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Frege's Proof of Referentiality

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Abstract I present a novel interpretation of Frege's attempt at *Grundgesetze* I §§29–31 to prove that every expression of his language has a unique reference. I argue that Frege's proof is based on a contextual account of reference, similar to but more sophisticated than that enshrined in his famous Context Principle. Although Frege's proof is incorrect, I argue that the account of reference on which it is based is of potential philosophical value, and I analyze the class of cases to which it may successfully be applied.

1 Introduction

In *Grundgesetze* I §§29–31 Frege attempts to prove that every expression of his language has a unique denotation.[1] But despite a large secondary literature, there has been little agreement about the proof—other than that it proceeds by induction and that Russell's paradox shows it must be flawed.[2] In this paper I develop a novel interpretation of Frege's proof which fits the text better than any of its predecessors and shows it to be less confused than often assumed.

The primary value of my interpretation lies in the light it sheds on reference to abstract objects, both in general and as understood by Frege. I argue that the proof of referentiality contains the most developed expression of a central idea of Frege's philosophy, namely, that it suffices for a proper name to refer that all *contexts* in which it can occur are meaningful. This idea is familiar from the celebrated Context Principle of *Grundlagen* (see Frege [9], pp. x, 71, 73, and 116) and forms the heart of Frege's attempt to solve the philosophical puzzle about how reference to and apprehension of abstract objects are possible. The contextual account of reference implicit in the proof of referentiality brings out new and interesting aspects of this idea.

However, the remarks about reference contained in the proof of referentiality are hard to interpret because they appear to be badly circular. Some commentators therefore deny that *Grundgesetze* contains a contextual account of reference at all.[3] Other

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commentators maintain that it does but are able to do so only at the cost of ascribing to Frege unappealing views, such as a substitutional understanding of the quantifiers, or to concede that the circularities of his contextual account are vicious.[4] Unlike the first group of commentators, I argue that the *Grundgesetze* proof of referentiality does involve a contextual account of reference to abstract objects; but unlike the latter, I deny that Frege had a substitutional understanding of the quantifiers or that the circularities of his contextual account are vicious. I argue that Frege made an ingenious, and so far overlooked, attempt at taming the circularities. This attempt is based on a clever procedure he devised for extending "the sphere of denoting names" step by step.

Why did Frege attempt to give a proof of referentiality? I believe the primary reason is his fundamental principle that in a scientifically perfect language, such as the Begriffsschrift, every name must have a unique denotation. But unlike the other names of the Begriffsschrift, the value-range names aren't assigned any denotation directly. All Frege does is lay down Basic Law V, which asserts that the value-range of a function $\Phi(\xi)$ is identical to that of $\Psi(\xi)$ just in case these functions are coextensive[5]

$$\text{(V)} \qquad \grave{\epsilon}.\Phi(\epsilon) = \grave{\epsilon}.\Psi(\epsilon) \leftrightarrow \forall x(\Phi(x) \leftrightarrow \Psi(x))$$

as well as a semantic counterpart of (V), which says that '$\grave{\epsilon}.\Phi(\epsilon) = \grave{\epsilon}.\Psi(\epsilon)$' is to denote the same as '$\forall x(\Phi(x) \leftrightarrow \Psi(x))$'. It is therefore of great importance for Frege to prove that this principle, along with some minor stipulations from §10, suffices to endow each value-range name with a unique denotation. I believe this is the main purpose of his proof.

But as Frege realized, the proof of referentiality would also establish the consistency of the logical theory developed in *Grundgesetze*. Thus, when he learned of Russell's paradox, Frege immediately conceded that "my explanations in sect. 31 do not suffice to secure a reference for my signs in all cases" (Frege [11], p. 132). Given Russell's paradox, we know that Frege's proof of referentiality must somehow be flawed.

My analysis brings out a good as well as a bad aspect of this flaw. What's good is that the proof fails only at a relatively late stage and that Frege's contextual account of reference and his procedure for extending the sphere of denoting names are logically sound and of potential philosophical value. Frege's mistake lies only in a faulty *application* of these ideas. In fact, Frege's proof strategy can correctly be applied to what we may call *predicative* abstraction principles, that is, to abstraction principles

$$\Sigma(\alpha) = \Sigma(\beta) \leftrightarrow \alpha \sim \beta$$

where the equivalence relation $\sim$ doesn't quantify over the entities introduced by the abstraction. What's bad is that these are the *only* abstraction principles to which Frege's proof strategy can correctly be applied. But most interesting abstraction principles, such as (V) and Hume's Principle, aren't predicative in this sense. So unless an alternative to Frege's proof strategy is possible, the potential philosophical payoff of his ideas will be severely limited.

The paper is organized as follows. In Section 2, I discuss Frege's theory of quantification and introduce some concepts that will be important to the ensuing discussion. In Section 3, I distinguish two interpretations of Frege's contextual criteria for names to refer and analyze the circularities that threaten these criteria. In Section 4, I defend one of these interpretations, and in Section 5, I analyze this interpretation in some detail. The next three sections are more exegetical: Section 6 discusses Frege's treatment of the basis case in §31 and argues that it favors my interpretation of the contextual criteria, and Sections 7 and 8 do likewise for Frege's treatment of the induction step in §30. Finally, in Section 9, I make some concluding remarks.

2 Quantification and Auxiliary Names

As is well known, in *Grundgesetze* Frege takes complete sentences to denote truth-values, and the truth-values to be objects. Concepts are accordingly regarded as first-level functions from objects in general to truth-values.[6] The quantifiers too are regarded as functions: the first-order quantifier, as a second-level function from first-level functions to truth-values, and the second-order quantifier, as the analogous third-level function. (In contrast, most contemporary expositions regard the quantifiers as *operators* that allow the formation of new well-formed formulas from ones already formed and whose associated semantic rules specify the conditions under which these new formulas are true.)

