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A Natural Deduction System for First Degree Entailment

ALLARD M. TAMMINGA and KOJI TANAKA

Abstract This paper is concerned with a natural deduction system for *First Degree Entailment (FDE)*. First, we exhibit a brief history of *FDE* and of combined systems whose underlying idea is used in developing the natural deduction system. Then, after presenting the language and a semantics of *FDE*, we develop a natural deduction system for *FDE*. We then prove soundness and completeness of the system with respect to the semantics. The system neatly represents the four-valued semantics for *FDE*.

1 Introduction

1.1 First degree entailment After being inspired by the work of Ackermann [1], Anderson and Belnap [2] started their investigation into a theory of implication: if . . . then ___. They developed a number of formal calculi of *entailment* which later came to be known as *Relevant Logics*.[1] In developing their systems, Anderson and Belnap encountered the difficulty of dealing with nested entailments. Consequently, they considered *FDE*, in which the antecedent φ and consequent ψ in an implicational sentence of the form $\varphi \to \psi$ are truth functional, that is, φ and ψ themselves do not contain implications. The purpose of the investigation into *FDE* is, then, to study the truth functional relationship between antecedent and consequent of implicational sentences.

Anderson and Belnap provide a Hilbert-style system and a Gentzen-style system for *FDE*. Although they give characteristic matrices, however, Anderson and Belnap do not provide any formal semantics for *FDE*. For this, we had to wait for Routley and Routley [11] and Dunn [5].[2] Routley and Routley provide a two-valued semantics for *FDE*. Although their semantics may be philosophically contentious, it serves as a basis for the semantics for various relevant logics.[3] However, in this paper, we are concerned only with Dunn's semantics, which is somewhat more intuitive. Together

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with a tableau system, Dunn presents an "intuitive" formal semantics for *FDE*. Classically, semantic evaluations of sentences are defined to be functions that assign to a formula exactly one truth value. For Dunn, however, evaluations are relations between a truth value and a formula. A formula may then take (relate to) no truth value or may take (relate to) multiple truth values.[4]

One feature of Dunn's semantics for *FDE* that we should take notice of is that truth and falsity are not mutually complementary. Truth and falsity are considered separately and are independent notions in Dunn's semantics. This feature plays an important role in developing a natural deduction system for *FDE* later in this paper.

1.2 Combined systems The idea of considering true and false formulas separately can also be found in the study of formal logics for 'assertion' and 'rejection'. Łukasiewicz was, to the best of our knowledge, the first to introduce both a sign for 'assertion' and a sign for 'rejection' into formal logic. Tracing back the history of the philosophy of logic, Łukasiewicz followed Brentano (1838–1917), who propounded a nonpropositional theory of judgment. Brentano [4] argued that

> As every judgement is based on an idea, the statement expressing a judgement necessarily contains a name [of the idea]. To this, another sign must come, a sign corresponding to the inner state which we call judging, that is, a sign completing the bare name to a sentence. And because this judging can be twofold, *viz.*, asserting or rejecting, the sign indicating it must be twofold too, one for affirmation and one for denial. These signs themselves do not mean anything, but in conjunction with a name, they are the expression of a judgement. Therefore, the most general scheme of a statement is 'A is' and 'A is not'.[5]

In the 1921 paper "Logika dwuwartościowa," later translated as "Two-valued logic," Łukasiewicz followed Brentano in adding to Frege's idea of assertion Brentano's idea of rejection. In his early works, Łukasiewicz argued that a proposition must be rejected if and only if it is false, parallel with Frege's condition for the assertion of a proposition.[6] Later, starting with *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Łukasiewicz redefined the concept of rejection to cover not only false propositions, but propositions which are false under at least one interpretation as well. Furthermore, he introduced syntactical techniques to *derive* all rejected, that is, nontautological, statements. By using the symbol '$\vdash$' for assertion (indicating tautologyhood) and '$\dashv$' for rejection (indicating nontautologyhood), what Łukasiewicz added to *classical propositional logic (CPL)* is the following:

Axiom	$\dashv p$, where p is a fixed propositional variable.
Detachment	If $\vdash \varphi \rightarrow \psi$ and $\dashv \psi$, then $\dashv \varphi$.
Substitution	If $\dashv \psi$ and ψ can be obtained from φ by substitution, then $\dashv \varphi$.

This system is first described in Łukasiewicz [8],[7] where Łukasiewicz also propounded a system of rejection for Aristotle's syllogistics, after some technical problems had been solved by Słupecki. Łukasiewicz also tried to construe systems of rejection for the *intuitionistic propositional logic (IPL)* and for his own version of modal logic. All these systems share one characteristic: they are all "combined systems," that is, they all include both a sign for 'assertion' and a sign for 'rejection'.

One of the advantages of combined systems over traditional ones worth mentioning in this paper is that metatheoretical results can be incorporated in the object language of the system under consideration. For instance, the disjunction property of *IPL* can be formulated in the object language of a proof system as follows:

$$\frac{\dashv \varphi \qquad \dashv \psi}{\dashv \varphi \vee \psi}\ .$$

Now, since in many of the combined systems, in particular the systems of Łukasiewicz, $\dashv \varphi$ is complemented by the failure of $\vdash \varphi$,[8] the concept of rejection contained in the systems is classical. Nonetheless, combined systems, prima facie, take the idea seriously that (possibly) false formulas be considered separately from true formulas. The idea of Dunn's semantics seems to have another home here.

2 Language and semantics

Definition 2.1 The alphabet of *FDE* consists of the following.

(i)	Propositional variables	$p_1, p_2, p_3, \ldots$
(ii)	Logical symbols	$\neg, \wedge, \vee$
(iii)	Auxiliary symbols	$), ($

$\Box$ denotes an empty sequence. $\mathcal{A}$ denotes the set of propositional variables.

Definition 2.2 The set of all formulas of *FDE*, denoted by $\mathcal{F}$, is the least set satisfying the following conditions:

(i) $\mathcal{A} \subset \mathcal{F}$,
(ii) $\varphi, \psi \in \mathcal{F} \implies (\varphi \wedge \psi), (\varphi \vee \psi) \in \mathcal{F}$,
(iii) $\varphi \in \mathcal{F} \implies \neg\varphi \in \mathcal{F}$.

Definition 2.3 Let $\mathcal{M} = \langle \mathcal{F}, \nu \rangle$ be an interpretation for the language where ν is an evaluation such that $\nu_{\mathcal{M}}$ is a function from $\mathcal{A}$ to $\wp(\{0, 1\})$. Then $\nu_{\mathcal{M}}$ is extended to an evaluation for all formulas φ and ψ by the following conditions:

(i) $1 \in \nu_{\mathcal{M}}(\varphi \wedge \psi) \iff 1 \in \nu_{\mathcal{M}}(\varphi)$ and $1 \in \nu_{\mathcal{M}}(\psi)$,
(ii) $0 \in \nu_{\mathcal{M}}(\varphi \wedge \psi) \iff 0 \in \nu_{\mathcal{M}}(\varphi)$ or $0 \in \nu_{\mathcal{M}}(\psi)$,
(iii) $1 \in \nu_{\mathcal{M}}(\varphi \vee \psi) \iff 1 \in \nu_{\mathcal{M}}(\varphi)$ or $1 \in \nu_{\mathcal{M}}(\psi)$,
(iv) $0 \in \nu_{\mathcal{M}}(\varphi \vee \psi) \iff 0 \in \nu_{\mathcal{M}}(\varphi)$ and $0 \in \nu_{\mathcal{M}}(\psi)$,
(v) $1 \in \nu_{\mathcal{M}}(\neg\varphi) \iff 0 \in \nu_{\mathcal{M}}(\varphi)$,
(vi) $0 \in \nu_{\mathcal{M}}(\neg\varphi) \iff 1 \in \nu_{\mathcal{M}}(\varphi)$.

