Notre Dame Journal of Formal Logic
Volume 36, Number 4, Fall 1995

Automorphisms of Countable Recursively Saturated Models of PA: a Survey

HENRYK KOTLARSKI

Abstract We give a survey of automorphisms of countable recursively saturated models of Peano Arithmetic.

Let me begin with the pre-history of the subject. The question whether PA has a model with a nontrivial automorphism (i.e., such that its elements cannot be "individualized") was due to Hasenjäger. It was solved positively by Ehrenfeucht and Mostowski [3]. Their result and the idea of indiscernibility is nowadays so well known that Hodges [5] writes "*today model theorists use it at least once a week*," so let me omit the statement of the Ehrenfeucht-Mostowski Theorem.

Another result I would like to put to the pre-history of the subject is

Theorem 1 (The Kueker-Reyes Theorem [23]) *Let $\mathcal{M}$ be a countable homogeneous model and $X \subseteq \mathcal{M}$. Assume that for every finite sequence $\bar{a}$ of elements of $\mathcal{M}$ there exist $b_1 \in X$ and $b_2 \notin X$ so that $(\mathcal{M}; \bar{a}, b_1) \simeq (\mathcal{M}; \bar{a}, b_2)$. Then X has continuum many automorphic images, i.e., $\{Y \subseteq \mathcal{M} : \exists g \in \mathrm{Aut}(\mathcal{M}) \quad Y = g * X\}$ is of power $2^{\aleph_0}$.*

In the early seventies Ressayre [27] and independently Barwise and Schlipf [2] introduced the notion of recursive saturation. Barwise and Schlipf used it to show that some weak system of *Second Order Arithmetic* (namely Δ^1_1-comprehension $+\Sigma^1_1$-choice scheme) is in fact a conservative extension of PA.

So let me state the definition.

Definition 2 A model $\mathcal{M}$ is *recursively saturated* iff for every recursive sequence $\varphi_0(v_0, \dots, v_{n-1}, v_n), \dots, \varphi_m(v_0, \dots, v_{n-1}, v_n), \dots$ and every sequence $b_0, \dots, b_{n-1}$ of elements of $\mathcal{M}$, if for every k

$$\mathcal{M} \models \exists x \bigwedge_{m \le k} \varphi_m(b_0, \dots, b_{n-1}, x)$$

then there exists $a \in \mathcal{M}$ so that for every m

$$\mathcal{M} \models \varphi_m(b_0, \dots, b_{n-1}, a).$$

Thus, this is the usual notion of saturation (to be more specific of ω-saturation because only finitely many parameters are allowed) but restricted to *recursive* types. I shall assume some familiarity with recursively saturated models (see Smoryński's survey [30] or Kaye's book [7]). Let me state only one nontrivial result.

Theorem 3 (The Barwise-Ressayre Theorem) *Let $\mathcal{M}$ be a countable and recursively saturated structure for the language L. Let $a_0, \ldots, a_{m-1}$ be elements of $\mathcal{M}$ and let T be a recursively enumerable theory in the language of the form $L \cup \{a_0, \ldots, a_{m-1}\} \cup \{X_0, \ldots, X_{r-1}\}$, where $X_0, \ldots, X_{r-1}$ are new relation symbols. Assume that $T \cup Th(\mathcal{M})_{\{a_0,\ldots,a_{m-1}\}}$ is consistent. Then there exist $A_0, \ldots, A_{r-1}$, which are interpretations of $X_0, \ldots, X_{r-1}$ respectively, so that $(\mathcal{M}, a_0, \ldots, a_{m-1}, A_0, \ldots, A_{r-1}) \models T$. Moreover, $A_0, \ldots, A_{r-1}$ may be chosen in such a way that the above-mentioned structure is recursively saturated.*

The property of countable recursively saturated models stated in the first part of this result is known as *resplendence*. The property stated in the moreover clause is known as *chronical resplendence*.

Schlipf [28] worked out applications of the notion of recursive saturation and resplendence in Model Theory eliminating the use of saturated models (whose existence needs some form of Generalized Continuum Hypothesis) or special models (whose existence is provable in *ZFC* set theory, but they are of quite large cardinality). This was possible because of

Theorem 4 (Barwise and Schlipf [2]) *Every structure $\mathcal{M}$ has an elementary extension of the same cardinality which is recursively saturated.*

Schlipf (in the same paper [28]) also showed

Theorem 5 *Every countable recursively saturated structure $\mathcal{M}$ has $2^{\aleph_0}$ automorphisms.*

Here and below $\mathrm{Aut}(\mathcal{M})$ denotes the group of all automorphisms of the structure $\mathcal{M}$. Schlipf derived the above result from the Kueker-Reyes Theorem and the Barwise-Ressayre Theorem.

All the results up to now, including the existence theorem for recursively saturated models were true for models for any theory. After the celebrated work on models of PA, done in mid-seventies, mainly by Kirby and Paris, it was soon noticed that countable recursively saturated models for this theory arise very naturally. Roughly speaking each construction of a model for PA either gives directly a recursively saturated model (for example the Arithmetized Completeness Theorem) or an inessential variant of the construction yields a recursively saturated model (e.g., the indicator construction). Thus countable recursively saturated models of PA are natural objects to study, and so are their automorphism groups. The obvious problems here are: (i) how does $\mathrm{Aut}(\mathcal{M})$ act on $\mathcal{M}$, and (ii) what are the properties of $\mathrm{Aut}(\mathcal{M})$ (either as an abstract group or as a topological group). I shall survey below all known results following the order of appearance of papers. This will show the reader the evolution of ideas.

Before stating the results we would like to point out that automorphisms groups of nonrecursively saturated models behave in a different manner. Thus Gaifman [4] proved that every group of the form $\mathrm{Aut}(X, <)$, where $(X, <)$ is a linear ordering, is isomorphic with Aut($\mathcal{M}$) for some $\mathcal{M} \models$ PA. In particular, for some model $\mathcal{M}$ of PA, Aut($\mathcal{M}$) is isomorphic with the group of order preserving permutations of rationals. We shall see below (Theorem 21) that such $\mathcal{M}$ cannot be countable recursively saturated.

At about 1980 Smoryński and I worked on automorphisms of countable recursively saturated models of PA. The first result I would like to mention is a condition which allows one to construct an automorphism of $\mathcal{M}$ which fixes (pointwise) the set $\{x \in \mathcal{M} : x < a\}$, a set of nonstandard finite cardinality.

