{(x,S)\in\mathcal{X}} [0 - \mathbb{P}(t^d > t_{\{0\}} | (\delta_m, S_m) = (x,S))] \cdot \mathrm{d}\mathbb{P}((x,S)|r_m \in [0,y_0]) - n^{-1/5} \\
&= \int_{(x,S)\in\mathcal{X}} [0 - \mathbb{P}(t^d > t_{\{0\}} | (\delta_m, S_m) = (x,S)), r_m \in [0,y_0]] \cdot \mathrm{d}\mathbb{P}((x,S)|r_m \in [0,y_0]) - n^{-1/5} \\
&= -\mathbb{P}(t^d > t_{\{0\}} | r_m \in [0,y_0]) - n^{-1/5} \\
&= -O(n^{-1/5}),
\end{aligned}
$$

where the first inequality is by Lemma 46, the second and fourth equalities are since conditioned on (δ_m, S_m), $\{(cost_t(\mathcal{H}^m), cost_t(\mathcal{H}^{m-1}))\}_{t\geq m+1}$ is independent on r_m, the second inequality is by (9) and the last equality by Lemma 45. This completes the proof of the first point of Lemma 24.

Next, we prove the second point of the lemma by providing a tighter lower bound on (9) when $m \leq c_1 n$. In the remainder of the proof, we consider a fixed $m \in \{1, \dots, c_1 n\}$.

First, we show that $t^w \geq n - n^{c_3}$. Note that if R is regular, then by Lemma 30, we have that for all $t \in [n - n^{c_3}]$, $s_{t,2} - s_{t,1} \leq 2\log(n)^4 n^{1-2c_3}$. Thus, for n large enough (and since we chose $c_3 > 4/5$), we have that $s_{t,2} - s_{t,1} < n^{-1/5}$. Since by definition of the instance, it is always the case that $s_{t,1} > y_0 = n^{-1/5}$, we thus have $s_{t,2} - s_{t,1} < s_{t,1}$. In addition, since $t \leq n - n^{c_3}$ and $c_3 > 4/5$, we have that for n large enough, $S_t \cap (0,1] \geq S_0 \cap (0,1] - (n - n^{c_3}) = n - (n^{4/5} + 4\log(n)^2\sqrt{n}) - (n - n^{c_3}) = n^{c_3} - n^{4/5} - 4\log(n)^2\sqrt{n} > 2$, thus $s_{t,2} \neq \emptyset$. Since $t^w = \min\{t \geq m : s_{t,2} - s_{t,1} > s_{t,1}, \text{ or } s_{t,2} = \emptyset\}$, we thus have $t \leq t^w$. Hence $t^w \geq n - n^{c_3}$.

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},t^w)-m\}} \delta_{t+m} | \delta_m, S_m, \text{R is regular}\Big) \\
&\qquad\qquad \geq \mathbb{E}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},n-n^{c_3})-m\}} \delta_{t+m} | \delta_m, S_m, \text{R is regular}\Big). \qquad (10)
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},n-n^{c_3})-m\}} \delta_{t+m} | \delta_m, S_m, \text{R is regular}\Big) \\
&\quad = \int_0^1 \mathbb{P}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},n-n^{c_3})-m\}} \delta_{t+m} \geq x | \delta_m, S_m, \text{R is regular}\Big) dx \\
&\quad \geq \sum_{i=0}^{d_1\log(n)} \int_{x\in I_i} \mathbb{P}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},n-n^{c_3})-m\}} \delta_{t+m} \geq x | \delta_m, S_m, \text{R is regular}\Big) dx \\
&\quad \geq \sum_{i=0}^{d_1\log(n)} (y_i - y_{i-1}) \cdot \mathbb{P}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},n-n^{c_3})-m\}} \delta_{t+m} \geq y_i | \delta_m, S_m, \text{R is regular}\Big) \\
&\quad \geq \sum_{i=0}^{d_1\log(n)} (y_i - y_{i-1}) \cdot \frac{\delta_m}{y_{i+1}} - n^{-\Omega(\log(n))} \\
&\quad = n^{-1/5} \cdot \frac{\delta_m}{(3/2)n^{-1/5}} + \sum_{i=1}^{d_1\log(n)} \frac{(3/2)^{i-1}n^{-1/5}}{2} \cdot \frac{\delta_m}{(3/2)^{i+1}n^{-1/5}} - n^{-\Omega(\log(n))} \\
&\quad = C\delta_m \log(n) - n^{-\Omega(\log(n))}, \qquad (11)
\end{aligned}
$$

for some constant $C > 0$. The first inequality is since $\bigsqcup_{i=1}^{d_1 \log(n)} I_i \subseteq [0,1]$, the second inequality is since $I_i = (y_{i-1}, y_i]$ and the third inequality results from Lemma 23. Finally, the second equality is since $y_i = (3/2)^i n^{-1/5}$ for all $i \in \{0, \dots, d_1 \log(n)\}$ and since $y_{-1} = 0$.

Thus, we get

$$
\begin{aligned}
&\mathbb{E}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},t^w)-m\}} \delta_{t+m}|\delta_m, S_m\Big)\\
&\geq \mathbb{E}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},t^w)-m\}} \delta_{t+m}|\delta_m, S_m, \text{R is regular}\Big)\mathbb{P}(\text{R is regular})\\
&\geq \mathbb{E}\Big(\max_{t\in\{0,\dots,\min(t_{\{0\}},n-n^{c_3})-m\}} \delta_{t+m}|\delta_m, S_m, \text{R is regular}\Big)\mathbb{P}(\text{R is regular})\\
&\geq (C\delta_m \log(n) - n^{-\Omega(\log(n))})(1-n^{-\Omega(\log(n))})\\
&= C\delta_m \log(n) - n^{-\Omega(\log(n))},
\end{aligned}
$$

where the second inequality is by (10) and the third one by (11) and the fact that R is regular with high probability by Lemma 22.

Combining this with (9) gives:

$$
\mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m))|\delta_m, S_m\Big] \geq \frac{1}{2}[C\delta_m \log(n) - n^{-\Omega(\log(n))} - \delta_m] - \mathbb{P}(t^d > t_{\{0\}}|\delta_m, S_m).
$$

Finally, similarly as for the first point, we get

$$
\begin{aligned}
&\mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)|r_m \in [0, y_0]]\\
&\geq \int_{(x,S)\in\mathcal{X}} \mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m))|\delta_m, S_m\Big] \cdot \mathrm{d}\mathbb{P}((x,S)|r_m \in [0,y_0]) - n^{-1/5}\\
&\geq \int_{(x,S)\in\mathcal{X}} \Big(\frac{1}{2}[Cx\log(n) - n^{-\Omega(\log(n))} - x] - \mathbb{P}(t^d > t_{\{0\}}|(\delta_m, S_m) = (x,S))\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathrm{d}\mathbb{P}((x,S)|r_m \in [0,y_0]) - n^{-1/5}\\
&\geq \mathbb{E}[\delta_m|r_m \in [0,y_0]] \cdot (\tfrac{1}{2}C\log(n) - 1) - n^{-\Omega(\log(n))} - \mathbb{P}(t^d > t_{\{0\}}|r_m \in [0,y_0]) - n^{-1/5}\\
&\geq \mathbb{E}[\delta_m|r_m \in [0,y_0]] \cdot (\tfrac{1}{2}C\log(n) - 1) - n^{-\Omega(\log(n))} - O(n^{-1/5})\\
&\geq \Big(\frac{n^{-1/5}}{4} - n^{-\Omega(\log(n))}\Big) \cdot (\tfrac{1}{2}C\log(n) - 1) - n^{-\Omega(\log(n))} - O(n^{-1/5})\\
&= \Omega(\log(n)n^{-1/5}),
\end{aligned}
$$

where the fourth inequality is by Lemma 45 and the fifth one by Lemma 44. This concludes the proof of the second point and the proof of the lemma.

□

4.3.5 Proof of Theorem 4

We are now ready to conclude the proof of our lower bound result.

Proof. Since $\mathcal{A}^0 = \mathcal{G}$ and $\mathcal{A}^n = \mathcal{A}$, we have that

$$
\begin{aligned}
&\mathbb{E}[cost(\mathcal{G})] - \mathbb{E}[cost(\mathcal{A})] \\
&= \sum_{m=1}^{n} \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^{m})] \\
&= \sum_{m=1}^{n} \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^{m}) | r_m \in (y_0, 1]]\mathbb{P}(r_m \in (y_0, 1]) \\
&+ \sum_{m=1}^{c_1 n} \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^{m}) | r_m \in [0, y_0]]\mathbb{P}(r_m \in [0, y_0]) \\
&+ \sum_{m=c_1 n+1}^{n} \mathbb{E}[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^{m}) | r_m \in [0, y_0]]\mathbb{P}(r_m \in [0, y_0]) \\
&\geq 0 + \sum_{m=1}^{c_1 n} C' \log(n) n^{-1/5} n^{-1/5} - \sum_{m=c_1 n+1}^{n} C n^{-1/5} n^{-1/5} \qquad \text{(for some constants } C, C' > 0\text{)} \\
&= n^{3/5}\Big(C'(\log(n)(c_1 - \tfrac{1}{n}) - C(1 - c_1 - \tfrac{1}{n})\Big) \\
&= \Omega(\log(n) n^{3/5}),
\end{aligned}
$$

where the inequality is by Lemma 24 and since $\mathbb{P}(r_m \in [0, y_0]) = \mathbb{P}(r_m \in [0, n^{-1/5}]) = n^{-1/5}$.

Thus, $\mathbb{E}[cost(\mathcal{G})] \geq \mathbb{E}[cost(\mathcal{A})] + \Omega(\log(n) n^{3/5}) = \Omega(\log(n) n^{3/5})$. Since by Lemma 25 we have $\mathbb{E}[OPT] = O(n^{3/5})$, we conclude that $\frac{\mathbb{E}[cost(\mathcal{G})]}{\mathbb{E}[OPT]} = \Omega(\log(n))$. □

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Appendix

A Auxiliary Lemmas

Throughout the paper, we will use the following version of Chernoff bounds.

Lemma 47. *(Chernoff Bounds) Let $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ with probability p_i and $X_i = 0$ with probability $1 - p_i$, and all X_i are independent. Let $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then*

- *Upper tail: $P(X \geq (1+\delta)\mu) \leq e^{-\delta^2\mu/(2+\delta)}$ for all $\delta > 0$.*
- *Lower tail: $P(X \leq (1-\delta)\mu) \leq e^{-\delta^2\mu/2}$ for all $\delta \in [0,1]$.*

In particular, we will repeatedly use the following lemma, which immediately follows from Chernoff bounds.

Lemma 48. *Let $X \sim \mathcal{B}(n,p)$ be a binomially distributed random variable with parameters n and p. Then,*

$$\mathbb{P}\big(X \geq \mathbb{E}[X] - \log(n)^2\sqrt{\mathbb{E}[X]}\big) \geq 1 - n^{-\Omega(\log(n))},$$

and if $np = \Omega(1)$,

$$\mathbb{P}\big(X \leq \mathbb{E}[X] + \log(n)^2\sqrt{\mathbb{E}[X]}\big) \geq 1 - n^{-\Omega(\log(n))}.$$

Proof. This results from a direct application of Chernoff bounds as stated in Lemma 47 with $\delta = \log(n)^2/\sqrt{\mathbb{E}[X]}$:

$$\mathbb{P}(X \leq \mathbb{E}[X](1 - \log(n)^2/\sqrt{\mathbb{E}[X]})) \leq e^{-\log(n)^4/2} = n^{-\Omega(\log(n))},$$

and if $np = \Omega(1)$,

$$\mathbb{P}(X \geq \mathbb{E}[X](1 + \log(n)^2/\sqrt{\mathbb{E}[X]}) \leq e^{-\log(n)^4/(2+\log(n)^2/\sqrt{\mathbb{E}[X]}))} = n^{-\Omega(\log(n))}$$

$$\text{(since } 1/\sqrt{\mathbb{E}[X]} = 1/\sqrt{np} = O(1)\text{).}$$

□

Finally, we recall the following classical inequality, which follows immediately from Jensen's inequality.

Lemma 49. *For any random variable Y: $\mathbb{E}[|Y - \mathbb{E}[Y]|] \leq std(Y)$.*

B Proof of the Hybrid Lemma (Lemma 5)

The objective of this section is to prove Lemma 5, that we restate below.

Lemma 5. ***(The Hybrid Lemma).*** *There exists a constant $C > 0$ such that for any online algorithm $\mathcal{A}$ that makes neighboring matches, for any instance with n servers $S = \{s_1, \ldots, s_n\}$ adversarially chosen, n requests $R = (r_1, \ldots, r_n)$ uniformly and independently drawn from $[0,1]$, and for any event E_m that depends only on S_{m-1}, r_m, we have*

$$\mathbb{E}\big[cost(\mathcal{H}_{\mathcal{A}}^{m-1}) - cost(\mathcal{H}_{\mathcal{A}}^{m})|E_m] \leq C \cdot \mathbb{E}[\big(1 + \log\big(\tfrac{1}{cost_m(\mathcal{A})}\big)\big)cost_m(\mathcal{A})\big|E_m].$$

Note that the proof globally follows the proof of the hybrid lemma (Lemma 5.1) in [GL12]. However, [GL12] considers a fixed deterministic sequence of requests and uses a coupling argument between a randomized greedy algorithm and an optimal offline matching, whereas we directly leverage the randomness of the input sequence to analyze the performance of hybrid algorithms between an online algorithm $\mathcal{A}$ and the standard deterministic greedy algorithm.

In the remainder of this section, for a given realization R of the sequence of requests and a fixed value of $m \in [n]$, we consider a simultaneous execution of $\mathcal{H}_{\mathcal{A}}^m$ and $\mathcal{H}_{\mathcal{A}}^{m-1}$ on the sequence R. To ease the exposition, we drop the reference to the algorithms in the indices. In particular, we write $S_t, s(r_t), \mathcal{N}(r_t)$ instead of $S_{\mathcal{H}_{\mathcal{A}}^m,t}, s_{\mathcal{H}_{\mathcal{A}}^m}(r_t), \mathcal{N}_{\mathcal{H}_{\mathcal{A}}^m}(r_t)$ to denote, respectively, the set of free server for $\mathcal{H}_{\mathcal{A}}^m$ just after matching r_t, the server to which $\mathcal{H}_{\mathcal{A}}^m$ matches r_t, and the set of servers neighboring r_t when r_t arrives. Similarly, we write $S'_t, s'(r_t), \mathcal{N}'(r_t)$ instead of $S_{\mathcal{H}_{\mathcal{A}}^{m-1},t}, s_{\mathcal{H}_{\mathcal{A}}^{m-1}}(r_t), \mathcal{N}_{\mathcal{H}_{\mathcal{A}}^{m-1}}(r_t)$ for the equivalent objects for $\mathcal{H}_{\mathcal{A}}^{m-1}$. Finally, to remove any ambiguity, we assume that all servers in S_0 are distinct (even if it means moving them an infinitesimal distance).

Before presenting the proof of Lemma 5, we introduce some useful lemmas. The first lemma holds for an arbitrary sequence of requests and first shows that at all time steps t, the sets S_t and S'_t of free servers for the two algorithms are all identical except for at most one server in each of these sets and that, if they each have such a unique server, there is no server in $S_t \cup S'_t$ that is in between the two unique servers (see Figure 1 for an illustration).

We let $g_t^L < g_t^R$ denote these at most two servers in the symmetric difference of S_t and S'_t, and let $\delta_t := g_t^R - g_t^L$ be the distance between these two servers. If $S_t = S'_t$, then we write $g_t^L = g_t^R = \emptyset$ and $\delta_t = 0$. We also define $s_t^L = \max\{s \in S_t : s < g_t^L\}$ and $s_t^R = \min\{s \in S_t : s > g_t^R\}$ (with the convention that $s_t^L, s_t^R = \emptyset$ when there are no such servers).

Lemma 6. *Let $\mathcal{A}$ be any online algorithm, S_0 be n arbitrary servers and R be n arbitrary requests. Let $(S_0, \ldots, S_n)$ and $(S'_0, \ldots, S'_n)$ denote the set of free servers for $\mathcal{H}_{\mathcal{A}}^m$ and $\mathcal{H}_{\mathcal{A}}^{m-1}$ at each time steps. Then, the following propositions hold for all $t \in \{m, \ldots, n\}$:*

1. ***Differ in at most one server.*** $|S_t \setminus S'_t| = |S'_t \setminus S_t| \leq 1$.
2. ***Consecutiveness of the different servers.*** *If $g_t^L, g_t^R \neq \emptyset$, there is no server $s \in S_t \cup S'_t$ such that $g_t^L < s < g_t^R$.*
3. ***The values.*** *If $t < n$ and $S_t \neq S'_t$ (and assuming without loss of generality that $S_t = S'_t \cup \{g_t^L\} \setminus \{g_t^R\}$), then the values of $s(r_{t+1}), s'(r_{t+1}), \delta_{t+1}, g_{t+1}^L, g_{t+1}^R$ and an upper bound on $\Delta cost_{t+1} := |cost_{t+1}(\mathcal{H}^{m-1}) - cost_{t+1}(\mathcal{H}^m)|$ are given in Tables 2, 3 and 4 (in Appendix B):*
   - *if $s_t^L \neq \emptyset, s_t^R \neq \emptyset$, the values are given in Table 2, where $d_t^L = g_t^L - s_t^L$ and $d_t^R = s_t^R - g_t^R$,*
   - *if $s_t^L = \emptyset, s_t^R \neq \emptyset$, the values are given in Table 3, where $d_t^R = s_t^R - g_t^R$,*
   - *if $s_t^R = \emptyset, s_t^L \neq \emptyset$, the values are given in Table 4, where $d_t^L = g_t^L - s_t^L$,*
   - *if $s_t^L = \emptyset, s_t^R = \emptyset$, then $S_{t+1} = S'_{t+1} = \emptyset, \delta_{t+1} = 0$, and $|cost_{t+1}(\mathcal{H}^{m-1}) - cost_{t+1}(\mathcal{H}^m)| \leq \delta_t$.*
4. ***Gap remains zero after disappearing.*** *If $\delta_t = 0$, then $\delta_{t'} = 0$ for all $t' \geq t$.*

Proof. First, note that since $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ both match $r_1, \ldots, r_{m-1}$ to exactly the same servers that $\mathcal{A}$ matches them to, we have that $S_t = S'_t$ for all $t \in [m-1]$.

We now show propositions $1, 2, 3$ by induction for $t \in \{m, \ldots, n\}$.

Figure 5: Illustration of Case 1 in the proof of Lemma 6

For $t = m$, first recall that $\mathcal{H}^m$ matches r_m to the same server as $\mathcal{A}$, while $\mathcal{H}^{m-1}$ matches r_m greedily. Let $s^L_{m-1} = \max\{s \in S_{\mathcal{A},m-1} : s \le r_m\}$ and $s^R_{m-1} = \min\{s \in S_{\mathcal{A},m-1} : s \ge r_m\}$. Since both $\mathcal{A}$ and greedy make neighboring matches, we have $s(r_m), s(r_m)' \in \{s^L_{m-1}, s^R_{m-1}\}$. Hence either $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ make the same matching decision for r_m, in which case we are done, or one algorithm matches r_m to s^L_{m-1} whereas the other matches it to s^R_{m-1}. In this last case, proposition 1 is satisfied, and by definition of g^L_m and g^R_m, we have $g^L_m = s^L_{m-1}$ and $g^R_m = s^R_{m-1}$. Now, by definition of s^L_{m-1}, s^R_{m-1}, there is no server $s \in S_m \cup S'_m$ such that $g^L_m < s < g^R_m$, which shows that proposition 2 is satisfied at time m.

Next, let $t \in \{m, \dots, n-1\}$ and assume that propositions $1, 2$ are satisfied at time t. We now show that proposition 3 is satisfied at time t and that propositions $1, 2$ are satisfied at time $t+1$. Recall that both algorithms match r_{t+1} by following the greedy criterion; hence, if $S_t = S'_t$, the result follows immediately. We now assume that $S_t \neq S'_t$. By the inductive hypothesis, we thus have that $|S_t \setminus S'_t| = |S_t \setminus S'_t| = 1$, with $S_t \Delta S'_t = \{g^L_t, g^R_t\}$, and that there is no server $s \in S_t \cup S'_t$ such that $g^L_t < s < g^R_t$. We assume without loss of generality that $S_t = S'_t \cup \{g^L_t\} \setminus \{g^R_t\}$. We will consider different cases depending on whether or not there is a free server s^L_t on the left of g^L_t and a free server s^R_t on the right of g^R_t. Recall that we defined $d^L_t = g^L_t - s^L_t$ when $s^L_t \neq \emptyset$ and $d^R_t = s^R_t - g^R_t$ when $s^R_t \neq \emptyset$.

$\mathbf{s^L_t}, \mathbf{s^R_t} \neq \emptyset$: We consider all possibles cases depending on the location of request r_{t+1} (see Figures 5,6,7,8,9,10)

- Case 1: $r_{t+1} \in s^L_t + [0, \frac{d^L_t}{2}]$. In this case, we have $\mathcal{N}(r_{t+1}) = \{s^L_t, g^L_t\}$, $\mathcal{N}(r_{t+1}) = \{s^L_t, g^R_t\}$, and it is immediate that $|r_{t+1} - s^L_t| \le |g^L_t - r_{t+1}|$ and that $|r_{t+1} - s^L_t| \le |g^R_t - r_{t+1}|$. Hence we get $s(r_{t+1}) = s'(r_{t+1}) = s^L_t$.

  Combining this with the induction hypothesis, we get: $S_{t+1} = S_t \setminus \{s^L_t\} = (S'_t \cup \{g^L_t\} \setminus \{g^R_t\}) \setminus \{s^L_t\} = (S'_t \setminus \{s^L_t\}) \cup \{g^L_t\} \setminus \{g^R_t\} = S'_{t+1} \cup \{g^L_t\} \setminus \{g^R_t\}$, which immediately implies that $g^L_{t+1} = g^L_t$, $g^R_{t+1} = g^R_t$ and $\delta_{t+1} = \delta_t$. In addition, since $s(r_{t+1}) = s'(r_{t+1})$, we have $\Delta\text{cost}_{t+1} = 0$.

- Case 2: $r_{t+1} \in s^L_t + [\frac{d^L_t}{2}, \frac{d^L_t + \delta_t}{2}]$. In this case, we have $\mathcal{N}(r_{t+1}) = \{s^L_t, g^L_t\}$, $\mathcal{N}'(r_{t+1}) = \{s^L_t, g^R_t\}$. Since $r_{t+1} \ge s^L_t + \frac{d^L_t}{2}$, we have $|r_{t+1} - s^L_t| \ge |g^L_t - r_{t+1}|$, thus we get $s(r_{t+1}) = g^L_t$, and since $r_{t+1} \le s^L_t + \frac{d^L_t + \delta_t}{2}$, we have $|r_{t+1} - s^L_t| \le |g^R_t - r_{t+1}|$, thus we get $s'(r_t) = s^L_t$.

  Combining this with the induction hypothesis, we get: $S_{t+1} = S_t \setminus \{g^L_t\} = (S'_t \cup \{g^L_t\} \setminus \{g^R_t\}) \setminus \{g^L_t\} = S'_t \setminus \{g^R_t\} = S'_{t+1} \cup \{s^L_t\} \setminus \{g^R_t\}$, which implies that $g^L_{t+1} = s^L_t$, $g^R_{t+1} = g^R_t$, and $\delta_{t+1} = g^R_t - s^L_t = (g^R_t - g^L_t) + (g^L_t - s^L_t) = \delta_t + d^L_t$. In addition, $\Delta\text{cost}_{t+1} = ||r_{t+1} - g^L_t| - |r_{t+1} - s^L_t|| \le |g^L_t - s^L_t| = d^L_t$.