Definition 2.4 Let $\Pi \subseteq \mathcal{F}$ and $\mathcal{M}$ be an interpretation. Then

(i) $1 \in \nu_{\mathcal{M}}(\Pi) := 1 \in \nu_{\mathcal{M}}(\varphi)$ for every $\varphi \in \Pi$,
(ii) $0 \in \nu_{\mathcal{M}}(\Pi) := 0 \in \nu_{\mathcal{M}}(\varphi)$ for every $\varphi \in \Pi$.

We are now in a position to define validity. Validity defined below incorporates the concept of Dunn's semantics for *FDE*. It concerns not only truth but also falsity as in Dunn's semantics.[9]

Definition 2.5 (*FDE* Validity) Let $\Pi, \Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. Then

(i) $\Pi; \Gamma \models \varphi; \Box \iff$ For all $\mathcal{M}$: if $1 \in \nu_{\mathcal{M}}(\Pi)$ and $0 \in \nu_{\mathcal{M}}(\Gamma)$, then $1 \in \nu_{\mathcal{M}}(\varphi)$,
(ii) $\Pi; \Gamma \models \Box; \varphi \iff$ For all $\mathcal{M}$: if $1 \in \nu_{\mathcal{M}}(\Pi)$ and $0 \in \nu_{\mathcal{M}}(\Gamma)$, then $0 \in \nu_{\mathcal{M}}(\varphi)$.

3 A natural deduction system While providing a Hilbert-style system and a Gentzen-style system and natural deduction systems for other relevant logics, Anderson and Belnap do not give any natural deduction systems for *FDE*. The first natural deduction system for *FDE* to be formally introduced, other than the system developed in this paper, will be by Priest [9].[10]

In this section, we introduce a natural deduction system for *FDE*, $\mathbf{ND_{FDE}}$. The system is developed by amalgamating the concept of Dunn's semantics and that of the combined systems. Instead of taking $\vdash \varphi$ to be an assertion of φ (a usual policy in combined systems), here it is semantically interpreted as: φ takes 'truth' as *a* truth value. Similarly, $\dashv \varphi$ is interpreted as: φ takes 'falsity' as *a* truth value.

The system $\mathbf{ND_{FDE}}$ is defined as follows.[11]

Definition 3.1 Derivations in the system $\mathbf{ND_{FDE}}$ are inductively generated as follows.

Basis: The proof tree with a single occurrence of an assumption $\vdash \varphi$ or $\dashv \varphi$ is a derivation.

Induction Step: Let $\mathcal{D}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ be derivations. Then they can be extended by the following rules:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \varphi\end{array}}{\dashv \neg\varphi}\,\neg I_{\vdash} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \dashv \varphi\end{array}}{\vdash \neg\varphi}\,\neg I_{\dashv} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \neg\varphi\end{array}}{\dashv \varphi}\,\neg E_{\vdash} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \dashv \neg\varphi\end{array}}{\vdash \varphi}\,\neg E_{\dashv}$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdash \varphi\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \vdash \psi\end{array}}{\vdash \varphi\wedge\psi}\,\wedge I_{\vdash} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \dashv \varphi_i\end{array}}{\dashv \varphi_0\wedge\varphi_1}\,\wedge I_{\dashv},\ i\in\{0,1\} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \varphi_0\wedge\varphi_1\end{array}}{\vdash \varphi_i}\,\wedge E_{\vdash},\ i\in\{0,1\}$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \dashv \varphi\wedge\psi\end{array} \quad \begin{array}{c}[\dashv \varphi]^u\\ \mathcal{D}_2\\ X\end{array} \quad \begin{array}{c}[\dashv \psi]^v\\ \mathcal{D}_3\\ X\end{array}}{X}\,\wedge E_{\dashv}^{u,v},\ \text{where either } X = \vdash \chi \text{ or } X = \dashv \chi$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \vdash \varphi_i\end{array}}{\vdash \varphi_0\vee\varphi_1}\,\vee I_{\vdash},\ i\in\{0,1\} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \dashv \varphi\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \dashv \psi\end{array}}{\dashv \varphi\vee\psi}\,\vee I_{\dashv} \quad \dfrac{\begin{array}{c}\mathcal{D}\\ \dashv \varphi_0\vee\varphi_1\end{array}}{\dashv \varphi_i}\,\vee E_{\dashv},\ i\in\{0,1\}$$

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdash \varphi\vee\psi\end{array} \quad \begin{array}{c}[\vdash \varphi]^u\\ \mathcal{D}_2\\ X\end{array} \quad \begin{array}{c}[\vdash \psi]^v\\ \mathcal{D}_3\\ X\end{array}}{X}\,\vee E_{\vdash}^{u,v},\ \text{where either } X = \vdash \chi \text{ or } X = \dashv \chi$$

where $[\vdash \varphi]$ and $[\dashv \varphi]$ are assumptions which are discharged by the application of the rules.

Lemma 3.2 *De Morgan rules of the following forms are available in* **ND**$_{\mathbf{FDE}}$ *(double lines indicate that the rules work both ways):*

$$\begin{array}{c}\mathcal{D}\\ \vdash \varphi\wedge\psi \\ \hline\hline \dashv \neg\varphi\vee\neg\psi\end{array}\, DeM_{\vdash} \qquad\qquad \begin{array}{c}\mathcal{D}\\ \dashv \varphi\wedge\psi \\ \hline\hline \vdash \neg\varphi\vee\neg\psi\end{array}\, DeM_{\dashv}$$

Proof: $DeM_{\vdash}$:

$$\dfrac{\dfrac{\dfrac{\vdash \varphi\wedge\psi}{\vdash \varphi}}{\dashv \neg\varphi} \qquad \dfrac{\dfrac{\vdash \varphi\wedge\psi}{\vdash \psi}\,\wedge E_{\vdash}}{\dashv \neg\psi}\,\neg I_{\vdash}}{\dashv \neg\varphi\vee\neg\psi}\,\vee I_{\dashv}$$

$$\dfrac{\dfrac{\dfrac{\dashv \neg\varphi\vee\neg\psi}{\dashv \neg\varphi}}{\vdash \varphi} \qquad \dfrac{\dfrac{\dashv \neg\varphi\vee\neg\psi}{\dashv \neg\psi}\,\vee E_{\dashv}}{\vdash \psi}\,\neg E_{\dashv}}{\vdash \varphi\wedge\psi}\,\wedge E_{\vdash}$$

$DeM_{\dashv}$:

$$\dfrac{\dashv \varphi\wedge\psi \qquad \dfrac{\dfrac{[\dashv \varphi]^{u}}{\vdash \neg\varphi}}{\vdash \neg\varphi\vee\neg\psi} \qquad \dfrac{\dfrac{[\dashv \psi]^{v}}{\vdash \neg\psi}\,\neg I_{\dashv}}{\vdash \neg\varphi\vee\neg\psi}\,\vee I_{\vdash}}{\vdash \neg\varphi\vee\neg\psi}\,\wedge E_{\dashv}^{u,v}$$

$$\dfrac{\vdash \neg\varphi\vee\neg\psi \qquad \dfrac{\dfrac{[\vdash \neg\varphi]^{u}}{\dashv \varphi}}{\dashv \varphi\wedge\psi} \qquad \dfrac{\dfrac{[\vdash \neg\psi]^{v}}{\dashv \psi}\,\neg E_{\vdash}}{\dashv \varphi\wedge\psi}\,\wedge I_{\dashv}}{\dashv \varphi\wedge\psi}\,\vee E_{\vdash}^{u,v}$$

$\square$

Definition 3.3 Let $\Pi \subseteq \mathcal{F}$. Then

(i) $\vdash \Pi \quad := \quad \{\vdash \varphi : \varphi \in \Pi\}$,
(ii) $\dashv \Pi \quad := \quad \{\dashv \varphi : \varphi \in \Pi\}$.