Lemma 6 (Kotlarski [19], Smoryński [31], Vencovska [unpublished]) *Let $\mathcal{M}$ be a countable recursively saturated model of* PA. *Let $a, b, c, d \in \mathcal{M}$ be such that*

1. $\mathcal{M} \models a < b,\ 2^a < b,\ 2^{2^a} < b, \ldots$
2. *For every formula φ* $\quad \mathcal{M} \models \forall x < b[\varphi(x, c) \Leftrightarrow \varphi(x, d)]$.

Then there exists $g \in \mathrm{Aut}(\mathcal{M})$ so that $gc = d$ and $\forall x < a\ gx = x$.

Smoryński's application was

Theorem 7 ([31]) *Let $\mathcal{M}$ be a countable recursively saturated model for* PA. *Let I be an initial segment of $\mathcal{M}$ which is closed under exponentiation. Then there exists $g \in \mathrm{Aut}(\mathcal{M})$ so that $g \restriction I = \mathrm{id}$ and $\forall b > I\ \exists c < b\ gc \neq c$.*

It is convenient to denote by $I_{\mathrm{fix}}(g)$ the set $\{x \in \mathcal{M} : \forall y < x\, g(y) = y\}$. Thus, Smoryński's result gives an exact condition for a cut I to be $I_{\mathrm{fix}}(g)$ for some $g \in \mathrm{Aut}(\mathcal{M})$.

My paper was published only in 1984. In order to state the results, let us define a subset $X \subseteq \mathcal{M}$, where $\mathcal{M} \models$ PA to be *closed* iff for every $b \in \mathcal{M} \setminus X$ there exists $g \in \mathrm{Aut}(\mathcal{M})$ with $gb \neq b$ and $g \restriction X = id$. Thus, this is the usual notion of a closed subset in the sense of the Galois theory.

Theorem 8 ([19]) *If X is an initial segment of a countable recursively saturated $\mathcal{M} \models$ PA and X is not closed then there exists $b \in \mathcal{M}$ so that $X = \mathcal{M}[b]$, where*

$$\mathcal{M}[b] = \{a \in \mathcal{M} : \text{ for every Skolem term } t(v)\ \mathcal{M} \models t(a) < b\}.$$

(Treat this notion as undefined if some $c > b$ is definable).

In order to see what this result means, observe at first that if $X \subseteq \mathcal{M}$ is closed then X is the universe of an elementary submodel of $\mathcal{M}$. Moreover, put, for $b \in \mathcal{M}$

$$\mathcal{M}(b) = \{a \in \mathcal{M} : \text{ there exists a Skolem term } t(v)\ \ \mathcal{M} \models a < t(b)\}.$$

Then it is easy to see that $\mathcal{M}(b)$ is the smallest elementary cut of $\mathcal{M}$ containing a. Also, $\mathcal{M}[a]$ is the greatest elementary cut of $\mathcal{M}$ which does not contain a. That is, there is no elementary cut of $\mathcal{M}$ between them. This justifies the name "gap" for their difference, i.e., we put

$$\mathrm{gap}(a) = \mathcal{M}(a) \setminus \mathcal{M}[a]$$

and call this set the *gap* around a. It is also easy to check that if $\mathcal{M}$ is a countable recursively saturated model of PA then

1. $\{\mathcal{M}(b) : b \in \mathcal{M}\}$ is ordered by inclusion in the order type 1+rationals.

2. The set of all elementary cuts of $\mathcal{M}$ is ordered in the order type of the Cantor set.

Thus, $\mathcal{M}$ has $2^{\aleph_0}$ elementary cuts, and all but countably many of them *are* closed because there are only countably many cuts of the form $\mathcal{M}[b]$.

The natural question whether cuts of the form $\mathcal{M}[b]$ are closed is settled in the following way: both may happen.

Theorem 9 ([19]) *There exists a recursive type $q(\cdot)$ such that for every $\mathcal{M} \models \mathrm{PA}$ and b realizing q in $\mathcal{M}$, $\mathcal{M}[b]$ is not closed.*

Theorem 10 ([19]) *There exists a recursive type $p(\cdot)$ such that for every recursively saturated $\mathcal{M} \models \mathrm{PA}$ and every b realizing p in $\mathcal{M}$, $\mathcal{M}[b]$ is closed.*

Note 11 *Theorem 9 may be proved using Gaifman's minimal types and hence a recursively saturated $\mathcal{M}$ has infinitely many essentially different nonclosed elementary cuts. It was recently verified (Piekart [26]) that under the same assumption $\mathcal{M}$ has also infinitely many essentially different closed elementary cuts.*

One more result should be mentioned in this place. Namely for $b \in \mathcal{M}$, $b > \mathcal{M}(0)$ we defined above

$$\mathrm{gap}(b) = \mathcal{M}(b) \setminus \mathcal{M}[b]$$

the *gap* around b. As pointed out above, the name is justified by the fact that sets of this form correspond to gaps in the Cantor set $\{\mathcal{N} : \mathcal{N} \prec \mathcal{M}$ is an initial segment$\}$. The result I have in mind is the following (I knew it at least in 1985, but it is published for the first time in §3 of Kaye, Kossak, Kotlarski [10].)

Theorem 12 (The Moving Gaps Lemma) *Let $\mathcal{M}$ be a countable recursively saturated model of* PA. *Let $a, b, c \in \mathcal{M}$ be such that $\mathcal{M}(a) < \mathcal{M}(b) < \mathcal{M}(c)$ and let $g \in \mathrm{Aut}(\mathcal{M})$ be such that $ga \neq a$. Then there exist $u, w \in \mathcal{M}$ with $\mathcal{M}(b) < \mathcal{M}(u) < \mathcal{M}(w) < \mathcal{M}(c)$ and either $gu > w$ or $gw < u$.*

This result has a clear topological meaning. It is as follows. If $ga \neq a$ then $\{\mathcal{N} \prec_{\mathrm{end}} \mathcal{M} : g * \mathcal{N} = \mathcal{N} \wedge a < \mathcal{N}\}$ is nowhere dense in the Cantor set mentioned above. In particular every nontrivial automorphism must move some gap, i.e., there exists $b \in \mathcal{M}$ so that $\mathrm{gap}(b) \neq g * \mathrm{gap}(b)$. Also: every nontrivial $g \in \mathrm{Aut}(\mathcal{M})$ must move (setwise) some elementary cut of $\mathcal{M}$. In many respects, for a countable recursively saturated $\mathcal{M}$, $\mathrm{Aut}(\mathcal{M})$ acts on $\mathcal{M}$ like $\mathrm{Aut}(\mathbb{Q}, <)$ acts on the ordering $(\mathbb{Q}, <)$ of rationals, the moving gaps lemma shows a drastic difference.

The next paper I would like to mention is Schmerl [29]. The result is as follows.