Strangely, Frege's explanations of the quantifiers seem constantly to confuse substitutional and objectual quantification. He seems not to distinguish between a quantified function's being "the True for every proper name" and "the True for every argument" but on the contrary uses these expressions as if they were interchangeable. But in contemporary expositions, the phrase "is true for every proper name" signals that the quantification is substitutional, and the phrase "is true for every argument," that it is objectual. For instance, in §8 Frege considers an equation which holds "whatever proper name we may substitute for" a variable and says this is the same as what he considered in §3. But in §3, the two functions flanking the identity-sign are said to "have always the same value for the same *argument*" (p. 36, my emphasis). Another example is found in §20, where Frege says that a formula of the form '$\forall f.F(f)$' is true if and only if the expression '$F(f)$' is "a name of the True, whatever function-name one may substitute" for the function variable 'f' (p. 71, my emphasis). But earlier in §20 Frege rehearsed the definition of what we would call objectual first-order quantification.

It is unlikely that this vacillation between expressions that *to us* signal different interpretations of the quantifiers reflects a conscious decision *on Frege's part* to make such a distinction. It is almost unbelievable that a thinker as careful as Frege should have made this distinction without explicitly announcing so. We must therefore seek an alternative interpretation of Frege's remarks which involves only one sort of quantification. This interpretation must explain how it can be permissible to treat the expressions "for every name" and "for every argument" as interchangeable.

One interpretation of this sort that has recently been defended is to regard Frege's quantifiers as substitutional. (See Resnik [24]; Dummett [6], pp. 215–22; and Hintikka and Sandu [17].) But apart from Frege's peculiar tendency to slide back and forth between the expressions "for every name" and "for every argument," evidence for this interpretation is scarce. In fact, some powerful arguments have been offered that Frege did not understand the quantifiers in this way (see Burgess [3] and Heck and Stanley [16]).

It is much more promising to seek an interpretation of the phrase "for every name" that is compatible with an objectual understanding of the quantifiers. An interpretation of this sort has been offered by Heck, who points out that Frege had a special category of names that allows us to interpret the phrase "for every name" as equivalent to "for every argument." (See Heck [14], especially Sections 2 and 3, and Heck [15].) These names are the uppercase Greek ones, which are introduced in a footnote to §5. Frege explains that he will use such letters "as if they were names denoting something, although I do not specify their denotation," and he adds that these names won't be part of the Begriffsschrift itself. I will adopt Heck's terminology and call these *auxiliary names*. Auxiliary names solve our problem: For when "name" is interpreted as "auxiliary name," the expressions "for every name" and "for every argument" will indeed be interchangeable.

Frege doesn't explicitly mention auxiliary *function* names. But there is little doubt that he accepted auxiliary names of all types. For instance, he uses the letters 'Φ', 'Ψ', and 'X' as function names, (see, for example, §§10 and 31) and because these are uppercase Greek letters[7] it follows from the footnote just mentioned that they are auxiliary names. Moreover, it is only by allowing auxiliary function names that the remark I quoted from §20 can be squared with an objectual interpretation of the second-order quantification-function.

As Heck argues, Frege's auxiliary names serve much the same purpose as assignments of values to free variables do in post-Tarskian logic.[8] For instance, Frege's reasoning in §17 must be understood as involving this kind of "free variable reasoning" (see Heck [14], Section 3). Auxiliary names also play an indispensable role in Frege's exposition of the Begriffsschrift. Consider, for instance, Frege's explanation of the horizontal function: "—Δ is the True if Δ is the True; on the other hand it is the False if Δ is not the True" (p. 38). This explanation would have been impossible without the use of auxiliary names or some other device corresponding to the assignment of values to free variables.

3 The Context Criteria of §29

We are now ready to delve into §29, which bears the title "When does a name denote something?" and gives a set of conditions for names of various categories to denote. As with the more familiar Context Principle from *Grundlagen*, the idea is that a name denotes if all contexts in which it can occur denote. I will therefore refer to the conditions of §29 as *the context criteria*. Although the context criteria play an essential role in the proof of referentiality in §§30–31, they aren't explained as clearly as one might have hoped. In this section I distinguish two kinds of interpretation. Much of the later work of this paper will be an attempt to adjudicate between these interpretations.

Typically, a proof of referentiality takes the notion of reference for granted, regarding it simply as a matter of a name's standing for some entity. Using an intuitive notion of reference, one first assumes that the basic vocabulary refers and then argues by induction that every well-formed name based on this vocabulary refers. But this is *not* the structure of Frege's proof. Frege provides an account of *what reference to abstract objects consists in* and uses this account to *argue* that his basic vocabulary refers. Not surprisingly, the context criteria play an essential role in both these tasks.

The criterion for a proper name to denote reads as follows:

> A proper name has denotation if the proper name that results from that proper name's filling the argument-places of a denoting name of a first-level function of one argument always has a denotation, and if the name of a first-level

> function of one argument that results from the proper name in question's filling the ξ-argument-places of a denoting name of a first-level function of two arguments always has a denotation, and if the same holds for the ζ-argument-places. (p. 84)

The criterion for a function name to denote is similar:[9]

> A name of a first-level function of one argument has a denotation . . . if the proper name that results from this function-name by its argument-places' being filled by a proper name always has a denotation if the name substituted denotes something. (p. 84)

Frege also states criteria for names of first-level functions of two arguments to denote and for names of second- and third-level functions. Since these criteria are obvious modifications of the two just quoted, I won't reproduce them here. But below I will formalize all the context criteria.

A conspicuous difference between the Context Principle of *Grundlagen* and the context criteria of *Grundgesetze* is that the former requires only that *sentential* contexts in which the name occurs denote, whereas the latter require that *all* contexts do. This change is necessitated by Frege's subsumption in *Grundgesetze* of the category of sentences under that of proper names.

I will call a name to which one of the context criteria is applied a *candidate name*. A candidate name is thus a name that is to be shown to denote. I will call the denoting names with which the context criteria tell us to combine the candidate name the *background names*. Expressed in this terminology, the context criteria tell us that a candidate name denotes if every name denotes which is formed by combining the candidate name in a permissible way with one or more background names.