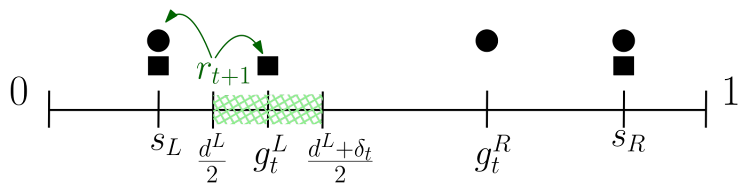

Figure 6: Illustration of Case 2 in the proof of Lemma 6

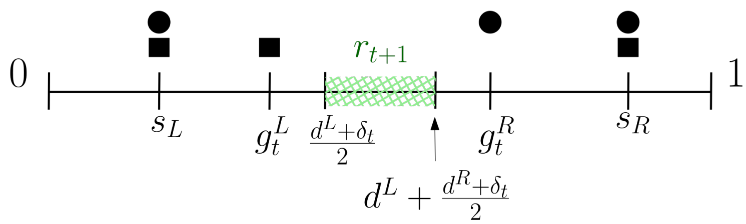

Figure 7: Illustration of Case 3 in the proof of Lemma 6

- Case 3: $r_{t+1} \in s_t^L + [\frac{d_t^L+\delta_t}{2}, d_t^L + \frac{d_t^R+\delta_t}{2}]$. In this case, we have $\mathcal{N}(r_{t+1}) \subseteq \{s_t^L, g_t^L, s_t^R\}$, $\mathcal{N}'(r_{t+1}) \subseteq \{s_t^L, g_t^R, s_t^R\}$, with $g_t^L \in \mathcal{N}(r_{t+1})$ and $g_t^R \in \mathcal{N}'(r_{t+1})$. since $r_{t+1} \geq s_t^L + \frac{d_t^L+\delta_t}{2} \geq s_t^L + \frac{d_t^L}{2}$, we have $|r_{t+1} - s_t^L| \geq |g_t^L - r_{t+1}|$, and since $r_{t+1} \leq s_t^L + d_t^L + \frac{d_t^R+\delta_t}{2}$, we have $|s_t^R - r_{t+1}| \geq |g_t^L - r_{t+1}|$, thus we get $s(r_{t+1}) = g_t^L$. Similarly, since $r_{t+1} \leq s_t^L + d_t^L + \frac{d_t^R+\delta_t}{2} \leq s_t^L + d_t^L + \delta_t + \frac{d_t^R}{2}$, we have $|s_t^R - r_{t+1}| \geq |g_t^R - r_{t+1}|$, and since $r_{t+1} \geq s_t^L + \frac{d_t^L+\delta_t}{2}$, we have $|s_t^L - r_{t+1}| \geq |g_t^R - r_{t+1}|$, thus we get $s'(r_{t+1}) = g_t^R$.

  Combining this with the induction hypothesis, we get: $S_{t+1} = S_t \setminus \{g_t^L\} = (S_t' \cup \{g_t^L\} \setminus \{g_t^R\}) \setminus \{g_t^L\} = S_t' \setminus \{g_t^R\} = S_{t+1}'$, which implies that $g_{t+1}^L = g_{t+1}^R = \emptyset$ and $\delta_{t+1} = 0$. In addition, $\Delta\text{cost}_{t+1} = ||r_{t+1} - g_t^L| - |r_{t+1} - g_t^R|| \leq |g_t^R - g_t^L| = \delta_t$.

- Case 4: $r_{t+1} \in s_t^L + [d_t^L + \frac{d_t^R+\delta_t}{2}, d_t^L + \delta_t + \frac{d_t^R}{2}]$. This case is symmetric to Case 2 by noting the one to one correspondence between $0, d_t^L, s_t^L, g_t^L$ and $1, d_t^R, s_t^R, g_t^R$. We get that $s(r_{t+1}) = s_t^R$ $s'(r_{t+1}) = g_t^R$, which implies $g_{t+1}^L = g_t^L$, $g_{t+1}^R = s_t^R$ and $\delta_{t+1} = s_t^R - g_t^L = (s_t^R - g_t^R) + (g_t^R - g_t^L) = d_t^R + \delta_t$. In addition, $\Delta\text{cost}_{t+1} \leq |s_t^R - g_t^R| = d_t^R$.

- Case 5: $r_{t+1} \in s_t^L + [d_t^L + \delta_t + \frac{d_t^R}{2}, d_t^L + \delta_t + d_t^R]$. This case is symmetric to Case 1 by noting the one to one correspondence between $0, d_t^L, s_t^L, g_t^L$ and $1, d_t^R, s_t^R, g_t^R$. We get that $s(r_{t+1}) = s'(r_{t+1}) = s_t^R$, which implies $\Delta\text{cost}_{t+1} = 0$, $g_{t+1}^L = g_t^L$, $g_{t+1}^R = g_t^R$ and $\delta_{t+1} = \delta_t$.

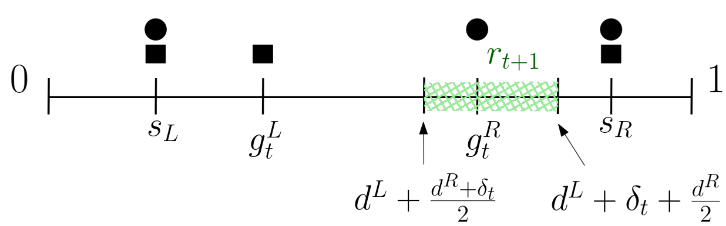

Figure 8: Illustration of Case 4 in the proof of Lemma 6

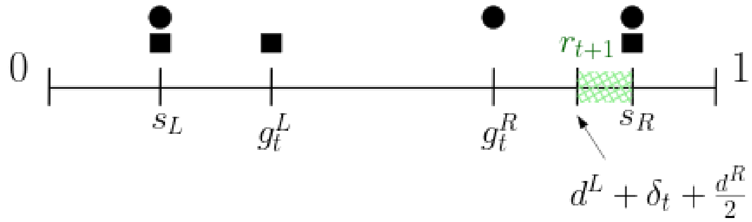

Figure 9: Illustration of Case 5 in the proof of Lemma 6

Figure 10: Illustration of Case 6 in the proof of Lemma 6

- Case 6: $r_{t+1} \in [0, s_t^L) \cup (s_t^R, 1]$. In this case, the free servers neighboring r_{t+1} are identical for $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$, thus $s(r_{t+1}) = s'(r_{t+1})$. By using the assumption that $|S_t \setminus S_t'| = |S_t \setminus S_t'| = 1$, we get that $|S_{t+1} \setminus S_{t+1}'| = |S_{t+1}' \setminus S_{t+1}| = 1$ and that $g_{t+1}^L = g_t^L$, $g_{t+1}^R = g_t^R$, $\delta_{t+1} = \delta_t$. In addition, since $s(r_{t+1}) = s'(r_{t+1})$, we have $\Delta\text{cost}_{t+1} = 0$.

Hence, in all cases, we have that propositions $1, 2$ hold at time $t+1$ and that the values of $g_{t+1}^L, g_{t+1}^R, \delta_{t+1}$ given in Table 2 hold.

$\mathbf{s_t^L} = \emptyset, \mathbf{s_t^R} \neq \emptyset$: We again consider all possibles cases depending on the location of request r_{t+1}. Note that the exact same argument as above shows that the value of $g_{t+1}^L, g_{t+1}^R, \delta_{t+1}$, and the upper bound on Δcost_{t+1} given in the last three columns of Table 3 are identical to those in the last three columns of Table 2, and that propositions $1, 2$ hold at time $t+1$ in these cases. We thus only need to show the result in the case $r_{t+1} \in [0, g_t^L + \frac{d_t^R + \delta_t}{2}]$.

In this case (see Figure 11), we have $\mathcal{N}(r_{t+1}) = \{g_t^L\}$, $\mathcal{N}'(r_{t+1}) = \{g_t^R\}$, hence we immediately get $s(r_{t+1}) = g_t^L, s'(r_{t+1}) = g_t^R$. Combining this with the induction hypothesis, we get: $S_{t+1} = S_t \setminus \{g_t^L\} = (S_t' \cup \{g_t^L\} \setminus \{g_t^R\}) \setminus \{g_t^L\} = S_t' \setminus \{g_t^R\} = S_{t+1}'$, which implies that $g_{t+1}^L = g_{t+1}^R = \emptyset$ and $\delta_{t+1} = 0$. In addition, $\Delta\text{cost}_{t+1} \leq |g_t^R - g_t^L| = \delta_t$. Hence, we have that propositions $1, 2$ hold at time $t+1$ and that the values of $g_{t+1}^L, g_{t+1}^R, \delta_{t+1}$ given in Table 3 hold.

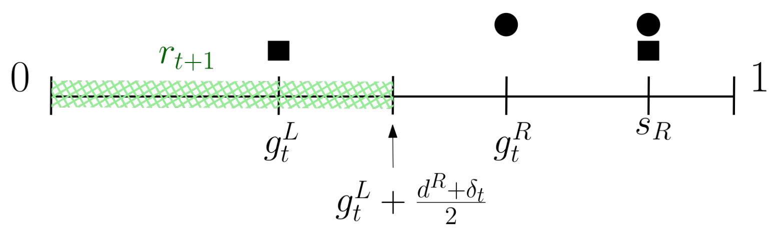

Figure 11: Illustration of the case $\mathbf{s_t^L} = \emptyset, \mathbf{s_t^R} \neq \emptyset$: and $r_{t+1} \in [0, g_t^L + \frac{d_t^R + \delta_t}{2}]$ in the proof of Lemma 6

$r_{t+1} - s_t^L \in \dots$	$[0, \frac{d_t^L}{2}]$	$[\frac{d_t^L}{2}, \frac{d_t^L+\delta_t}{2}]$	$[\frac{d_t^L+\delta_t}{2}, d_t^L + \frac{d_t^R+\delta_t}{2}]$	$[d_t^L + \frac{d_t^R+\delta_t}{2}, d_t^L + \delta_t + \frac{d_t^R}{2}]$	$[d_t^L + \delta_t + \frac{d_t^R}{2}, d_t^L + \delta_t + d_t^R]$	$[-s_t^L, 0) \cup (s_t^R - s_t^L, 1 - s_t^L]$
$s(r_{t+1})$	s_t^L	g_t^L	g_t^L	s_t^R	s_t^R	$\in [0, s_t^L) \cup (s_t^R, 1]$
$s'(r_{t+1})$	s_t^L	s_t^L	g_t^R	g_t^R	s_t^R	$\in [0, s_t^L) \cup (s_t^R, 1]$
g_{t+1}^L	g_t^L	s_t^L	$\emptyset$	g_t^L	g_t^L	g_t^L
g_{t+1}^R	g_t^R	g_t^R	$\emptyset$	s_t^R	g_t^R	g_t^R
δ_{t+1}	δ_t	$\delta_t + d_t^L$	0	$\delta_t + d_t^R$	δ_t	δ_t
$\Delta\text{cost}_{t+1} \leq$	0	d_t^L	δ_t	d_t^R	0	0

Table 2: Values of $\delta_{t+1}, g_{t+1}^L, g_{t+1}^R$, and upper bound on Δcost_{t+1} when $s_t^L, s_t^R \neq \emptyset$.

$r_{t+1} \in \dots$	$[0, g_t^L + \frac{d_t^R+\delta_t}{2}]$	$[g_t^L + \frac{d_t^R+\delta_t}{2}, g_t^L + \delta_t + \frac{d_t^R}{2}]$	$[g_t^L + \delta_t + \frac{d_t^R}{2}, g_t^L + \delta_t + d_t^R]$	$(s_t^R, 1]$
$s(r_{t+1})$	g_t^L	s_t^R	s_t^R	$\in (s_t^R, 1]$
$s'(r_{t+1})$	g_t^R	g_t^R	s_t^R	$\in (s_t^R, 1]$
g_{t+1}^L	$\emptyset$	g_t^L	g_t^L	g_t^L
g_{t+1}^R	$\emptyset$	s_t^R	g_t^R	g_t^R
δ_{t+1}	0	$\delta_t + d_t^R$	δ_t	δ_t
$\Delta\text{cost}_{t+1} \leq$	δ_t	d_t^R	0	0

Table 3: Values of $\delta_{t+1}, g_{t+1}^L, g_{t+1}^R$, and upper bound on Δcost_{t+1} when $s_t^L = \emptyset, s_t^R \neq \emptyset$.

$\mathbf{s_t^R} = \emptyset, \mathbf{s_t^L} \neq \emptyset$: This case is symmetric to the case $s_t^R = \emptyset, s_t^L \neq \emptyset$, by noting the one to one correspondence between $0, d_t^L, s_t^L, g_t^L$ and $1, d_t^R, s_t^R, g_t^R$.

$\mathbf{s_t^L} = \emptyset, \mathbf{s_t^R} = \emptyset$: In this case, we have $S_t = \{g_t^L\}$ and $S'_t = \{g_t^R\}$. Hence, whatever the value of r_{t+1}, we get that $S_{t+1} = S'_{t+1} = \emptyset$ and $\delta_{t+1} = 0$. In addition, we have $\Delta\text{cost}_{t+1} \leq |g_t^R - g_t^L| = \delta_t$.

This concludes the proof that proposition 3 is satisfied at time t and that propositions $1, 2$ are satisfied at time $t+1$. Hence the three first propositions of the lemma hold for all $t \in \{m, \dots, n\}$.

Finally, we show proposition 4. Note that if $\delta_t = 0$ for some $t \in \{m, \dots, n\}$, then by definition of δ_t, we have $S_t = S'_t$. Since both $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ match greedily $r_{t+1}, \dots, r_n$, we get $S_j = S'_j$ for all $j \in \{t, \dots, n\}$, which shows that $\delta_j = 0$ for $j \in \{t, \dots, n\}$. □

$r_{t+1} \in \dots$	$[0, s_t^L)$	$[g_t^R - (\delta_t + d_t^L), g_t^R - (\delta_t + \frac{d_t^L}{2})]$	$[g_t^R - (\delta_t + \frac{d_t^L}{2}), g_t^R - \frac{d_t^L+\delta_t}{2}]$	$[g_t^R - \frac{d_t^L+\delta_t}{2}, 1]$
$s(r_{t+1})$	$\in [0, s_t^L)$	s_t^L	g_t^L	g_t^L
$s'(r_{t+1})$	$\in [0, s_t^L)$	s_t^L	s_t^L	g_t^R
g_{t+1}^L	g_t^L	g_t^L	s_t^L	$\emptyset$
g_{t+1}^R	g_t^R	g_t^R	g_t^R	$\emptyset$
δ_{t+1}	δ_t	δ_t	$\delta_t + d_t^L$	0
$\Delta\text{cost}_{t+1} \leq$	0	0	d_t^L	δ_t

Table 4: Values of $\delta_{t+1}, g_{t+1}^L, g_{t+1}^R$, and upper bound on Δcost_{t+1} when $s_t^R = \emptyset, s_t^L \neq \emptyset$

In the remainder of this section, we assume that the structural properties proved in Lemma 6 hold. By using the third and fourth propositions of Lemma 6, we now upper bound the total difference of cost incurred during the simultaneous execution of $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ for an arbitrary sequence of requests R.

Lemma 50. *For any arbitrary sequence of n requests in $[0,1]$, we have*

$$\textit{cost}(\mathcal{H}^{m-1}) - \textit{cost}(\mathcal{H}^m) \leq 2 \max_{t \in \{m,\dots,n-1\}} \delta_t.$$

Proof. Since $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ both match $r_1, \dots, r_{m-1}$ to exactly the same servers as $\mathcal{A}$, we first have that $\text{cost}_t(\mathcal{H}^{m-1}) - \text{cost}_t(\mathcal{H}^m) = 0$ for all $t \in [m-1]$. Then, since $\mathcal{H}^m$ matches r_m to the same server as $\mathcal{A}$ while $\mathcal{H}^{m-1}$ matches r_m greedily, we have that $|r_m - s'(r_m)| = \min\{|r_m - s| : s \in S_{\mathcal{A},m-1}\} \leq |r_m - s(r_m)|$. Thus, $\text{cost}_m(\mathcal{H}^{m-1}) - \text{cost}_m(\mathcal{H}^m) \leq 0$.

Next, we define $t_0 := \min\{t \geq m : \delta_t = 0\}$, i.e., the first time step where the two sets of free servers become identical again. Note that by the fourth point of Lemma 6, we have that $\delta_t = 0$ for any $t \geq t_0$, which, by definition of δ, implies that $S_t = S'_t$ for any $t \geq t_0$. We deduce that $\text{cost}_t(\mathcal{H}^{m-1}) - \textit{cost}_t(\mathcal{H}^m) = 0$ for any $t \in [t_0 + 1, \dots, n]$.

Finally, by a direct inspection of all possible cases enumerated in the third point of Lemma 6, we get that for all $t \in \{m, \dots, t_0 - 1\}$, $\text{cost}_t(\mathcal{H}^{m-1}) - \text{cost}_t(\mathcal{H}^m) \leq \delta_t - \delta_{t-1}$, and we get that $\textit{cost}_{t_0}(\mathcal{H}^{m-1}) - \textit{cost}_{t_0}(\mathcal{H}^m) \leq \delta_{t_0-1}$.

Putting everything together, we obtain

$$\begin{aligned}\text{cost}(\mathcal{H}^{m-1}) - \text{cost}(\mathcal{H}^m) &= \sum_{t=1}^{n} (\text{cost}_t(\mathcal{H}^{m-1}) - \text{cost}_t(\mathcal{H}^m)) \\ &\leq 0 + \sum_{t=m+1}^{t_0-1} (\delta_t - \delta_{t-1}) + \delta_{t_0-1} + 0 = 2\delta_{t_0-1} - \delta_m \leq 2 \max_{t \in \{m,\dots,n-1\}} \delta_t.\end{aligned}$$

□

In the remainder of the section, we consider the more specific case where the requests in R are sampled uniformly at random in $[0,1]$. In the following lemma, we show that for any initial set of servers, if we have at time m that $\delta_m = x$, then the probability that the distance δ_t between the (potential) extra server of $\mathcal{H}^m$ and the (potential) extra server of $\mathcal{H}^{m-1}$ will ever exceed y at any time $t \in \{m, \dots, n-1\}$ is upper bounded by $\frac{x}{y}$.

Lemma 51. *Let $x \in (0,1]$ and $y \in [x,1]$. Then, for any initial set S of $n-m$ arbitrary servers in $[0,1]$, we have*

$$\mathbb{P}_R\Big(\max_{\{m,\dots,n-1\}} \delta_t \geq y | \delta_m = x, S_m = S\Big) \leq \frac{x}{y}.$$

Proof. We show by downward induction on j that for any $j \in \{m, \dots, n-1\}$, any $x \in (0,1]$, $y \in [x,1]$, and any arbitrary set S of $n-j$ arbitrary servers in $[0,1]$,

$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n-1\}} \delta_t \geq y | \delta_j = x, S_j = S\Big) \leq \frac{x}{y}.$$

We first show the base case, which is for $j = n-1$. For any $x \in (0,1]$, $y \in [x,1]$, and any arbitrary server s in $[0,1]$, it is immediate that

$$\mathbb{P}_R\Big(\delta_{n-1} \geq y | \delta_{n-1} = x, S_{n-1} = \{s\}\Big) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \leq \frac{x}{y}.$$

Next, let $j \in \{m, \dots, n-2\}$, and assume that for all $x \in (0,1]$, $y \in [x,1]$, and any set S of $n-(j+1)$ arbitrary servers in $[0,1]$, we have

$$\mathbb{P}_R\Big(\max_{t\in\{j+1,\dots,n-1\}} \delta_t \geq y | \delta_{j+1} = x, S_{j+1} = S\Big) \leq \frac{x}{y}.$$

Now, consider some arbitrary $x \in (0,1]$, $y \in [x,1]$, and S some arbitrary set of $n-j-1$ servers in $[0,1]$, and assume that $\delta_j = x$ and $S_j = S$.

Since $\delta_j = x \neq 0$, we have $S_j \neq S'_j$. Furthermore, by Lemma 6, we have $|S_j \setminus S'_j| = |S'_j \setminus S_j| = 1$ with $S_j \Delta S'_j = \{g_j^L, g_j^R\}$. We assume without loss of generality that $S'_j = S_j \cup \{g_j^R\} \setminus \{g_j^L\}$. Recall that we defined $s_j^L = \max\{s \in S_j : s \leq g_j^L\}$, $s_j^R = \min\{s \in S_j : s \geq g_j^R\}$, and $d_j^L = g_j^L - s_j^L$, $d_j^R = s_j^R - g_j^R$.

First, we note the following proposition: for any $r \in [0,1]$, letting $\chi(x,S,r)$ and $T(x,S,r)$ be the value of δ_{j+1} and S_{j+1} assuming that $\delta_j = x, S_j = S$ and $r_{j+1} = r$, we have

$$\begin{aligned}
&\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n-1\}} \delta_t \geq y | \delta_j = x, S_j = S, r_{j+1} = r\Big) \\
&= \mathbb{P}_R\Big(\max_{t\in\{j,\dots,n-1\}} \delta_t \geq y | \delta_j = x, S_j = S, \delta_{j+1} = \chi(x,S,r), S_{j+1} = T(x,S,r), r_{j+1} = r\Big) \\
&= \mathbb{P}_R\Big(\max_{t\in\{j+1,\dots,n-1\}} \delta_t \geq y | \delta_j = x, S_j = S, \delta_{j+1} = \chi(x,S,r), S_{j+1} = T(x,S,r), r_{j+1} = r\Big) \\
&= \mathbb{P}_R\Big(\max_{t\in\{j+1,\dots,n-1\}} \delta_t \geq y | \delta_{j+1} = \chi(x,S,r), S_{j+1} = T(x,S,r)\Big) \\
&\leq \frac{\chi(x,S,r)}{y},
\end{aligned} \tag{12}$$

where the second equality is since we conditioned on $\delta_j = x < y$, the third equality is since conditioned on S_{j+1}, δ_{j+1}, we have that $\{\delta_t\}_{t\in\{j+1,\dots,n\}}$ is independent on r_{j+1}, S_j, δ_j, and the inequality is by the induction hypothesis.

We now enumerate all possible cases depending on request r_{j+1}. We start by the case where $s_j^L, s_j^R \neq \emptyset$. By Lemma 6, the values of $\chi(x,S,r_{j+1})$ are the one given in Table 2.

- Case 1: $r_{j+1} \in s_j^L + [0, \frac{d_j^L}{2}]$. We have, by Table 2, that $\chi(x, S, r_{j+1}) = \delta_j = x$. Thus, by (12), we get

$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n\}} \delta_t \geq y | \delta_j = x, S_j = S, r_{j+1} \in s_j^L + [0, \tfrac{d_j^L}{2}]\Big) \leq \frac{x}{y}.$$

- Case 2: $r_{j+1} \in s_j^L + [\frac{d_j^L}{2}, \frac{d_j^L+\delta_j}{2}]$. We have $\chi(x, S, r_{j+1}) = \delta_j + d_j^L = x + d_j^L$. Thus, by (12), we get

$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n\}} \delta_t \geq y | \delta_j = x, S_j = S, r_{j+1} \in s_j^L + [\tfrac{d_j^L}{2}, \tfrac{d_j^L+\delta_j}{2}]\Big) \leq \frac{x + d_j^L}{y}.$$

- Case 3: $r_{j+1} \in s_j^L + [\frac{d_j^L+\delta_j}{2}, d_j^L + \frac{d_j^R+\delta_j}{2}]$. We have that $\chi(x, S, r_{j+1}) = 0$. Thus, by (12), we get

$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n\}} \delta_t \geq y | \delta_j = x, S_j = S, r_{j+1} \in s_j^L + [\tfrac{d_j^L+\delta_j}{2}, d_j^L + \tfrac{d_j^R+\delta_j}{2}]\Big) \leq 0.$$

- Case 4: $r_{j+1} \in s_j^L + [d_j^L + \frac{d_j^R+\delta_j}{2}, d_j^L + \delta_j + \frac{d_j^R}{2}]$. We have $\chi(x, S, r_{j+1}) = \delta_j + d_j^R = x + d_j^R$. Thus, by (12), we get

$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n\}} \delta_t \geq y | \delta_j = x, S_j = S, r_{j+1} \in s_j^L + [d_j^L + \tfrac{d_j^R+\delta_j}{2}, d_j^L + \delta_j + \tfrac{d_j^R}{2}]\Big) \leq \frac{x + d_j^R}{y}.$$

- Case 5: $r_{j+1} \in s_j^L + [d_j^L + \delta_j + \frac{d_j^R}{2}, d_j^L + \delta_j + d_j^R]$. We have $\chi(x, S, r_{j+1}) = \delta_j = x$. Thus, by (12), we get

$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n\}} \delta_t \geq y | \delta_j = x, S_j = S, r_{j+1} \in s_j^L + [d_j^L + \delta_j + \tfrac{d_j^R}{2}, d_j^L + \delta_j + d_j^R]\Big) \leq \frac{x}{y}.$$

- Case 6: $r_{j+1} \in [0, s_j^L) \cup (s_j^R, 1]$. We have $\chi(x, S, r_{j+1}) = \delta_j = x$. Thus, by (12), we get

$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n\}} \delta_t \geq y | \delta_j = x, S_j = S, r_{j+1} \in [0, s_j^L) \cup (s_j^R, 1]\Big) \leq \frac{x}{y}.$$

By combining the six cases above and using that $\delta_j = x$, we get

$$\begin{aligned}
&\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n-1\}} \delta_t \geq y | \delta_j = x, S_j = S\Big) \\
&\leq \mathbb{P}(r_{j+1} \in s_j^L + [0, \tfrac{d_j^L}{2}]) \cdot \frac{x}{y} + \mathbb{P}(r_{j+1} \in s_j^L + [\tfrac{d_j^L}{2}, \tfrac{d_j^L+\delta_j}{2}]) \cdot \frac{x + d_j^L}{y} + 0 \\
&+ \mathbb{P}(r_{j+1} \in s_j^L + [d_j^L + \tfrac{d_j^R+\delta_j}{2}, d_j^L + \delta_j + \tfrac{d_j^R}{2}]) \cdot \frac{x + d_j^R}{y} \\
&+ \mathbb{P}(r_{j+1} \in s_j^L + [d_j^L + \delta_j + \tfrac{d_j^R}{2}, d_j^L + \delta_j + d_j^R]) \cdot \frac{x}{y} + \mathbb{P}(r_{j+1} \in [0, s_j^L) \cup (s_j^R, 1]) \cdot \frac{x}{y} \\
&= \frac{d_j^L}{2} \cdot \frac{x}{y} + \frac{x}{2} \cdot \frac{x + d_j^L}{y} + \frac{x}{2} \cdot \frac{x + d_j^R}{y} + \frac{d_j^R}{2} \cdot \frac{x}{y} + (1 - (d_j^L + d_j^R + x)) \cdot \frac{x}{y} \\
&= \frac{x}{y} \cdot \Big(\frac{d_j^L}{2} + \frac{x + d_j^L}{2} + \frac{x + d_j^R}{2} + \frac{d_j^R}{2} + (1 - (d_j^L + d_j^R + x))\Big) \\
&= \frac{x}{y}.
\end{aligned}$$

We now consider the case where $s_j^L = \emptyset, s_j^R \neq \emptyset$. By Lemma 6, the values of $\chi(x, S, r_{j+1})$ and $T(x, S, r_{j+1})$ are the one given in Table 3. We consider four different cases.