Theorem 13 ([29]) *Let $\mathcal{M}$ be a countable and recursively saturated model of* PA. *Then $\mathcal{M}$ is generated by a set of order indiscernibles. In fact, if $(X, <)$ is any given countable linear order with no greatest element then $\mathcal{M}$ is generated by a set of order indiscernibles of order type of $(X, <)$.*

We obtain the following as an immediate corollary.

Corollary 14 *If $\mathcal{M}$ is a countable and recursively saturated model of* PA *then the group* $\mathrm{Aut}(\mathbb{Q}, <)$ *is embeddable in* $\mathrm{Aut}(\mathcal{M})$.

Schmerl's proof of Theorem 13 heavily depends on the combinatorial theorem due to Abramson-Harrington [1] (earlier this result was obtained by Nešetríl and Rödl). The more familiar constructions, also the ones involving minimal types (cf. [4]) do not yield a recursively saturated model.

In order to state the next result we need a definition (Kirby [11]). If $\mathcal{M} \models PA$ and $I \subset_{\text{end}} \mathcal{M}$ we say that $\mathbb{N} \downarrow$ *codes* I *in* $\mathcal{M}$ iff there exists a function $f \in \mathcal{M}$ such that $I = \{x \in \mathcal{M} : \forall n \in \mathbb{N} \;\; x < f(n)\}$.

Theorem 15 (Kossak, Kotlarski [14]) *Let $\mathcal{M}$ be a countable recursively saturated model for* PA *and let $J \prec_{\text{end}} \mathcal{M}$. Assume $\mathbb{N}$ does not $\downarrow$ code J in $\mathcal{M}$. Then for $g \in \text{Aut}(J)$, g is extendable to some $\hat{g} \in \text{Aut}(\mathcal{M})$ iff for every $X \subseteq J$ if $X = J \cap b$ for some $b \in \mathcal{M}$ then $g * X, g^{-1} * X$ are also both of the form $J \cap b$ for some $b \in \mathcal{M}$ (i.e., g, g^{-1} send subsets of J coded in $\mathcal{M}$ onto sets that are still coded in $\mathcal{M}$). Here by $J \cap b$ we denote the intersection of J with the set (coded by) b.*

Note 16 *Recently Kossak [15] showed that the special assumption $\mathcal{M}$ does not $\downarrow$ code J in $\mathcal{M}$ is necessary. But of course only countably many $J \prec_{\text{end}} \mathcal{M}$ are $\downarrow$ coded in $\mathcal{M}$ by $\mathbb{N}$ (because there are only countably many possibilities for the coding function).*

For many years it was not known whether $\text{Aut}(\mathcal{M})$ depended on $\mathcal{M}$ at all (provided $\mathcal{M}$ is a countable recursively saturated model of PA). Kaye [8] attacked this problem, the problem of recovering at least some information about $\mathcal{M}$ from $\text{Aut}(\mathcal{M})$. His result is the following theorem.

Theorem 17 (Kaye [8]) *Let $\mathcal{M}, \mathcal{N}$ be countable recursively saturated models for the same complete extension of* PA. *Then $\mathcal{M}, \mathcal{N}$ are isomorphic iff there exists an isomorphism $j : \text{Aut}(\mathcal{M}) \to \text{Aut}(\mathcal{N})$ such that for every $f, g \in \text{Aut}(\mathcal{M})$, $(\mathcal{M}, f, g)$ is recursively saturated iff $(\mathcal{N}, jf, jg)$ is recursively saturated.*

This was the first result in this direction, nowadays a much stronger result (due to Kossak and Schmerl) is known, we shall speak about it later.

Another result of [8] which is interesting is as follows.

Theorem 18 (Kaye [8]) *If $\mathcal{M}$ is a countable and recursively saturated structure then F_ω, the free group with $\aleph_0$ generators, is embeddable in $\text{Aut}(\mathcal{M})$ as a dense subset.*

The topology for $\text{Aut}(\mathcal{M})$ is defined like in other algebraic considerations. A subbasis is $\{U_a^b : a, b \in \mathcal{M}\}$, where $U_a^b = \{g \in \text{Aut}(\mathcal{M}) : ga = b\}$. In the case of models of PA this family is in fact a basis (because of the pairing function in PA).

Theorem 18 is an extension of a result of Macpherson [25].

The next paper I would like to mention in this survey is due to Kossak and Schmerl [17]. They proved the following theorem.

Theorem 19 *Let $\mathcal{M} \models$ PA be countable and recursively saturated. Then there exists a countable recursively saturated $\mathcal{N} \succ_{\text{end}} \mathcal{M}$ such that id is the only element of $\text{Aut}(\mathcal{M})$ which extends to some element of $\text{Aut}(\mathcal{N})$.*

The proof of Theorem 19 is based on the notion of a *minimal* satisfaction class, i.e., such satisfaction class S that $(\mathcal{M}, S)$ is pointwise definable.

The next paper I would like to mention here is Kaye, Kossak, Kotlarski [10], where several results are proved. The first is the moving gaps lemma, i.e., Theorem 12 above.

As an immediate corollary we get the following theorem.

Theorem 20 *If $\mathcal{M}$ is a countable recursively saturated model of* PA *then* $\mathrm{Aut}(\mathcal{M})$ *is embeddable into* $\mathrm{Aut}(\mathbb{Q}, <)$ *as a dense subset.*

The topology for $\mathrm{Aut}(\mathbb{Q}, <)$ is defined exactly as for $\mathrm{Aut}(\mathcal{M})$ as above. The idea of the proof of Theorem 20 is as follows. Pick a minimal type $q(\cdot)$ realized in $\mathcal{M}$ (see [4] for more in this direction) and let $q^{\mathcal{M}} = \{b \in \mathcal{M} : b \text{ realizes } q\}$. It is easy to check that $q^{\mathcal{M}}$ is ordered in the order type of $(\mathbb{Q}, <)$. Then for $g \in \mathrm{Aut}(\mathcal{M})$ we put $\hat{g} = g \restriction q^{\mathcal{M}}$. Again it is easy to check that $\hat{g} \in \mathrm{Aut}(q^{\mathcal{M}}, <)$ and the function $g \mapsto \hat{g}$ is a group homomorphism. It follows from the moving gaps lemma that it is one-to-one. Finally, minimality of q ensures that the image is dense, because by [4] every minimal type is indiscernible.

Thus, Theorem 20 and Corollary 14 suggest that $\mathrm{Aut}(\mathcal{M})$ and $\mathrm{Aut}(\mathbb{Q}, <)$ being mutually embeddable should be isomorphic. This is not so:

Theorem 21 (Lascar, [10]) *If $\mathcal{M} \models$* PA *is countable and recursively saturated then* $\mathrm{Aut}(\mathcal{M})$ *is not isomorphic to* $\mathrm{Aut}(\mathbb{Q}, <)$.