The context criteria, as expressed in §29, give rise to two questions. The first question is how to understand the background names. One possibility is that they are *auxiliary* names (in the sense of Section 2). If so, the context criteria will say that a candidate name denotes just in case all names denote which are formed by combining the candidate name with denoting name of the correct syntactic category, whether these names are part of the Begriffsschrift or not.[10] Another possibility is that the background names are *nonauxiliary*. If so, the context criteria will say that a candidate name denotes just in case it forms denoting names when correctly combined with names already recognized as denoting. I will refer to these as, respectively, an *auxiliary* and a *nonauxiliary* interpretation of the context criteria.[11] Note that both kinds of interpretation are compatible with Frege's formulation of the context criteria in §29. Moreover, by granting, as I did in Section 2, that Frege's explanations of the quantifiers involve auxiliary names, I don't thereby commit myself to an auxiliary reading of the context criteria.

The second question is how the context criteria can avoid vicious circularity. For some time now it has been clear that Frege's much criticized[12] subsumption of sentences under proper names introduces a circularity into the context criteria. (See, for example, Dummett [4], p. 645 and Dummett [5], p. 409.) For when this is done, many of the contexts that must be shown to denote will *themselves* belong to the category of proper names. This means there is an essential occurrence of the notion of a proper name's denoting on the right-hand side of the context criteria. Since this circularity arises only in *Grundgesetze*, I will refer to it as *the new circularity*.

I will now show that the context criteria contain another, rarely noticed, circularity, which cannot be blamed on the *Grundgesetze* subsumption of sentences under proper names but which threatens even the original Context Principle of *Grundlagen*.[13] This circularity arises as follows. The Context Principle says it suffices for a proper name to refer that all contexts in which it can occur are meaningful. But *which* contexts are we talking about? Restricting oneself to *some isolated contexts* seems insufficient if the proper name is to refer in anything like the ordinary sense.[14] But considering *all contexts whatsoever* would include sentences whose meaninglessness isn't the fault of the proper name but of the context. What we want to say is rather that, for a proper name to refer, it must yield a meaningful sentence when combined with any *referring* predicate.[15] But for a predicate to refer, it must yield a meaningful sentence when combined with any referring proper name. Thus, the desired criterion for a proper name to refer appeals to the notion of a predicate's referring, which in turn appeals to that of a proper name's referring. Since this circularity threatens even the Context Principle of *Grundlagen*, I will refer to it as *the old circularity*. The old circularity is inherited by the contextual account of reference developed in *Grundgesetze*.

To get clearer on these circularities, it is useful to formalize the context criteria. I begin by introducing the following indices on the word 'denotes':

'denotes$_0$' applies to proper names;
'denotes$_1$' applies to one-place function names of first type;
'denotes$_{11}$' applies to two-place function names of first type;
'denotes$_2$' applies to function names of second type;
'denotes$_3$' applies to function names of third type.

Using Gothic letters to range over syntactic entities, the context criteria can be formalized as follows.

(CC$'_0$) $\mathfrak{a}$ denotes$_0$ iff, whenever $\mathfrak{f}$ denotes$_1$, $\ulcorner\mathfrak{f}(\mathfrak{a})\urcorner$ denotes$_0$, and whenever $\mathfrak{g}$ denotes$_{11}$, $\ulcorner\mathfrak{g}(\mathfrak{a}, \zeta)\urcorner$ and $\ulcorner\mathfrak{g}(\xi, \mathfrak{a})\urcorner$ denote$_1$

(CC$_1$) $\mathfrak{f}$ denotes$_1$ iff, whenever $\mathfrak{a}$ denotes$_0$, $\ulcorner\mathfrak{f}(\mathfrak{a})\urcorner$ denotes$_0$

(CC$'_{11}$) $\mathfrak{g}$ denotes$_{11}$ iff, whenever $\mathfrak{a}$ denotes$_0$, $\ulcorner\mathfrak{g}(\xi, \mathfrak{a})\urcorner$ and $\ulcorner\mathfrak{g}(\mathfrak{a}, \zeta)\urcorner$ denote$_1$

(CC$_2$) $\mathfrak{F}$ denotes$_2$ iff, whenever $\mathfrak{f}$ denotes$_1$, $\ulcorner\mathfrak{F}(\mathfrak{f})\urcorner$ denotes$_0$

(CC$_3$) $\mathfrak{X}$ denotes$_3$ iff, whenever $\mathfrak{F}$ denotes$_2$, $\ulcorner\mathfrak{X}(\mathfrak{F})\urcorner$ denotes$_0$

Note that (CC$'_0$) and (CC$'_{11}$) can be simplified as follows:[16]

(CC$_0$) $\mathfrak{a}$ denotes$_0$ iff, whenever $\mathfrak{f}$ denotes$_1$, $\ulcorner\mathfrak{f}(\mathfrak{a})\urcorner$ denotes$_0$

(CC$_{11}$) $\mathfrak{g}$ denotes$_{11}$ iff, whenever $\mathfrak{a}$ and $\mathfrak{b}$ denote$_0$, $\ulcorner\mathfrak{g}(\mathfrak{a}, \mathfrak{b})\urcorner$ denotes$_0$

Henceforth, I will always use these simplifications. Finally, note that (CC$_2$) and (CC$_3$) can be expressed solely in terms of denotation$_0$:

(CC$'_2$) $\mathfrak{F}$ denotes$_2$ iff, whenever $\mathfrak{f}$ is such that (whenever $\mathfrak{a}$ denotes$_0$, $\ulcorner\mathfrak{f}(\mathfrak{a})\urcorner$ denotes$_0$), then $\ulcorner\mathfrak{F}(\mathfrak{f})\urcorner$ denotes$_0$

and likewise for (CC$_3$).

The circularities can now be given a precise characterization. The old circularity arises because the notions of denotation that apply to function names, among them denotation$_1$, have been characterized in terms of denotation$_0$, although this notion in turn has been characterized in terms of denotation$_1$. This circularity cannot be dismissed as merely a result of Frege's subsumption of the sentences under proper names. For even if a special notion of sentence denotation were employed, say denotation$_s$, the notion of denotation$_0$ would still remain in the *antecedents* of the conditionals on the right-hand side of the biconditionals. This means that the old circularity is independent of Frege's controversial subsumption of sentences under proper names and thus that it applies to the original Context Principle of *Grundlagen*. All the subsumption of sentences under proper names in *Grundgesetze* does is slightly complicate the situation by introducing the notion of denotation$_0$ in the *consequents* of the relevant conditionals as well as in their antecedents. This is the new circularity.