- Case 1: $r_{j+1} \in [0, g_j^L + \frac{d_j^R+\delta_j}{2}]$. We have, by Table 3 that $\chi(x, S, r_{j+1}) = 0$. Thus, by (12), we get
$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n\}} \delta_j \geq y | \delta_j = x, S_j = S, r_{j+1} \in [0, g_j^L + \tfrac{d_j^R+\delta_j}{2}]\Big) \leq 0.$$

- Cases 2,3,4: the upper bounds we get in the cases $r_{j+1} \in [g_j^L + \frac{d_j^R+\delta_j}{2}, d_j^L + \delta_j + \frac{d_j^R}{2}]$, $r_{j+1} \in [g_j^L + \delta_j + \frac{d_j^R}{2}, g_j^L + \delta_j + d_j^R]$ and $r_{j+1} \in (s_j^R, 1]$ are identical as the ones in Cases 4,5,6 when $s_j^L, s_j^R \neq \emptyset$.

By combining the four cases above, we get

$$\begin{aligned}
&\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n-1\}} \delta_j \geq y | \delta_j = x, S_j = S\Big) \\
&\leq 0 + \mathbb{P}(r_{j+1} \in [g_j^L + \tfrac{d_j^R+\delta_j}{2}, d_j^L + \delta_j + \tfrac{d_j^R}{2}]) \cdot \frac{x + d_j^R}{y} \\
&+ \mathbb{P}(r_{j+1} \in [g_j^L + \delta_j + \tfrac{d_j^R}{2}, g_j^L + \delta_j + d_j^R]) \cdot \frac{x}{y} + \mathbb{P}(r_{j+1} \in (s_j^R, 1]) \cdot \frac{x}{y} \\
&= \frac{x}{2} \cdot \frac{x + d_j^R}{y} + \frac{d_j^R}{2} \cdot \frac{x}{y} + (1 - (d_j^L + d_j^R + x)) \cdot \frac{x}{y} \\
&= \frac{x}{y} \cdot \Big(\frac{x + d_j^R}{2} + \frac{d_j^R}{2} + (1 - (d_j^L + d_j^R + x))\Big) \\
&\leq \frac{x}{y}.
\end{aligned}$$

The case $s_j^L \neq \emptyset, s_j^R = \emptyset$ is similar to the case above and we conclude in the same way. Finally, in the case $s_j^L, s_j^R = \emptyset$, we have by Lemma 6 that for any r_{j+1}, we have $\delta_{j+1} = 0$. Thus, by (12), we get
$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n-1\}} \delta_t \geq y | \delta_t = x, S_j = S\Big) \leq 0.$$

Hence, in all cases, we have shown that
$$\mathbb{P}_R\Big(\max_{t\in\{j,\dots,n-1\}} \delta_j \geq y | \delta_j = x, S_j = S\Big) \leq \frac{x}{y},$$
which concludes the inductive case and the proof of the lemma. □

Lemma 52. *Let E_m any event that depends only on S_{m-1}, r_m. Then, for a constant $C > 0$:*
$$\mathbb{E}_R\Big[\delta_m(1 + \log(1/\delta_m)) | E_m, \delta_m > 0\Big] \cdot \mathbb{P}(\delta_m > 0) \leq C\mathbb{E}_R\Big[\big(1 + \log\big(\tfrac{1}{cost_m(\mathcal{A})}\big)\big) cost_m(\mathcal{A}) \big| E_m\Big]$$

Proof. We first condition on E_m and on $\{\delta_m > 0\}$.

Recall that $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ both match $r_1, \dots, r_{m-1}$ to the same servers as $\mathcal{A}$, and that $\mathcal{H}^m$ matches r_m to the same server as $\mathcal{A}$, while $\mathcal{H}^{m-1}$ matches r_m greedily. Hence, we have that $|r_m - s(r_m)| = \text{cost}_m(\mathcal{A})$ and that $|r_m - s'(r_m)| = \min\{|r_m - s| : s \in S_{\mathcal{A},m-1}\} \leq |r_m - s(r_m)|$. Thus,
$$\delta_m = |s(r_m) - s'(r_m)| \leq |r_m - s(r_m)| + |r_m - s'(r_m)| \leq 2|r_m - s(r_m)| = 2\text{cost}_m(\mathcal{A}).$$

Note that $x \longmapsto x\log(\frac{1}{x})$ reaches its maximum value over $(0,1]$ at $x = 1/e$ (with $\frac{1}{e}\log(\frac{1}{1/e}) = \frac{1}{e}$) and is non-decreasing on $(0, 1/e]$. Since $\delta_m \in (0,1]$, we thus have

$$\delta_m(1+\log(1/\delta_m)) \leq \quad 2\text{cost}_m(\mathcal{A}) + \begin{cases} 2\text{cost}_m(\mathcal{A})\log(1/(2\text{cost}_m(\mathcal{A}))) & \text{if } 2\text{cost}_m(\mathcal{A}) \in (0,1/e] \\ 1/e \leq 2\text{cost}_m(\mathcal{A}) & \text{if } 2\text{cost}_m(\mathcal{A}) \in [1/e,1]. \end{cases}$$
$$\leq \quad 2\text{cost}_m(\mathcal{A})\big(1+\log(1/(2\text{cost}_m(\mathcal{A})))+1\big).$$

As a result, we get that for some $C > 0$,

$$\begin{aligned} &\mathbb{E}_R\Big[\delta_m(1+\log(1/\delta_m))|E_m, \delta_m > 0\Big] \cdot \mathbb{P}(\delta_m > 0) \\ &\leq C\mathbb{E}_R\Big[\big(1+\log\big(\tfrac{1}{\text{cost}_m(\mathcal{A})}\big)\big)\text{cost}_m(\mathcal{A})\big|E_m, \delta_m > 0\Big] \cdot \mathbb{P}(\delta_m > 0) \\ &\leq C\mathbb{E}_R\Big[\big(1+\log\big(\tfrac{1}{\text{cost}_m(\mathcal{A})}\big)\big)\text{cost}_m(\mathcal{A})\big|E_m\Big]. \end{aligned}$$

□

We are now ready to present the proof of the hybrid lemma.

Proof of Lemma 5.

Proof. Let S be an arbitrary set of n servers in $[0,1]$ and R a sequence of n requests drawn uniformly at random from $[0,1]$. In the remainder of the proof, we consider a simultaneous execution of $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ with initial set of servers S and requests R. Let E_m be any event that depends only on r_m and S_{m-1}.

Conditioning on event E_m and on the variables δ_m, S_m, we have

$$\begin{aligned} &\mathbb{E}_R[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)|E_m, \delta_m, S_m] \\ &\leq 2\mathbb{E}_R\big[\max_{t\in\{m,\dots,n-1\}} \delta_t|E_m, \delta_m, S_m\big] && \text{Lemma 50} \\ &= \mathbb{1}_{\delta_m>0} \cdot 2\mathbb{E}_R\big[\max_{t\in\{m,\dots,n-1\}} \delta_t|E_m, \delta_m, S_m\big] && \text{Lemma 6 (proposition 4)} \\ &\leq \mathbb{1}_{\delta_m>0} \cdot 2\Big(\delta_m + \int_{\delta_m}^1 \mathbb{P}_R\big[\max_{t\in\{m,\dots,n-1\}} \delta_t \geq y|E_m, \delta_m, S_m\big]\text{dy}\Big) \\ &= \mathbb{1}_{\delta_m>0} \cdot 2\Big(\delta_m + \int_{\delta_m}^1 \mathbb{P}_R\big[\max_{t\in\{m,\dots,n-1\}} \delta_t \geq y|\delta_m, S_m\big]\text{dy}\Big) && \max_{t\geq m}\delta_t \perp\!\!\!\perp (S_{m-1}, r_m) \text{ when } |\delta_m, S_m \\ &\leq \mathbb{1}_{\delta_m>0} \cdot 2\Big(\delta_m + \int_{\delta_m}^1 \frac{\delta_m}{y}\text{dy}\Big) && \text{Lemma 51} \\ &= \mathbb{1}_{\delta_m>0} \cdot 2\delta_m(1+\log(1/\delta_m)). && (13) \\ & && (14) \end{aligned}$$

By the tower rule, we conclude that

$$\begin{aligned} &\mathbb{E}_R[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)|E_m] \\ &= \mathbb{E}_R[\mathbb{E}_R[cost(\mathcal{H}^{m-1}) - cost(\mathcal{H}^m)|E_m, \delta_m, S_m]|E_m] \\ &\leq \mathbb{E}_R[\mathbb{1}_{\delta_m>0} \cdot 2\delta_m(1+\log(1/\delta_m))|E_m] && \text{by (13)} \\ &= 2\mathbb{E}_R[\delta_m(1+\log(1/\delta_m))|E_m, \delta_m > 0] \cdot \mathbb{P}(\delta_m > 0) \\ &\leq 2C\mathbb{E}_R[\big(1+\log\big(\tfrac{1}{\text{cost}_m(\mathcal{A})}\big)\big)\text{cost}_m(\mathcal{A})\big|E_m]. && \text{Lemma 52} \end{aligned}$$

□

C Missing Analysis from Section 3

Lemma 9. *For all $t \in [n]$, if r_t is matched at level ℓ, then we have $cost_t(\mathcal{A}^H)\log(1/cost_t(\mathcal{A}^H)) \leq 2^{\ell-\ell_0}(\log(2)(\ell_0-\ell)+1)$.*

Proof. Let $\ell \in \{0,\dots,\ell_0\}$. Since by construction, the intervals of $\mathcal{I}_\ell$ have length at most $2^{\ell-\ell_0}$, the cost incurred by $\mathcal{A}^H$ when matching a request r_t at level ℓ satisfies: $\text{cost}_t(\mathcal{A}^H) \leq 2^{\ell-\ell_0}$.

Since $x \longmapsto x\log(1/x)$ reaches its maximum value over $(0,1]$ at $x = 1/e$ (with $\frac{1}{e}\log(\frac{1}{1/e}) = \frac{1}{e}$) and is non-decreasing on $(0, 1/e]$, we conclude that if r_t is matched at level ℓ,

$$\begin{aligned}
\text{cost}_t(\mathcal{A}^H)\log(1/\text{cost}_t(\mathcal{A}^H)) &\leq \begin{cases} 2^{\ell-\ell_0}\log(1/2^{\ell-\ell_0}) & \text{if } 2^{\ell-\ell_0} \in (0,1/e] \\ 1/e \leq 2^{\ell-\ell_0} & \text{if } 2^{\ell-\ell_0} \in [1/e,1] \end{cases} \\
&\leq 2^{\ell-\ell_0}\log(1/2^{\ell-\ell_0}) + 2^{\ell-\ell_0} \\
&= 2^{\ell-\ell_0}(\log(2)(\ell_0-\ell)+1).
\end{aligned}$$

□

The excess supply setting. In the remainder of this section, we present the proof that for any constant $\epsilon > 0$,in the *fully random ϵ-excess model*, the expected cost of greedy is upper bounded by a constant. We restate below the main lemma of this section, and we give the proof at the end of the section.

Lemma 15. *Let $\epsilon > 0$ be a constant. There exists a constant $C''_\epsilon > 0$ such that in the fully random ϵ-excess model, we have $\mathbb{E}[cost(\mathcal{G})] \leq C''_\epsilon$.*

We first introduce some notations that will be used through this section.

- For any $\ell, m \in [0,1]$, we let $x_{(\ell,m)} = |\{t \in [n-1] : r_t \in (\ell,m)\}|$ be the number of requests out of the $n-1$ first requests that arrived in the interval (ℓ,m).
- For any $\ell, m \in [0,1]$, we let $y_{(\ell,m)} = |\{t \in [n(1+\epsilon)] : s_t \in (\ell,m)\}|$ be the total number of servers that lie in the interval (ℓ,m).

We now prove a couple of lemmas. The first one upper bounds the probability that for some given $z \in [\frac{4(1+\epsilon/4)}{\epsilon n}, 1]$, there exists an interval of length large enough w.r.t. z such that $r_n \in I$ and such that when r_n arrives, the number of servers and of requests that arrived strictly before r_n and lying in I are equal.

Lemma 12. *Let $\epsilon > 0$ be a constant. There are constants C_ϵ, C'_ϵ such that, in the fully random ϵ-excess model, we have that for all $z \in [\frac{4+\epsilon}{\epsilon n}, 1]$,*

$$\mathbb{P}(\exists \ell, m \in [0,1] : x_{(\ell,m)} = y_{(\ell,m)}, (r_n - \ell \geq z \text{ or } \ell = 0), (m - r_n \geq z \text{ or } m = 1) \mid r_n) \leq C'_\epsilon e^{-nzC_\epsilon}.$$

Proof. In the remainder of the proof, we condition on the random variable r_n. We start by discretizing the interval $[0,1]$, and first let

$$j_0 = \begin{cases} \max\{j \in \mathbb{Z}_{\geq 0} : r_n - z - \frac{j}{n} \geq 0\} & \text{if } r_n \geq z \\ -1 & \text{otherwise.} \end{cases}$$

and

$$k_0 = \begin{cases} \min\{k \in \mathbb{Z}_{\geq 0}, r_n + z + \frac{k}{n} \leq 1\} & \text{if } 1 - r_n \geq z \\ -1 & \text{otherwise.} \end{cases}$$

Consider the case $j_0, k_0 \neq -1$. For all $j \in \{0, \dots, j_0\}$, we let $\ell_j := r_n - z - \frac{j}{n}$, and for all $k \in \{0, \dots, k_0\}$, we let $m_k := r_n + z + \frac{k}{n}$. We also let $\ell_{j_0+1} = 0$, and $m_{k_0+1} = 1$.

Now, consider any pair $(j,k) \in \{0,\dots,j_0\} \times \{0,\dots,k_0\}$. First, note that for any realization R of the sequence of requests, and for any $\ell \in [\ell_{j+1}, \ell_j], m \in [m_k, m_{k+1}]$, we have

$$x_{(\ell,m)} \leq x_{(\ell_{j+1},m_{k+1})} \quad \text{and} \quad y_{(\ell,m)} \geq y_{(\ell_j,m_k)}. \tag{15}$$

Then, note that $x_{(\ell_{j+1},m_{k+1})}$ follows a binomial distribution $\mathcal{B}(n-1, m_{k+1} - \ell_{j+1})$ and that $y_{(\ell_j,m_k)}$ follows a binomial distribution $\mathcal{B}(n(1+\epsilon), m_k - \ell_j)$. Thus, by Chernoff bounds, we get that for some $C_\epsilon > 0$,

$$\begin{aligned} \mathbb{P}(x_{(\ell_{j+1},m_{k+1})} \geq (n-1)(m_{k+1} - \ell_{j+1})(1+\epsilon/4)) \leq \; & e^{-(n-1)(m_{k+1}-\ell_{j+1})\frac{(\epsilon/4)^2}{(2+\epsilon/4)}} \\ \leq \; & e^{-(n-1)(m_k-\ell_j)C_\epsilon}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} \mathbb{P}(y_{(\ell_j,m_k)} \leq n(1+\epsilon)(m_k - \ell_j)\big(1 - \tfrac{\epsilon}{4(1+\epsilon)}\big)) \leq \; & e^{-\frac{n(1+\epsilon)(m_k-\ell_j))(\frac{\epsilon}{4(1+\epsilon)})^2}{2}} \\ \leq \; & e^{-(n-1)(m_k-\ell_j)C_\epsilon}. \end{aligned} \tag{17}$$

Next, since $z \geq \frac{4(1+\epsilon/4)}{\epsilon n}$, we have that

$$\begin{aligned} n(1+\epsilon)(m_k - \ell_j)(1 - \tfrac{\epsilon}{4(1+\epsilon)})) &- (n-1)(m_{k+1} - \ell_{j+1})(1+\epsilon/4) \\ &\geq n(1+\epsilon)(m_k - \ell_j)(1 - \tfrac{\epsilon}{4(1+\epsilon)})) - n(m_{k+1} - \ell_{j+1})(1+\epsilon/4) \\ &= n(m_k - \ell_j)(1 + 3\epsilon/4)) - n(m_k - \ell_j + \tfrac{2}{n})(1+\epsilon/4) \\ &= n(m_k - \ell_j)\epsilon/2 - 2(1+\epsilon/4) \\ &\geq nz\epsilon/2 - 2(1+\epsilon/4) \\ &\geq 0. \end{aligned} \tag{18}$$

Hence, if we have that $x_{(\ell_j,m_k)} < (n-1)(m_{k+1} - \ell_{j+1})(1+\epsilon/4)$ and that $y_{(\ell_j,m_k)} > n(1+\epsilon)(m_k - \ell_j)\big(1 - \frac{\epsilon}{4(1+\epsilon)}\big)$, then, we obtain by (15) and (18) that for all $\ell \in [\ell_{j+1}, \ell_j], m \in [m_k, m_{k+1}]$:

$$x_{(\ell,m)} \leq x_{(\ell_{j+1},m_{k+1})} < (n-1)(m_{k+1} - \ell_{j+1})(1+\epsilon/4) \leq n(1+\epsilon)(m_k - \ell_j)\big(1 - \tfrac{\epsilon}{4(1+\epsilon)}\big) < y_{(\ell_j,m_k)} \leq y_{(\ell,m)}.$$

Thus, we have:

$$\begin{aligned}
&\mathbb{P}(\exists \ell \in [\ell_{j+1}, \ell_j], m \in [m_k, m_{k+1}] : x_{(\ell,m)} = y_{(\ell,m)} |\, r_n) \\
&\leq \mathbb{P}(x_{(\ell_{j+1}, m_{k+1})} \geq (n-1)(m_{k+1} - \ell_{j+1})(1+\epsilon/4) |\, r_n) \\
&\qquad\qquad\qquad\qquad + \mathbb{P}(y_{(\ell_j, m_k)} \leq n(1+\epsilon)(m_k - \ell_j)(1 - \tfrac{\epsilon}{4(1+\epsilon)}) |\, r_n) \\
&\leq 2e^{-(n-1)(m_k - \ell_j)C_\epsilon},
\end{aligned}$$

where the last inequality is by (16) and (17), and since $x_{(\ell_j, m_k)}$ and $y_{(\ell_j, m_k)}$ are independent of r_n.

By union bound over all $(j,k) \in \{0, \ldots, j_0\} \times \{0, \ldots, k_0\}$, we obtain that for some constant $C'_\epsilon > 0$:

$$\begin{aligned}
&\mathbb{P}(\exists \ell, m \in [0,1] : x_{(\ell,m)} = y_{(\ell,m)}, (r_n - \ell \geq z \text{ or } \ell = 0), (m - r_n \geq z \text{ or } m = 1) \mid r_n) \\
&\qquad = \mathbb{P}\Big(\bigcup_{j,k \in \{0,\ldots,j_0\} \times \{0,\ldots,k_0\}} \{\exists \ell \in [\ell_{j+1}, \ell_j], m \in [m_k, m_{k+1}] : x_{(\ell,m)} = y_{(\ell,m)}\} |\, r_n \Big) \\
&\qquad \leq \sum_{j,k \in \{0,\ldots,j_0\} \times \{0,\ldots,k_0\}} \mathbb{P}(\exists \ell \in [\ell_{j+1}, \ell_j], m \in [m_k, m_{k+1}] : x_{(\ell,m)} = y_{(\ell,m)} |\, r_n) \\
&\qquad \leq \sum_{j,k \in \{0,\ldots,j_0\} \times \{0,\ldots,k_0\}} 2e^{-(n-1)(m_k - \ell_j)C_\epsilon} \\
&\qquad = 2e^{-2(n-1)zC_\epsilon} \sum_{j,k \in \{0,\ldots,j_0\} \times \{0,\ldots,k_0\}} e^{-(n-1)(\frac{j+k}{n})C_\epsilon} \\
&\qquad = C'_\epsilon e^{-2nzC_\epsilon},
\end{aligned} \tag{19}$$

where the last inequality holds since for any $C_\epsilon > 0$, $\sum_{j=1,\ldots,+\infty, k=1,\ldots,+\infty} e^{-(j+k)C_\epsilon}$ converges.

Next, we consider the case where $k_0 = -1$ or $j_0 = -1$. Now, by a similar argument as above, we have that if $j_0 = -1$ and $k_0 \neq -1$, then for all $k \in \{0, \ldots, k_0\}$,

$$\mathbb{P}(\exists m \in [m_k, m_{k+1}] : x_{(0,m)} = y_{(0,m)} |\, r_n) \leq 2e^{-n(m_k - 0)C_\epsilon},$$

and if $k_0 = -1$ and $j_0 \neq -1$, then for all $j \in \{0, \ldots, j_0\}$,

$$\mathbb{P}(\exists \ell \in [\ell_{j+1}, \ell_j] : x_{(\ell,1)} = y_{(\ell,1)} |\, r_n) \leq 2e^{-n(1-\ell_j)C_\epsilon}.$$

We conclude in a similar way as in (19) that there are $C_\epsilon, C'_\epsilon > 0$ such that

$$\mathbb{P}(\exists \ell, m \in [0,1] : x_{(\ell,m)} = y_{(\ell,m)}, (r_n - \ell \geq z \text{ or } \ell = 0), (m - r_n \geq z \text{ or } m = 1) \mid r_n) \leq C'_\epsilon e^{-nzC_\epsilon}.$$

Finally, if $j_0 = -1$ and $k_0 = -1$, note that $1 - z < r_n < z$. In this case, since $x_{(0,1)} = n - 1$ and $y_{(0,1)} = n(1+\epsilon)$, we simply have

$$\begin{aligned}
&\mathbb{P}(\exists \ell, m \in [0,1] : x_{(\ell,m)} = y_{(\ell,m)}, (r_n - \ell \geq z \text{ or } \ell = 0), (m - r_n \geq z \text{ or } m = 1) \mid r_n) \\
&\qquad\qquad\qquad\qquad = \mathbb{P}(x_{(0,1)} = y_{(0,1)} |\, r_n) = 0 \leq C'_\epsilon e^{-nzC_\epsilon}.
\end{aligned}$$

□

Next, we upper bound the expected cost incurred by $\mathcal{G}$ while matching the n^{th} request.