The property distinguishing these groups is:

$$\text{for every open } H < \mathrm{Aut}(\mathbb{Q}, <), \{K < \mathrm{Aut}(\mathbb{Q}, <) : H < K\} \text{ is finite.}$$

Then $\mathrm{Aut}(\mathcal{M})$ fails to have this property. In order to eliminate topology one uses the result due to Truss [32]: $\mathrm{Aut}(\mathbb{Q}, <)$ has the *small index property*, (i.e., for every $H < \mathrm{Aut}(\mathbb{Q}, <)$, H is open iff the index $[\mathrm{Aut}(\mathbb{Q}, <) : H]$ is countable). A small trick allows one to use the small index property of $\mathrm{Aut}(\mathbb{Q}, <)$ here, this property of $\mathrm{Aut}(\mathcal{M})$ is not needed in this place.

The first result showing that $\mathrm{Aut}(\mathcal{M})$, as a topological group, depends on $\mathcal{M}$, is due to Kaye. In order to state it we need a definition (due to Kirby [11]). A cut J of $\mathcal{M}$ is *strong* in $\mathcal{M}$ iff for every function $f \in \mathcal{M}$ with $J \subseteq \mathrm{Dom}(f)$ there exists $u \in \mathcal{M} \setminus J$ such that $\forall a \in J \;\; f(a) > J \Rightarrow \;\; f(a) > u$. It is also known from [11] that there exist (countable recursively saturated) models in which $\mathbb{N}$ is strong and models in which it is not strong.

Theorem 22 (Kaye, [10]) *Let $\mathcal{M}$ be a countable recursively saturated model of* PA. *Then $\mathbb{N}$ is strong in $\mathcal{M}$ iff there exists $g \in \mathrm{Aut}(\mathcal{M})$ and an open subgroup H of* $\mathrm{Aut}(\mathcal{M})$ *such that*

$$\forall f \in \mathrm{Aut}(\mathcal{M}) \quad f^{-1} \circ g \circ f \notin H.$$

This result shows that $\mathrm{Aut}(\mathcal{M})$ as a topological group depends on $\mathcal{M}$. Later we shall see that the topology can be eliminated in this place.

Kaye derived Theorem 22 from a result which connects the strength of $\mathbb{N}$ in $\mathcal{M}$ with the existence of $g \in \mathrm{Aut}(\mathcal{M})$ with few fixpoints. This connection was obtained, in three different forms, by Kaye in Oxford, Kossak and me in Warsaw and by Schmerl in Storrs. Rather than stating this result in the most general form (we would need some definitions to do this), I shall state several related results.

Theorem 23 ([10]) *Let $\mathcal{M} \models \mathrm{PA}$ be countable and recursively saturated. Then the following are equivalent:*

1. *$\mathbb{N}$ is strong in $\mathcal{M}$*
2. *there exists $g \in \mathrm{Aut}(\mathcal{M})$ which moves all undefinable elements of $\mathcal{M}$*
3. *there exists $g \in \mathrm{Aut}(\mathcal{M})$ and a type q which is realized in $\mathcal{M}$ and g moves all elements realizing q*
4. *for some $a \in \mathcal{M}$ there exists $g \in \mathrm{Aut}(\mathcal{M})$ with $ga = a$ and g moves all elements undefinable from a*
5. *for all $a \in \mathcal{M}$ there exists $g \in \mathrm{Aut}(\mathcal{M})$ with $ga = a$ and g moves all elements undefinable from a.*

Let me derive Theorem 22 from Theorem 23. Assume that $\mathbb{N}$ is strong in $\mathcal{M}$. Pick $a \in \mathcal{M}$, a undefinable. By Theorem 23, point 2, pick $g \in \mathrm{Aut}(\mathcal{M})$ which moves all undefinable elements of $\mathcal{M}$. Let also $H = \mathrm{Aut}(\mathcal{M})_a = \{f \in \mathrm{Aut}(\mathcal{M}) : fa = a\}$. I claim that g, H have the desired property. Indeed, let $f \in \mathrm{Aut}(\mathcal{M})$ be given. Put $b = fa$. Then b is undefinable (bacause a is) and hence $gb \neq b$. Thus $f^{-1}gb \neq f^{-1}b = a$, so $f^{-1}gf \notin H$.

For the converse assume that $\mathbb{N}$ is not strong in $\mathcal{M}$. Pick $g \in \mathrm{Aut}(\mathcal{M})$ and and any open $H \subseteq \mathrm{Aut}(\mathcal{M})$. Pick a with $\mathrm{Aut}(\mathcal{M})_a \subseteq H$ (it is easy to see that every open subgroup H of $\mathrm{Aut}(\mathcal{M})$ must contain a subgroup of this form, a *basic* subgroup). By Theorem 23, point 3, g fixes some b with $\mathrm{tp}(b) = \mathrm{tp}(a)$. By homogeneity of $\mathcal{M}$, there exists $f \in \mathrm{Aut}(\mathcal{M})$ with $fa = b$. Then $f^{-1}gfa = f^{-1}b = a$, so $f^{-1}gf \in \mathrm{Aut}(\mathcal{M})_a \subseteq H$.

Theorem 23 as stated admits some generalizations and variants.

Theorem 24 ([10]) *If $\mathcal{M} \models \mathrm{PA}$ is countable and recursively saturated, $I \prec_{\mathrm{end}} \mathcal{M}$, then I is strong in $\mathcal{M}$ iff there exists*

$$g \in \mathrm{Aut}(\mathcal{M}) \text{ with } \forall b \in \mathcal{M} \;\; gb = b \text{ iff } b \in I.$$

Further generalizations of Theorem 23 (and Theorem 15) are due to Kossak [12]. In order to state these results we need a definition. Let $\mathcal{M} \models \mathrm{PA}$ and $f \in \mathrm{Aut}(\mathcal{M})$. For $b \in \mathcal{M}$, the set

$$(\inf\{f^n b : n \in \mathbb{Z}\},\ \sup\{f^n b : b \in \mathbb{Z}\}),$$

is called the f-interval (around b). Observe that if $f \in \mathrm{Aut}(\mathcal{M})$ is such that $(\mathcal{M}, f)$ is recursively saturated then there are infinitely many f-intervals.

Theorem 25 ([12]) *If $\mathbb{N}$ is strong in a countable and recursively saturated $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$ then there exists $f \in \mathrm{Aut}(\mathcal{M})$ such that $\mathcal{M}$ is the union of $\mathbb{N}$ and one f-interval.*

This result admits further generalization.