4 Extending the Sphere of Denoting Names

Frege is perfectly aware that the context criteria appear circular. In the opening sentences of §30 he therefore writes that the criteria cannot

> be regarded as definitions of the phrases "have a denotation" or "denote something," because their application always presupposes that we have already recognized some names as denoting. They can serve only in *the extension step by step of the sphere of [denoting] names*. (p. 85, my emphasis)

His idea seems to be, roughly, that if it can be established that some names denote, the context criteria will allow us to demonstrate that other names too denote. Although the technical details of this procedure for extending the sphere of denoting names are less than obvious, the procedure was no doubt intended to be crucial to the proof of referentiality; in fact, Frege appeals to it three times in the course of the proof (see pp. 85, 87, and 89).

In this section I argue that Frege's procedure for extending the sphere of denoting names favors a nonauxiliary reading of the context criteria (that is, a reading that takes the background names to be ordinary rather than auxiliary). The first argument is textual, the second, systematic. These arguments will be reinforced in later sections, where I will present the particular nonauxiliary reading that I favor and argue that this reading accords well with Frege's treatment of the basis case and the induction step.

The most informative characterization Frege gives of the procedure for extending the sphere of denoting names reads as follows.

> We start from the fact that the names of the truth-values denote something, namely, either the True or the False. We then gradually widen the sphere of names to be recognized as succeeding in denoting by showing that those to be adopted, *together with those already adopted*, form denoting names by way of one's appearing at fitting argument-places of the other. (p. 87, my emphasis)

Note what Frege says about how a candidate name can be shown to denote: by showing that all permissible combinations of it with the names "already adopted" denote. So the background names must be among the names already adopted. It is hard to reconcile this requirement with the auxiliary reading of the context criteria. For unlike ordinary names, auxiliary names are never part of the Begriffsschrift itself.[17] So this is strong textual evidence in favor of the nonauxiliary reading. Had Frege intended the auxiliary reading, he would have characterized the background names differently, for instance, as "every denoting name of the appropriate category."

Before presenting the systematic argument we need a definition. Say that a function name $\mathfrak{f}$ is *nonexpansive*, relative to a language $\mathcal{L}$, just in case $\mathfrak{f}$ is so defined that for every denoting proper name 'Δ'[18], $\ulcorner \mathfrak{f}\Delta \urcorner$ reduces to some proper name $\mathfrak{b}$ in $\mathcal{L}$. More precisely, there are mutually exclusive and jointly exhaustive conditions $C_1, \ldots, C_n$ and proper names $\mathfrak{b}_{\mathcal{N}}, \ldots, \mathfrak{b}_{\mathfrak{n}} \in \mathcal{L}$ such that if Δ satisfies C_i, then $\ulcorner \mathfrak{f}\Delta \urcorner$ reduces to $\mathfrak{b}_{\mathfrak{i}}$. If this condition isn't met, the function name is *expansive* relative to $\mathcal{L}$. Mutatis mutandis for function names of higher types. Here are some examples. The name of Frege's horizontal function, $-\!\!-\xi$, is nonexpansive relative to the languages we are interested in, since it is defined to have the True as value for the True as argument and the False for all other arguments.[19] The same goes for the name of the quantification function, $\forall x.\varphi x$. For when '$\Phi\xi$' is a denoting function name, the proper name '$\forall x.\Phi x$' denotes the True or the False according as '$\Phi\Delta$' denotes the True for every argument Δ or not. However, the name of the value-range function, $\grave{\epsilon}.\varphi\epsilon$, is expansive relative to the languages we are interested in, since its definition doesn't allow any reduction of names of the form '$\grave{\epsilon}.\Phi\epsilon$' to names not of this form.

The systematic argument aims to show that, on the auxiliary reading of the context criteria, Frege's extension procedure cannot deal with expansive function names. Let $\mathcal{L}$ be a language all of whose names have been shown to denote. Let's attempt to add to this language a new proper name $\mathfrak{a}$. To show that $\mathfrak{a}$ denotes$_0$ we must, according to the auxiliary reading of (CC_0), show that for every denoting$_1$ function name '$\Phi\xi$', the complex name $\ulcorner \Phi\mathfrak{a} \urcorner$ denotes$_0$. If '$\Phi\xi$' is nonexpansive relative to $\mathcal{L}$, this is easy. However, if '$\Phi\xi$' is expansive, we will need to apply (CC_0) a second time: we will need to verify that for any denoting$_1$ one-place function name '$\Psi\xi$', the name $\ulcorner \Psi(\Phi\mathfrak{a}) \urcorner$ denotes$_0$. But now we realize we are going round in a circle. So on the auxiliary reading, it is impossible ever to establish that a new proper name denotes.

To show that a new function name $\mathfrak{f}$ denotes, we must, according to the auxiliary reading of (CC_1), verify that the name $\ulcorner \mathfrak{f}\Delta \urcorner$ denotes whenever 'Δ' is a denoting proper name. When $\mathfrak{f}$ is nonexpansive, we know that no matter what the proper name 'Δ' denotes, the name $\ulcorner \mathfrak{f}\Delta \urcorner$ denotes. This means that Frege's procedure works for nonexpansive function names. However, when the function name $\mathfrak{f}$ *is* expansive relative to $\mathcal{L}$, we're thrown back to the impossible task of showing that a new proper name denotes. In particular, Frege's procedure fails for the name of the important value-range function $\grave{\epsilon}.\varphi\epsilon$, since this name is expansive.