Lemma 13. *Let $\epsilon > 0$ be a constant. There is a constant C''_ϵ such that, in the fully random ϵ-excess model, we have $\mathbb{E}[cost_n(\mathcal{G})] \leq \frac{C''_\epsilon}{n}$.*

Proof. To exclude any ambiguity, we condition on the event that all servers are distinct and that no server or requests are at positions 0 and 1, which occurs almost surely.

In the remainder of the proof, we condition on the variable r_n and let $s_n^L = \max\{s \in S_{n-1} : s \leq r_n\}$ and $s_n^R = \min\{s \in S_{n-1} : s \geq r_n\}$ denote the nearest available servers on the left and on the right of r_n when r_n arrives; with the convention that $s_n^L = 0$ and $s_n^L = 1$ if there are no such servers.

Now, let $z \in [\frac{4(1+\epsilon/4)}{\epsilon n}, 1]$ and assume that $\text{cost}_n(\mathcal{G}) \geq z$. Since $\mathcal{G}$ matches r_n to the closest available server, we must have $r_n - s_n^L \geq z$ or $s_n^L = 0$, and $s_n^R - r_n \geq z$ or $s_n^R = 1$. In addition, by definition of s_n^L and s_n^R, we have that $(s_n^L, s_n^R) \cap S_{n-1} = \emptyset$. Now, recall that all requests $r_1, \ldots, r_{n-1}$ have been matched each time to the closest available server. Moreover, s_n^L was either available when r_j arrives, but r_j was not matched to it, or $s_n^L = 0$; similarly for s_n^R. Hence, for all $j \in [n-1]$, if $r_j \notin (s_n^L, s_n^R)$, then $s_\mathcal{G}(r_j) \notin (s_n^L, s_n^R)$. Similarly, if $r_j \in (s_n^L, s_n^R)$, then $s_\mathcal{G}(r_j) \in (s_n^L, s_n^R)$. Therefore,

$$|\{j \in [n-1] : s_\mathcal{G}(r_j) \in (s_n^L, s_n^R)\}| = |\{j \in [n-1] : r_j \in (s_n^L, s_n^R)\}|.$$

In addition, since $(s_n^L, s_n^R) \cap S_{n-1} = \emptyset$, all servers in $(s_n^L, s_n^R) \cap S_0$ must have been matched to some request before time $n-1$, hence

$$|\{j \in [n-1] : s_\mathcal{G}(r_j) \in (s_n^L, s_n^R)\}| = |\{j \in [n(1+\epsilon)] : s_j \in (s_n^L, s_n^R)\}|.$$

By combining the two previous equalities and by definition of $x_{(s_n^L, s_n^R)}$, and $y_{(s_n^L, s_n^R)}$, we get that

$$x_{(s_n^L, s_n^R)} = |\{j \in [n-1] : r_j \in (s_n^L, s_n^R)\}| = |\{j \in [n(1+\epsilon)] : s_j \in (s_n^L, s_n^R)\}| = y_{(s_n^L, s_n^R)}.$$

Since we have that $r_n - s_n^L \geq z$ or $s_n^L = 0$, and $s_n^R - r_n \geq z$ or $s_n^R = 1$, we thus have that

$$\begin{aligned}
&\mathbb{P}(\text{cost}_n(\mathcal{G}) \geq z \mid r_n) \\
&\quad \leq \mathbb{P}(\exists \ell, m \in [0,1] : x_{(\ell,m)} = y_{(\ell,m)}, (r_n - \ell \geq z \text{ or } \ell = 0), (m - r_n \geq z \text{ or } m = 1) \mid r_n).
\end{aligned} \tag{20}$$

Now, by Lemma 12, we have that for some constants $C_\epsilon, C'_\epsilon > 0$:

$$\mathbb{P}(\exists \ell, m \in [0,1] : x_{(\ell,m)} = y_{(\ell,m)}, (r_n - \ell \geq z \text{ or } \ell = 0), (m - r_n \geq z \text{ or } m = 1) \mid r_n) \leq C'_\epsilon e^{-nzC_\epsilon}.$$

Combining this with (20) and by the law of total probability, we get $\mathbb{P}(\text{cost}_n(\mathcal{G}) \geq z) \leq C'_\epsilon e^{-nzC_\epsilon}$. Hence, we obtain

$$\begin{aligned}
\mathbb{E}[cost_n(\mathcal{G})] &\leq \tfrac{4(1+\epsilon/4)}{\epsilon n} + \int_{z=\frac{4(1+\epsilon/4)}{\epsilon n}}^{1} \mathbb{P}(cost_n(\mathcal{G}) \geq z) \mathrm{d}z \\
&\leq \tfrac{4(1+\epsilon/4)}{\epsilon n} + \int_{z=\frac{4(1+\epsilon/4)}{\epsilon n}}^{1} C'_\epsilon e^{-nzC_\epsilon} \mathrm{d}z \\
&\leq \tfrac{4(1+\epsilon/4)}{\epsilon n} + \frac{C'_\epsilon}{C_\epsilon n} && \text{(for some } C_\epsilon > 0). \\
&= \frac{C''_\epsilon}{n} && \text{(for some } C''_\epsilon > 0).
\end{aligned}$$

□

We would like to underscore that a simple application of Chernoff bounds between all initial pairs of servers locations would only lead to a weaker version of the above lemma, involving polylogarithmic terms. Since our objective was to present a sharp analysis of greedy, we introduced the more refined analysis presented above.

We next show that, because of servers getting less and less dense as requests arrive, the expected cost at each step of the greedy algorithm is nondecreasing.

Lemma 14. *Let $\epsilon > 0$ be a constant. Then, in the fully random ϵ-excess model, we have that for all $i \in [n-1]$, $\mathbb{E}[cost_i(\mathcal{G})] \leq \mathbb{E}[cost_{i+1}(\mathcal{G})]$.*

Proof. Recall that $S_0 \supseteq \ldots \supseteq \mathcal{S}_n$ denote the sequence of set of free servers obtained during the execution of $\mathcal{G}$. Since for all $i \in [n-1]$, $\mathcal{S}_i \subseteq \mathcal{S}_{i-1}$, we have that for all $S \subseteq [0,1]$ with $|S| = n-(i-1)$ and $S' \subseteq S$ such that $|S'| = |S| - 1$:

$$\begin{aligned}\mathbb{E}[cost_i(\mathcal{G})|S_{i-1} = S] &= \mathbb{E}_{r_i \sim \mathcal{U}[0,1]}[\min_{s \in S} |r_i - s|] \\ &\leq \mathbb{E}_{r_{i+1} \sim \mathcal{U}[0,1]}[\min_{s \in S'} |r_{i+1} - s|] \\ &= \mathbb{E}[cost_{i+1}(\mathcal{G})|S_i = S'] \\ &= \mathbb{E}[cost_{i+1}(\mathcal{G})|S_i = S', S_{i-1} = S],\end{aligned}$$

where the last equation holds since conditioning on S_i, the matching decision for r_{i+1} is independent of S_{i-1}.

Hence, by applying a first time the tower rule over S_{i-1}, we get $\mathbb{E}[cost_i(\mathcal{G})] \leq \mathbb{E}[cost_{i+1}(\mathcal{G})|S_i = S']$, and by applying it a second time over S_i, we get

$$\mathbb{E}[cost_i(\mathcal{G})] \leq \mathbb{E}[cost_{i+1}(\mathcal{G})].$$ □

We are now ready to prove the main lemma of this section.

Lemma 15. *Let $\epsilon > 0$ be a constant. There exists a constant $C''_\epsilon > 0$ such that in the fully random ϵ-excess model, we have $\mathbb{E}[cost(\mathcal{G})] \leq C''_\epsilon$.*

Proof. Using Lemma 13 and Lemma 14, we conclude that

$$\mathbb{E}[cost(\mathcal{G})] = \sum_{i=1}^{n} \mathbb{E}[cost_i(\mathcal{G})] \leq n \cdot \mathbb{E}[cost_n(\mathcal{G})] \leq C''_\epsilon.$$ □

D Missing Analysis from Section 4.1

Lemma 17. *In the random requests model, for any online algorithm $\mathcal{A}$ and any time step $t \in [n]$, we have that $\mathbb{P}(cost_t(\mathcal{A}) \geq 1/n^4) \geq 1 - 2/n^3$ and $\mathbb{E}[cost_t(\mathcal{A})] \geq \frac{1}{2(n+1)}$.*

Proof. We first lower bound the probability that $\text{cost}_t(\mathcal{A}) \geq 1/n^4$. Conditioning on S_{t-1}, we have

$$\begin{aligned}
\mathbb{P}(\text{cost}_t(\mathcal{A}) < 1/n^4 \mid S_{t-1}) &\leq \mathbb{P}(\exists s \in S_{t-1} : |r_t - s| < 1/n^4 \mid S_{t-1}) \\
&= \mathbb{P}\Big(\bigcup_{s \in S_{t-1}} \{r_t \in [\max(0, s - 1/n^4), \min(1, s + 1/n^4)]\} \mid S_{t-1}\Big) \\
&\leq \sum_{s \in S_{t-1}} \mathbb{P}(r_t \in [\max(0, s - 1/n^4), \min(1, s + 1/n^4)] \mid S_{t-1}) \\
&\leq \sum_{s \in S_{t-1}} 2/n^4 \\
&\leq 2/n^3.
\end{aligned}$$

Thus, $\mathbb{P}(\text{cost}_t(\mathcal{A}) < 1/n^4) \leq 2/n^3$.

Next, we lower bound $\mathbb{E}[\text{cost}_t(\mathcal{A})]$. We condition on S_{t-1} and let $0 \leq s_{t,1} \leq \ldots \leq s_{t,n} \leq 1$ denote the ordered servers of S_{t-1}. By convention, we also write $s_{t,0} = 0$, $s_{t,n+1} = 1$. Then,

$$\begin{aligned}
\mathbb{E}[\text{cost}_t(\mathcal{A}) | S_{t-1}] &= \sum_{i=0}^{n} \mathbb{P}(r_t \in [s_{t,i}, s_{t,i+1}]) \mathbb{E}[\text{cost}_t(\mathcal{A}) | r_t \in [s_{t,i}, s_{t,i+1}]] \\
&= \sum_{i=0}^{n} \frac{(s_{t,i+1} - s_{t,i})^2}{2}.
\end{aligned}$$

Since $\sum_{i=0}^{n}(s_{t,i+1} - s_{t,i}) = s_{t,n+1} - s_{t,0} = 1$, the above sum is minimized when $s_{t,i+1} - s_{t,i} = 1/(n+1)$ for all i, and the minimum value is $\frac{1}{2(n+1)}$. By the tower law, we deduce that $\mathbb{E}[cost_t(\mathcal{A})] \geq \frac{1}{2(n+1)}$. ☐

In the remainder of this section, we demonstrate the existence of a $O(1)$-competitive algorithm for the random requests model that makes neighboring matches. We first show that we can always transform any algorithm into an algorithm which satisfies this last property without increasing the total cost.

Lemma 53. *For any online algorithm $\mathcal{A}$, there exists an algorithm $\mathcal{A}'$ that makes neighboring matches such that $\mathbb{E}[cost(\mathcal{A})'] \leq \mathbb{E}[cost(\mathcal{A})]$.*

Proof. We show the result by induction on t. Let $t_0 \geq 0$ and suppose that $\mathcal{A}$ is an online algorithm that makes neighboring matches for all $t \leq t_0$, but does not necessarily make neighboring matches when $t > t_0$. Without loss of generality, assume that r_{t_0+1} is matched by $\mathcal{A}$ to an available server $s \in S_{\mathcal{A},t_0}$ such that $r_{t_0+1} \leq s$. Now, let $s' = \min\{z \in S_{\mathcal{A},t_0} : z \geq r_{t_0+1}\}$ denote the closest available server on the right of r_{t_0+1}, and let $j \in \{t_0 + 1, \ldots n\}$ be such that $\mathcal{A}$ matches request r_j to s'. We define the algorithm $\mathcal{A}'$ that matches all requests r_t to exactly the same servers $\mathcal{A}$ matches them to for all $t \neq j, t_0 + 1$, matches r_{t_0+1} to s' and matches r_j to s (this is a valid construction since s' is available when r_{t_0+1} arrives and s is available when r_j arrives). Then by construction, $\mathcal{A}'$ makes neighboring matches for all $t \leq t_0 + 1$.

We now analyse the cost of $\mathcal{A}'$. Since $\mathcal{A}$ and $\mathcal{A}'$ match all requests other than r_{t_0+1} and r_j to the same servers, they incur the same cost for these requests. Now, we consider three cases: if $r_j \leq s'$, then

$$\begin{aligned}
\text{cost}_{t_0+1}(\mathcal{A}) + \text{cost}_j(\mathcal{A}) &= |r_{t_0+1} - s| + |r_j - s'| = (s - r_{t_0+1}) + (s' - r_j) \\
&= (s' - r_{t_0+1}) + (s - r_j) = |r_{t_0+1} - s'| + |r_j - s| = \text{cost}_{t_0+1}(\mathcal{A}') + \text{cost}_j(\mathcal{A}'),
\end{aligned}$$

if $s' \leq r_j \leq s$, then

$$\begin{aligned}\text{cost}_{t_0+1}(\mathcal{A}) + \text{cost}_j(\mathcal{A}) &= |r_{t_0+1} - s| + |r_j - s'| \\ &= (s - r_{t_0+1}) + (r_j - s')| = (s' - r_{t_0+1}) + (s - r_j) + 2(r_j - s') \\ &\geq (s' - r_{t_0+1}) + (s - r_j) = |r_{t_0+1} - s'| + |r_j - s| = \text{cost}_{t_0+1}(\mathcal{A}') + \text{cost}_j(\mathcal{A}'),\end{aligned}$$

and if $r_j \geq s$, then

$$\begin{aligned}\text{cost}_{t_0+1}(\mathcal{A}) + \text{cost}_j(\mathcal{A}) &= |r_{t_0+1} - s| + |r_j - s'| \\ &= (s - r_{t_0+1}) + (r_j - s')| = (s' - r_{t_0+1}) + (r_j - s) + 2(s - s') \\ &\geq (s' - r_{t_0+1}) + (r_j - s) = |r_{t_0+1} - s'| + |r_j - s| = \text{cost}_{t_0+1}(\mathcal{A}') + \text{cost}_j(\mathcal{A}'),\end{aligned}$$

Hence, in all cases, $\mathcal{A}'$ achieves a lower cost than $\mathcal{A}$ for requests $\{r_{t_0+1}, r_j\}$.

Therefore, $\mathcal{A}'$ makes neighboring matches and is such that $\mathbb{E}[cost(\mathcal{A})'] \leq \mathbb{E}[cost(\mathcal{A})]$. □

Next, we show that a simple adaptation of the algorithm Fair-Bias from [GGPW19] is a $O(1)$-competitive algorithm for the random requests model. We first recall a result from [GGPW19].

Lemma 54 (Theorem 4.6. in [GGPW19])**.** *Given a tree metric (S, d) with a server at each of the $n = |S| \geq 2$ points. Algorithm Fair-Bias is 9-competitive if the requests are drawn from a distribution $\mathcal{D}$ over the servers' locations.*

Note that in the above lemma, the requests have support in the servers' locations, whereas, in the random requests model, we consider uniform requests in $[0, 1]$. We show in the following lemma that we can nevertheless derive from Algorithm Fair-Bias a $O(1)$-competitive algorithm $\mathcal{A}$ in the random requests model, and such that $\mathcal{A}$ makes neighboring matches.

Lemma 19. *In the random requests model, there exists a $O(1)$-competitive algorithm that makes neighboring matches.*

Proof. To ease the notations, we write $\mathcal{F}_b$ to denote the algorithm Fair-bias in the remainder of the proof.

Given an instance of the random requests model with set of servers S and a realization of the requests sequence $R = \{r_1, \dots, r_n\}$, we let $\tilde{R} = \{\tilde{r_1}, \dots, \tilde{r_n}\}$ be such that for all $i \in [n]$, $\tilde{r_i} = \arg\min_{s \in S} |r_i - s|$ is the closest server location to r_i. We consider the algorithm $\mathcal{A}$ that matches the requests in R exactly to the same servers $\mathcal{F}_b$ matches the requests in $\tilde{R}$. In order to analyze $\mathcal{A}$, we now define a distribution $\mathcal{D}$ over the servers locations, such that for all $s \in S$,

$$\mathbb{P}_{r \sim \mathcal{D}}(r = s) = \mathbb{P}_{r \sim \mathcal{U}([0,1])}\big(\arg\min_{s' \in S} |r - s'| = s\big).$$

Note that since for all $i \in [n]$, $r_i \sim \mathcal{U}([0, 1])$, we have, by construction, that $\tilde{R} \sim \mathcal{D}$ when $R \sim \mathcal{U}([0, 1])$.

We now show that $\mathcal{A}$ has a constant competitive ratio. Let R be the realization of the requests and $\tilde{R}$ the corresponding transformed requests. We let M_R be an optimal offline matching for R and $\texttt{OPT}_R$ be the cost of this matching. We also let $s_{M_R}(i)$ be the server M matches i to. In addition, let $\texttt{OPT}_{\tilde{R}}$ denote the cost of an optimal offline matching for $\tilde{R}$. Now, the cost of the matching returned by $\mathcal{A}$ satisfies:

$$\text{cost}_R(\mathcal{A}) = \sum_{i=1}^{n} |r_i - s_{\mathcal{A}}(r_i)| \leq \sum_{i=1}^{n} (|r_i - \tilde{r_i}| + |\tilde{r_i} - s_{\mathcal{A}}(r_i)|) = \sum_{i=1}^{n} |r_i - \tilde{r_i}| + \text{cost}_{\tilde{R}}(\mathcal{F}_b). \tag{21}$$

Since for all $i \in [n]$, we have that $|r_i - \tilde{r}_i| = \min_{s \in S} |r_i - s| \leq |r_i - s_{M_R}(i)|$, we immediately get

$$\sum_{i=1}^{n} |r_i - \tilde{r}_i| \leq \texttt{OPT}_R. \tag{22}$$

In addition, by considering the matching $\{(\tilde{r}_i, s_{M_R}(i))\}_{i \in [n]}$ for $\tilde{R}$, we get

$$\texttt{OPT}_{\tilde{R}} \leq \sum_{i=1}^{n} |\tilde{r}_i - s_{M_R}(i)| \leq \sum_{i=1}^{n} (|\tilde{r}_i - r_i| + |r_i - s_{M_R}(i)|) \leq \sum_{i=1}^{n} 2|r_i - s_{M_R}(i)| = 2\texttt{OPT}_R.$$

Hence, $\mathbb{E}_{R \sim \mathcal{U}([0,1])}[\texttt{OPT}_{\tilde{R}}] \leq 2 \cdot \mathbb{E}_{R \sim \mathcal{U}([0,1])}[\texttt{OPT}_R]$. Combining this with the fact that fair-bias is 9-competitive by Lemma 54, and using the definition of the distribution $\mathcal{D}$, we obtain

$$\begin{aligned}\mathbb{E}_{R \sim \mathcal{U}([0,1])}[\text{cost}_{\tilde{R}}(\mathcal{F}_b)] = \mathbb{E}_{\tilde{R} \sim \mathcal{D}}[\text{cost}_{\tilde{R}}(\mathcal{F}_b)] \leq 9 \cdot \mathbb{E}_{\tilde{R} \sim \mathcal{D}}[\texttt{OPT}_{\tilde{R}}] \\ = 9 \cdot \mathbb{E}_{R \sim \mathcal{U}([0,1])}[\texttt{OPT}_{\tilde{R}}] \leq 18 \cdot \mathbb{E}_{R \sim \mathcal{U}([0,1])}[\texttt{OPT}_R].\end{aligned} \tag{23}$$

Hence, taking the expectation over $R \sim \mathcal{U}([0,1])$ on both sides of (21) and combining it with (22) and (23), we finally obtain

$$\mathbb{E}_{R \sim \mathcal{U}([0,1])}[\text{cost}_R(\mathcal{A})] \leq (1 + 18) \cdot \mathbb{E}_{R \sim \mathcal{U}([0,1])}[\texttt{OPT}_R],$$

which shows that $\mathcal{A}$ is $O(1)$-competitive in the random requests model. Using Lemma 53, we can then transform $\mathcal{A}$ into a $O(1)$-competitive algorithm $\mathcal{A}'$ that makes neighboring matches, which completes the proof of the lemma. □

E Hierarchical Greedy is $\Omega(n^{1/4})$ in the Random Requests Model

In this section, we show that in the random requests model, the Hierarchical Greedy algorithm proposed in [Kan21] is $\Omega(n^{1/4})$ competitive on the line. We first introduce the instance on which this lower bound is achieved. To ease the presentation, we define an instance $\mathcal{J}_{2n}$ with $2n$ servers and $2n$ requests and where the servers and requests are in $[0, 2]$. Note that by as simple scaling argument, this instance can be cast as an instance of the random requests model with n servers and requests in $[0, 1]$.

Description of the instance $\mathcal{J}_{2n}$. We define the set of servers S_0 as follows: there are $n - n^{3/4}$ servers uniformly spread in the interval $[0, 1 - n^{-1/4}]$, there are no servers in the interval $(1 - n^{-1/4}, 1)$, there are $n^{3/4}$ servers at position $1 + n^{-1/4}$, and the remaining n servers are uniformly spread in the interval $[1, 2]$. More precisely, we let $s_j = \frac{j}{n}$ for all $j \in [n - n^{3/4}]$, $s_j = 1 + n^{-1/4}$ for all $j \in \{n - n^{3/4} + 1, \ldots, n\}$ and $s_j = \frac{j}{n}$ for all $j \in \{n + 1, \ldots, 2n\}$. The sequence of requests R contains $2n$ requests sampled uniformly at random in $[0, 2]$. We note that, interestingly, the servers are almost uniform since a $1 - o(1)$ fraction of the servers are uniformly spread in the interval $[0, 2]$. In other words, the Hierarchical Greedy algorithm is not robust to a small perturbation of the servers.

Lemma 55. *The expected value of the optimal offline matching for the instance $\mathcal{J}_{2n}$ satisfies:* $\mathbb{E}[\mathit{OPT}] = O(\sqrt{n})$.

Proof. For a given realization R of the requests sequence, we partition the requests into $R_1 = \{r \in R : r \in [0, 1 - n^{-1/4}]\}$, $R_2 = \{r \in R : r \in (1 - n^{-1/4}, 1)\}$ and $R_3 = \{r \in R : r \in [1, 2]\}$. We also let $\overline{R}_1$ be the first $n - n^{3/4}$ elements of R_1, or $\overline{R}_1 = R_1$ if $|R_1| < n - n^{3/4}$; we let $\overline{R}_2$ be the first $n^{3/4}$ elements of R_2, or $\overline{R}_2 = R_2$ if $|R_2| < n^{3/4}$, and we let $\overline{R}_3$ be the first n elements of R_3, or $\overline{R}_3 = R_3$ if $|R_3| < n$.

We now define the following matching M, where for all $r \in R$, $s_M(r)$ denotes the server to which r is matched and for all $\tilde{R} \subseteq R$, $M|_{\tilde{R}}$ denotes the restriction of M to requests in $\tilde{R}$:

- $M|_{\overline{R}_1}$ is an optimal matching between $\overline{R}_1$ and $S_0 \cap [0, 1 - n^{-1/4}]$.
- For all $r \in \overline{R}_2$, $s_M(r) = 1 + n^{-1/4}$.
- $M|_{\overline{R}_3}$ is an optimal matching between $\overline{R}_3$ and $S_0 \cap [1, 2]$.
- The remaining requests are matched arbitrarily to the remaining free servers.

Note that M is well defined since $|\overline{R}_1| \le n - n^{3/4} = |S_0 \cap [0, 1 - n^{-1/4}]|$, $|\overline{R}_2| \le n^{3/4} = |S_0 \cap \{1 + n^{-1/4}\}|$, and $|\overline{R}_3| \le n = |S_0 \cap [1, 2]|$.