Theorem 26 ([12]) *Let $\mathcal{M} \models \mathrm{PA}$ be countable and recursively saturated. Let $I \prec_{\mathrm{end}} \mathcal{M}$ and let $g \in \mathrm{Aut}(I)$. Assume that I is strong in $\mathcal{M}$. Suppose also that g, g^{-1} send coded subsets of I onto coded subsets. Then g extends to $f \in \mathrm{Aut}(\mathcal{M})$ such that $\mathcal{M} \setminus I$ is an f-interval.*

Let me tell about more recent work on this topic. A very substantial result is due to Kaye [9]. He describes all closed normal subgroups of $\mathrm{Aut}(\mathcal{M})$.

A cut I of $\mathcal{M}$ is called *invariant* iff every $g \in \mathrm{Aut}(\mathcal{M})$ fixes I setwise. It is easy to check that if $\mathcal{M}$ is countable and recursively saturated then I is invariant iff at least one of

$$I = \sup\{b_n : n \in \mathbb{N}\}$$

or

$$I = \inf\{b_n : n \in \mathbb{N}\}$$

for some (not necessarily coded) sequence b_n of definable elements. Moreover it is easy to check that if $H = \mathrm{Aut}(\mathcal{M})_{(I)}$, the pointwise stabilizer of I, then H is normal in $\mathrm{Aut}(\mathcal{M})$ iff I is invariant. Also obviously each pointwise stabilizer is closed.

Theorem 27 (Kaye [9]) *If H is a closed normal subgroup of* $\mathrm{Aut}(\mathcal{M})$, *where $\mathcal{M}$ is a countable recursively saturated model of* PA *then $H = \mathrm{Aut}(\mathcal{M})_{(I)}$ for some invariant $I \subseteq_{\mathrm{end}} \mathcal{M}$.*

Corollary 28 *Under the assumption of Theorem 27, $\mathcal{M} \models$ True Arithmetic,* $\mathrm{Th}(\mathbb{N})$, *iff Aut($\mathcal{M}$) has only two closed normal subgroups (the trivial ones).*

The idea of Kaye's proof of Theorem 27 is as follows. Given a normal subgroup H of $\mathrm{Aut}(\mathcal{M})$, we define

$$I_{\mathrm{fix}}(H) = \sup\{a \in \mathcal{M} : \forall x < a \forall h \in H\, h(x) = x\}$$

and then a lot of work is needed to show that this cut has the desired property. Corollary 28 follows at once from the remark that a model of true arithmetic has only trivial invariant cuts, contrary to models with nonstandard definable elements.

Another major result which follows is due to Lascar [24].

Theorem 29 ([24]) *If $\mathbb{N}$ is strong in a countable recursively saturated model $\mathcal{M}$ of* PA *then* $\mathrm{Aut}(\mathcal{M})$ *has the small index property.*

Corollary 30 *If $\mathcal{M}$, $\mathcal{N}$ are countable recursively saturated models of* PA *and $\mathbb{N}$ is strong in one of them but not in the second then* $\mathrm{Aut}(\mathcal{M})$ *and Aut($\mathcal{N}$) are not isomorphic as abstract groups.*

This corollary was a solution of an outstanding open problem. It shows also that models given by Corollary 22 have no isomorphic automorphism groups (as abstract groups).

The heart of the matter in the proof of Lascar's result is as follows. At first there is no reason for $\mathrm{Aut}(\mathcal{M})$ to be a compact group. Nevertheless it is a topological group with a good property, namely it is a Baire space (this remark is just a reformulation of the 'back and forth' method of constructing automorphisms). Say that $f \in \mathrm{Aut}(\mathcal{M})$ is *generic* iff the set of its conjugates is comeager in $\mathrm{Aut}(\mathcal{M})$. Lascar proves that if $\mathbb{N}$ is strong in $\mathcal{M}$ then generic automorphisms do exist. As a matter of fact he works with generic tuples of automorphisms, indeed, $\mathrm{Aut}(\mathcal{M})^n$ is also a Baire space for each natural number n. Then he refers to Hodges et al. [6].

By the way, it is easy to check that under our usual assumption on $\mathcal{M}$ almost all automorphisms move elements which are arbitrarily low above $\mathbb{N}$. More precisely, $\{g \in \mathrm{Aut}(\mathcal{M}) : \exists b > \mathbb{N}\, \forall x < b\; gx = x\}$ is of the first category in $\mathrm{Aut}(\mathcal{M})$.

The next paper we shall speak about is Kossak, Kotlarski, Schmerl [16]. Let me say something about its content.

Call a subgroup H of G *strongly maximal* iff for all $f, g \in G \setminus H$ there exist $\alpha, \beta \in H$ so that $f = \alpha \circ g^{-1} \circ \beta$ or $f = \alpha \circ g \circ \beta$. (In other words, $H \cup \{f\}$ generates G in one step for $f \notin H$. In group theoretic terminology, the double coset index of H in G is 3).

It was noticed in [10] that under our usual assumption on $\mathcal{M}$, $\mathrm{Aut}(\mathcal{M})$ has open strongly maximal subgroups. In [16] we show that $\mathrm{Aut}(\mathcal{M})$ has basic strongly maximal subgroups, basic maximal but not strongly maximal subgroups. Moreover we show that if I is an elementary cut of M which is of the form $I_b = \sup\{b_n : n \in \mathbb{N}\}$, where $b \in \mathcal{M}$ is *quickly increasing* (i.e., for every n and every Skolem term $t(\cdot)$, $t(b_n) < b_{n+1}$), then the setwise stabilizer of I is strongly maximal. Similarly for cuts of the form $I^b = \inf\{b_n : n \in \mathbb{N}\}$ with quickly decreasing b. Moreover we study another class of open subgroups of $\mathrm{Aut}(\mathcal{M})$, namely *gap stabilizers*, that is groups of the form $\mathrm{Aut}(\mathcal{M})_{\{\mathrm{gap}(a)\}}$. We show that such subgroups may be maximal and may be not maximal. (It is not known if subgroups of this form may be strongly maximal; it seems possible to construct one which is maximal but not strongly maximal). Finally we show that if $\mathbb{N}$ is strong in $\mathcal{M}$ then every open $H \subset \mathrm{Aut}(\mathcal{M})$ extends to a maximal one. (To be more precise, we proved it under an additional assumption that $\mathcal{M}$ satisfies *True Arithmetic*, Lascar eliminated this assumption). Another result of [16] which should be mentioned here is: if q is an unbounded and 2-indiscernible type then q is Gaifman-minimal (this was known before for unbounded 4-indiscernible types).