But perhaps one doesn't have to go all the way to the full nonauxiliary reading of the context criteria to deal with this difficulty. Perhaps there is an intermediate option in the form of a *mixed reading* that interprets some context criteria in one way and some in the other. To avoid the difficulty I've just pointed to, the mixed reading would have to read (CC_0) in the nonauxiliary way. But perhaps one can still maintain an auxiliary reading of the criteria for function names. This suggestion has been defended by Heck in an important recent paper.[20]

Although this mixed reading avoids the difficulty discussed above and is of considerable independent interest,[21] it is implausible as an interpretation of Frege. §29 provides no evidence that different context criteria are to be understood differently and substantial evidence that they are not. Since (CC_0) is the second item on Frege's list of four context criteria, the mixed reading is committed to the view that the one criterion to be understood in the nonauxiliary way is flanked by criteria to be understood in the auxiliary way. But if so, it would be extremely strange that Frege says nothing to alert his reader that he is going back and forth between completely different understandings of the background names. This omission would be extremely uncharacteristic of a philosopher as careful as Frege. Moreover, in Section 6, I will argue that the basis case in §31 indicates that Frege understood the context criteria for function names too in the nonauxiliary way. This provides further evidence against the mixed reading.

5 A Nonauxiliary Analysis of the Context Criteria

I will now present the nonauxiliary analysis of the context criteria that I favor. I will argue that on this analysis, Frege's procedure for extending the sphere of denoting names looks relatively promising. Since we've just seen that this isn't so when the context criteria are read in the auxiliary way, my analysis provides strong support for the nonauxiliary reading.

The key to my analysis is to note that on the nonauxiliary reading, the context criteria become language-relative: A candidate name will be said to denote just in case every name denotes which results from combining the candidate name with *background names from a certain language*. I will make this language-relativity explicit by means of an upper index, specifying the language in question. Letting $\mathcal{L}_t$ be the subset of a language $\mathcal{L}$ consisting of names of type t, my proposal is that the context criteria be understood as follows:

($\mathrm{CC}_0^{\mathcal{L}}$) $\mathfrak{a}$ denotes$_0^{\mathcal{L}}$ iff, whenever $\mathfrak{f} \in \mathcal{L}_1$, $\ulcorner \mathfrak{f}(\mathfrak{a}) \urcorner$ denotes$_0^{\mathcal{L}}$

($\mathrm{CC}_1^{\mathcal{L}}$) $\mathfrak{f}$ denotes$_1^{\mathcal{L}}$ iff, whenever $\mathfrak{a} \in \mathcal{L}_0$, $\ulcorner \mathfrak{f}(\mathfrak{a}) \urcorner$ denotes$_0^{\mathcal{L}}$

($\mathrm{CC}_{11}^{\mathcal{L}}$) $\mathfrak{g}$ denotes$_{11}^{\mathcal{L}}$ iff, whenever $\mathfrak{a}, \mathfrak{b} \in \mathcal{L}_0$, $\ulcorner \mathfrak{g}(\mathfrak{a}, \mathfrak{b}) \urcorner$ denotes$_0^{\mathcal{L}}$

($\mathrm{CC}_2^{\mathcal{L}}$) $\mathfrak{F}$ denotes$_2^{\mathcal{L}}$ iff, whenever $\mathfrak{f} \in \mathcal{L}_1$, $\ulcorner \mathfrak{F}(\mathfrak{f}) \urcorner$ denotes$_0^{\mathcal{L}}$

($\mathrm{CC}_3^{\mathcal{L}}$) $\mathfrak{X}$ denotes$_3^{\mathcal{L}}$ iff, whenever $\mathfrak{F} \in \mathcal{L}_2$, $\ulcorner \mathfrak{X}(\mathfrak{F}) \urcorner$ denotes$_0^{\mathcal{L}}$

The notion of denotation$^{\mathcal{L}}$ should be understood as denotation with respect to the language $\mathcal{L}$. A name denotes simpliciter when it denotes with respect to the language to which it belongs.[22]

This interpretation of the context criteria accords well with Frege's characterization of his procedure for extending the sphere of denoting names. For on this interpretation, what we need to do in order to extend the sphere of denoting names is precisely to show that "those [names] to be adopted, together with those already adopted, form denoting names by way of one's appearing at fitting argument-places of the other" (p. 87).

When the context criteria are formalized in this way, the old circularity disappears because the notion of a name's denoting no longer occurs in the antecedents of the conditionals on the right-hand side of the context criteria. But the new circularity remains, since 'denotes$_0^{\mathcal{L}}$' still occurs in the consequents of these conditionals. However, I claim that this remaining circularity is innocent. This will become apparent when I now go on to describe the procedure for extending the sphere of denoting names.

Assume that all names of the language $\mathcal{L}$ have been shown to denote, where $\mathcal{L}$ is the class wf(P) of well-formed names based on a class P of primitives. Let's consider how a new name $\mathfrak{n}$ (of any type) can be added to the sphere of denoting names. In the remainder of this section I will discuss this problem in a somewhat abstract way. In Sections 6 and 8 I will argue that Frege's own treatment of the basis case and the induction step fits this abstract characterization. This will provide some concrete examples.

First there is a *basis case* in which we need to show that $\mathfrak{n}$ satisfies the appropriate criterion ($\mathrm{CC}_t^{\mathcal{L}}$). This immediately leads to the problem posed by the new circularity: we need to show that every complex proper name $\mathfrak{a}$ referred to on the right-hand side of the relevant biconditional denotes$_0^{\mathcal{L}}$, and it seems this can be done only by applying the context criteria a second time, this time to $\mathfrak{a}$, and thus starting a regress. Admittedly, invoking the context criteria again is one way of showing that $\mathfrak{a}$ denotes$_0^{\mathcal{L}}$. But there is a second and much simpler way as well, namely, by showing that $\mathfrak{a}$ reduces to some proper name $\mathfrak{b} \in \mathcal{L}$, which by assumption we know to denote. This second option provides a basis in which to ground the iterated application of the context criteria associated with the first option. The new circularity is therefore benign. In fact, in Frege's own proof we never need to invoke the context criteria more than twice. The reason is that the only names of first-order functions considered in the proof of referentiality are *predicates*—that is, names of functions from objects in general to truth-values—and thus nonexpansive.[23] This ensures that all names resulting from a second invocation of the context criteria will denote.