Definition 3.4 (*FDE* Derivability) Let $\Pi, \Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. Then

(i) $\Pi; \Gamma \mapsto \varphi; \Box \iff$ There is a derivation of $\vdash \varphi$ from $\vdash \Pi \cup \dashv \Gamma$ in **ND**$_{\mathbf{FDE}}$,
(ii) $\Pi; \Gamma \mapsto \Box; \varphi \iff$ There is a derivation of $\dashv \varphi$ from $\vdash \Pi \cup \dashv \Gamma$ in **ND**$_{\mathbf{FDE}}$.

4 Soundness

Lemma 4.1 *Let* $\Pi_i, \Gamma_i \subseteq \mathcal{F}$ *for* $i \in \{1, 2, 3\}$ *and* $\varphi, \psi, \chi \in \mathcal{F}$. *Then*

$$
\begin{array}{rlcl}
\text{(i)} & \Pi; \Gamma \models \varphi; \Box, & \textit{if} & \varphi \in \Pi \\
\text{(ii)} & \Pi; \Gamma \models \Box; \varphi, & \textit{if} & \varphi \in \Gamma \\
\text{(iii)} & \Pi; \Gamma \models \varphi; \Box & \Longrightarrow & \Pi; \Gamma \models \Box; \neg\varphi \\
\text{(iv)} & \Pi; \Gamma \models \Box; \varphi & \Longrightarrow & \Pi; \Gamma \models \neg\varphi; \Box \\
\text{(v)} & \Pi; \Gamma \models \neg\varphi; \Box & \Longrightarrow & \Pi; \Gamma \models \Box; \varphi \\
\text{(vi)} & \Pi; \Gamma \models \Box; \neg\varphi & \Longrightarrow & \Pi; \Gamma \models \varphi; \Box \\
\text{(vii)} & \left.\begin{array}{l} \Pi_1; \Gamma_1 \models \varphi; \Box \\ \Pi_2; \Gamma_2 \models \psi; \Box \end{array}\right\} & \Longrightarrow & \Pi_1, \Pi_2; \Gamma_1, \Gamma_2 \models \varphi \wedge \psi; \Box \\
\text{(viii)} & \Pi; \Gamma \models \Box; \varphi & \Longrightarrow & \Pi; \Gamma \models \Box; \varphi \wedge \psi \\
\text{(ix)} & \Pi; \Gamma \models \Box; \psi & \Longrightarrow & \Pi; \Gamma \models \Box; \varphi \wedge \psi \\
\text{(x)} & \Pi; \Gamma \models \varphi \wedge \psi; \Box & \Longrightarrow & \Pi; \Gamma \models \varphi; \Box \\
\text{(xi)} & \Pi; \Gamma \models \varphi \wedge \psi; \Box & \Longrightarrow & \Pi; \Gamma \models \psi; \Box \\
\text{(xii)} & \left.\begin{array}{l} \Pi_1; \Gamma_1 \models \Box; \varphi \wedge \psi \\ \Pi_2; \Gamma_2, \varphi \models \chi; \Box \\ \Pi_3; \Gamma_3, \psi \models \chi; \Box \end{array}\right\} & \Longrightarrow & \Pi_1, \Pi_2, \Pi_3; \Gamma_1, \Gamma_2, \Gamma_3 \models \chi; \Box \\
\text{(xiii)} & \left.\begin{array}{l} \Pi_1; \Gamma_1 \models \Box; \varphi \wedge \psi \\ \Pi_2; \Gamma_2, \varphi \models \Box; \chi \\ \Pi_3; \Gamma_3, \psi \models \Box; \chi \end{array}\right\} & \Longrightarrow & \Pi_1, \Pi_2, \Pi_3; \Gamma_1, \Gamma_2, \Gamma_3 \models \Box; \chi \\
\text{(xiv)} & \Pi; \Gamma \models \varphi; \Box & \Longrightarrow & \Pi; \Gamma \models \varphi \vee \psi; \Box \\
\text{(xv)} & \Pi; \Gamma \models \psi; \Box & \Longrightarrow & \Pi; \Gamma \models \varphi \vee \psi; \Box \\
\text{(xvi)} & \left.\begin{array}{l} \Pi_1; \Gamma_1 \models \Box; \varphi \\ \Pi_2; \Gamma_2 \models \Box; \psi \end{array}\right\} & \Longrightarrow & \Pi_1, \Pi_2; \Gamma_1, \Gamma_2 \models \Box; \varphi \vee \psi \\
\text{(xvii)} & \Pi; \Gamma \models \Box; \varphi \vee \psi & \Longrightarrow & \Pi; \Gamma \models \Box; \varphi \\
\text{(xviii)} & \Pi; \Gamma \models \Box; \varphi \vee \psi & \Longrightarrow & \Pi; \Gamma \models \Box; \psi \\
\text{(xix)} & \left.\begin{array}{l} \Pi_1; \Gamma_1 \models \varphi \vee \psi; \Box \\ \Pi_2, \varphi; \Gamma_2 \models \chi; \Box \\ \Pi_3, \psi; \Gamma_3 \models \chi; \Box \end{array}\right\} & \Longrightarrow & \Pi_1, \Pi_2, \Pi_3; \Gamma_1, \Gamma_2, \Gamma_3 \models \chi; \Box \\
\text{(xx)} & \left.\begin{array}{l} \Pi_1; \Gamma_1 \models \varphi \vee \psi; \Box \\ \Pi_2, \varphi; \Gamma_2 \models \Box; \chi \\ \Pi_3, \psi; \Gamma_3 \models \Box; \chi \end{array}\right\} & \Longrightarrow & \Pi_1, \Pi_2, \Pi_3; \Gamma_1, \Gamma_2, \Gamma_3 \models \Box; \chi.
\end{array}
$$

Proof: (xii) Suppose that $\mathcal{M}$ is an interpretation such that $1 \in \nu_{\mathcal{M}}(\Pi_1, \Pi_2, \Pi_3)$ and $0 \in \nu_{\mathcal{M}}(\Gamma_1, \Gamma_2, \Gamma_3)$. Then, as $\Pi_1; \Gamma_1 \models \Box; \varphi \wedge \psi$, we have $0 \in \nu_{\mathcal{M}}(\varphi \wedge \psi)$. Therefore, $0 \in \nu_{\mathcal{M}}(\varphi)$ or $0 \in \nu_{\mathcal{M}}(\psi)$. Suppose $0 \in \nu_{\mathcal{M}}(\varphi)$. Then, as $\Pi_2; \Gamma_2, \varphi \models \chi; \Box$, we have $1 \in \nu_{\mathcal{M}}(\chi)$. Suppose $0 \in \nu_{\mathcal{M}}(\psi)$. Then, as $\Pi_3; \Gamma_3, \psi \models \chi; \Box$, we have $1 \in \nu_{\mathcal{M}}(\chi)$. Hence $1 \in \nu_{\mathcal{M}}(\chi)$. Therefore, $\Pi_1, \Pi_2, \Pi_3; \Gamma_1, \Gamma_2, \Gamma_3 \models \Box; \chi$.