The notion of a strongly maximal subgroup of $\mathrm{Aut}(\mathcal{M})$ has demonstrated its importance in a paper [20] by Kaye and me, where we use it to identify (by topological-group theoretic means) basic subgroups among maximal open ones, in the case when $\mathcal{M}$ is countable recursively saturated and satisfies *True Arithmetic*. The essential reason lies in the fact that a strongly maximal subgroup of $\mathrm{Aut}(\mathcal{M})$ gives an additional structure on the family of its conjugates, the ordering in the order type of $(\mathbb{Q}, <)$. Moreover (and this is the heart of the matter) this linear ordering turns out to be isomorphic with some subfamily of all elementary cuts of $\mathcal{M}$. This subfamily is rich enough so that its properties are almost the same as of the family of all elementary cuts of $\mathcal{M}$. Granted this we were able to identify gaps stabilizers and then use the Kaye's Closed Normal Subgroup Theorem (i.e., Theorem 27 above) and some "covering a gap" idea to identify subgroups of the form $\mathrm{Aut}_a(\mathcal{M})$, with $\mathrm{tp}(a)$ 2-indiscernible. These ideas worked for models of $\mathrm{Th}(\mathbb{N})$, but the trick of a subgroup *preceded* by G (invented by Kossak and Schmerl in the next paper I shall tell a few words about) allows one to do the same for any (countable recursively saturated) model of PA.

Recently it was shown that if $\mathcal{M} \models$ True Arithmetic and is countable recursively saturated then the lattice of subgroups of $\mathrm{Aut}(\mathcal{M})$ is rather strange. Indeed, in [21] we give examples of open nonmaximal subgroups of $\mathrm{Aut}(\mathcal{M})$ which extend to a maximal one uniquely.

The most important paper in this direction is due to Kossak and Schmerl [18], entitled "The automorphism group of an arithmetically saturated model of Peano arithmetic." They found a way to encode the standard system of $\mathcal{M}$ under the assumption that $\mathcal{M}$ is a countable and recursively saturated model of PA in which $\mathbb{N}$ is strong and obtained the following theorem.

Theorem 31 *If $\mathcal{M}$, $\mathcal{N}$ are countable recursively saturated models of* PA *in which* $\mathbb{N}$ *is strong and have isomorphic automorphism groups and satisfy the same completion of* PA, *then* $\mathcal{M} \simeq \mathcal{N}$.

To be more specific, they encode the standard system of $\mathcal{M}$ by $\mathrm{Aut}(\mathcal{M})$ as a topological group and then apply the Lascar's result, Theorem 29.

As pointed out above, Piekart and I constructed nontrivial open subgroups of $\mathrm{Aut}(\mathcal{M})$ which extend to a maximal one uniquely. In her Ph.D. thesis Piekart strengthens this considerably. If $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$ and $\mathrm{Aut}_a(\mathcal{M})$ is basic open maximal subgroups of $G = \mathrm{Aut}(\mathcal{M})$ then she shows a tree $\mathbb{N}^{<\mathbb{N}}$ of open subroups whose only maximal extension is G_a. She also shows the same result for nonbasic open subgroups of G, which are of the form $G_{\{I^b\}}$ or $G_{\{I_b\}}$.

Kossak and Bamber [13] show that for a countable PA, $\mathrm{Aut}(\mathcal{M})$ is not divisible. It follows that $\mathrm{Aut}(\mathcal{M})$ is not elementary equivalent with $\mathrm{Aut}(\mathbb{Q}, <)$, this strengthens Theorem 21. Also, Bamber shows that all cyclic subgroups of $\mathrm{Aut}(\mathcal{M})$ are closed.

In the new paper [15] by Kossak and me we give some more information about extendability of automorphism to greater models. Thus we show that elementarity is essential in Theorem 19. That is we show that every (not necessarily recursively saturated) model $\mathcal{M}$ of PA has end extensions $\mathcal{K}$ which are recursively saturated models of PA and each automorphism of $\mathcal{M}$ extends to $\mathcal{K}$. Moreover, given $n \in \mathbb{N}$, $\mathcal{K}$ may be chosen to be Σ_n-elementary. Another result of [15] gives a sufficient condition on the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$ which ensures that every automorphism of $\mathcal{M}$, sending coded subsets to coded ones, extends to $\mathcal{K}$. To be more exact, at first there is no problem in defining coded sets in the case of cofinal extensions. That is if we are given $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$, every subset of $\mathcal{M}$ of the form $a \cap \mathcal{M}$, $a \in \mathcal{K}$, is called *coded*. Say that the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$ has the *automorphism extension property* iff for every $g \in \mathrm{Aut}(\mathcal{M})$, such that g, g^{-1} send coded sets to coded ones, g extends to $\mathcal{K}$. The natural question is: what additional assumptions on the extension are needed in order to ensure the automorphism extension property?

Definition 32 An extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the *covering property* iff for every $\gamma \in \mathcal{M}$ there exists a sequence $\langle E_n : n \in \mathbb{N}\rangle$ which is coded in $\mathcal{R}$, is increasing with respect to inclusion and

1. $E_n \in \mathcal{M}$ for all $n \in \mathbb{N}$
2. $\{x \in \mathcal{M} : \mathcal{M} \models x < \gamma\} = \{x \in \mathcal{R} : \exists n \in \mathbb{N}\ \mathcal{R} \models x \in E_n\}$.
3. for every set $e \in \mathcal{R}$ and $n \in \mathbb{N}$, the intersection $e \cap E_n$ is in $\mathcal{M}$.

It is convenient to think of the sequence $\langle E_n\rangle$, the sequence *covering* $(< \gamma)$ in such a way that every standard n, $E_n \subseteq \mathcal{M}$, and for nonstandard j, E_j add no new elements of $\mathcal{M}$ below γ. The last condition may be thought of as some sort of comprehension, also for sets in $\mathcal{R} \setminus \mathcal{M}$. From a more technical point of view the assumption that the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ has the covering property plays the role of the additional assumption ($\mathbb{N}$ does not $\downarrow$ code $\mathcal{M}$ in $\mathcal{R}$) in Theorem 15. To be more exact it is an analogue of "$\mathbb{N}$ codes $\mathcal{M}$ from below in $\mathcal{R}$".

Theorem 33 *Let the extension $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{R}$ have the covering property. Assume that $\mathcal{M}$ is recursively saturated and $\mathcal{R}$ is countable. Then this extension has the automorphisms extension property.*

In particular, it follows that every countable recursively saturated model $\mathcal{M}$ of PA has a countable cofinal extension $\mathcal{K}$ such that every $g \in \mathrm{Aut}(\mathcal{M})$ such that g, g^{-1} send coded subsets to coded ones, extends to $\mathcal{K}$. Indeed, stronger:

Lemma 34 *If $\mathcal{M} \prec_{\mathrm{cof}} \mathcal{K}$ are countable models then there exists a countable $\mathcal{R} \succ_{\mathrm{cof}} \mathcal{K}$ such that the extension $\mathcal{M} \prec \mathcal{R}$ has the covering property.*

Lemma 34 is proved more or less by means of the procedure which allows one to extend a given model to a saturated one, but an auxiliary notion of the *strong covering property* is used.