Assume the basis case has been carried out. We therefore add $\mathfrak{n}$ to our language by letting $P' = P \cup \{\mathfrak{n}\}$ and $\mathcal{L}' = \mathrm{wf}(P')$. What remains before $\mathfrak{n}$ can be taken up into the sphere of denoting names is an *induction step* in which we apply the criteria ($\mathrm{CC}_t^{\mathcal{L}'}$) to show that every name in $\mathcal{L}'$ denotes$^{\mathcal{L}'}$ (and thus denotes simpliciter). One easily sees that it suffices to show that every *proper* name in $\mathcal{L}'$ denotes$_0^{\mathcal{L}'}$. In order to show this latter, we again run into the problem posed by the new circularity. But again, this circularity is benign because a name can always be shown to denote$_0^{\mathcal{L}'}$ by showing that it reduces to some name already accepted into the sphere of denoting names.

This two-step procedure must be repeated as many times as we desire to extend our sphere of denoting names. As we've seen, Frege begins with a language consisting of just the names of the truth-values, which are assumed to denote. He then successively adds the names of the connectives, of the quantifiers, and of the value-range function. In the former two cases, it is comparatively easy to carry out the two steps of Frege's procedure. In the case of the name of the value-range function, the procedure is supposed to go through because this function name is introduced by means of an abstraction principle, which gives an explicit criterion of identity for the new objects it introduces.

But the presence of function names of higher type gives rise to a complication, to which I now turn. This complication is best brought out by considering some examples. There are two kinds of cases: adding a name that is expansive, and adding one that isn't. I begin with the latter case since it is easier. Assume we want to add a new function name $\tilde{\mathfrak{f}}$ with the property that for every $\mathfrak{a} \in \mathcal{L}_0$, $\ulcorner \tilde{\mathfrak{f}}\mathfrak{a} \urcorner$ reduces to some $\mathfrak{b} \in \mathcal{L}_0$. By ($\mathrm{CC}_1^{\mathcal{L}}$), this means that $\tilde{\mathfrak{f}}$ denotes$_1^{\mathcal{L}}$. The basis case is therefore complete, and we can let $P' = P \cup \{\tilde{\mathfrak{f}}\}$. Then we need to carry out the induction step. As remarked above, it suffices to show that every *proper* name in $\mathcal{L}'$ denotes$_0^{\mathcal{L}'}$. This seems easy to prove; for whenever $\tilde{\mathfrak{f}}$ is applied to some name $\mathfrak{a} \in \mathcal{L}_0$, the resulting name $\ulcorner \tilde{\mathfrak{f}}\mathfrak{a} \urcorner$ will reduce to some $\mathfrak{b} \in \mathcal{L}_0$, and application of further primitive function names won't disrupt this property of reducing to some proper name already in $\mathcal{L}$.

However, things aren't as easy as they seem. The argument just given works only for function names of *first* type. But P' may also contain function names of *higher* type. To guarantee that proper names involving such function names denote, we need to require more of these function names, when we introduce them, than that they satisfy the ordinary nonauxiliary context criteria. As an example of what can go wrong if we don't, assume $\mathcal{L}_1$ is empty. Then trivially, any given function name $\mathfrak{F}$ of second type denotes$_2^{\mathcal{L}}$. Assume we now go on to show that $\mathfrak{f}$ denotes$_0^{\mathcal{L}}$ and therefore add $\mathfrak{f}$ to the language by letting $P' = P \cup \{\mathfrak{f}\}$. The problem then arises that we have no guarantee that $\ulcorner\mathfrak{F}(\mathfrak{f})\urcorner$ denotes$_0^{\mathcal{L}'}$. To deal with this problem, we need to require of every function name $\mathfrak{F}$ of second type which we want to introduce that $\mathfrak{F}$ be such that, whatever function name 'Φ' we may in the future come to recognize as denoting, $\ulcorner\mathfrak{F}(\Phi)\urcorner$ denotes. Since the open-ended generality of this condition corresponds to dropping upper indices on the notion of a name's denoting, the condition can be formalized as

(OE$_2$) $\mathfrak{F}$ is such that, whenever 'Φ' denotes$_1$, $\ulcorner\mathfrak{F}(\Phi)\urcorner$ denotes$_0$.

The corresponding condition for function names of third type is

(OE$_3$) $\mathfrak{X}$ is such that, whenever 'M' satisfies (OE$_2$), $\ulcorner\mathfrak{X}(M)\urcorner$ denotes$_0$.

Let's now consider the addition of an expansive name. A simple example is the addition of a new proper name $\mathfrak{a}$. Assume we have shown that for every $\mathfrak{f} \in \mathcal{L}_1$, $\ulcorner\mathfrak{f}(\mathfrak{a})\urcorner$ reduces to some $\mathfrak{b} \in \mathcal{L}_0$. By (CC$_0^{\mathcal{L}}$), this means that $\mathfrak{a}$ denotes$_0^{\mathcal{L}}$, which completes the basis case. So we can add $\mathfrak{a}$ to the sphere of denoting names. Then we need to carry out the induction step. As above, it suffices to show that every proper name in $\mathcal{L}'$ denotes$_0^{\mathcal{L}'}$. As long as we limit ourselves to function names of first type, this is easy: any combination of $\mathfrak{a}$ with such function names will be equivalent to some proper name in $\mathcal{L}$ and thus a fortiori denote$_0^{\mathcal{L}'}$. But for this argument to work for function names of higher types as well, these function names must be shown to satisfy the open-ended conditions stated above.

In Section 6, I will argue that Frege was aware of the complication discussed in the previous three paragraphs. Moreover, since the only names of higher type he was interested in were those of the quantifiers and of the value-range function, he may reasonably have thought the complication to be manageable. For the names of the quantifiers are nonexpansive, and the name of the value-range function is fairly well controlled by the semantic counterpart of Basic Law V. (For details, see Section 6.) So I conclude that on the nonauxiliary analysis I have proposed, Frege's extension procedure looks relatively promising, whereas on the alternative auxiliary interpretation, it is easily seen to be doomed.