The other cases can be proved analogously. $\Box$

Theorem 4.2 (Soundness of $\mathbf{ND}_{\mathrm{FDE}}$**)** *Let* $\Pi, \Gamma \subseteq \mathcal{F}$ *and* $\varphi \in \mathcal{F}$. *Then*

$$
\begin{array}{rlcl}
\text{(i)} & \Pi; \Gamma \mapsto \varphi; \Box & \Longrightarrow & \Pi; \Gamma \models \varphi; \Box, \\
\text{(ii)} & \Pi; \Gamma \mapsto \Box; \varphi & \Longrightarrow & \Pi; \Gamma \models \Box; \varphi.
\end{array}
$$

Proof: The proof is by induction on the depth of derivation. All that needs to be checked is that the rules preserve truth and falsity in the appropriate way. This can be shown using Lemma 4.1. □

5 Completeness We now prove the completeness theorem for $\mathbf{ND}_{\mathrm{FDE}}$. Priest [9] demonstrates techniques to prove completeness theorems for natural deduction systems for various relevant and paraconsistent logics.[12] Although Priest defines validity and derivability in a standard way, his techniques provide some insights into the structure of the proof for the theorem. Here we adapt his techniques in our proof.

Definition 5.1 Let $\Pi, \Gamma \subseteq \mathcal{F}$. Then $\langle \Pi; \Gamma \rangle$ is a theory, if $\langle \Pi; \Gamma \rangle$ is closed under deducibility, that is, if both

(i) $\Pi; \Gamma \mapsto \varphi; \Box \quad \Longrightarrow \quad \varphi \in \Pi$,
(ii) $\Pi; \Gamma \mapsto \Box; \varphi \quad \Longrightarrow \quad \varphi \in \Gamma$.

Definition 5.2 Let $\langle \Pi; \Gamma \rangle$ be a theory. Then $\langle \Pi; \Gamma \rangle$ is dual prime, if $\langle \Pi; \Gamma \rangle$ has both the disjunction property and the conjunction property, that is, if both

(i) $\varphi \vee \psi \in \Pi \quad \Longrightarrow \quad \varphi \in \Pi$ or $\psi \in \Pi$,
(ii) $\varphi \wedge \psi \in \Gamma \quad \Longrightarrow \quad \varphi \in \Gamma$ or $\psi \in \Gamma$.

Lemma 5.3 *Let $\Pi, \Gamma \subseteq \mathcal{F}$ and $\varphi, \psi \in \mathcal{F}$. Let $\langle \Pi; \Gamma \rangle$ be a dual prime theory. Then*

(i) $\varphi \wedge \psi \in \Pi \quad \Longleftrightarrow \quad \varphi \in \Pi$ and $\psi \in \Pi$,
(ii) $\varphi \wedge \psi \in \Gamma \quad \Longleftrightarrow \quad \varphi \in \Gamma$ or $\psi \in \Gamma$,
(iii) $\varphi \vee \psi \in \Pi \quad \Longleftrightarrow \quad \varphi \in \Pi$ or $\psi \in \Pi$,
(iv) $\varphi \vee \psi \in \Gamma \quad \Longleftrightarrow \quad \varphi \in \Gamma$ and $\psi \in \Gamma$,
(v) $\varphi \in \Pi \quad \Longleftrightarrow \quad \neg\varphi \in \Gamma$,
(vi) $\neg\varphi \in \Pi \quad \Longleftrightarrow \quad \varphi \in \Gamma$.

Proof:

(i) Suppose $\varphi \wedge \psi \in \Pi$. Then $\Pi; \Gamma \mapsto \varphi \wedge \psi; \Box$. So $\Pi; \Gamma \mapsto \varphi; \Box$ and $\Pi; \Gamma \mapsto \psi; \Box$ by $\wedge E_{\vdash}$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \in \Pi$ and $\psi \in \Pi$. Suppose $\varphi \in \Pi$ and $\psi \in \Pi$. By $\wedge I_{\vdash}$, $\Pi; \Gamma \mapsto \varphi \wedge \psi; \Box$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \wedge \psi \in \Pi$.
(ii) Suppose $\varphi \wedge \psi \in \Gamma$. By dual primeness, $\varphi \in \Gamma$ or $\psi \in \Gamma$. Suppose $\varphi \in \Gamma$ or $\psi \in \Gamma$. By $\wedge I_{\dashv}$, $\Pi; \Gamma \mapsto \Box; \varphi \wedge \psi$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \wedge \psi \in \Gamma$.
(iii) Suppose $\varphi \vee \psi \in \Pi$. By dual primeness, $\varphi \in \Pi$ or $\psi \in \Pi$. Suppose $\varphi \in \Pi$ or $\psi \in \Pi$. By $\vee I_{\vdash}$, $\Pi; \Gamma \mapsto \varphi \vee \psi; \Box$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \vee \psi \in \Pi$.
(iv) Suppose $\varphi \vee \psi \in \Gamma$. By $\vee E_{\dashv}$, $\Pi; \Gamma \mapsto \Box; \varphi$ and $\Pi; \Gamma \mapsto \Box; \psi$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \in \Gamma$ and $\psi \in \Gamma$. Suppose $\varphi \in \Gamma$ and $\psi \in \Gamma$. By $\vee I_{\dashv}$, $\Pi; \Gamma \mapsto \Box; \varphi \vee \psi$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \vee \psi \in \Gamma$.
(v) Suppose $\varphi \in \Pi$. By $\neg I_{\vdash}$, $\Pi; \Gamma \mapsto \Box; \neg\varphi$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\neg\varphi \in \Gamma$. Suppose $\neg\varphi \in \Gamma$. By $\neg E_{\dashv}$, $\Pi; \Gamma \mapsto \varphi; \Box$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \in \Pi$.
(vi) Suppose $\neg\varphi \in \Pi$. By $\neg E_{\vdash}$, $\Pi; \Gamma \mapsto \Box; \varphi$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\varphi \in \Gamma$. Suppose $\varphi \in \Gamma$. By $\neg I_{\dashv}$, $\Pi; \Gamma \mapsto \neg\varphi; \Box$. Since $\langle \Pi; \Gamma \rangle$ is a theory, $\neg\varphi \in \Pi$.

□

Definition 5.4 Let $\Pi, \Gamma, \Delta, \Sigma \subseteq \mathcal{F}$. Then

(i) $\Pi; \Gamma \mapsto \Delta; \Box \Longleftrightarrow$ There are $\delta_1, \ldots, \delta_n \in \Delta$ such that $\Pi; \Gamma \mapsto \delta_1 \vee \cdots \vee \delta_n; \Box$,
(ii) $\Pi; \Gamma \mapsto \Box; \Sigma \Longleftrightarrow$ There are $\sigma_1, \ldots, \sigma_n \in \Sigma$ such that $\Pi; \Gamma \mapsto \Box; \sigma_1 \wedge \cdots \wedge \sigma_n$.