Now, for all $r \in \overline{R}_2$, since $r \in (1 - n^{-1/4}, 1)$, we have $|r - s_M(r)| = |1 + n^{-1/4} - r| \le 2n^{-1/4}$, hence, letting $\text{cost}(M)$ denote the cost of the matching M, we have

$$\mathbb{E}[\text{cost}(M|_{\overline{R}_2})] = \mathbb{E}[\sum_{r \in \overline{R}_2} |s_M(r) - r|] \le \mathbb{E}[|\overline{R}_2|] \cdot 2n^{-1/4} \le n^{3/4} \cdot 2n^{-1/4} = 2n^{1/2}. \tag{24}$$

Next, note that the requests in $\overline{R}_1$ are uniform i.i.d. in $[0, 1 - n^{-1/4}]$ and the servers in $S_0 \cap [0, 1 - n^{-1/4}]$ are uniformly spread in $[0, 1 - n^{-1/4}]$. Similarly, the requests in $\overline{R}_3$ are uniform i.i.d. in $[1, 2]$ and the servers in $S_0 \cap [1, 2]$ are uniformly spread in $[1, 2]$. Hence by Lemma 11, we have that

$$\mathbb{E}[\text{cost}(M|_{\overline{R}_1})] + \mathbb{E}[\text{cost}(M|_{\overline{R}_3})] = O(\sqrt{n}). \tag{25}$$

Now, note that $|R_1| = |\{r \in R : r \in [0, 1 - n^{-1/4}]\}|$ follows a binomial distribution $\mathcal{B}(2n, (1 - n^{-1/4})/2)$, $|R_2| = |\{r \in R : r \in (1 - n^{-1/4}, 1)\}|$ follows a binomial distribution $\mathcal{B}(2n, n^{-1/4}/2)$ and $|R_3| = |\{r \in R : r \in [1, 2]\}|$ follows a binomial distribution $\mathcal{B}(2n, 1/2)$. Hence, by Lemma 49, we get

$$\begin{aligned}
\mathbb{E}[|R_1 \setminus \overline{R}_1|] &= \mathbb{E}[\max(0, |R_1| - n - n^{3/4})] \le \mathbb{E}[||R_1| - n - n^{3/4}|] \\
&\le \sqrt{(n - n^{3/4}) \cdot (1 - (1 - n^{-1/4})/2)} = O(\sqrt{n}), \\
\mathbb{E}[|R_2 \setminus \overline{R}_2|] &= \mathbb{E}[\max(0, |R_2| - n^{3/4})] \le \mathbb{E}[||R_2| - n^{3/4}|] \le \sqrt{n^{3/4} \cdot (1 - n^{-1/4}/2)} = O(\sqrt{n}), \\
\mathbb{E}[|R_3 \setminus \overline{R}_3|] &= \mathbb{E}[\max(0, |R_3| - n] \le \mathbb{E}[||R_3| - n|] \le \sqrt{n(1 - 1/2)} = O(\sqrt{n}).
\end{aligned}$$

Since for all $r \in R$, we have $|s_M(r) - r| \le 1$, we get

$$\mathbb{E}[\text{cost}(M|_{(R_1 \setminus \overline{R}_1) \cup (R_2 \setminus \overline{R}_2)) \cup (R_3 \setminus \overline{R}_3)})] \le \mathbb{E}[|R_1 \setminus \overline{R}_1|] + \mathbb{E}[|R_2 \setminus \overline{R}_2|] + \mathbb{E}[|R_3 \setminus \overline{R}_3|] = O(\sqrt{n}). \tag{26}$$

Combining (24), (25) and (26), we get

$$\begin{aligned}
\mathbb{E}[\texttt{OPT}] \le \mathbb{E}[\text{cost}(M)] = \mathbb{E}[\text{cost}(M|_{(R_1 \setminus \overline{R}_1) \cup (R_2 \setminus \overline{R}_2) \cup (R_3 \setminus \overline{R}_3)}] + \mathbb{E}[\text{cost}(M|_{\overline{R}_1})] \\
+ \mathbb{E}[\text{cost}(M|_{\overline{R}_2})] + \mathbb{E}[\text{cost}(M|_{\overline{R}_3})] = O(\sqrt{n}). \quad \square
\end{aligned}$$

Lemma 56. *The expected cost of the matching returned by algorithm $\mathcal{A}^H$ on instance $\mathcal{J}_{2n}$ satisfies:* $\mathbb{E}[cost(\mathcal{A}^H, \mathcal{J}_{2n})] = \Omega(n^{3/4})$.

Proof. For a given realization R of the requests sequence, we let $R_1 = \{r \in R : r \in [0,1]\}$. We also let $\overline{R}_1$ be the first $n - n^{3/4}$ elements of R_1, or $\overline{R}_1 = R_1$ if $|R_1| < n - n^{3/4}$. Now, note that $|R_1| = |\{r \in R : r \in [0,1]\}|$ follows a binomial distribution $\mathcal{B}(2n, 1/2)$ with mean n. Hence,

$$\mathbb{E}[|R_1 \setminus \overline{R}_1|] \geq n^{3/4}\mathbb{P}(|R_1 \setminus \overline{R}_1| \geq n^{3/4}) = n^{3/4}\mathbb{P}(|R_1| \geq n) = \frac{1}{2}n^{3/4}. \tag{27}$$

Next, note that the Hierarchical Greedy algorithm matches a request $r \in [0,1]$ to a server in $(1,2]$ only if $[0,1]$ has no more available servers. Hence, since $\overline{R}_1$ contains at most $n - n^{3/4}$ requests and there are initially $n - n^{3/4}$ servers in $[0,1]$, all requests in $\overline{R}_1$ will be matched to servers in $[0,1]$. Now, if $|R_1 \setminus \overline{R}_1| > 0$, then $|\overline{R}_1| = n - n^{3/4} = S_0 \cap [0,1]$; hence, when any request $r \in |R_1 \setminus \overline{R}_1|$ arrives, all the servers in $[0,1]$ have already been matched to a request in $\overline{R}_1$. Therefore, r is matched by $\mathcal{A}^H$ to a server $s_{\mathcal{A}^H}(r)$ in $(1,2]$. Noting that the requests in $R_1 \setminus \overline{R}_1$ are uniform in $[0,1]$, we thus have, for any t such that $r_t \in |R_1 \setminus \overline{R}_1|$,

$$\mathbb{E}[cost_t(\mathcal{A}^H, r_t)|r_t \in R_1 \setminus \overline{R}_1] = \mathbb{E}[s_{\mathcal{A}^H}(r_t) - r_t|r_t \in R_1 \setminus \overline{R}_1] \geq 1 - \mathbb{E}[r_t|r_t \in R_1 \setminus \overline{R}_1] = \frac{1}{2}. \tag{28}$$

Thus,

$$\begin{aligned}
\mathbb{E}[cost(\mathcal{A}^H, \mathcal{J}_{2n})] &= \sum_{t \in [n]} \mathbb{E}[cost_t(\mathcal{A}^H, r_t)] \\
&\geq \sum_{t \in [n]} \mathbb{E}[cost_t(\mathcal{A}^H, r_t)|r_t \in R_1 \setminus \overline{R}_1]\mathbb{P}(r_t \in R_1 \setminus \overline{R}_1) \\
&\geq \sum_{t \in [n]} \frac{1}{2}\mathbb{P}(r_t \in R_1 \setminus \overline{R}_1) && \text{(by (28))} \\
&= \frac{1}{2}\mathbb{E}[\sum_{t \in [n]} \mathbb{1}_{r_t \in R_1 \setminus \overline{R}_1}] \\
&= \frac{1}{2}\mathbb{E}[|R_1 \setminus \overline{R}_1|] \\
&= \frac{1}{4}n^{3/4}. && \text{(by (27))}
\end{aligned}$$

□

By combining Lemmas 56 and 55, we get the following result.

Lemma 57. *For online matching on the line in the random requests model, the Hierarchical Greedy algorithm $\mathcal{A}^H$ achieves an $\Omega(n^{1/4})$-competitive ratio.*

F Missing Analysis from Section 4.3

F.1 Missing analysis from Section 4.3.1

Fact 26. *Let $\tilde{n} := n - 4\log(n)^2\sqrt{n}/(1 - n^{-1/5})$. For any $I \subseteq [n^{-1/5}, 1]$, we have $|S_0 \cap I| \in [\tilde{n}|I| - 1, \tilde{n}|I| + 3]$*

Proof. Let $x, y \in [0,1]$ such that $x \le y$ and let $k, j \in \mathbb{N}$ be such that $\frac{k}{\tilde{n}} \le x \le \frac{k+1}{\tilde{n}}$ and $\frac{j-1}{\tilde{n}} \le y \le \frac{j}{\tilde{n}}$. Then by construction of $\mathcal{I}_n$, the number of servers in the interval $[x, y]$ is in $\{j-k-1, j-k, j-k+1\}$, and by definition of k, j, we have $j-k-2 \le \tilde{n}(y-x) \le j-k$. Hence, for all $z > \tilde{n}(y-x)+3$ or $z < \tilde{n}(y-x)-1$, we have $z \notin \{j-k-1, j-k, j-k+1\}$; hence $|S_0 \cap [x,y]| \in [\tilde{n}(y-x)-1, \tilde{n}(y-x)+3]$. $\square$

Lemma 30. *Assume that the sequence of requests is regular. Then, for n large enough and for all $t \in [n - n^{c_3}]$ and $j \in [m_t - 1]$, we have $s_{t,j+1} - s_{t,j} \le 2\log(n)^4 n^{1-2c_3}$.*

Proof. Note that if the statement of the lemma holds for $t = n - n^{c_3}$, then it holds for all $t \in [n - n^{c_3}]$ since $S_t \supseteq S_{n-n^{c_3}}$ when $t \le n - n^{c_3}$. Hence it suffices to consider the case $t = n - n^{c_3}$.

Now, consider $s, s' \in S_{n-n^{c_3}} \cap (0,1]$ such that $s' - s > 2\log(n)^4 n^{1-2c_3}$. Note that if there exists $s'' \in S_{n-n^{c_3}}$ such that $s < s'' < s'$, we are done. In the remainder of the proof, we show that there is such an s''. By definition of $\mathcal{H}^m$, each request is either matched greedily, or it is matched to 0. Hence, for all $j \in [n - n^{c_3}]$, if $r_j \notin (s, s')$, then $s_{\mathcal{H}^m}(r_j) \notin (s, s')$ (since r_j is closer to either s or s' than any point in (s, s'), and both s and s' are available when r_j arrives). Similarly, by the greediness of $\mathcal{H}^m$ for requests $r > y_0$ and since $s > y_0$ (by definition of S_0 and since $S_{n-n^{c_3}} \subseteq S_0$), if $r_j \in (s, s')$, then $s_{\mathcal{H}^m}(r_j) \in (s, s')$ for all $j \le n - n^{c_3}$. Therefore,

$$|\{j \in [n - n^{c_3}] : s_{\mathcal{H}^m}(r_j) \in (s, s')\}| = |\{j \in [n - n^{c_3}] : r_j \in (s, s')\}|.$$

Note that $d^+((s, s')) \cdot n^{1-2c_3} \ge (s' - s)n^{1-2c_3} \ge 2\log(n)^4 n^{1-2c_3}(n - n^{c_3}) = \Omega(1)$. Hence, by applying the first regularity condition with $t = 0$, $t' = n - n^{c_3}$, and $(d, d') = d^+((s, s'))$, we have

$$\begin{aligned}
|\{j \in [n - n^{c_3}] :\ r_j \in (s, s')\}| &\le |\{j \in [n - n^{c_3}] :\ r_j \in d^+((s, s'))\}| \\
&\le (n - n^{c_3}) \cdot |d^+((s, s'))| + \log(n)^2 \sqrt{(n - n^{c_3}) \cdot |d^+((s, s'))|} \\
&\le (n - n^{c_3})(s' - s + \tfrac{2}{n}) + \log(n)^2 \sqrt{(n - n^{c_3})(s' - s + \tfrac{2}{n})} \\
&\le (n - n^{c_3})(s' - s) + \log(n)^2 \sqrt{(n - n^{c_3})(s' - s)} + 4\log(n)^2.
\end{aligned}$$

By combining the previous inequality with the previous equality, we obtain that

$$|\{j \in [n - n^{c_3}] : s_{\mathcal{H}^m}(r_j) \in (s, s')\}| \le (n - n^{c_3})(s' - s) + \log(n)^2 \sqrt{(n - n^{c_3})(s' - s)} + 4\log(n)^2. \quad (29)$$

Now, since we assumed $s' - s > 2\log(n)^4 n^{1-2c_3}$, we have, for n large enough and since $c_3 \in (4/5, 1)$, that $\left(1 - 4\log(n)^2 n^{1/2-c_3}/(1 - n^{-1/5}) - (4\log(n)^2 + 1)/((s' - s)n^{c_3})\right)^{-1} \le 2$, thus we also have

$$\begin{aligned}
(s' - s) &> 2\log(n)^4 n^{1-2c_3} \\
&\ge 2\log(n)^4 \frac{(n - n^{c_3})}{n^{2c_3}} \\
&\ge \log(n)^4 \frac{(n - n^{c_3})}{n^{2c_3}(1 - 4\log(n)^2 n^{1/2-c_3}/(1 - n^{-1/5}) - (4\log(n)^2 + 1)/((s' - s')n^{c_3})} \\
&= \log(n)^4 \frac{(n - n^{c_3})}{(n^{c_3} - 4\log(n)^2 n^{1/2}/(1 - n^{-1/5}) - (4\log(n)^2 + 1)/(s' - s))^2} \\
&= \log(n)^4 \frac{(n - n^{c_3})}{((n - 4\log(n)^2 n^{1/2}/(1 - n^{-1/5}) - (4\log(n)^2 + 1)/(s' - s)) - (n - n^{c_3}))^2} \\
&= \log(n)^4 \frac{(n - n^{c_3})}{(\tilde{n} - (4\log(n)^2 + 1)/(s' - s) - (n - n^{c_3}))^2},
\end{aligned}$$

which implies that $(s'-s)^2(\tilde{n}-(4\log(n)^2+1)/(s'-s)-(n-n^{c_3}))^2 > \log(n)^4(n-n^{c_3})(s'-s)$. By taking the square root on both sides and reorganizing the terms, this gives

$$(s'-s)\tilde{n}-1 > (n-n^{c_3})(s'-s)+\log(n)^2\sqrt{(n-n^{c_3})(s'-s)}+4\log(n)^2.$$

Combining this with (29) and using that $(s'-s)\tilde{n}-1 \le |S_0 \cap (s,s')|$ by Fact 26, we obtain

$$|\{j \in [n-n^{c_3}] | s_{\mathcal{H}^m}(r_j) \in (s,s')\}| < (s'-s)\tilde{n}-1 \le |S_0 \cap (s,s')|,$$

Hence,

$$|S_{n-n^{c_3}} \cap (s,s')| = |S_0 \cap (s,s') \setminus \{j \in [n-n^{c_3}] | s_{\mathcal{H}^m}(r_j) \in (s,s')\}| > 0.$$

Thus, there exists $s'' \in S_{n-n^{c_3}}$ such that $s < s'' < s'$, which concludes the proof. □

F.2 Missing analysis from Section 4.3.3

Lemma 34. *Let $m \le c_1 n$ and $i \in [d_1\log(n)]$. Assume that R is regular and that $t_{i-1} \le n-(1-c_2)^{i-1}n$. Then, $t_{i-1} < t_i$.*

Proof. We treat separately the cases $i=1$ and $i>1$. The case $i=1$ is immediate since by construction of the instance, $I_0 \cap S_0 = (0, n^{-1/5}] \cap S_0 = \emptyset$ whereas $I_1 \cap S_0 = (n^{-1/5}, \frac{3}{2}n^{-1/5}] \cap S_0 \neq \emptyset$, which implies $t_0 = 0 < t_1$.

Next, assume $i \in \{2, \dots, d_1\log(n)\}$. By definition of t_{i-1}, we have that $(y_{i-2}, y_{i-1}] \cap S_{t_{i-1}-1} \neq \emptyset$, which implies that $s_{t_{i-1}-1,1} \le y_{i-1}$. In addition, because of the assumptions and by Lemma 33, we have $t_{i-1}-1 < n-(1-c_2)^{i-1}n \le n-n^{c_3}$. Hence, by applying Lemma 30 with $t = t_{i-1}$, we deduce $(y_{i-1}, y_{i-1}+2\log(n)^4 n^{1-2c_3}] \cap S_{t_{i-1}-1} \neq \emptyset$. Then, since $c_3 > 3/4$ and since we assumed n sufficiently large, we have that $y_{i-1}+2\log(n)^4 n^{1-2c_3} = (3/2)^{i-1}n^{-1/5}+2\log(n)^4 n^{1-2c_3} \le (3/2)^i n^{-1/5} = y_i$. Thus, we get that $(y_{i-1}, y_i] \cap S_{t_{i-1}-1} \neq \emptyset$. By definition of t_i, this implies that $t_i > t_{i-1}-1$. In addition, since I_{i-1} and I_i are disjoint, at most one of I_{i-1} and I_i can be depleted at each time step and we have that $t_{i-1} \neq t_i$. We conclude that $t_i > t_{i-1}$. □

Lemma 35. *Let $m \le c_1 n$ and $i \in [d_1\log(n)]$. Assume that R is regular, that $t_0 < \ldots < t_{i-1} \le n-(1-c_2)^{i-1}n$ and that $t_{i-1} < t_i$. Let $\overline{t_i} := \min(t_i, t_{i-1}+c_2(n-t_{i-1}))$. Then,*

$$|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \notin I_i\}| = \tilde{O}(\sqrt{n}).$$

Proof. Let $i \in [d_1\log(n)]$. We first upper bound $|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) > y_i\}|$.

Since $\overline{t_i} = \min(t_i, t_{i-1}+c_2(n-t_{i-1})) \le t_i$, we have that $(y_{i-1}, y_i] \cap S_{\overline{t_i}-1} \neq \emptyset$ by definition of t_i, which implies that $s_{\overline{t_i}-1,1} \le y_i$. Now, using the definition of $\overline{t_i}$, the assumption that $t_{i-1} \le n-(1-c_2)^{i-1}n$, and Lemma 33, we have $\overline{t_i}-1 \le t_{i-1}+c_2(n-t_{i-1})-1 = c_2 n + t_{i-1}(1-c_2)-1 < n-(1-c_2)^i n \le n-n^{c_3}$. Hence, by Lemma 30, applied with $t = \overline{t_i}-1$, we obtain that there is $s \in (y_i - 2\log(n)^4 n^{1-2c_3}, y_i] \cap S_{\overline{t_i}-1}$. We let s be such a server. Then, by the greediness of $\mathcal{H}^m$ for all requests $r > y_0$, we have that for all $j \in [\overline{t_i}]$, if $r_j \le y_i - 2\log(n)^4 n^{1-2c_3}$, then $s_{\mathcal{H}^m}(r_j) \le s \le y_i$ (since r_j is closer to s than any point in $(s, y_i]$, and s is available when r_j arrives).

Now, note that since we assumed $c_3 > 3/4$ and n large enough, we have $y_i - 2\log(n)^4 n^{1-2c_3} \ge (3/2)^i n^{-1/5} - 2\log(n)^4 n^{-1/4} > (3/2)^{i-1}n^{-1/5} = y_{i-1}$. Hence, we can write $I_i = (y_{i-1}, y_i -$

$2\log(n)^4 n^{1-2c_3}] \cup (y_i - 2\log(n)^4 n^{1-2c_3}, y_i]$. Since we have shown that $s_{\mathcal{H}^m}(r_j) \le y_i$ for all j such that $r_j \le y_i - 2\log(n)^4 n^{1-2c_3}$, we thus obtain

$$\begin{aligned}
&|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) > y_i\}| \\
&\quad = |\{j \in [\overline{t_i}] : r_j \in (y_i - 2\log(n)^4 n^{1-2c_3}, y_i], s_{\mathcal{H}^m}(r_j) > y_i\}| \\
&\quad \le |\{j \in [\overline{t_i}] : r_j \in (y_i - 2\log(n)^4 n^{1-2c_3}, y_i]\}| \\
&\quad \le |\{j \in [n] : r_j \in d^+([y_i - 2\log(n)^4 n^{1-2c_3}, y_i])\}| \\
&\quad \le |d^+([y_i - 2\log(n)^4 n^{1-2c_3}, y_i])| \cdot n + \log^2(n)\sqrt{|d^+([y_i - 2\log(n)^4 n^{1-2c_3}, y_i])| \cdot n} \\
&\quad \le (2\log(n)^4 n^{1-2c_3} + 2/n) \cdot n + \log^2(n)\sqrt{(2\log(n)^4 n^{1-2c_3} + 2/n) \cdot n} \\
&\quad = \tilde{O}(\sqrt{n}). \qquad (30)
\end{aligned}$$

where the third inequality is by the second regularity condition, applied with $t = 0$, $t' = n$, $[d, d'] = d^+([y_i - 2\log(n)^4 n^{1-2c_3}, y_i])$, which satisfy the condition $(t - t')(d - d') = n \cdot 2\log(n)^4 n^{1-2c_3} = \Omega(1)$ since $c_3 < 1$. The fourth inequality is by definition of $d^+(\cdot)$, and the fifth is since $c_3 > 3/4$.

Next, we upper bound $|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \le y_{i-1}\}|$. We treat separately the cases where $j \in [t_{i-1}]$ and $j \in \{t_{i-1} + 1, \overline{t_i}\}$.