The final result of [15] was mentioned previously, that is the additional assumption that $\mathbb{N}$ does not $\downarrow$ code $\mathcal{N}$ in $\mathcal{M}$ in the extension theorem (i.e., Theorem 15) is essential.

In the last few months the problem of describing the action of $\mathrm{Aut}(\mathcal{M})$ inside $\mathcal{M}(0)$ was attacked. Let us say a few words about it. Let $a \in \mathcal{M}(0)$ be undefinable. Consider two cuts associated with a:

$$I_a^- = \sup\{u \in \mathcal{M} : u \text{ is definable and } u < a\}$$

and

$$I_a^+ = \inf\{u \in \mathcal{M} : u \text{ is definable and } ua\}.$$

The set difference of these, i.e., $\Omega_a = I_a^+ \setminus I_a^-$ is called the *interstice* around a. This notion was isolated already in [10], the name "interstice" was introduced by Bamber. Granted this, one introduces the notion of an *intersticial gap*, i.e., a gap inside an interstice. This notion has many properties of the usual gaps, provided $\mathbb{N}$ is strong in the model considered. In [10] we gave an erroneous "proof" of the appropriate version of the moving gaps lemma. Recently Bamber gave a sufficient condition on an interstice for the moving gaps lemma to hold and I constructed interstices for which this lemma fails. (Both results require $\mathbb{N}$ to be strong in $\mathcal{M}$.) But it is too early to give more about the action of $\mathrm{Aut}(\mathcal{M})$ inside $\mathcal{M}(0)$. What is expected is to obtain a linear ordering, given by means of $\mathrm{Aut}(\mathcal{M})$, which would allow one to recover the ordering of gaps of $\mathcal{M}$ (both: ordinary gaps above $\mathcal{M}(0)$ and intersticial ones). But trying to extend the material of [20] does not seem to work. The main difficulty in the case of countable recursively saturated models of PA is that in this case subgroups H of $\mathrm{Aut}(\mathcal{M})$ with $J(H)$ invariant exist, as shown by Piekart and me [22].

I would like to stress that in almost every result the countability assumption is essential. Indeed, Kossak and Schmerl [17] constructed ω_1-like recursively saturated *rigid* models of PA.

Let me pose some open problems on automorphisms of countable recursively saturated models of PA. Some of them are almost directly stated in the text.

1. Several results depend on the assumption that $\mathbb{N}$ is strong in the model considered. Indeed, it seems that the notion of a countable recursively saturated

model in which $\mathbb{N}$ is strong is a much better notion to work with than without this last part of the assumption. Is this assumption needed in the Lascar's result, i.e., Theorem 29? The same question for the Kossak-Schmerl encoding of $\mathrm{SSy}(\mathcal{M})$, theorem 31. Also: does every open subgroup of $\mathrm{Aut}(\mathcal{M})$ extend to a maximal one? As pointed out in our comments to [16], the answer is positive if $\mathbb{N}$ is strong in $\mathcal{M}$. I also don't know any nonopen subgroup of $\mathrm{Aut}(\mathcal{M})$ with no maximal extension.

2. More or less nothing is known about the connections of $\mathrm{Th}(\mathcal{M})$ and $\mathrm{Aut}(\mathcal{M})$. The only exception is Corollary 28. But how to distinguish $2^{\aleph_0}$ false complete extensions of PA by automorphisms groups?
3. Kaye described all closed normal subgroups in his theorem (Theorem 27 above). Many not closed normal subgroups are known. Thus, if I is an invariant cut in $\mathcal{M}$ then $G_{(>I)} = \{g \in \mathrm{Aut}(\mathcal{M}) : \exists b > I\; g{\restriction} < b = \mathrm{id}\}$ is normal. Another is the subgroup $\langle \mathrm{RSA}(\mathcal{M})\rangle$, the subgroup generated by $\{g \in \mathrm{Aut}(\mathcal{M}) : (\mathcal{M}, g)$ is recursively saturated in the expanded language$\}$. Are these all normal subgroups? Is $\langle \mathrm{RSA}(\mathcal{M})\rangle = G_{(>\mathbb{N})}$?
4. I believe that one can pose several problems connected with Piekart's work mentioned above. At first for which other classes of structures the phenomenon of the existence of nontrivial open subgroups with unique extension to a maximal subgroup occurs? In the case of countable recursively saturated models of PA does every open maximal subgroup of $\mathcal{M}$ have such subgroups? This is not known even in the case of $\mathcal{M} \models \mathrm{Th}(\mathbb{N})$. Presumably at least in the case of a basic maximal subgroup G_a the ordering of rationals is embeddable in $\{H < G_a : G_a$ is the unique extension of $H\}$.

Acknowledgments Preparation of this paper was supported by The State Committee For Scientific Research (Poland), KBN, grant number 2 2029 92 03.

REFERENCES

[1] Abramson, F., and L. Harrington, "Models without indiscernibles," *The Journal of Symbolic Logic*, vol. 43 (1978), pp. 572–600. Zbl 0391.03027 MR 80a:03045 0

[2] Barwise, J., and J. Schlipf, "On recursively saturated models of arithmetic," pp. 42–55 in *Model Theory and Algebra: a memorial tribute to A. Robinson*, Springer Lecture Notes in Mathematics, vol. 498, edited by D. H. Saracino and V. Weipsfennig, Springer-Verlag, Berlin, 1976. Zbl 0343.02031 MR 53:12934 0, 4

[3] Ehrenfeucht, A., and A. Mostowski, "Models of axiomatic theories admitting automorphisms," *Fundamenta Mathematicæ* vol. 43 (1956), pp. 50–68. Zbl 0073.00704 MR 18,863a 0

[4] Gaifman, H., "Models and types of Peano Arithmetic," *Annals of Mathematical Logic*, vol. 9 (1976), pp. 223–306. Zbl 0332.02058 MR 53:10577 0, 0, 0, 0

[5] Hodges, W., "Models built on linear orderings," *Annals of Discrete Mathematics*, vol. 23 (1984), pp. 207–234. Zbl 0553.03020 MR 86h:03053 0

[6] Hodges, W., I. Hodkinson, D. Lascar, and S. Shelah, "The small index property for ω-stable ω-categorical theory and for the random graph," *Journal of the London Mathematical Society*, vol. 48 (1993), pp. 204–218. Zbl 0788.03039 MR 94d:03063 0