Lemma 5.5 *Let* $\Pi, \Gamma, \Delta \subseteq \mathcal{F}$ *such that* $\Pi; \Gamma \not\mapsto \Delta; \Box$. *Then there are sets* $\Pi^* \supseteq \Pi$, $\Gamma^* \supseteq \Gamma$ *and* $\Delta^* \supseteq \Delta$ *such that*

(i) $\Pi^*; \Gamma^* \not\mapsto \Delta^*; \Box$,
(ii) $\langle \Pi^*; \Gamma^* \rangle$ *is a theory,*
(iii) $\langle \Pi^*; \Gamma^* \rangle$ *is dual prime.*

Proof: Assume that $\Pi; \Gamma \not\mapsto \Delta; \Box$ for $\Pi, \Gamma, \Delta \subseteq \mathcal{F}$. Let $\chi_0, \chi_2, \chi_4, \ldots$ be an enumeration of $\mathcal{F}$. Let $m \in \{0, 2, 4, \ldots\}$. We define by recursion the sequence $\langle \Pi_n; \Gamma_n; \Delta_n \rangle$ $(n \in \omega)$ as follows:

$$\langle \Pi_0; \Gamma_0; \Delta_0 \rangle \quad := \quad \langle \Pi; \Gamma; \Delta \rangle$$

$$\langle \Pi_{m+1}; \Gamma_{m+1}; \Delta_{m+1} \rangle \quad := \begin{cases} \langle \Pi_m, \chi_m; \Gamma_m; \Delta_m \rangle, & \textit{if} \\ \quad \Pi_m, \chi_m; \Gamma_m \not\mapsto \Delta_m; \Box & \\ & \\ \langle \Pi_m; \Gamma_m; \Delta_m, \chi_m \rangle, & \textit{if} \\ \quad \Pi_m, \chi_m; \Gamma_m \mapsto \Delta_m; \Box. & \end{cases}$$

$$\langle \Pi_{m+2}; \Gamma_{m+2}; \Delta_{m+2} \rangle \quad := \begin{cases} \langle \Pi_{m+1}; \Gamma_{m+1}, \chi_m; \Delta_{m+1} \rangle, & \textit{if} \\ \quad \Pi_{m+1}; \Gamma_{m+1}, \chi_m \not\mapsto \Delta_{m+1}; \Box & \\ & \\ \langle \Pi_{m+1}; \Gamma_{m+1}; \Delta_{m+1}, \neg\chi_m \rangle, & \textit{if} \\ \quad \Pi_{m+1}; \Gamma_{m+1}, \chi_m \mapsto \Delta_{m+1}; \Box. & \end{cases}$$

We define the following by means of the sequence defined above thus:

$$\langle \Pi^*; \Gamma^*; \Delta^* \rangle \quad := \quad \langle \textstyle\bigcup_{n\in\omega} \Pi_n; \bigcup_{n\in\omega} \Gamma_n; \bigcup_{n\in\omega} \Delta_n \rangle.$$

(i) We show that $\Pi^*; \Gamma^* \not\mapsto \Delta^*; \Box$ by induction on the construction of $\langle \Pi^*; \Gamma^*; \Delta^* \rangle$.
Basis: $n=0$. Then $\Pi_0; \Gamma_0 \not\mapsto \Delta_0; \Box$ by assumption.
Induction Hypothesis: $\Pi_n; \Gamma_n \not\mapsto \Delta_n; \Box$.
Induction Step: We must show that $\Pi_{n+1}; \Gamma_{n+1} \not\mapsto \Delta_{n+1}; \Box$. There are two cases: (a) $n+1 = m+1$ for some $m \in \{0, 2, 4, \ldots\}$, and (b) $n+1 = m+2$ for some $m \in \{0, 2, 4, \ldots\}$.

(a) Suppose that $n+1 = m+1$ for some $m \in \{0, 2, 4, \ldots\}$. Then there are two cases based on the construction of $\langle \Pi_{m+1}; \Gamma_{m+1}; \Delta_{m+1} \rangle$ from $\langle \Pi_m; \Gamma_m; \Delta_m \rangle$.

(a') $\langle \Pi_{m+1}; \Gamma_{m+1}; \Delta_{m+1} \rangle = \langle \Pi_m, \chi_m; \Gamma_m; \Delta_m \rangle$. By the construction, it must be that $\Pi_m, \chi_m; \Gamma_m \not\mapsto \Delta_m; \Box$. Hence $\Pi_{m+1}; \Gamma_{m+1} \not\mapsto \Delta_{m+1}; \Box$. Therefore, $\Pi_{n+1}; \Gamma_{n+1} \not\mapsto \Delta_{n+1}; \Box$.

(a'') $\langle \Pi_{m+1}; \Gamma_{m+1}; \Delta_{m+1} \rangle = \langle \Pi_m; \Gamma_m; \Delta_m, \chi_m \rangle$. By the construction, it must be that $\Pi_m, \chi_m; \Gamma_m \mapsto \Delta_m; \Box$. Suppose that Π_{m+1};

$\Gamma_{m+1} \mapsto \Delta_{m+1}; \Box$. Then $\Pi_m; \Gamma_m \mapsto \Delta_m, \chi_m; \Box$. By an application of $\vee E_\vdash$, we have that $\Pi_m; \Gamma_m \mapsto \Delta_m; \Box$, that is, $\Pi_n; \Gamma_n \mapsto \Delta_n; \Box$, contrary to the Induction Hypothesis.

(b) Suppose that $n+1 = m+2$ for some $m \in \{0, 2, 4, \ldots\}$. Then there are two cases based on the construction of $\langle \Pi_{m+2}; \Gamma_{m+2}; \Delta_{m+2} \rangle$ from $\langle \Pi_{m+1}; \Gamma_{m+1}; \Delta_{m+1} \rangle$.

(b') $\langle \Pi_{m+2}; \Gamma_{m+2}; \Delta_{m+2} \rangle = \langle \Pi_{m+1}; \Gamma_{m+1}, \chi_m; \Delta_{m+1} \rangle$. By the construction, it must be that $\Pi_{m+1}; \Gamma_{m+1}, \chi_m \not\mapsto \Delta_{m+1}; \Box$. Hence Π_{m+2}; $\Gamma_{m+2} \not\mapsto \Delta_{m+2}; \Box$. Therefore, $\Pi_{n+1}; \Gamma_{n+1} \not\mapsto \Delta_{n+1}; \Box$.

(b'') $\langle \Pi_{m+2}; \Gamma_{m+2}; \Delta_{m+2} \rangle = \langle \Pi_{m+1}; \Gamma_{m+1}; \Delta_{m+1}, \neg\chi_m \rangle$. By the construction, it must be that $\Pi_{m+1}; \Gamma_{m+1}, \chi_m \mapsto \Delta_{m+1}; \Box$. Suppose that Π_{m+2}; $\Gamma_{m+2} \mapsto \Delta_{m+2}; \Box$. Then $\Pi_{m+1}; \Gamma_{m+1} \mapsto \Delta_{m+1}, \neg\chi_m; \Box$. By applications of $\neg E_\vdash$ and $\vee E_\vdash$, $\Pi_{m+1}; \Gamma_{m+1} \mapsto \Delta_{m+1}; \Box$. Therefore, $\Pi_n; \Gamma_n \mapsto \Delta_n; \Box$, contrary to the Induction Hypothesis.

By (a) and (b), $\Pi_{n+1}; \Gamma_{n+1} \not\mapsto \Delta_{n+1}; \Box$. Hence $\Pi_n; \Gamma_n \not\mapsto \Delta_n; \Box$ for all n by induction. Therefore, $\Pi^*; \Gamma^* \not\mapsto \Delta^*; \Box$.

(ii) We show that $\langle \Pi^*; \Gamma^* \rangle$ is a theory. Assume that $\Pi^*; \Gamma^* \mapsto \varphi; \Box$. Now suppose that $\varphi \notin \Pi^*$. Then by the construction, for some $m \in \{0, 2, 4, \ldots\}$ where $\varphi = \chi_m$, it is not the case that $\Pi_m, \chi_m; \Gamma_m \not\mapsto \Delta_m; \Box$. So $\Pi_m, \chi_m; \Gamma_m \mapsto \Delta_m; \Box$. Hence $\varphi \in \Delta_{m+1} \subseteq \Delta^*$. Thus $\Pi^*; \Gamma^* \mapsto \Delta^*; \Box$, contrary to (i) proved above.