I will end this section by taking a slightly different perspective on what we have discussed. Thus far I have, like Frege himself, focused on the procedure for extending the sphere of denoting names. This procedure represents the context criteria in what we may call their *dynamic use*. Is it possible to give an informative *static* characterization of what the context criteria say? Frege seems to think it isn't, for he denies that the context criteria can be regarded as definitions of the phrase 'has a denotation'. However, if we reintroduce a distinction between objects and truth-values—and thus also between proper names and sentences—it is possible to do better. We can now give an informative static characterization of what the context criteria say: It suffices for a name to denote that all sentences in which it can occur have been assigned a unique truth-value, and that these assignments have been made in accordance with the laws governing the logical vocabulary involved in the sentences.[24] I will refer to this as *Frege's contextual account of reference*. Note that this contextual account is more general than Frege's procedure for stepwise extending the sphere of denoting names: the procedure is just one way of showing the contextual account to be satisfied. In Section 9, I will discuss whether there are alternative ways of showing this.

What Frege's procedure for stepwise extending the sphere of denoting names does is in effect to construct larger and larger *term models* by successively adding new pieces of syntax. When a new name is added by carrying out the basis case and the induction step, we settle the truth-value of every sentence involving this name. Let $\sim$ be the equivalence relation that holds between two proper names $\mathfrak{a}$ and $\mathfrak{b}$ of the resulting language $\mathcal{L}$ when $\ulcorner \mathfrak{a} = \mathfrak{b} \urcorner$ has been deemed true. The first-order domain that results from carrying out Frege's procedure is then modeled by the set of equivalence classes of such names, $\mathcal{L}_0 / \sim$.[25] The domain of first-level concepts is modeled by its powerset (and mutatis mutandis for other higher types). For recall from Section 2 that Frege's quantifiers are to be interpreted objectually. Thus, $\ulcorner \forall f. \mathfrak{F}(f) \urcorner$ is true just in case $\ulcorner \mathfrak{F}(\Phi) \urcorner$ is true for every denoting auxiliary function name '$\Phi\xi$'.[26]

6 The Basis Case in §31

In §31 Frege argues that his primitive logical signs denote. As we've seen, his strategy is, first, to assume that the names of the truth-values denote, and then, to use the context criteria to "gradually widen the sphere of names to be recognized as succeeding in denoting" (p. 87). I'll now discuss his treatment of the various names he wants to add. I'll argue that this too supports the nonauxiliary reading of the context criteria. In fact, on this reading, all of Frege's arguments *look* correct and most of them in fact *are* so.

The first new names Frege seeks to introduce are those of the functions $—\xi$ and $\neg\xi$. To show that these names denote, Frege says, "we have only to show that those names succeed in denoting that result from our putting for 'ξ' "a name of a truth-value" (p. 87). And this follows immediately from his definitions of these functions: Each of the composite names in question will denote a truth-value.

The interesting question, however, is why Frege takes the only relevant background names to be names of truth-values. Two different answers are possible. If the context criteria are read in the nonauxiliary way, the answer is simply that the names of the truth-values are the only proper names accepted so far.[27] If, on the other hand, the context criteria are read in the auxiliary way, one would expect as background names not just the names of the truth-values but any denoting proper names whatsoever.

However, as Heck points out, this expectation is justified only if we take Frege to be working with a universal domain. (See [14], Section 5.) If instead we attribute to Frege a domain consisting of just the two truth-values, it will suffice—even on the auxiliary reading—to consider just the names of the truth-values. Moreover, Heck claims that, at this stage of the proof, Frege is indeed operating with a domain containing just the two truth-values. As evidence Heck adduces a parenthetical remark that immediately follows Frege's statement that it suffices to consider the names of the truth-values. Here Frege writes that "we have not yet recognized other objects"

(p. 87), which Heck interprets as stating that other objects have not yet been *admitted into the domain*.

Although this defense of the auxiliary reading goes some way, I still find it problematic. It cannot explain *why* Frege limits his domain to just the two truth-values. On my nonauxiliary reading, this fact has an obvious explanation: Frege's domain is syntactically characterized, and since the syntax adopted so far is limited, so will be the domain. But on the competing auxiliary reading there is no reason why the domain should be restricted in this way. Since the function names '$—\xi$' and '$\neg\xi$' are nonexpansive, the proper names that result from substituting denoting names for 'ξ' will always denote either the True or the False, which proves that these function names denote. Moreover, since this proof is completely obvious and works no matter what the domain contains, it would be completely gratuitous to restrict the domain as Frege does.[28]

Next Frege introduces the two function names '$\xi \to \zeta$' and '$\xi = \zeta$'. These two cases are analogous to the previous two; in particular, the only background names Frege considers are names of truth-values. So by the same reasoning as in the previous paragraphs, I claim that here too Frege understood the relevant context criterion in the nonauxiliary way and gave a correct argument for the basis case.

Next in line are the names of the first- and second-order quantifiers. Since these two cases are parallel, Frege discusses only the former. The background names he considers are of the form '$\Phi\xi$'. Are these names auxiliary or not? The choice of a uppercase Greek letter suggests the former.[29] This may appear to conflict with my interpretation of Frege's extension procedure, which reads the context criteria in the nonauxiliary way. But in fact, it doesn't. For as we saw in Section 4, in order to guarantee that the induction step will go through, one must, even on my interpretation, require that function names of higher type satisfy conditions much like the auxiliary reading of the context criteria. What Frege needs to show is that, for every function name '$\Phi(\xi)$' we may in the future come to adopt as denoting, $\ulcorner\forall x.\Phi(x)\urcorner$ denotes. And this is precisely what Frege does: he gives an argument intended to be valid at any stage of any extension of the sphere of denoting names. The argument is the obvious one. If '$\Phi(\xi)$' denotes$_1$, then it is either the case that for every denoting proper name 'Δ', '$\Phi(\Delta)$' denotes the True, or not. In the former case, $\ulcorner\forall x.\Phi(x)\urcorner$ denotes the True; in the latter, the False. This argument is valid—provided that some range of quantification has been specified. In particular, the argument is valid at the present stage of the proof, where the range of quantification is just the set of truth-values. However, we will see later that the condition that a range of quantification be specified can be problematic.