First, consider the case $j \in [t_{i-1}]$. If $i = 1$, then by construction of the instance, $I_0 \cap S_0 = \emptyset$, hence $t_0 = 0$ and we have the trivial identity $|\{j \in [t_0] : r_j \in I_1, s_{\mathcal{H}^m}(r_j) > y_1\}| = 0$. Now, for $i > 1$, by definition of t_{i-1}, we have that $(y_{i-2}, y_{i-1}] \cap S_{t_{i-1}-1} \ne \emptyset$, which implies that $s_{t_{i-1}-1,1} \le y_{i-1}$. Since by assumption and by Lemma 33, we have $t_{i-1} - 1 < n - (1 - c_2)^{i-1} n \le n - n^{c_3}$, we obtain, by applying Lemma 30 at time $t = t_{i-1} - 1$ and by a similar argument as in (30):

$$|\{j \in [t_{i-1}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \le y_{i-1}\}| \le |\{j \in [n] : r_j \in d^+([y_{i-1}, y_{i-1} + 2\log(n)^4 n^{1-2c_3}])\}| = \tilde{O}(\sqrt{n}) \qquad (31)$$

Now, for all $j \in \{t_{i-1} + 1, \overline{t_i}\}$, since $j \ge t_{i-1} + 1$ and $t_0 < \ldots < t_{i-1}$ by assumption, we have that $(0, y_{i-1}] \cap S_{j-1} = \left(\bigcup_{k \in [i-1]} (y_{k-1}, y_k]\right) \cap S_{j-1} = \emptyset$ by definition of $t_0, \ldots, t_{i-1}$; and since $j \le \overline{t_i} \le t_i$, we have that $(y_{i-1}, y_i] \cap S_{j-1} \ne \emptyset$ by definition of t_i. Hence, if $r_j \in I_i$, we either have $s_{\mathcal{H}^m}(r_j) > y_{i-1}$ or $s_{\mathcal{H}^m}(r_j) = 0$. By the greediness of $\mathcal{H}^m$ for all $r > n^{-1/5}$ and since $|y_i - r_j| \le |y_i - y_{i-1}| = |(3/2)^i n^{-1/5} - (3/2)^{i-1} n^{-1/5}| = 1/2(3/2)^{i-1} n^{-1/5} \le |r_j - 0|$ for any $r_j \in I_i$, we have that $s_{\mathcal{H}^m}(r_j) > y_{i-1}$ for any $r_j \in I_i$. Hence, we get that

$$|\{j \in \{t_{i-1} + 1, \overline{t_i}\} : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \le y_{i-1}\}| = 0.$$

Combining this with (31), we obtain:

$$|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \le y_{i-1}\}| = \tilde{O}(\sqrt{n}). \qquad (32)$$

Finally, from (30) and (32), we get

$$\begin{aligned}
&|\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \notin I_i\}| \\
&= |\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) > y_i\}| + |\{j \in [\overline{t_i}] : r_j \in I_i, s_{\mathcal{H}^m}(r_j) \le y_{i-1}\}| = \tilde{O}(\sqrt{n}). \qquad \square
\end{aligned}$$

Lemma 36. *Let $m \le c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular, that $t_0 < \ldots < t_{i-1}$ and that $t_{i-1} < t_i$. Let $\overline{t_i} := \min(t_i, t_{i-1} + c_2(n - t_{i-1}))$. Then,*

$$\begin{aligned}
&|\{j \in \{t_{i-1} + 1 + c_1(n - t_{i-1}), \ldots, \overline{t_i}\} : r_j \in [\tfrac{3}{4} y_{i-1}, y_{i-1}], s_{\mathcal{H}^m}(r_j) \in I_i\}| \\
&\ge \frac{1}{2}(\overline{t_i} - t_{i-1} - c_1(n - t_{i-1}))|I_i| - \tilde{O}(\sqrt{n}).
\end{aligned}$$

Proof. Consider $j \in \{t_{i-1}+1+c_1(n-t_{i-1}),\dots,\overline{t_i}\}$ and assume that $r_j \in [\frac{3}{4}y_{i-1}, y_{i-1}]$. Since $j \geq t_{i-1}+1$ and $t_0 < \dots < t_{i-1}$ by assumption, we have $j > t_\ell$ for all $\ell \in [i-1]$. Thus, by definition of $t_0,\dots,t_{i-1}$, we have that $(0, y_{i-1}] \cap S_{j-1} = \left(\bigcup_{k\in[i-1]}(y_{k-1}, y_k]\right) \cap S_{j-1} = \emptyset$. In addition, since $j \leq \overline{t_i} \leq t_i$, we have that $(y_{i-1}, y_i] \cap S_{j-1} \neq \emptyset$ by definition of t_i. Thus, by the greediness of $\mathcal{H}^m$ for all $j \geq t_{i-1}+1+c_1(n-t_{i-1}) \geq c_1 n \geq m$, and since $r_j \in [\frac{3}{4}y_{i-1}, y_{i-1}]$, we either have $s_{\mathcal{H}^m}(r_j) \in (y_{i-1}, y_i]$ or $s_{\mathcal{H}^m}(r_j) = 0$. Now, since $(y_{i-1}, y_i] \cap S_{j-1} \neq \emptyset$, and since for any $s \in (y_{i-1}, y_i] \cap S_{j-1}$, we have

$$|s - r_j| \leq |y_i - \tfrac{3}{4}y_{i-1}| = (3/2)^{i-1} n^{-1/5}[\tfrac{3}{2} - \tfrac{3}{4}] = |\tfrac{3}{4}y_{i-1}| \leq |r_j - 0|,$$

we must have $s_{\mathcal{H}^m}(r_j) \in (y_{i-1}, y_i]$. Hence,

$$\begin{aligned} &|\{j \in \{t_{i-1}+c_1(n-t_{i-1}),\dots,\overline{t_i}\} : r_j \in [\tfrac{3}{4}y_{i-1}, y_{i-1}]\}| \\ &= |\{j \in \{t_{i-1}+c_1(n-t_{i-1}),\dots,\overline{t_i}\} : r_j \in [\tfrac{3}{4}y_{i-1}, y_{i-1}], s_{\mathcal{H}^m}(r_j) \in (y_{i-1}, y_i]\}|. \end{aligned} \tag{33}$$

Now, since we assumed that the sequence of requests is regular, by applying the first regularity condition with $t = t_{i-1}+c_1(n-t_{i-1})$, $t' = \overline{t_i}$ and $[d, d'] = d^-([\frac{3}{4}y_{i-1}, y_{i-1}]))$, we have that

$$\begin{aligned} |\{j \in &\{t_{i-1}+1+c_1(n-t_{i-1}),\dots,\overline{t_i}\} : r_j \in [\tfrac{3}{4}y_{i-1}, y_{i-1}]\} \\ &\geq |\{j \in \{t_{i-1}+1+c_1(n-t_{i-1}),\dots,\overline{t_i}\} : r_j \in d^-([\tfrac{3}{4}y_{i-1}, y_{i-1}])\}| \\ &\geq d^-([\tfrac{3}{4}y_{i-1}, y_{i-1}]) \cdot (\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}) - 1) \qquad (34) \\ &\qquad\qquad - \log(n)^2\sqrt{d^-([\tfrac{3}{4}y_{i-1}, y_{i-1}]) \cdot (\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}) - 1)}. \\ &\geq (y_{i-1} - \tfrac{3}{4}y_{i-1} - 2/n)(\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}) - 1) \\ &\qquad\qquad - \log(n)^2\sqrt{(y_{i-1} - \tfrac{3}{4}y_{i-1} - 2/n)(\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}) - 1)}. \\ &\geq (\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}))\frac{y_{i-1}}{4} - \tilde{O}(\sqrt{n}). \qquad (35) \end{aligned}$$

By combining (35) and (33), and noting that $|I_i| = (3/2)^i n^{-1/5} - (3/2)^{i-1} n^{-1/5} = (3/2)^{i-1} n^{-1/5} \cdot \frac{1}{2} = \frac{y_{i-1}}{2}$, we finally obtain that

$$\begin{aligned} &|\{j \in \{t_{i-1}+c_1(n-t_{i-1}),\dots,\overline{t_i}\} : r_j \in [\tfrac{3}{4}y_{i-1}, y_{i-1}], s_{\mathcal{H}^m}(r_j) \in (y_{i-1}, y_i]\}| \\ &\geq \frac{1}{2}(\overline{t_i} - t_{i-1} - c_1(n-t_{i-1}))|I_i| - \tilde{O}(\sqrt{n}). \end{aligned}$$

□

Lemma 38. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular and that $t_1 < t_2$. Then, $c_1 n < t_1$.*

Proof. First, note that, since t_1 is the time at which I_1 is depleted, we have, using Fact 26, that

$$|\{j \in [t_1] : s_{\mathcal{H}^m}(r_j) \in I_1\}| = |S_0 \cap I_1| \geq |I_1|\tilde{n} - 1. \tag{36}$$

In particular, $t_1 \geq |\{j \in [t_1] : s_{\mathcal{H}^m}(r_j) \in I_1\}| \geq |I_1|\tilde{n} - 1$; thus for all $i \geq 0$, we have

$$t_1|I_i| = \Omega(|I_1||I_i|\tilde{n}) = \Omega(1). \tag{37}$$

Next, we upper bound $|\{j \in [t_1] : s_{\mathcal{H}^m}(r_j) \in I_1\}|$. Since $t_1 < t_2$ by assumption, for all $j \in [t_1]$, we have that $(y_1, y_2] \cap S_{j-1} \neq \emptyset$ by definition of t_2. Hence, by the greediness of $\mathcal{H}^m$ for all requests

$r > y_0$, if $r_j > y_2$, we have $s_{\mathcal{H}^m}(r_j) > y_1$ (since r_j is closer to any $s \in (y_1, y_2] \cap S_{j-1}$ than any point in $[0, y_1]$). We thus get

$$|\{j \in [t_1] \; | s_{\mathcal{H}^m}(r_j) \in I_1\}| \leq |\{j \in [t_1] : r_j \leq y_2\}|$$

Now, note that by (37), we have $t_1 |d^+(I_i)| = \Omega(1)$ for all i. Hence, by applying the second regularity condition with $t = 0$, $t' = t_1$, and $[d, d'] = d^+(I_0), d^+(I_1), d^+(I_2)$, respectively, we get

$$\begin{aligned}
&|\{j \in [t_1] : r_j \leq y_2\}| \\
&\qquad \leq |\{j \in [t_1] : r_j \in d^+(I_0)\}| + |\{j \in [t_1] : r_j \in d^+(I_1)\}| + |\{j \in [t_1] : r_j \in d^+(I_2)\}| \\
&\qquad \leq (d^+(I_0) + d^+(I_1) + d^+(I_2))t_1 + \log(n)^2(\sqrt{d^+(I_0)t_1} + \sqrt{d^+(I_1)t_1} + \sqrt{d^+(I_2)t_1}) \\
&\qquad \leq (|I_0| + |I_1| + |I_2| + 6/n)t_1 + \log(n)^2(\sqrt{|I_0|t_1 + 2/n} + \sqrt{|I_1|t_1 + 2/n} + \sqrt{|I_2|t_1 + 2/n}) \\
&\qquad \leq (|I_0| + |I_1| + |I_2|)t_1 + \tilde{O}(\sqrt{n}) \\
&\qquad = \frac{9}{2}|I_1|t_1 + \tilde{O}(\sqrt{n}),
\end{aligned}$$

where the first inequality is since $(0, y_2] = I_0 \cup I_1 \cup I_2$, and the equality is since $|I_1| = \frac{n^{-1/5}}{2}$ and $|I_0 \cup I_1 \cup I_2| = (3/2)^2 n^{-1/5}$. Hence, by combining the two previous inequalities, we obtain

$$|\{j \in [t_1] \; | s_{\mathcal{H}^m}(r_j) \in I_1\}| \leq \frac{9}{2}|I_1|t_1 + \tilde{O}(\sqrt{n}). \tag{38}$$

Combining (36) and (38), and reorganizing the terms, we get

$$t_1 \geq \frac{2\tilde{n}}{9} - \tilde{O}(\sqrt{n}/|I_1|) = \frac{2(n - 4\log(n)^2\sqrt{n}/(1 - n^{-1/5}))}{9} - \tilde{O}(\sqrt{n}/n^{-1/5}) = \frac{2n}{9} - \tilde{O}(n^{7/10}).$$

Hence, since we chose $c_1 < 2/9$, and since we assumed n sufficiently large, we have $t_1 > c_1 n$. $\square$

Lemma 39. *Let $m \leq c_1 n$ and $i \in [d_1 \log(n)]$. Assume that R is regular and that $c_1 n < t_1$, then $c_1 n < t_{\{0\}}$.*

Proof. Note that by definition of t_1 and since we assumed $c_1 n < t_1$, we have that for all $j \in [c_1 n]$, $S_j \cap I_1 \neq \emptyset$. Hence, by the greediness of $\mathcal{H}^m$ for all requests $r > y_0$, if $r_j > y_1$, we have $s_{\mathcal{H}^m}(r_j) > y_0$ (since r_j is closer to any $s \in (y_0, y_1] \cap S_{j-1}$ than to the servers at location 0). We thus get

$$|\{j \in [c_1 n] \; | s_{\mathcal{H}^m}(r_j) = 0\}| \leq |\{j \in [c_1 n] \; | s_{\mathcal{H}^m}(r_j) \in [0, y_0)\}| \leq |\{j \in [c_1 n] : r_j \leq y_1\}|. \tag{39}$$

Now, note that we have $c_1 n |d^+(I_j)| = \Omega(1)$ for $j = 0, 1$. Hence, by applying the second regularity condition with $t = 0$, $t' = c_1 n$, and $[d, d'] = d^+(I_0), d^+(I_1)$, respectively, we get

$$\begin{aligned}
|\{j \in [c_1 n] : r_j \leq y_1\}| &\leq |\{j \in [c_1 n] : r_j \in d^+(I_0)\}| + |\{j \in [c_1 n] : r_j \in d^+(I_1)\}| \\
&\leq (d^+(I_0) + d^+(I_1))c_1 n + \log(n)^2(\sqrt{d^+(I_0)c_1 n} + \sqrt{d^+(I_1)c_1 n}) \\
&\leq (|I_0| + |I_1|)c_1 n + \tilde{O}(\sqrt{n}) \\
&= c_1(3/2)n^{-1/5} \cdot n + \tilde{O}(\sqrt{n}), \\
&< n^{4/5} \\
&\leq |S_0 \cap \{0\}|,
\end{aligned}$$

where the fourth inequality is since we set $c_1 < 2/3$ and since we assumed n large enough and the last one by definition of the instance. Hence, combining this with (39), we get $|S_{c_1 n} \cap \{0\}| = |S_0 \cap \{0\}| - |\{j \in [c_1 n] \; | s_{\mathcal{H}^m}(r_j) = 0\}| > 0$. By definition of $t_{\{0\}}$, we deduce that $t_{\{0\}} > c_1 n$. $\square$

F.3 Missing analysis from Section 4.3.4

In the following, we write $\mathcal{N}(r_t) = \{\max\{z \in S_{t-1} : z \leq r_t\}, \min\{z \in S_{t-1} : z \geq r_t\}\}$ and $\mathcal{N}(r_t)' = \{\max\{z \in S'_{t-1} : z \leq r_t\}, \min\{z \in S'_{t-1} : z \geq r_t\}\}$ to denote the servers in S_t and S'_t which are either closest on the left or closest on the right to r_t. We also write $s(r_t)$ and $s'(r_t)$ to denote the servers to which r_t is matched by $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$, respectively.

Lemma 20. *Let R be n arbitrary requests and S_0 be n arbitrary servers. Then, for all $t \in \{0, \ldots, m-1\}$, we have $S_t = S'_t$, and for all $t \geq m$, either $S_t = S'_t$ or $S'_t = S_t \cup \{0\} \setminus \{\min\{s \in S_t : s > 0\}\}$ (and $\{s \in S_t : s > 0\} \neq \emptyset$).*

Proof. It is immediate that $S_t = S'_t$ for all $t \in \{0, \ldots, m-1\}$ since $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ make the same matching decisions until time $m-1$.

Next, we show that either $S_m = S'_m$, or $\{s > 0 | s \in S_m\} \neq \emptyset$ and $S'_m = S_m \cup \{0\} \setminus \{s_{m,1}\}$. We consider different cases depending on the location of request r_m.

- **Case 1:** $r_m \in (n^{-1/5}, 1]$ **or (**$r_m \in [0, n^{-1/5}]$ **and** $S_{m-1} \cap \{0\} = \emptyset$**).** In this case, both $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ match r_m greedily. Since we also have $S_{m-1} = S'_{m-1}$, we get $S_m = S'_m$.

- **Case 2:** $r_m \in [0, n^{-1/5}]$ **and** $S_{m-1} \cap \{0\} \neq \emptyset$**.** In this case, $\mathcal{H}^m$ matches r_m to 0, i.e. $s(r_m) = 0$, while $\mathcal{H}^{m-1}$ matches r_m greedily. Note that $S'_{m-1} \cap (0, n^{-1/5}] \subseteq S'_0 \cap (0, n^{-1/5}] = \emptyset$, hence $s'_{m-1,1} = \min\{s > 0 : s \in S_{m-1'}\} \geq n^{-1/5}$ and we thus have $\mathcal{N}'(r_m) \subseteq \{0, s'_{m-1,1}\}$. Since $\mathcal{H}^{m-1}$ matches r_m greedily, we get that $s(r_m)' \in \{0, s'_{m-1,1}\}$.

  We now consider two cases:
  (1) $s'(r_m) = 0$. In this case, we have $s'(r_m) = s(r_m) = 0$, hence $S_m = S'_m$.

  (2) $s'(r_m) = s'_{m-1,1}$. In this case, we have $s(r_m) = 0$ and $s'(r_m) = s'_{m-1,1} = \min\{s > 0 : s \in S'_{m-1}\} = \min\{s > 0 : s \in S_{m-1}\} = \min\{s > 0 : s \in S_m \cup \{0\}\} = \min\{s > 0 : s \in S_m\} = s_{m,1}$. Hence $S'_m = S'_{m-1} \setminus \{s'(r_m)\} = S'_{m-1} \setminus \{s_{m,1}\} = S_{m-1} \setminus \{s_{m,1}\} = S_m \cup \{0\} \setminus \{s_{m,1}\}$.

Hence we either have that $S_m = S'_m$ or that $S'_m = S_m \cup \{0\} \setminus \{s_{m,1}\}$. Now, we show by induction on t that for all $t \in \{m, \ldots, n\}$, we either have that $S_t = S'_t$ or that $S'_t = S_t \cup \{0\} \setminus \{s_{t,1}\}$.

Fix $t \in \{m, \ldots, n-1\}$. If $S_t = S'_t$, it is immediate that $S_{t+1} = S'_{t+1}$ and we are done. We now assume that $S'_t = S_t \cup \{0\} \setminus \{s_{t,1}\}$. We thus have that $S'_t = S_t \cup \{g^L_t\} \setminus \{g^R_t\}$ with $g^L_t = 0, g^R_t = s_{t,1}$. To get the values of S_{t+1}, S'_{t+1}, we apply the third point of Lemma 6, noting that we have here $g^L_t = 0, g^R_t = s_{t,1}, s^L_t = 0, s^R_t = s_{t,2}, d^L_t = |g^L_t - s^L_t| = 0, d^R_t = |s^R_t - g^R_t| = s_{t_2} - s_{t_1}$. We enumerate below all possible values of S_{t+1}, S'_{t+1} by reporting the values given in Tables 2, 3 and 4 (note that the roles of S_t and S'_t are reversed since $S'_t = S_t \cup \{g^L_t\} \setminus \{g^R_t\}$ here instead of $S_t = S'_t \cup \{g^L_t\} \setminus \{g^R_t\}$ as in the statement of Lemma 6).

- **Case 1:** $s_{t,2} \neq \emptyset$ **and** $S_t \cap \{0\} \neq \emptyset$**.** In this case, the values of S_{t+1}, S'_{t+1} are obtained by using Table 2. There are three possible cases: (1) $S_{t+1} = S'_{t+1}$ (Column 4 of Table 2) (2) $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t,1}\}$ and $s_{t,1} \in S_{t+1}$ (Column 2,3,6,7) (3) $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t,2}\}$ and $s_{t,1} \notin S_{t+1}$ (Column 5).

- **Case 2:** $S_t \cap \{0\} = \emptyset$ **and** $s_{t,2} \neq \emptyset$**.** In this case, the values of S_{t+1}, S'_{t+1} are obtained by using Table 3. There are three possible cases: (1) $S_{t+1} = S'_{t+1}$ (Column 2) (2) $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t,1}\}$ and $s_{t,1} \in S_{t+1}$ (Column 4,5) (3) $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t,2}\}$ and $s_{t,1} \notin S_{t+1}$ (Column 3).

- **Case 3: $s_{t,2} = \emptyset$ and $S_t \cap \{0\} \neq \emptyset$.** In this case, the values of S_{t+1}, S'_{t+1} are obtained by using Table 4. There are two possible cases: (1) $S_{t+1} = S'_{t+1}$ (Column 5) (2) $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t,1}\}$ and $s_{t,1} \in S_{t+1}$ (Column 2,3,4).

- **Case 4: $s_{t,2} = \emptyset$ and $S_t \cap \{0\} = \emptyset$.** From Lemma 6, we get $S_{t+1} = S'_{t+1}$.

In all cases, we get that either (1) $S_{t+1} = S'_{t+1}$, (2) $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t,1}\}$ and $s_{t,1} \in S_{t+1}$ or (3) $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t,2}\}$ and $s_{t,1} \notin S_{t+1}$. If case (2) holds, and since $s_{t,1} \in S_{t+1}$, note that $s_{t+1,1} = \min\{s \in S_{t+1} : s > 0\} = s_{t,1}$, and if case (3) holds, since $s_{t,1} \notin S_{t+1}$, note that $s_{t+1,1} = \min\{s \in S_{t+1} : s > 0\} = \min\{s \in S_t \setminus \{s_{t,1}\} : s > 0\} = s_{t,2}$. In all cases, we have that either $S_{t+1} = S'_{t+1}$, or $S'_{t+1} = S_{t+1} \cup \{0\} \setminus \{s_{t+1,1}\}$, which concludes the inductive case and the proof.

□

Lemma 41. *All following properties hold at any time $t \in \{m, \dots, n-1\}$:*

1. *if $\delta_t = 0$, then for all $t' \geq t$, we have $\delta_{t'} = 0$ and $\Delta cost_{t+1} = 0$,*
2. *if $S_t \cap \{0\} \neq \emptyset$, then $\Delta cost_{t+1} \geq 0$.*
3. *if $S_t \cap \{0\} \neq \emptyset$, $\delta_t \neq 0$ and $|S_t \cap (\delta_t, 1]| \geq 1$, then the values of (δ_{t+1}, S_{t+1}) and the expected value of $\Delta cost_{t+1}$ conditioning on (δ_t, S_t) and on r_{t+1} are as given in Table 1, where $w_t := s_{t,2} - s_{t,1}$ and where we write $\mathbb{E}[\Delta cost_{t+1}|...]$ instead of $\mathbb{E}[\Delta cost_{t+1}|(\delta_t, S_t), S_t \cap \{0\} \neq \emptyset, \delta_t \neq 0, |S_t \cap (\delta_t, 1]| \geq 1, r_{t+1} \in \dots]$.*
4. *if $\delta_{t+1} \neq \delta_t$, then $S_{t+1} = S_t \setminus \{\delta_t\}$.*
5. *$\mathbb{1}_{S_t \cap \{0\} = \emptyset, \delta_t \neq 0} \cdot \mathbb{E}[\Delta cost_{t+1}|(\delta_t, S_t)] \geq -\mathbb{1}_{S_t \cap \{0\} = \emptyset, \delta_t \neq 0} \cdot \mathbb{P}(\delta_{t+1} = 0|(\delta_t, S_t))$.*

Proof. In the following, we consider a fixed $t \in \{m, \dots, n-1\}$. We start by the proof of Point 1.

Proof of Point 1. By definition of δ, if $\delta_t = 0$, then $S_t = S'_t$. Since both $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ match all requests $r_{t+1}, \dots, r_n$ greedily, it is immediate that $S_j = S'_j$ for all $j \geq t$, which also implies that $\delta_j = 0$ and $\Delta \text{cost}_{j+1} = 0$ for all $j \geq t$.

We now show points 2,3,4,5. First, note that by Lemma 20, we have that either $S_t = S'_t$ or $S'_t = S_t \cup \{0\} \setminus \{s_{t,1}\}$. Since all properties follow immediately when $S_t = S'_t$, we assume in the following that $S'_t = S_t \cup \{0\} \setminus \{s_{t,1}\}$.

Proof of Point 2. Assume that $S_t \cap \{0\} \neq \emptyset$. Then, $S'_t \cap [0,1] = (S_t \cup \{0\} \setminus \{s_{t,1}\}) \cap [0,1] \subseteq S_t \cap [0,1]$, hence, for any value of $r_{t+1} \in [0,1]$, we have, by definition of the process:

$$\begin{aligned}\text{cost}_{t+1}(\mathcal{H}^m) = |r_{t+1} - s(r_{t+1})| &= \min_{s \in S_t \cap [0,1]} |r_{t+1} - s| \\ &\leq \min_{s \in S'_t \cap [0,1]} |r_{t+1} - s| = |r_{t+1} - s'(r_{t+1})| = \text{cost}_{t+1}(\mathcal{H}^{m-1}).\end{aligned}$$

Thus, $\Delta \text{cost}_{t+1} = \text{cost}_t(\mathcal{H}^{m-1}) - \text{cost}_t(\mathcal{H}^m) \geq 0$.

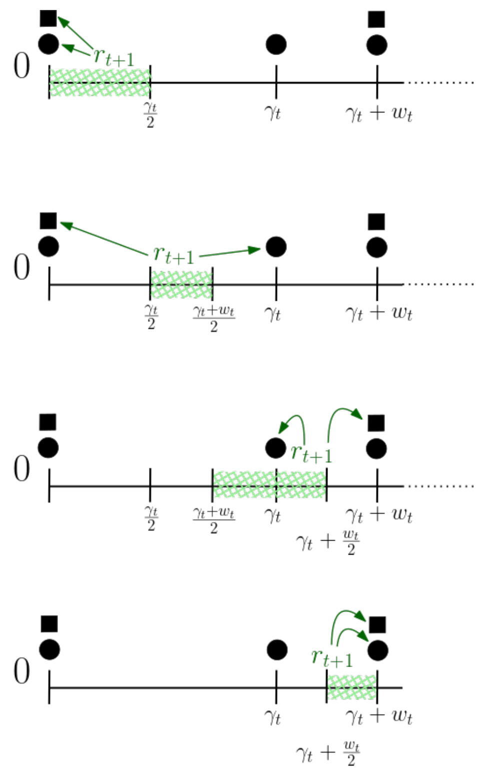

Figure 12: Illustration of the different cases in Point 3 of Lemma 41.

Proof of point 3. In the remainder of this paragraph, we condition on the variables (δ_t, S_t) and we assume that $S_t \cap \{0\} \neq \emptyset$, $\delta_t \neq 0$ and $|S_t \cap (\delta_t, 1]| \geq 1$.

To get the values of (δ_{t+1}, S_{t+1}) depending on the location of r_{t+1}, we apply the third point of Lemma 6, by noting that we have in this case $g_t^L = 0, g_t^R = s_{t,1}, s_t^L = 0, s_t^R = s_{t,2}, d_t^L = |g_t^L - s_t^L| = 0, d_t^R = |s_t^R - g_t^R| = s_{t_2} - s_{t_1} = w_t$. The values given in Table 1 are thus directly reported from Table 2 (see Figure 12 for an illustration of the different cases).