Assume that $\Pi^*; \Gamma^* \mapsto \Box; \varphi$, or equivalently, $\Pi^*; \Gamma^* \mapsto \neg\varphi; \Box$ by $\neg I_\dashv$. Now suppose that $\varphi \notin \Gamma^*$. Then by the construction, for some $m \in \{0, 2, 4, \ldots\}$ where $\varphi = \chi_m$, it is not the case that $\Pi_{m+1}; \Gamma_{m+1}, \chi_m \not\mapsto \Delta_{m+1}; \Box$. So $\Pi_{m+1}; \Gamma_{m+1}$, $\chi_m \mapsto \Delta_{m+1}; \Box$. Hence $\neg\varphi \in \Delta_{m+2} \subseteq \Delta^*$. Thus $\Pi^*; \Gamma^* \mapsto \Delta^*; \Box$, contrary to (i) proved above.

(iii) We show that $\langle \Pi^*; \Gamma^* \rangle$ is dual prime. Assume that $\varphi \vee \psi \in \Pi^*$. Then $\Pi^*; \Gamma^* \mapsto \varphi \vee \psi; \Box$. Now suppose that $\varphi \notin \Pi^*$ and $\psi \notin \Pi^*$. By the construction, for some $m \in \{0, 2, 4, \ldots\}$ where $\varphi = \chi_m$ and $n \in \{0, 2, 4, \ldots\}$ where $\psi = \chi_n$, it is not the case that $\Pi_m, \chi_m; \Gamma_m \not\mapsto \Delta_m; \Box$, nor that $\Pi_n, \chi_n; \Gamma_n \not\mapsto \Delta_n; \Box$. So $\Pi_m, \chi_m; \Gamma_m \mapsto \Delta_m; \Box$ and $\Pi_n, \chi_n; \Gamma_n \mapsto \Delta_n; \Box$. Hence $\varphi \in \Delta_{m+1} \subseteq \Delta^*$ and $\psi \in \Delta_{n+1} \subseteq \Delta^*$. Therefore $\Pi^*; \Gamma^* \mapsto \Delta^*; \Box$, contrary to (i) proved above.

Assume that $\varphi \wedge \psi \in \Gamma^*$. Then $\Pi^*; \Gamma^* \mapsto \Box; \varphi \wedge \psi$, or equivalently, $\Pi^*; \Gamma^* \mapsto \neg\varphi \vee \neg\psi; \Box$ by $DeM_\dashv$. Now suppose that $\varphi \notin \Gamma^*$ and $\psi \notin \Gamma^*$. By the construction, for some $m \in \{0, 2, 4, \ldots\}$ where $\varphi = \chi_m$ and $n \in \{0, 2, 4, \ldots\}$ where $\psi = \chi_n$, it is not the case that $\Pi_{m+1}; \Gamma_{m+1}, \chi_m \not\mapsto \Delta_{m+1}; \Box$, nor that $\Pi_{n+1}; \Gamma_{n+1}, \chi_n \not\mapsto \Delta_{n+1}; \Box$. So $\Pi_{m+1}; \Gamma_{m+1}, \chi_m \mapsto \Delta_{m+1}; \Box$ and $\Pi_{n+1}; \Gamma_{n+1}, \chi_n \mapsto \Delta_{n+1}; \Box$. Hence $\neg\varphi \in \Delta_{m+2} \subseteq \Delta^*$ and $\neg\psi \in \Delta_{n+2} \subseteq \Delta^*$. Therefore $\Pi^*; \Gamma^* \mapsto \Delta^*; \Box$, contrary to (i) proved above.

$\Box$

Lemma 5.6 *Let* $\Pi, \Gamma, \Sigma \subseteq \mathcal{F}$ *such that* $\Pi; \Gamma \not\mapsto \Box; \Sigma$. *Then there are sets* $\Pi^* \supseteq \Pi$, $\Gamma^* \supseteq \Gamma$ *and* $\Sigma^* \supseteq \Sigma$ *such that*

(i) $\Pi^*; \Gamma^* \not\mapsto \Box; \Sigma^*$,

(ii) $\langle \Pi^*; \Gamma^* \rangle$ *is a theory,*

(iii) $\langle \Pi^*; \Gamma^* \rangle$ *is dual prime.*

Proof: Assume that $\Pi; \Gamma \not\mapsto \Box; \Sigma$ for $\Pi, \Gamma, \Sigma \subseteq \mathcal{F}$. Let $\chi_0, \chi_2, \chi_4, \ldots$ be an enumeration of $\mathcal{F}$. Let $m \in \{0, 2, 4, \ldots\}$. We define by recursion the sequence $\langle \Pi_n; \Gamma_n; \Sigma_n \rangle$ $(n \in \omega)$ as follows:

$$\langle \Pi_0; \Gamma_0; \Sigma_0 \rangle \quad := \quad \langle \Pi; \Gamma; \Sigma \rangle$$

$$\langle \Pi_{m+1}; \Gamma_{m+1}; \Sigma_{m+1} \rangle \quad := \quad \begin{cases} \langle \Pi_m, \chi_m; \Gamma_m; \Sigma_m \rangle, & \textit{if} \\ \quad \Pi_m, \chi_m; \Gamma_m \not\mapsto \Box; \Sigma_m & \\ & \\ \langle \Pi_m; \Gamma_m; \Sigma_m, \neg\chi_m \rangle, & \textit{if} \\ \quad \Pi_m, \chi_m; \Gamma_m \mapsto \Box; \Sigma_m. & \end{cases}$$

$$\langle \Pi_{m+2}; \Gamma_{m+2}; \Sigma_{m+2} \rangle \quad := \quad \begin{cases} \langle \Pi_{m+1}; \Gamma_{m+1}, \chi_m; \Sigma_{m+1} \rangle, & \textit{if} \\ \quad \Pi_{m+1}; \Gamma_{m+1}, \chi_m \not\mapsto \Box; \Sigma_{m+1} & \\ & \\ \langle \Pi_{m+1}; \Gamma_{m+1}; \Sigma_{m+1}, \chi_m \rangle, & \textit{if} \\ \quad \Pi_{m+1}; \Gamma_{m+1}, \chi_m \mapsto \Box; \Sigma_{m+1}. & \end{cases}$$

Then (i), (ii), and (iii) can be proved as in Lemma 5.5. □

Lemma 5.7 *Let* $\Pi, \Gamma \subseteq \mathcal{F}$ *and* $\varphi \in \mathcal{F}$. *Then*

$$\Pi; \Gamma \models \varphi; \Box \quad \Longrightarrow \quad \Pi; \Gamma \mapsto \varphi; \Box.$$

Proof: Suppose that $\Pi; \Gamma \not\mapsto \varphi; \Box$ for $\Pi, \Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. By applying Lemma 5.5 with $\{\varphi\}$ as Δ, there is a dual prime theory $\langle \Pi^*; \Gamma^* \rangle$ for $\Pi^* \supseteq \Pi$ and $\Gamma^* \supseteq \Gamma$ and $\Delta^* \supseteq \Delta$, such that $\Pi^*; \Gamma^* \not\mapsto \Delta^*; \Box$.

Let $\mathcal{M}$ $(= \langle \mathcal{F}, \nu \rangle)$ be an interpretation and $p \in \mathcal{A}$. We define an evaluation ν as:

$$\begin{array}{lcl} 1 \in \nu_{\mathcal{M}}(p) & \Longleftrightarrow & p \in \Pi^*, \\ 0 \in \nu_{\mathcal{M}}(p) & \Longleftrightarrow & p \in \Gamma^*. \end{array}$$

It is then asserted that the above conditions extend to all formulas:

$$\begin{array}{lcl} 1 \in \nu_{\mathcal{M}}(\varphi) & \Longleftrightarrow & \varphi \in \Pi^*, \\ 0 \in \nu_{\mathcal{M}}(\varphi) & \Longleftrightarrow & \varphi \in \Gamma^*. \end{array}$$

The assertion is proved by structural induction on φ.