[7] Kaye, R., "Models of Peano Arithmetic," *Oxford Logic Guides*, Oxford University Press, Oxford, 1991. Zbl 0744.03037 MR 92k:03034 0

[8] Kaye, R., "The automorphism group of a countable recursively saturated structure," *Proceedings of the London Mathematical Society*, vol. 65 (1992), pp. 225–244. Zbl 0763.20002 MR 93g:03026 0, 17, 0, 18

[9] Kaye, R., "A Galois correspondence for models of PA," pp. 293–312 in *Automorphism of first order structures*, edited by R. Kaye and D. Macpherson, Oxford University Press, Oxford, 1994. Zbl 0824.03015 MR 1325480 0, 27

[10] Kaye, R., R. Kossak, and H. Kotlarski, "Automorphisms of recursively saturated models of arithmetic," *Annals of Pure and Applied Logic*, vol. 55 (1991), pp. 67–91. Zbl 0748.03023 MR 93c:03047 0, 0, 21, 22, 23, 24, 0, 0, 0

[11] Kirby, L., "Initial segments of models of arithmetic," Ph.D. Thesis, University of Manchester, 1977. 0, 0, 0

[12] Kossak, R., "Exercises in 'back and forth'," in *Proceedings of the Ninth Easter Conference on Model Theory*, vol. 112, edited by H. Wolter, Fachbereich Mathematik der Humboldt Uniwersität zu Berlin, 1991. Zbl 0753.03014 0, 25, 26

[13] Kossak, R., and N. Bamber, "On two questions concerning the automorphism groups of countable recursively saturated models of PA, unpublished. 0

[14] Kossak, R., and H. Kotlarski, "Results on automorphisms of recursively saturated models of PA," *Fundamenta Mathematicæ*, vol. 129 (1988), pp. 9–15. Zbl 0662.03027 MR 89h:03064 15

[15] Kossak, R., and H. Kotlarski, "On extending automorphisms of countable recursively saturated models of PA," forthcoming in *Fundamenta Mathematicæ*. Zbl 0905.03025 MR 97d:03050 16, 0, 0, 0

[16] Kossak, R., H. Kotlarski, and J. Schmerl, "On maximal subgroups of the automorphism group of a countable recursively saturated models of PA," *Annals of Pure and Applied Logic*, vol. 65 (1993), pp. 125–148. Zbl 0796.03043 MR 94m:03061 0, 0, 0, 1

[17] Kossak, R., and J. Schmerl, "Minimal satisfaction classes with an application," *Notre Dame Journal of Formal Logic*, vol. 32 (1991), pp. 392–398. Zbl 0748.03024 MR 92h:03051 0, 0

[18] Kossak, R., and J. Schmerl, "The automorphism group of an arithmetically saturated model of Peano arithmetic," forthcoming in *Journal of the London Mathematical Society*. Zbl 0905.03024 MR 97a:03048 0

[19] Kotlarski, H., "On elementary cuts in recursively saturated models of arithmetic," *Fundamenta Mathematicæ*, vol. 120 (1984), pp. 205–222. Zbl 0572.03016 MR 86f:03056 6, 8, 9, 10

[20] Kotlarski, H., and R. Kaye, "Automorphisms of models of True Arithmetic: recognizing some basic open subgroups," *Notre Dame Journal of Formal Logic*, vol. 35 (1994), pp. 1–14. Zbl 0824.03016 MR 95h:03094 0, 0

[21] Kotlarski, H., and B. Piekart, "Automorphisms of models of True Arithmetic: subgroups which extend to a maximal one uniquely," *Mathematical Logic Quarterly*, vol. 40 (1994), pp. 95–102. Zbl 0806.03026 MR 95e:03112 0

[22] Kotlarski, H.,and B. Piekart, "Automorphisms of countable recursively saturated models of PA: open subgroups and invariant cuts," *Mathematical Logic Quarterly*, vol. 41 (1995), pp. 138–142. Zbl 0816.03018 MR 96e:03043 0

[23] Kueker, D., "Back and forth arguments in infinitary logic," pp. 17–71 in *In memoriam Carol Karp*, Springer Lecture Notes in Mathematics, vol. 429, edited by D. W. Kueker, Springer-Verlag, Berlin, 1969. Zbl 0316.02018 MR 57:2905 1

[24] Lascar, D., "The small index property and recursively saturated models of Peano Arithmetic, in *Automorphisms of the First Order Structures* edited by R. Kaye and D. Macpherson, Oxford University Press, Oxford, 1994. Zbl 0824.03014 MR 1 325 479 0, 29

[25] Macpherson, D., "Groups of automorphisms of $\aleph_0$-categorical structures," *Oxford Quarterly Journal of Mathematics*, vol. 37 (1986), pp. 449–465. Zbl 0611.03014 MR 88d:20007 0

[26] Piekart, B., "On closed elementary cuts in recursively saturated models of Peano Arithmetic," *Notre Dame Journal Formal Logic*, vol. 34 (1993), pp. 223–230. Zbl 0783.03016 MR 94j:03077 11

[27] Ressayre, J. P., "Models with compactness properties relative to an admissible language," *Annals of Mathematical Logic*, vol. 7 (1976), pp. 31–55. Zbl 0376.02032 MR 57:5735 0

[28] Schlipf, J., "Toward model theory through recursive saturation," *The Journal of Symbolic Logic*, vol. 43 (1978), pp. 183–206. Zbl 0409.03019 MR 58:10399 0, 0

[29] Schmerl, J., "Recursively saturated models generated by indiscernibles," *Notre Dame Journal Formal Logic*, vol. 26 (1985), pp. 95–105. Zbl 0487.03018 MR 87g:03038 0, 13

[30] Smoryński, C., "Recursively saturated nonstandard models of arithmetic," *The Journalof Symbolic Logic*, vol. 46 (1981), pp. 259–256. Zbl 0501.03044 MR 84b:03054 0

[31] Smoryński C., "Back and forth inside a recursively saturated model of arithmetic," pp. 273–278 in *Logic Colloquium '80*, edited by D. van Dalen, North-Holland, Amsterdam, 1982. Zbl 0503.03033 MR 83m:03041 6, 7

[32] Trussa, J., "Infinite permutation groups: subgroups of small index," *Journal of Algebra*, vol. 120 (1989), pp. 494–515. Zbl 0666.20002 MR 90c:20004 0

Instytut Matematyki i Fizyki
ul. Orlicz-Dreszera 19/21
08-110 Siedlce
Poland
email: hkl@impan.impan.gov.pl