Next, we give a lower bound on the expected value of Δcost_{t+1} depending on r_{t+1}. Note that, since $S_t \cap \{0\} \neq \emptyset$, we already have that $\Delta\text{cost}_{t+1} \geq 0$ from Point 2 and we can fill the corresponding values in Table 1. We thus only need to refine the lower bound on Δcost_{t+1} in the case $r_{t+1} \in [\frac{\delta_t+w_t}{2}, \delta_t + \frac{w_t}{2}]$ and $w_t \leq \delta_t$. To ease the exposition, we let $\mathcal{E}_t$ be the set of all (δ, S) that satisfy the assumptions of the third point of the lemma (i.e., $\mathcal{E}_t = \{(\delta, S) \in [0,1] \times [0,1]^{n-t} : S \cap \{0\} \neq \emptyset,$

$\delta \neq 0$ and $|S \cap (\delta, 1]| \geq 1\}$). Since $s'(r_{t+1}) = \delta_t + w_t$, we have

$$\begin{aligned}
&\mathbb{E}[\mathrm{cost}'_{t+1} \,|(\delta_t, S_t), (\delta_t, S_t) \in \mathcal{E}_t, w_t \leq \delta_t, r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \\
&= \mathbb{E}[|(\delta_t + w_t) - r_{t+1}| \,|(\delta_t, S_t), (\delta_t, S_t) \in \mathcal{E}_t, w_t \leq \delta_t, r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \\
&= \mathbb{E}[(\delta_t + w_t) - r_{t+1} | (\delta_t, S_t), (\delta_t, S_t) \in \mathcal{E}_t, w_t \leq \delta_t, r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \\
&= \frac{w_t}{2} + \mathbb{E}[(\delta_t + \tfrac{w_t}{2}) - r_{t+1} | (\delta_t, S_t), (\delta_t, S_t) \in \mathcal{E}_t, w_t \leq \delta_t, r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \\
&= \frac{w_t}{2} + \frac{(\delta_t + \frac{w_t}{2} - \frac{\delta_t + w_t}{2})}{2} \\
&= \frac{w_t}{2} + \frac{\delta_t}{4},
\end{aligned}$$

and since since $s(r_{t+1}) = \delta_t$, we have

$$\begin{aligned}
&\mathbb{E}[\mathrm{cost}_{t+1} \,|(\delta_t, S_t), (\delta_t, S_t) \in \mathcal{E}_t, w_t \leq \delta_t, \; r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \\
&= \mathbb{E}[|\delta_t - r_{t+1}| \,|(\delta_t, S_t), (\delta_t, S_t) \in \mathcal{E}_t, w_t \leq \delta_t, r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \\
&\leq \frac{(\delta_t + \frac{w_t}{2}) - \frac{\delta_t + w_t}{2}}{2} \\
&= \frac{\delta_t}{4},
\end{aligned}$$

where the inequality is since $\delta_t \in [\frac{\delta_t + w_t}{2}, \delta_t + \frac{w_t}{2}]$ when $w_t \leq \delta_t$.

Hence,

$$\mathbb{E}[\Delta \mathrm{cost}_{t+1} | (\delta_t, S_t), (\delta_t, S_t) \in \mathcal{E}_t, w_t \leq \delta_t, r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \geq \left((\frac{w_t}{2} + \frac{\delta_t}{4}) - \frac{\delta_t}{4} \right) \geq \frac{w_t}{2}.$$

Proof of point 4.

By assumption, we have $\delta_t = \min\{s > 0 : s \in S_t\} \in S_t$. Now, assume that $s(r_{t+1}) \neq \delta_t$. Then, whatever the value of $s'(r_{t+1})$, we have that $\delta_t \notin (S_t \cup \{0\} \setminus \{\delta_t\}) \setminus \{s'(r_{t+1})\}$, whereas $\delta_t \in S_t \setminus \{s(r_{t+1})\}$. Thus, $S'_{t+1} = (S_t \cup \{0\} \setminus \{\delta_t\}) \setminus \{s'(r_{t+1})\} \neq S_t \setminus \{s(r_{t+1})\} = S_{t+1}$, and by definition of δ, we get $\delta_{t+1} = \min\{s > 0 : s \in S_{t+1}\} = \min\{s > 0 : s \in S_t \setminus \{s(r_{t+1})\}\} = \delta_t$. By contraposition, if $\delta_{t+1} \neq \delta_t$, then we must have $s(r_{t+1}) = \delta_t$, and we thus get $S_{t+1} = S_t \setminus \{s(r_{t+1})\} = S_t \setminus \{\delta_t\}$.

Proof of point 5. We condition on the variables (δ_t, S_t) and assume that $S_t \cap \{0\} = \emptyset$ and $\delta_t \neq 0$. We first show that $\delta_{t+1} \neq 0$ if and only if $s'(r_{t+1}) \neq 0$.

- $\Leftarrow$: Assume that $s'(r_{t+1}) \neq 0$. Since $0 \in (S_t \cup \{0\} \setminus \{s_{t,1}\}) \setminus \{s'(r_{t+1})\}$, and $0 \notin S_t$ by assumption, we have that whatever the value of $s(r_{t+1})$, $S'_{t+1} = (S_t \cup \{0\} \setminus \{s_{t,1}\}) \setminus \{s'(r_{t+1})\} \neq S_t \setminus \{s(r_{t+1})\} = S_{t+1}$. Thus, by construction of δ, we get $\delta_{t+1} \neq 0$.
- $\Rightarrow$: Assume, by contrapositive, that $s'(r_{t+1}) = 0$. Since by assumption, $S_t \cap \{0\} = \emptyset$, we have that $S_t = \{s_{t,1}, s_{t,2}\} \cup (S_t \cap (s_{t,2}, 1])$. Now, since $s'(r_{t+1}) = \arg\min_{s \in S_t \cup \{0\} \setminus \{s_{t,1}\}} |s - r_{t+1}|$, we have that for all $s \in S_t \cap [s_{t,2}, 1]$, $|r_{t+1} - 0| \leq |r_{t+1} - s|$; thus, if $r_{t+1} \geq s_{t,1}$, we have that $|r_{t+1} - s_{t,1}| \leq |r_{t+1} - 0| \leq |r_{t+1} - s|$, and if $r_{t+1} \leq s_{t,1}$, it is immediate that for all $s \in S_t \cap [s_{t,2}, 1]$, $|s_{t,1} - r_{t+1}| \leq |s - r_{t+1}|$. Hence, we get that $s(r_{t+1}) = \arg\min_{s \in S_t} |r_{t+1} - s| = s_{t,1}$, which immediately implies that $S_{t+1} = S'_{t+1}$, from which we deduce $\delta_{t+1} = 0$.

We now show that $\Delta\text{cost}_{t+1} \geq 0$ when $s'(r_{t+1}) \neq 0$. Since $s'(r_{t+1}) \in S_t \cup \{0\} \setminus \{s_{t,1}\}$ and $s'(r_{t+1}) \neq 0$, we have $s'(r_{t+1}) \in S_t$. Thus, $|s(r_{t+1}) - r_{t+1}| = \min_{s \in S_t} |s - r_{t+1}| \leq |s'(r_{t+1}) - r_{t+1}|$, and we deduce $\Delta\text{cost}_{t+1} = |s'(r_{t+1}) - r_{t+1}| - |s(r_{t+1}) - r_{t+1}| \geq 0$.

Hence, we have shown that $\delta_{t+1} \neq 0$ if and only if $s'(r_{t+1}) \neq 0$, and that if $s'(r_{t+1}) \neq 0$, then $\Delta\text{cost}_{t+1} \geq 0$. Using that it is always the case that $\Delta\text{cost}_{t+1} \in [-1, 1]$, we get:

$$\begin{aligned}
&\mathbb{1}_{S_t \cap \{0\} = \emptyset, \delta_t \neq 0} \cdot \mathbb{E}[\Delta\text{cost}_{t+1} | (\delta_t, S_t)] \\
&= \mathbb{1}_{S_t \cap \{0\} = \emptyset, \delta_t \neq 0} \cdot \Big(\mathbb{E}[\Delta\text{cost}_{t+1} | (\delta_t, S_t), \delta_{t+1} \neq 0] \mathbb{P}(\delta_{t+1} \neq 0 | (\delta_t, S_t)) \\
&\qquad\qquad + \mathbb{E}[\Delta\text{cost}_{t+1} | (\delta_t, S_t), \delta_{t+1} = 0] \mathbb{P}(\delta_{t+1} = 0 | (\delta_t, S_t)) \Big) \\
&\geq 0 - \mathbb{1}_{S_t \cap \{0\} = \emptyset, \delta_t \neq 0} \cdot \mathbb{P}(\delta_{t+1} = 0 | (\delta_t, S_t)),
\end{aligned}$$

which concludes the proof of the fifth point and the proof of Lemma 41. □

Lemma 43. *Conditioning on the gap δ_m and available servers S_m, and for all $y \in [\delta_m, 1]$, we have*

$$\mathbb{P}\Big(\min(t_{(0,y]}, t_{\{0\}}) \leq \min(t^d, t_{\{0\}}) \Big| \delta_m, S_m \Big) \geq \frac{\delta_m}{y},$$

Proof. For all $j \in \{m, \dots, n-1\}$, we define the auxiliary stopping times: $t^{j,d} = \min\{t \geq j : \delta_t = 0\}$, $t^j_{\{0\}} = \min\{t \geq j : S_t \cap \{0\} = \emptyset\}$ and $t^j_{(0,y]} = \min\{t \geq j : S_t \cap \{(0, y]\} = \emptyset\}$. To ease the presentation, we write $\underline{t^j_y}$ and $\underline{t^{j,d}}$ instead of $\min(t^j_{\{0\}}, t^j_{[0,y)})$ and $\min(t^j_{\{0\}}, t^{j,d})$.

We now show by downward induction on j that for any $j \in \{m, \dots, n\}$, any pair (x, S) with $x \in [0, 1]$ and with S a set of $n - j$ arbitrary servers in $[0, 1]$ such that either $x = 0$ or $x = \min\{s \in S : s > 0\}$, and any $y \in (x, 1]$, we have:

$$\mathbb{P}_R\Big(\underline{t^j_y} \leq \underline{t^{j,d}} | \delta_j = x, S_j = S \Big) \geq \frac{x}{y}.$$

We first show the base case, which is for $j = n$. The only valid pair of (x, S) is $(0, \emptyset)$, and it is immediate that for any $y \in (0, 1]$, we have

$$\mathbb{P}_R\Big(\underline{t^n_y} \leq \underline{t^{n,d}} | \delta_n = 0, S_n = \emptyset \Big) = 1 \geq \frac{x}{y}.$$

Next, let $j \in \{m, \dots, n-1\}$, and assume that for any pair (x, S) with $x \in [0, 1]$ and with S a set of $n - (j+1)$ arbitrary servers in $[0, 1]$ such that either $x = 0$ or $x = \min\{s \in S : s > 0\}$, and any $y \in (x, 1]$, we have

$$\mathbb{P}_R\Big(\underline{t^{j+1}_y} \leq \underline{t^{j+1,d}} | \delta_{j+1} = x, S_{j+1} = S \Big) \geq \frac{x}{y}.$$

Now, consider some arbitrary pair (x, S) with $x \in [0, 1]$ and with S a set of $n - j$ arbitrary servers in $[0, 1]$ such that either $x = 0$ or $x = \min\{s \in S : s > 0\}$, and some arbitrary $y \in (x, 1]$, and assume that $\delta_j = x$ and $S_j = S$.

We first consider the case where $x > 0$, $|(x, y] \cap S| \geq 1$ and $S \cap \{0\} \neq \emptyset$.

First, note that since $S_j \cap \{0\} \neq \emptyset$, $\delta_j = x > 0$ and $S_j \cap (0, y] \supseteq \{x\} \neq \emptyset$, we have that $\underline{t^j_y}, \underline{t^{j,d}} \geq j + 1$. We deduce the following proposition: for any $r \in [0, 1]$, letting $\chi(x, S, r)$ and

$T(x,S,r)$ be the value of δ_{j+1} and S_{j+1} assuming that $\delta_j = x, S_j = S$ and $r_{j+1} = r$, we have

$$\begin{aligned}
&\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S, r_{j+1} = r\Big) \\
&= \mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S, \delta_{j+1} = \chi(x,S,r), S_{j+1} = T(x,S,r), r_{j+1} = r\Big) \\
&= \mathbb{P}_R\Big(\underline{t^{j+1}_y} \le \underline{t^{j+1,d}} | \delta_j = x, S_j = S, \delta_{j+1} = \chi(x,S,r), S_{j+1} = T(x,S,r), r_{j+1} = r\Big) \\
&= \mathbb{P}_R\Big(\underline{t^{j+1}_y} \le \underline{t^{j+1,d}} | \delta_{j+1} = \chi(x,S,r), S_{j+1} = T(x,S,r)\Big) \\
&\ge \frac{\chi(x,S,r)}{y},
\end{aligned} \tag{40}$$

where the second equality is since $\underline{t^j_y}, \underline{t^{j,d}} \ge j+1$, the third equality is since conditioned on S_{j+1}, δ_{j+1}, we have that $\{(\delta_t, S_t)\}_{t\in\{j+1,\dots,n\}}$ is independent on r_{j+1}, S_j, δ_j, and the inequality is by the induction hypothesis.

We now enumerate five different cases depending on request r_{j+1}. Since we assumed that $|(x,y] \cap S| \ge 1$, $S \cap \{0\} \ne \emptyset$ and $\delta_j = x \ne 0$, we have by Lemma 41 that the values of $\chi(x,S,r_{j+1})$ are the one given in Table 1, with $s^R = \min\{s \in S : s > x\}$ and $w = s^R - x$.

- Case 1: $r_{j+1} \in [0, \frac{x}{2}]$. Then, from Table 1, we have $\chi(x,S,r_{j+1}) = \delta_j = x$. Thus, by using (40), we get
$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S, r_{j+1} \in [0, \tfrac{x}{2}]\Big) \ge \frac{x}{y}.$$

- Case 2: $r_{j+1} \in [\frac{x}{2}, \frac{x+w}{2}]$. Trivially, we have
$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S, r_{j+1} \in [\tfrac{x}{2}, \tfrac{x+w}{2}]\Big) \ge 0.$$

- Case 3: $r_{j+1} \in [\frac{x+w}{2}, x + \frac{w}{2}]$. Then, from Table 1, we have $\chi(x,S,r_{j+1}) = s^R$. Thus, by using (40), we get
$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S, r_{j+1} \in [\tfrac{x+w}{2}, x + \tfrac{w}{2}]\Big) \ge \frac{s^R}{y} = \frac{x+w}{2}.$$

- Case 4: $r_{j+1} \in [x + \frac{w}{2}, x+w]$. Then, from Table 1, we have $\chi(x,S,r_{j+1}) = x$.

  Thus, by using (40), we get
$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S, r_{j+1} \in [x + \tfrac{w}{2}, x+w]\Big) \ge \frac{x}{y}.$$

- Case 5: $r_{j+1} \in [x+w, 1]$. Then, from Table 1, we have that $\chi(x,S,r_{j+1}) = x$. Thus, by using (40), we get
$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S, r_{j+1} \in [x+w, 1]\Big) \ge \frac{x}{y}.$$

By combining the five cases above, we get

$$\begin{aligned}
&\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S\Big) \\
&\ge \mathbb{P}(r_{j+1} \in [0, \tfrac{x}{2}]) \cdot \frac{x}{y} + 0 + \mathbb{P}(r_{j+1} \in [\tfrac{x+w}{2}, x + \tfrac{w}{2}]) \cdot \frac{x+w}{y} \\
&+ \mathbb{P}(r_{j+1} \in [x + \tfrac{w}{2}, x + w]) \cdot \frac{x}{y} + \mathbb{P}(r_{j+1} \in [x + w, 1]) \cdot \frac{x}{y} \\
&= \frac{x}{2} \cdot \frac{x}{y} + \frac{x}{2} \cdot \frac{x+w}{y} + \frac{w}{2} \cdot \frac{x}{y} + (1 - (x + w)) \cdot \frac{x}{y} \\
&= \frac{x}{y} \cdot \Big(\frac{x}{2} + \frac{x+w}{2} + \frac{w}{2} + (1 - (x + w))\Big) \\
&= \frac{x}{y}.
\end{aligned}$$

It remains to show the inductive case when either $x = 0$, $|(x, y] \cap S| = 0$, or $S \cap \{0\} = \emptyset$. Note that if $x = 0$, it is immediate that

$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S\Big) \ge 0 = \frac{x}{y},$$

and if $S \cap \{0\} = \emptyset$, then $\underline{t^j_y} = \underline{t^{j,d}} = j$ and we have

$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S\Big) = 1 \ge \frac{x}{y}.$$

Finally, if $|(x, y] \cap S| = 0$ and $x > 0$, then

$$S_j \cap (0, y] = (S \cap (0, x]) \cup (S \cap (x, y]) = \{x\} \cup \emptyset = \{\delta_j\},$$

where the second equality is since $\min\{s > 0 | s \in S\} = x$ and the assumption that $|S \cap (x, y]| = 0$. Now, we have

$$\begin{aligned}
\underline{t^{j,d}} &:= \min\{t \ge j : \delta_t = 0 \text{ or } S_t \cap \{0\} = \emptyset\} \\
&\ge \min\{t \ge j : \delta_t \ne \delta_j \text{ or } S_t \cap \{0\} = \emptyset\} \\
&= \min\{t \ge j : S_t = S_{t-1} \setminus \{\delta_j\} \text{ or } S_t \cap \{0\} = \emptyset\} \\
&= \min\{t \ge j : S_t \cap (0, y] = \emptyset \text{ or } S_t \cap \{0\} = \emptyset\} \\
&=: \underline{t^j_y},
\end{aligned}$$

where the inequality is since $\delta_j = x > 0$, the second equality is from the fourth point of Lemma 41, and the third equality is since $S_j \cap (0, y] = \{\delta_j\}$. Hence we also get

$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S\Big) = 1 \ge \frac{x}{y}.$$

Hence, in all possible cases, we have shown that

$$\mathbb{P}_R\Big(\underline{t^j_y} \le \underline{t^{j,d}} | \delta_j = x, S_j = S\Big) \ge \frac{x}{y},$$

which concludes the inductive case. We conclude the proof by applying the above inequality with $j = m$. □

Lemma 44.

1. *If $\delta_m > 0$, then $r_m \in [0, y_0]$.*
2. *For all $m \in [n]$, $\delta_m \in [0, 2n^{-1/5}]$.*
3. *For all $m \in [c_1 n]$, $\mathbb{E}[\delta_m | r_m \in [0, y_0]] \geq \frac{n^{-1/5}}{4} - n^{-\Omega(\log(n))}$.*

Proof. **Point 1.** By Lemma 20, we have that $S_{m-1} = S'_{m-1}$. Now, if $r_m \in (y_0, 1]$, then both $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ match r_m greedily. Hence, $S_m = S'_m$, which, by definition of δ, implies that $\delta_m = 0$. By contraposition, if $\delta_m > 0$, we must have $r_m \in [0, y_0]$.

Point 2. It is immediate that $\delta_m = 0$ when $r_m \in (y_0, 1]$. Now, if $r_m \in [0, y_0]$, then either $S_m \cap \{0\} = \emptyset$ or $S_m \cap \{0\} \neq \emptyset$. In the first case, r_m is matched greedily by both $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ and we get $S_m = S'_m$ and $\delta_m = 0$. In the second case, we first have, by definition of $\mathcal{H}^m$, that $s(r_m) = 0$. Then, by the greediness of $\mathcal{H}^{m-1}$ for r_m, we get $|s(r_m)' - r_m| \leq |r_m - 0|$, which implies $s(r_m)' \leq 2r_m$. Hence, $\delta_m = s(r'_m) - 0 \leq 2r_m \leq 2y_0 = 2n^{-1/5}$. We conclude that for all $m \in [n]$, $\delta_m \in [0, 2n^{-1/5}]$.

Point 3. Fix $m \leq c_1 n$. We first show that if R is regular and $r_m \in [\frac{3}{4}y_0, y_0]$, we have that $\delta_m = s_{m-1,1} \geq n^{-1/5}$.

Assume that the sequence of requests is regular and consider $r_m \in [\frac{3}{4}y_0, y_0]$. We start by showing that $s(r_m)' = s_{m-1,1}$ and $s(r_m) = 0$. Since $m \leq c_1 n$, we have by Lemma 32 that $m < t_1 = \min\{t \geq 0 : S_t \cap I_1 = \emptyset\}$. Hence, $S_{m-1} \cap [n^{-1/5}, (3/2)n^{-1/5}] \neq \emptyset$. Since $S_{m-1} \cap [0, n^{-1/5}] \subseteq S_0 \cap [0, n^{-1/5}] = \emptyset$, we thus have $s'_{m-1,1} = s_{m-1,1} \in [n^{-1/5}, (3/2)n^{-1/5}]$. Since $r_m \in [\frac{3}{4}y_0, y_0] = [\frac{3}{4}n^{-1/5}, n^{-1/5}]$, we deduce that $\mathcal{N}'(r_m) \subseteq \{0, s_{m-1,1}\}$. Now, note that

$$|s_{m-1,1} - r_m| \leq |(3/2)n^{-1/5} - r_m| \leq |(3/2)n^{-1/5} - (3/4)n^{-1/5}| = (3/4)n^{-1/5} \leq |r_m - 0|.$$

Since $\mathcal{H}^{m-1}$ matches r_m greedily, we get that $s'(r_m) = s_{m-1,1}$.

On the other hand, $\mathcal{H}^m$ follows $\mathcal{A}$ for matching r_m. Note that by Lemma 32, we have that $m < t_{\{0\}}$, hence $S'_{m-1} \cap \{0\} = S_{m-1} \cap \{0\} \neq \emptyset$. Since $r_m \in [0, n^{-1/5}]$, we get by definition of $\mathcal{A}$ that $s(r_m) = 0$.

Since $s(r_m) = 0$ and $s(r'_m) = s_{m-1,1}$, we deduce that $S_m = S_{m-1} \setminus \{0\} \neq S_{m-1} \setminus \{s_{m-1,1}\} = S'_{m-1} \setminus \{s_{m-1,1}\} = S'_m$, hence, by definition of δ, we get: $\delta_m = s_{m,1} = s_{m-1,1} \geq n^{-1/5}$. We have thus shown that if R is regular and $r_m \in [\frac{3}{4}y_0, y_0]$, then $\delta_m \geq n^{-1/5}$. As a result,

$$\begin{aligned}
&\mathbb{E}[\delta_m | r_m \in [0, y_0]] \geq n^{-1/5}\mathbb{P}(\delta_m \geq n^{-1/5} | r_m \in [0, y_0]) \\
&\geq n^{-1/5}\mathbb{P}(r_m \in [\tfrac{3}{4}y_0, y_0], \text{R is regular} | r_m \in [0, y_0]) \\
&= n^{-1/5}(\mathbb{P}(r_m \in [\tfrac{3}{4}y_0, y_0] | r_m \in [0, y_0]) - \mathbb{P}(r_m \in [\tfrac{3}{4}y_0, y_0], \text{R is not regular} | r_m \in [0, y_0])) \\
&\geq n^{-1/5}(\mathbb{P}(r_m \in [\tfrac{3}{4}y_0, y_0] | r_m \in [0, y_0]) - \mathbb{P}(\text{R is not regular} | r_m \in [0, y_0])) \\
&= n^{-1/5}(\mathbb{P}(r_m \in [\tfrac{3}{4}y_0, y_0] | r_m \in [0, y_0]) - \mathbb{P}(r_m \in [0, y_0], \text{R is not regular}) / \mathbb{P}(r_m \in [0, y_0])) \\
&\geq n^{-1/5}(\mathbb{P}(r_m \in [\tfrac{3}{4}y_0, y_0] | r_m \in [0, y_0]) - \mathbb{P}(\text{R is not regular}) / \mathbb{P}(r_m \in [0, y_0])) \\
&= \frac{1}{4}n^{-1/5} - n^{-\Omega(\log(n))},
\end{aligned}$$

where the last equality holds since R is regular with high probability by Lemma 22. □

Lemma 45. *For all $m \in [n]$ and $i \in [d_1 \log n]$,*

1. *if R is regular, then $t_{(0,y_i]} \leq t_{\{0\}}$.*
2. *$\mathbb{P}(t^d > t_{\{0\}} | r_m \in [0, y_0]) = O(n^{-1/5})$.*

Proof. We first assume that the requests sequence is regular, and we show that there is no $t \in [n]$ such that $s_{t,1} < 1/2$ and $S_t \cap \{0\} = \emptyset$.