Basis: By assumption:

$$\begin{array}{lcl} 1 \in \nu_{\mathcal{M}}(\varphi) & \Longleftrightarrow & \varphi \in \Pi^*, \\ 0 \in \nu_{\mathcal{M}}(\varphi) & \Longleftrightarrow & \varphi \in \Gamma^*. \end{array}$$

Induction Hypothesis: For all ψ with fewer logical operators than φ:

$$\begin{array}{lcl} 1 \in \nu_{\mathcal{M}}(\psi) & \Longleftrightarrow & \psi \in \Pi^*, \\ 0 \in \nu_{\mathcal{M}}(\psi) & \Longleftrightarrow & \psi \in \Gamma^*. \end{array}$$

Induction Step: There are six cases based on the connectives in φ.

$$\begin{array}{llll} 1 \in v_{\mathcal{M}}(\psi_1 \wedge \psi_2) & \Longleftrightarrow & 1 \in v_{\mathcal{M}}(\psi_1) \text{ and } 1 \in v_{\mathcal{M}}(\psi_2) & \text{by Definition 2.3} \\ & \Longleftrightarrow & \psi_1 \in \Pi^* \text{ and } \psi_2 \in \Pi^* & \text{by Induction Hypothesis} \\ & \Longleftrightarrow & \psi_1 \wedge \psi_2 \in \Pi^* & \text{by Lemma 5.3.} \end{array}$$

$$\begin{array}{llll} 0 \in v_{\mathcal{M}}(\psi_1 \wedge \psi_2) & \Longleftrightarrow & 0 \in v_{\mathcal{M}}(\psi_1) \text{ or } 0 \in v_{\mathcal{M}}(\psi_2) & \text{by Definition 2.3} \\ & \Longleftrightarrow & \psi_1 \in \Gamma^* \text{ or } \psi_2 \in \Gamma^* & \text{by Induction Hypothesis} \\ & \Longleftrightarrow & \psi_1 \wedge \psi_2 \in \Gamma^* & \text{by Lemma 5.3.} \end{array}$$

Similarly, we have

$$\begin{array}{lll} 1 \in v_{\mathcal{M}}(\psi_1 \vee \psi_2) & \Longleftrightarrow & \psi_1 \vee \psi_2 \in \Pi^*, \\ 0 \in v_{\mathcal{M}}(\psi_1 \vee \psi_2) & \Longleftrightarrow & \psi_1 \vee \psi_2 \in \Gamma^*, \\ 1 \in v_{\mathcal{M}}(\neg\psi) & \Longleftrightarrow & \neg\psi \in \Pi^*, \\ 0 \in v_{\mathcal{M}}(\neg\psi) & \Longleftrightarrow & \neg\psi \in \Gamma^*. \end{array}$$

Hence the evaluation conditions defined above hold for all formulas by induction. Since $\Pi^*; \Gamma^* \not\mapsto \Delta^*; \Box$, we have that $\varphi \notin \Pi^*$. By the above conditions then, $1 \notin v_{\mathcal{M}}(\varphi)$. But $1 \in v_{\mathcal{M}}(\psi)$ and $0 \in v_{\mathcal{M}}(\chi)$ for all $\psi \in \Pi^*$ and $\chi \in \Gamma^*$. Hence $\Pi^*; \Gamma^* \not\models \varphi; \Box$. □

Lemma 5.8 *Let $\Pi, \Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. Then*

$$\Pi; \Gamma \models \Box; \varphi \quad \Longrightarrow \quad \Pi; \Gamma \mapsto \Box; \varphi.$$

Proof: Suppose that $\Pi; \Gamma \not\mapsto \Box; \varphi$ for $\Pi, \Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. By applying Lemma 5.6 with $\{\varphi\}$ as Σ, there is a dual prime theory $\langle \Pi^*; \Gamma^* \rangle$ for $\Pi^* \supseteq \Pi$ and $\Gamma^* \supseteq \Gamma$ and $\Sigma^* \supseteq \Sigma$, such that $\Pi^*; \Gamma^* \not\mapsto \Box; \Sigma^*$.

Let $\mathcal{M}$ ($= \langle \mathcal{F}, v \rangle$) be an interpretation and $p \in \mathcal{A}$. We define an evaluation v as:

$$\begin{array}{lll} 1 \in v_{\mathcal{M}}(p) & \Longleftrightarrow & p \in \Pi^*, \\ 0 \in v_{\mathcal{M}}(p) & \Longleftrightarrow & p \in \Gamma^*. \end{array}$$

It is then asserted that the above conditions extend to all formulas:

$$\begin{array}{lll} 1 \in v_{\mathcal{M}}(\varphi) & \Longleftrightarrow & \varphi \in \Pi^*, \\ 0 \in v_{\mathcal{M}}(\varphi) & \Longleftrightarrow & \varphi \in \Gamma^*. \end{array}$$

This assertion is proved as in Lemma 5.7. Since $\Pi^*; \Gamma^* \not\mapsto \Box; \Sigma^*$, we have that $\varphi \notin \Gamma^*$. By the above conditions, then, $0 \notin v_{\mathcal{M}}(\varphi)$. But $1 \in v_{\mathcal{M}}(\psi)$ and $0 \in v_{\mathcal{M}}(\chi)$ for all $\psi \in \Pi^*$ and $\chi \in \Gamma^*$. Hence $\Pi^*; \Gamma^* \not\models \Box; \varphi$. □

Theorem 5.9 (Completeness of **ND**$_{\text{FDE}}$) *Let $\Pi, \Gamma \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. Then*

(i) $\Pi; \Gamma \models \varphi; \Box \quad \Longleftrightarrow \quad \Pi; \Gamma \mapsto \varphi; \Box$,
(ii) $\Pi; \Gamma \models \Box; \varphi \quad \Longleftrightarrow \quad \Pi; \Gamma \mapsto \Box; \varphi$.

Proof: The result follows from Theorem 4.2, Lemma 5.7, and Lemma 5.8. □

6 Rejection eliminated? Although the system $\mathbf{ND}_{\mathrm{FDE}}$ captures the underlying idea of Dunn's semantics, one might argue that the introduction of rejected formulas is theoretically redundant. The argument runs as follows. $\mathbf{ND}_{\mathrm{FDE}}$ takes $\dashv$ as a falsity operator understood semantically. So stating $\dashv \varphi$ amounts to stating that φ is false. But then $\dashv \varphi$ is just $\vdash \neg\varphi$. Hence '$\dashv$' may be replaced by '$\vdash \neg$'. Once we have adopted this convention, '$\vdash$' can be dropped from the system, since we need not indicate the status (asserted or rejected) of a formula anymore. For example, the rule $\neg I_{\vdash}$ becomes

$$\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}}{\neg\neg\varphi}\,\neg\neg I$$

and $\wedge E_{\dashv}^{u,v}$ becomes

$$\dfrac{\begin{matrix}\\ \mathcal{D}_1\\ \neg(\varphi\wedge\psi)\end{matrix}\qquad\begin{matrix}[\neg\varphi]^u\\ \mathcal{D}_2\\ \chi\end{matrix}\qquad\begin{matrix}[\neg\psi]^v\\ \mathcal{D}_3\\ \chi\end{matrix}}{\chi}\,\neg\wedge E^{u,v}.$$

Moreover, if we add the De Morgan rules as primitive, there will be some rules of inference which are redundant. For example, $\neg\wedge E^{u,v}$ in the new system will be a special case of $\vee E^{u,v}$. The resulting system will then be that of Priest [9], as can easily be checked.[13]

These changes give rise to changes to the definitions of validity and derivability as well. Since every (rejected) formula in Γ in our definition of validity, that is, Definition 2.5, can be incorporated into Π by placing '$\neg$' in front of the formulas under consideration, validity is defined standardly. Similarly, derivability is defined standardly. Then soundness and completeness can be established as in [9].

The fact that $\mathbf{ND}_{\mathrm{FDE}}$ collapses under the proposed substitution to a standard system, such as Priest's, however, does not imply the inferiority of the system presented in this paper, as there are some obvious advantages of our combined system over the standard ones. First, $\mathbf{ND}_{\mathrm{FDE}}$ visually reflects the underlying idea of Dunn's semantics: truth and falsity are evaluated separately. Second, because of the introduction of both asserted and rejected formulas in our proof system, our system, contrary to Priest's, does not have any rules for combinations of logical operators: each operator has two introduction rules and two elimination rules, according to the status (asserted or rejected) of the formula which serves as a premise in the application of a rule. Rules which necessitate combinations of operators obscure the meanings of the operators. In constructing a proof tree in our system, at each step only the principal operator needs to be considered. This procedure makes the construction of proofs intuitive and mechanical, which is the main purpose of formal logics.

Third, $\mathbf{ND}_{\mathrm{FDE}}$ has conjunction elimination rules which have the same forms as disjunction elimination rules. Standardly, the disjunction elimination rule includes subproof trees, while the conjunction elimination rule does not. So they have different forms. In $\mathbf{ND}_{\mathrm{FDE}}$, the conjunction elimination rule, $\wedge E_{\vdash}$, has the same form as the disjunction elimination rule, $\vee E_{\dashv}$, and $\wedge E_{\dashv}$ does the same as $\vee E_{\vdash}$. Thus the elimination rules for conjunction and disjunction are dual. This feature of the system, therefore, provides symmetric proofs which capture the semantics in a natural way without any technical complications.

Finally, our definition of validity may be extended to capture more general consequence relations as follows.

Definition 6.1 Let $\Pi, \Gamma, \Sigma, \Delta \subseteq \mathcal{F}$ and $\varphi \in \mathcal{F}$. Then

(i) $\Pi; \Gamma; \Sigma; \Delta \models \varphi; \Box; \Box; \Box \iff$ For all $\mathcal{M}$: if $1 \in v_{\mathcal{M}}(\Pi)$ and $0 \in v_{\mathcal{M}}(\Gamma)$ and $1 \notin v_{\mathcal{M}}(\Sigma)$ and $0 \notin v_{\mathcal{M}}(\Delta)$, then $1 \in v_{\mathcal{M}}(\varphi)$,

(ii) $\Pi; \Gamma; \Sigma; \Delta \models \Box; \varphi; \Box; \Box \iff$ For all $\mathcal{M}$: if $1 \in v_{\mathcal{M}}(\Pi)$ and $0 \in v_{\mathcal{M}}(\Gamma)$ and $1 \notin v_{\mathcal{M}}(\Sigma)$ and $0 \notin v_{\mathcal{M}}(\Delta)$, then $0 \in v_{\mathcal{M}}(\varphi)$,

(iii) $\Pi; \Gamma; \Sigma; \Delta \models \Box; \Box; \varphi; \Box \iff$ For all $\mathcal{M}$: if $1 \in v_{\mathcal{M}}(\Pi)$ and $0 \in v_{\mathcal{M}}(\Gamma)$ and $1 \notin v_{\mathcal{M}}(\Sigma)$ and $0 \notin v_{\mathcal{M}}(\Delta)$, then $1 \notin v_{\mathcal{M}}(\varphi)$,

(iv) $\Pi; \Gamma; \Sigma; \Delta \models \Box; \Box; \Box; \varphi \iff$ For all $\mathcal{M}$: if $1 \in v_{\mathcal{M}}(\Pi)$ and $0 \in v_{\mathcal{M}}(\Gamma)$ and $1 \notin v_{\mathcal{M}}(\Sigma)$ and $0 \notin v_{\mathcal{M}}(\Delta)$, then $0 \notin v_{\mathcal{M}}(\varphi)$.

Proof-theoretical characterizations of the above consequence relations have yet to be investigated. However, it does not seem impossible to give a proof theory in the style of Konikowska [6]. Moreover, these general consequence relations may be studied in the context of many logics other than *FDE* as well.

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NOTES

1. Or *Relevance Logics*. '*Relevant* logics' is often preferred by Australian relevan*t* logicians. '*Relevance* logics', on the other hand, is preferred by American relevan*ce* logicians.

2. The paper is included in Anderson, Belnap, and Dunn [3] which is the second volume of Anderson and Belnap [2].

3. See Priest and Sylvan [10].

4. Some *Paraconsistent Logics* are developed based on this idea. Unsurprisingly, *FDE* is often considered to be a paraconsistent logic, as well as a relevant logic.

5. Brentano writes: "Da jedem Urteil eine Vorstellung zugrunde liegt, so wird die Aussage als Ausdruck des Urteils notwendig einen Namen enthalten. Dazu wird aber noch ein anderes Zeichen kommen müssen, das demjenigen inneren Zustand entspricht, den wir eben Urteilen nennen, d.h. ein Zeichen, das den bloßen Namen zum Satz ergänzt. Und da dieses Urteilen von doppelter Art sein kann, nämlich ein Anerkennen oder Verwerfen, so wird auch das Zeichen dafür ein doppeltes sein müssen, eines für die Bejahung und eines für die Verneinung. Für sich allein bedeuten diese Zeichen nichts [...], aber in Verbindung mit einem Namen sind sie Ausdruck eines Urteils. Das allgemeinste Schema der Aussage lautet daher: *A* ist ($A\,+$) und *A* ist nicht ($A\,-$)." ([4], pp. 97–98.)

6. Łukasiewicz writes: "I wish to assert truth and only truth, and to reject falsehood and only falsehood." (Łukasiewicz [7], p. 91)

7. For a synopsis of the history of theories of rejection for *CPL*, the reader may have recourse to Tamminga [13].

8. For a discussion of this feature of combined systems, see [13].

9. Standardly, validity for *FDE* is defined as in classical logic as follows:

$$\Pi \models \varphi \quad \iff \quad \text{For all } \mathcal{M}: \text{ if } 1 \in \nu_{\mathcal{M}}(\Pi), \text{ then } 1 \in \nu_{\mathcal{M}}(\varphi).$$

10. After the development of Dunn's semantics, the history of *FDE* is largely anecdotal. For this reason, it is uncertain whether the system provided by Priest will be the first. However, there do not seem to be any published papers that introduce natural deduction systems for *FDE*. This claim was suggested in conversations with Dunn and Priest.

11. The notational conventions used here are a slight modification of those of Troelstra and Schwichtenberg [15].

12. A completeness proof for a classical natural deduction system can be found in Tennant [14].

13. Smullyan [12] shows a similar result for a classical tableaux system.

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ILLC/Department of Philosophy
University of Amsterdam
Nieuwe Doelenstraat 15
1012 CP Amsterdam
THE NETHERLANDS
email: tamminga@hum.uva.nl

Department of Philosophy
University of Queensland
St. Lucia Brisbane QLD 4072
AUSTRALIA
email: bluesky@arts.uq.edu.au