Assume by contradiction that there is such a t. Since $s_{t,1}$ is available at time t, we have that for all $i \in [t]$, $s_{t,1}$ is available when request r_i arrives. In addition, recall that $\mathcal{H}^m$ either matches each request to 0, or matches it greedily, and it matches a request r to 0 only if $r \leq y_0$. Since by definition of the instance, $s_{t,1} > y_0$, we get that there is no $i \in [t]$ such that $r_i < s_{t,1}$ and $s(r_i) \geq s_{t,1}$ (since r_i is closer to $s_{t,1}$ than any other server $s > s_{t,1}$ and $s_{t,1}$ is available when r_i arrives) and there is no $i \in [t]$ such that $r_i \geq s_{t,1}$ and $s(r_i) < s_{t,1}$. Hence,

$$|\{i \in [t] : r_i \in [0, s_{t,1}]| = |i \in [t] : s(r_i) \in [0, s_{t,1}]\}|.$$

In addition, since $s_{t,1} = \min\{s > 0 : s \in S_t\}$ and since we assumed that $S_t \cap \{0\} = \emptyset$, we have $[0, s_{t,1}) \cap S_t = S_t \cap \{0\} = \emptyset$, hence all servers in $[0, s_{t,1})$ have been matched to some request before time t and we have $|i \in [t] : s(r_i) \in [0, s_{t,1}]| = |S_0 \cap [0, s_{t,1})|$. Let $d^+ = \min\{j/n : j \in [n], j/n > s_{t,1}\}$. We get

$$\begin{aligned}
&|\{i \in [n] : r_i \in [0, d^+]\}| \\
&\geq |\{i \in [t] : r_i \in [0, d^+]\}| \\
&\geq |\{i \in [t] : r_i \in [0, s_{t,1}]\}| \\
&= |S_0 \cap [0, s_{t,1})| \\
&\geq |S_0 \cap [0, n^{-1/5}]| + |S_0 \cap [n^{-1/5}, d^+ - 1/n)| \\
&\geq [n^{4/5} + 4\log(n)^2\sqrt{n}] + [(d^+ - 1/n - n^{-1/5})\tilde{n} - 1] \\
&= [n^{4/5} + 4\log(n)^2\sqrt{n}] + [(d^+ - 1/n - n^{-1/5})(n - 4\log(n)^2\sqrt{n}/(1 - n^{-1/5})) - 1] \\
&= d^+ n + 4\log(n)^2\sqrt{n}(1 - (d^+ - 1/n - n^{-1/5})/(1 - n^{-1/5})) - 2 \\
&= d^+ n + 4\log(n)^2\sqrt{n}(1 - (1/2 - n^{-1/5})/(1 - n^{-1/5})) - 2 \\
&> d^+ n + \log n^2\sqrt{d^+ n},
\end{aligned}$$

where the fourth inequality is by definition of the instance and by Fact 26, and the fourth equality since $d^+ \leq s_{t,1} + 1/n \leq 1/2 + 1/n$ and the last inequality since $d^+ \leq 1/2$. Hence, the second regularity condition from Definition 29 is not satisfied for $t = 0, t' = n$ and $d = 0, d' = d^+$. Thus, if R is regular, then there is no $t \in [n]$ such that $s_{t,1} < 1/2$ and $S_t \cap \{0\} = \emptyset$.

On the way, we deduce the following equation, that will be used in the proof of the second part of the lemma.

$$\begin{aligned}
&\mathbb{P}(\exists t \in [n] : s_{t,1} < 1/2 \text{ and } S_t \cap \{0\} = \emptyset | r_m \in [0, y_0]) \\
&\qquad \leq \mathbb{P}(\text{R is not regular} | r_m \in [0, y_0]) \leq \mathbb{P}(\text{R is not regular})/\mathbb{P}(r_m \in [0, y_0]) = n^{-\Omega(\log(n))},
\end{aligned} \tag{41}$$

where the last inequality holds since R is regular with high probability by Lemma 22.

Now, we assume that R is regular and we show that for all $i \in [d_1 \log(n)]$, $t_{(0,y_i]} \le t_{\{0\}}$. Note that if $t_{(0,y_i]} > t_{\{0\}}$, then, by definition of $t_{(0,y_i]}$, we have that $S_{t_{\{0\}}} \cap (0, y_i] \neq \emptyset$, which implies that $s_{t_{\{0\}},1} \le y_i < 1/2$. By definition of $t_{\{0\}}$, we also have $S_{t_{\{0\}}} \cap \{0\} = \emptyset$. This contradicts the fact that there is no $t \in [n]$ such that $s_{t,1} < 1/2$ and $S_t \cap \{0\} = \emptyset$. Hence, we have $t_{(0,y_i]} \le t_{\{0\}}$, which concludes the proof of the first part of the lemma.

Next, we show that $t^d \le t_{\{0\}}$ with high probability. First, note that by Lemma 51, we have $\mathbb{P}(\max_{t\in[n]} \delta_t \ge 1/2 \mid \delta_m) \le 2\delta_m$. Since by Lemma 44, we have that for all $m \in [n]$, $\delta_m \le 2n^{-1/5}$, we get

$$\mathbb{P}(\max_{t\in[n]} \delta_t \ge 1/2 | r_m \in [0, y_0]) \le 4n^{-1/5}. \tag{42}$$

Hence, we have

$$\begin{aligned}
&\mathbb{P}(t^d > t_{\{0\}} | r_m \in [0, y_0]) \\
&= \mathbb{P}(t^d > t_{\{0\}},\ \delta_{t_{\{0\}}} \ge 1/2 | r_m \in [0, y_0]) + \mathbb{P}(t^d > t_{\{0\}},\ \delta_{t_{\{0\}}} < 1/2 | r_m \in [0, y_0]) \\
&\le \mathbb{P}(\delta_{t_{\{0\}}} \ge 1/2 | r_m \in [0, y_0]) + \mathbb{P}(\delta_{t_{\{0\}}} < 1/2, \delta_{t_{\{0\}}} > 0, S_{t_{\{0\}}} \cap \{0\} = \emptyset | r_m \in [0, y_0]) \\
&\le \mathbb{P}(\delta_{t_{\{0\}}} \ge 1/2 | r_m \in [0, y_0]) + \mathbb{P}(s_{t_{\{0\}},1} < 1/2, S_{t_{\{0\}}} \cap \{0\} = \emptyset | r_m \in [0, y_0]) \\
&\le \mathbb{P}(\max_{t\in[n]} \delta_t \ge 1/2 | r_m \in [0, y_0]) + \mathbb{P}(\exists t \in [n] : s_{t,1} < 1/2 \text{ and } S_t \cap \{0\} = \emptyset | r_m \in [0, y_0]) \\
&\le 4n^{-1/5} + n^{-\Omega(\log(n))} \\
&= O(n^{-1/5}),
\end{aligned}$$

where the first inequality holds since by definition of t^d, if $t^d > t_{\{0\}}$, then $\delta_{t_{\{0\}}} > 0$, and since we always have $S_{t_{\{0\}}} \cap \{0\} = \emptyset$ by definition of $t_{\{0\}}$. The second inequality holds since by definition of δ, if $\delta_{t_{\{0\}}} > 0$, then $\delta_{t_{\{0\}}} = s_{t_{\{0\}},1}$. The last inequality is by (42) and (41).

□

Lemma 21. *For all $m \in [n]$, we have that*

$$\mathbb{E}\Big[\sum_{t=m+1}^{n} (cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^m) | \delta_m, S_m\Big] \ge \frac{1}{2}\mathbb{E}\Big[\max_{t\in\{0,\dots,\min(t_{\{0\}},t_w)-m\}} \delta_{t+m} - \delta_m | \delta_m, S_m\Big] - \mathbb{P}(t^d > t_{\{0\}} | \delta_m, S_m),$$

where $s_{t,1} = \min\{s > 0 : s \in S_t\}$ and $s_{t,2} = \min\{s > s_{t,1} : s \in S_t\}$; $t_w := \min\{t \ge m : s_{t,2} - s_{t,1} > s_{t,1},\ \text{or } s_{t,2} = \emptyset\}$, $t^d = \min\{t \ge m : \delta_t = 0\}$ and $t_{\{0\}} := \min\{t \ge m |\ S_t \cap \{0\} = \emptyset\}$.

Proof. We analyse the difference of cost between $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ for all requests $r_{m+1}, \dots, r_n$. We consider in the following paragraphs some time steps $t \ge m$ and we omit to mention this condition throughout the proof.

We start by some preliminary notational considerations. We first recall that $w_t = s_{t,2} - s_{t,1}$. Also, note that for all $t \in \mathbb{N}$, we have $S_{t+1} \subseteq S_t$; thus if $S_t \cap \{0\} = \emptyset$, then $S_{t+1} \cap \{0\} = \emptyset$. Hence, by definition of $t_{\{0\}}$, we first have that $S_t \cap \{0\} \neq \emptyset$ if and only if $t \le t_{\{0\}} - 1$. Then, by definition, we have that if $\delta_t \neq 0$, then $\delta_t = s_{t,1}$. Thus, if $\delta_t \neq 0$ and $w_t \le \delta_t$, we have that $|S_t \cap (\delta_t, 1]| = |S_t \cap (s_{t,1}, 1]|$ and that $s_{t,2} - s_{t,1} = w_t \le \delta_t = s_{1,t}$, hence we have that $t \le t^w - 1$. In addition, since $\delta_t \neq 0$, by the first point of Lemma 41, we have that $\delta'_t \neq 0$ for all $t' \le t$; hence we have that $t \le t^d - 1$. Reciprocally, and by a similar argument, if $t \le t^w - 1$ and $t \le t^d - 1$, then we have that $\delta_t \neq 0$, $|S_t \cap (\delta_t, 1]| \geq 1$ and $w_t \leq \delta_t$. Hence, we have that $\delta_t \neq 0$, $|S_t \cap (\delta_t, 1]| \geq 1$ and $w_t \leq \delta_t$ if and only if $t \leq t^w - 1$ and $t \leq t^d - 1$. Therefore,

$$\mathbb{1}_{S_t \cap \{0\} \neq \emptyset, \delta_t \neq 0, |S_t \cap (\delta_t, 1]| \geq 1, w_t \leq \delta_t} = \mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \quad \text{and} \quad \mathbb{1}_{S_t \cap \{0\} \neq \emptyset} = \mathbb{1}_{t \leq t_{\{0\}}-1}.$$

Lower bound on $\mathbb{E}[\Delta\mathbf{cost}_{t+1}]$ in the case where $t \leq t_{\{0\}} - 1$. For all $t \leq t_{\{0\}} - 1$ we have $S_t \cap \{0\} \neq \emptyset$; thus, by the second point of Lemma 41, we have that $\Delta\text{cost}_{t+1} \geq 0$. Hence,

$$\mathbb{E}[\mathbb{1}_{t \leq t_{\{0\}}-1} \cdot \Delta\text{cost}_{t+1} | (\delta_t, S_t)] \geq 0.$$

If we further have that $t \leq \min(t^w, t^d, t_{\{0\}})-1$, then, as argued above, we first have that $S_t \cap \{0\} \neq \emptyset$, $\delta_t \neq 0$ and $|S_t \cap (\delta_t, 1]| \geq 1$, thus the assumptions of the third point of Lemma 41 are satisfied. In addition, we have that $w_t \leq \delta_t$. Therefore, by inspecting all possible cases given in Table 1, we obtain:

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \cdot \Delta\text{cost}_{t+1} | (\delta_t, S_t)] \\
&\quad \geq 0 + \mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \cdot \mathbb{E}[\Delta\text{cost}_{t+1} | (\delta_t, S_t), r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{P}(r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}] | (\delta_t, S_t)) \\
&\quad = \mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \cdot \frac{w_t}{2} \cdot \mathbb{P}(r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]),
\end{aligned}$$

where the last equality is since $w_t \leq \delta_t$ and by inspecting the corresponding case in Table 1, and since r_{t+1} is independent of (δ_t, S_t).

By combining the two previous inequalities, we get

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{t \leq t_{\{0\}}-1} \cdot \Delta\text{cost}_{t+1} | (\delta_t, S_t)] \\
&= \mathbb{E}[\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \cdot \Delta\text{cost}_{t+1} | (\delta_t, S_t)] + \mathbb{E}[\mathbb{1}_{\min(t^w, t^d, t_{\{0\}})-1 \leq t \leq t_{\{0\}}-1} \cdot \Delta\text{cost}_{t+1} | (\delta_t, S_t)] \\
&\geq \mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \cdot \frac{w_t}{2} \cdot \mathbb{P}(r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]) + 0.
\end{aligned}$$

Note that conditioning on (δ_t, S_t), we have that Δcost_{t+1} is independent of (δ_m, S_m). Thus, for any $(x, S) \in \mathcal{X}$, first conditioning on (δ_m, S_m), then applying the tower law gives:

$$\begin{aligned}
&\mathbb{E}[\mathbb{1}_{t \leq t_{\{0\}}-1} \cdot \Delta\text{cost}_{t+1} | (\delta_m, S_m)] \\
&\quad = \mathbb{E}[\mathbb{E}[\mathbb{1}_{t \leq t_{\{0\}}-1} \cdot \Delta\text{cost}_{t+1} | (\delta_t, S_t), (\delta_m, S_m)] \; | (\delta_m, S_m)] \\
&\quad = \mathbb{E}[\mathbb{E}[\mathbb{1}_{t \leq t_{\{0\}}-1} \cdot \Delta\text{cost}_{t+1} | (\delta_t, S_t)] \; | (\delta_m, S_m)] \\
&\quad \geq \mathbb{E}[\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \cdot \frac{w_t}{2} | (\delta_m, S_m)] \cdot \mathbb{P}(r_{t+1} \in [\tfrac{\delta_t + w_t}{2}, \delta_t + \tfrac{w_t}{2}]) \\
&\quad = \mathbb{E}[\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1} \cdot \frac{w_t}{2} | (\delta_m, S_m)] \cdot \mathbb{E}[\mathbb{1}_{r_{t+1} \in [\frac{\delta_t + w_t}{2}, \delta_t + \frac{w_t}{2}]} | (\delta_m, S_m)] \\
&\quad = \mathbb{E}[\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1, r_{t+1} \in [\frac{\delta_t + w_t}{2}, \delta_t + \frac{w_t}{2}]} \cdot \frac{w_t}{2} | (\delta_m, S_m)], \\
&\quad = \mathbb{E}[\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1, r_{t+1} \in [\frac{\delta_t + w_t}{2}, \delta_t + \frac{w_t}{2}]} \cdot \frac{\delta_{t+1} - \delta_t}{2} | (\delta_m, S_m)],
\end{aligned} \tag{43}$$

where the third equality uses that r_{t+1} is independent of (δ_m, S_m), and the fourth equality holds since r_{t+1} and (δ_t, S_t) are independent, which implies that r_{t+1} and $\{t \leq \min(t^w, t^d, t_{\{0\}}) - 1\}$ are independent. To see the last equality, note that if $t \leq \min(t^w, t^d, t_{\{0\}})-1$, then, as argued above, the assumptions of the third point of Lemma 41 are satisfied. Since we also have $r_{t+1} \in [\frac{\delta_t + w_t}{2}, \delta_t + \frac{w_t}{2}]$, we get from Table 1 that $\delta_{t+1} = \delta_t + w_t$.

Next, if the assumptions of the third point of Lemma 41 are satisfied, by inspecting all possible cases given in Table 1, we have that $r_{t+1} \in [\frac{\delta_t + w_t}{2}, \delta_t + \frac{w_t}{2}]$ if and only if $\delta_{t+1} \neq \delta_t$ and $\delta_{t+1} \neq 0$. Thus,

$$\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1, r_{t+1} \in [\frac{\delta_t + w_t}{2}, \delta_t + \frac{w_t}{2}]} = \mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1, \delta_{t+1} \neq \delta_t, \delta_{t+1} \neq 0}. \tag{44}$$

In addition, by definition of t^d and by the first point of Lemma 41, we have that $\delta_{t+1} \neq 0$ if and only if $t \leq t^d - 2$, thus

$$\mathbb{1}_{t \leq \min(t^w, t^d, t_{\{0\}})-1, \delta_{t+1} \neq \delta_t, \delta_{t+1} \neq 0} = \mathbb{1}_{t \leq \min(t^w, t^d - 1, t_{\{0\}})-1, \delta_{t+1} \neq \delta_t}. \tag{45}$$

Hence, by combining (43), (44) and (45), we get that for all $t \in \mathbb{N}$,

$$\mathbb{E}[\mathbb{1}_{t \leq t_{\{0\}}-1} \cdot \Delta \text{cost}_{t+1} | (\delta_m, S_m)] \geq \mathbb{E}[\mathbb{1}_{t \leq \min(t^w, t^d - 1, t_{\{0\}})-1, \delta_{t+1} \neq \delta_t} \cdot \frac{\delta_{t+1} - \delta_t}{2} | (\delta_m, S_m)]. \tag{46}$$

Lower bound on $\mathbb{E}[\Delta \text{cost}_{t+1}]$ in the case where $t \geq t_{\{0\}}$. Note that from the first point of Lemma 41, if $\delta_t = 0$, then $\Delta \text{cost}_{t+1} = 0$. In addition, recall that $S_t \cap \{0\} = \emptyset$ when $t \geq t_{\{0\}}$. Hence, using the fifth point of Lemma 41, we get

$$\begin{aligned}
E[\mathbb{1}_{t \geq t_{\{0\}}} \cdot \Delta \text{cost}_{t+1} | (\delta_t, S_t)] &= \mathbb{E}[\mathbb{1}_{t \geq t_{\{0\}}, \delta_t \neq 0} \cdot \Delta \text{cost}_{t+1} | (\delta_t, S_t)] + \mathbb{E}[\mathbb{1}_{t \geq t_{\{0\}}, \delta_t = 0} \cdot \Delta \text{cost}_{t+1} | (\delta_t, S_t)] \\
&= \mathbb{E}[\mathbb{1}_{t \geq t_{\{0\}}, \delta_t \neq 0} \cdot \Delta \text{cost}_{t+1} | (\delta_t, S_t)] \\
&\geq -\mathbb{1}_{t \geq t_{\{0\}}, \delta_t \neq 0} \cdot \mathbb{P}(\delta_{t+1} = 0 | (\delta_t, S_t)) \\
&= -\mathbb{1}_{t \geq t_{\{0\}}, \delta_t \neq 0} \cdot E[\mathbb{1}_{\delta_{t+1} = 0} | (\delta_t, S_t)] \\
&= -\mathbb{E}[\mathbb{1}_{t \geq t_{\{0\}}, \delta_t \neq 0, \delta_{t+1} = 0} | (\delta_t, S_t)].
\end{aligned}$$

Note that by the first point of Lemma 41 and by definition of t^d, we have that $\delta_t \neq 0, \delta_{t+1} = 0$ if and only if $\delta_{t+1} = t^d$. Thus, $\mathbb{1}_{t \geq t_{\{0\}}, \delta_t \neq 0, \delta_{t+1} = 0} = \mathbb{1}_{t \geq t_{\{0\}}, \delta_{t+1} = t^d}$. Finally, by the tower law, we get:

$$\mathbb{E}[\mathbb{1}_{t \geq t_{\{0\}}} \cdot \Delta \text{cost}_{t+1} | (\delta_m, S_m)] \geq -\mathbb{E}[\mathbb{1}_{t \geq t_{\{0\}}, \delta_{t+1} = t^d} | (\delta_m, S_m)]. \tag{47}$$

Concluding the proof of Lemma 21. We lower bound the difference of costs for matching requests $r_{m+1},\dots,r_n$:

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=m+1}^{n} cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^{m})|\delta_m, S_m\right]\\
&= \mathbb{E}\left[\sum_{t=m}^{n-1} \Delta\text{cost}_{t+1}|(\delta_m, S_m)\right]\\
&= \mathbb{E}\left[\sum_{t=m}^{n-1} \mathbb{1}_{t\le t_{\{0\}}-1}\cdot \Delta\text{cost}_{t+1}|(\delta_m, S_m)\right] + \mathbb{E}\left[\sum_{t=m}^{n-1} \mathbb{1}_{t\ge t_{\{0\}}}\cdot \Delta\text{cost}_{t+1}|(\delta_m, S_m)\right]\\
&\ge \mathbb{E}\left[\sum_{t=m}^{n-1} \mathbb{1}_{t\le \min(t^w, t^d-1, t_{\{0\}})-1, \delta_{t+1}\ne\delta_t}\cdot \frac{\delta_{t+1}-\delta_t}{2}\middle|(\delta_m, S_m)\right]\\
&\qquad\qquad - \mathbb{E}\left[\sum_{t=m}^{n-1} \mathbb{1}_{t\ge t_{\{0\}},\, \delta_{t+1}=t^d}|(\delta_m, S_m)\right]\\
&= \mathbb{E}\left[\sum_{t=m}^{\min(t^w, t^d-1, t_{\{0\}})-1} \mathbb{1}_{\delta_{t+1}\ne\delta_t}\cdot \frac{\delta_{t+1}-\delta_t}{2}\middle|(\delta_m, S_m)\right] - \mathbb{P}(t^d > t_{\{0\}}|(\delta_m, S_m))\\
&= \frac{1}{2}\mathbb{E}\left[\delta_{\min(t^w, t^d-1, t_{\{0\}})} - \delta_m\middle|(\delta_m, S_m)\right] - \mathbb{P}(t^d > t_{\{0\}}|(\delta_m, S_m)),
\end{aligned}$$

where the inequality is by (46) and (47).

In addition, note that by construction of the process, for all $t\in\mathbb{N}$, we have that $S_{t+1}\subseteq S_t$; hence $s_{t,1}\le s_{t+1,1}$. Then, for all $t\le t^d-2$, we have that $\delta_t, \delta_{t+1}\ne 0$, hence by construction, we have $\delta_t = s_{t,1}$ and $\delta_{t+1} = s_{t+1,1}$. Thus, we get that $\delta_t\le\delta_{t+1}$. Hence, we have that $\delta_{\min(t^w,t^d-1,t_{\{0\}})} = \max_{t\in\{0,\dots,\min(t^w,t^d-1,t_{\{0\}})\}}\delta_t$. In addition, since by the first point of Lemma 41, we have that $\delta_t = 0$ for all $t\ge t^d$, we get that $\max_{t\in\{0,\dots,\min(t^w,t^d-1,t_{\{0\}})\}}\delta_t = \max_{t\in\{0,\dots,\min(t^w,t_{\{0\}})\}}\delta_t$. Therefore,

$$\mathbb{E}\left[\sum_{t=m+1}^{n} cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^{m})|\delta_m, S_m\right] \ge \frac{1}{2}\mathbb{E}\left[\max_{t\in\{0,\dots,\min(t^w,t_{\{0\}})\}}\delta_t - \delta_m\middle|(\delta_m, S_m)\right] - \mathbb{P}(t^d > t_{\{0\}}|(\delta_m, S_m)).$$

□

Lemma 46. *For all $m\in[n]$, $\mathbb{E}\left[\sum_{t=1}^{m}(cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^{m}))|r_m\in[0,y_0]\right]\ge -n^{-1/5}$.*

Proof. Since $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ both follow $\mathcal{A}$ for the first $m-1$ requests, it is immediate that

$$\mathbb{E}\left[\sum_{t=1}^{m-1} cost_t(\mathcal{H}^{m-1}) - cost_t(\mathcal{H}^{m})|\delta_m, S_m\right] = 0.$$

We now lower bound the cost of matching request r_m. We consider two cases:
(1) If $r_m\in(y_0,1]$ or ($r_m\in[0,y_0]$ and $S_m\cap\{0\}=\emptyset$), then both $\mathcal{H}^m$ and $\mathcal{H}^{m-1}$ match r_m greedily. Since $S_{m-1} = S'_{m-1}$, we get $cost_m(\mathcal{H}^{m-1}) = cost_m(\mathcal{H}^{m})$.

(2) If ($r_m \in [0, y_0]$ and $S_m \cap \{0\} \neq \emptyset$), then $s(r_m) = 0$, hence $cost_m(\mathcal{H}^{m-1}) - cost_m(\mathcal{H}^m) \geq -cost_m(\mathcal{H}^m) = -|r_m - 0| \geq -y_0 = -n^{-1/5}$.

In both cases,

$$\mathbb{E}[cost_m(\mathcal{H}^{m-1}) - cost_m(\mathcal{H}^m)|\delta_m, S_m] \geq -n^{-1/5}.$$

Combining the two above equations concludes the proof. □