Finally, Frege turns to the basis case he regards as "less simple" (p. 88), namely, that of the name of the value-range function. [30] With the name '$\grave{\epsilon}.\varphi\epsilon$', Frege writes,

> we are introducing not merely a new function-name, but simultaneously answering to every name of a first-level function of one argument, a new proper name (value-range-name); in fact not just for those [function-names] known already, but in advance for all such that may be introduced in the future. (p. 88)

Clearly, Frege intends to show that '$\grave{\epsilon}.\varphi\epsilon$' satisfies more than just ($CC_2^{\mathcal{L}}$). For he says that we must inquire whether the value-range name '$\grave{\epsilon}.\Phi\epsilon$' denotes whenever '$\Phi\xi$' is a denoting one-place function name, whether or not this function name has been accepted yet. (A value-range name '$\grave{\epsilon}.\Phi\epsilon$' is said to be *proper*[31] when '$\Phi\xi$'

denotes.) But as remarked in the previous paragraph, this is just what one would expect on my analysis.

What makes this case harder than others is that the name '$\grave{\epsilon}.\varphi\epsilon$' is expansive: proper value-range names aren't definitionally equivalent to any names accepted so far. So in order to prove that proper value-range names denote, we need to invoke the context criteria a second time. Frege describes the work that needs to be done as follows:

> We must examine whether a proper value-range-name placed in the argument-places of "$—\xi$" and "$\neg\xi$" yields a denoting proper name, and further whether, placed in the ξ-argument-places or in the ζ-argument-places of "$\xi \rightarrow \zeta$" and "$\xi = \zeta$", it always forms a denoting name of a first-level function of one argument. (p. 88)

Here there is no doubt that Frege interprets the context criterion for proper names in the nonauxiliary way.[32] This is hardly surprising; for we saw in Section 4 that on the auxiliary reading it is impossible ever to prove that a new proper name denotes.

Frege's argument begins with a series of reductions. First, since '$\neg\xi$' and '$\xi \rightarrow \zeta$' contain horizontals in front of their argument-places, these cases reduce to that of the horizontal.[33] Next, since the horizontal, '$—\xi$', denotes the same as '$\xi = (\xi = \xi)$', the case of the former reduces to that of the latter. It follows that all we need to consider is the name of the identity-function. By applying the relevant context criterion, this leads to the question whether '$\xi = \grave{\epsilon}.\Phi\epsilon$' denotes. And "to that end," Frege writes, "it is to be asked in turn whether all proper names denote something that results from our putting in the argument-place [indicated by 'ξ'] either a name of a truth-value or a proper value-range-name" (p. 88). He claims that all such complex proper names have been assigned a denotation, either by the semantic counterpart of Basic Law V, according to which '$\grave{\epsilon}.\Phi(\epsilon) = \grave{\epsilon}.\Psi(\epsilon)$' denotes the same as '$\forall x(\Phi(x) \leftrightarrow \Psi(x))$', or by the stipulations from §10, which identify the True with $\grave{\epsilon}.—\epsilon$ and the False with $\grave{\epsilon}(\epsilon = \neg\forall x(x = x))$. This argument certainly *looks* correct. For it looks like the criterion of identity for proper value-range names can be read off directly from the abstraction principle (V) that governs the value-range function. However, as we'll see in Section 8 there is a subtle mistake here.

First, however, we must ask why it suffices to fill the argument of '$\xi = \grave{\epsilon}.\Phi\epsilon$' with names of truth-values and proper value-range names. As at the beginning of this section, two different answers are possible. One answer is that Frege understood the context criterion for function names in the nonauxiliary way and that these are the only proper names accepted so far. The other answer is that he understood this criterion in the auxiliary way but that he restricted the domain to truth-values and value-ranges. (Heck defends this reading in [14], p. 460.) As before, I favor the former answer. In order to limit the domain to one containing nothing but truth-values and value-ranges, Frege would need a general concept of value-range. But at this point, he doesn't possess any such concept. Rather, his argument is supposed to introduce value-ranges, to explain what they are. And this introduction is mediated by the introduction of the appropriate syntax.

To sum up, I've claimed that §31 favors the nonauxiliary reading of the context criteria. I've argued that this is clearly the case with the first four function names Frege introduces and with the proper value-range names. And I've explained how the apparent exceptions represented by the names of the quantifiers and of the value-range function are in fact just what one would expect on the nonauxiliary analysis

I presented in Section 4. Moreover, we've seen that all of Frege's arguments *look* correct and that most of them in fact *are* so.

7 The Two Ways of Forming New Names in §30

§30 of *Grundgesetze* serves two distinct purposes: first, to give a recursive definition of the well-formed expressions of the Begriffsschrift, and second, to carry out the induction step. Oddly enough, Frege doesn't clearly distinguish between these two purposes. Moreover, his recursive definition of the class of well-formed expressions seems oddly cumbersome compared to our contemporary way of specifying this class. In this section I will explain how these oddities aren't so much shortcomings on Frege's part as natural consequences of an important theoretical commitment of his. However, for reasons of clarity I will distinguish between the two purposes of §30. I will discuss the first purpose in this section, and the second, in Section 8.

Assume the primitive names we accept are proper names $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$; one-place function names of first type $\mathfrak{f}_1, \ldots, \mathfrak{f}_l$; two-place function names of first type $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$; functions names of second type $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$; and finally, the name '$\forall f.\mu_\beta(f(\beta))$' of the second-order quantification function.[34] (Since Frege regards the logical connectives as functions, there is no need to list these separately.)

All these primitive names are *complete*[35] in the sense that they contain no free variables and purport to stand for specific entities. This is no accident: the Begriffsschrift has no room for free variables. Contemporary formal languages, on the other hand, typically allow free variables. When free variables are available, the formation rules can be very simple: just combine any expressions of matching types. But as we will see shortly, things are less simple without free variables. The problem is that, with the simple formation rules just alluded to, a lot of closed expressions can be reached only by a detour through open formulas, which won't be available when free variables are disallowed. In fact, by requiring that all expressions be complete, Frege in effect requires that his syntactic notion of well-formedness track the semantic notion of having denotation. This explains why he doesn't distinguish between the two purposes of §30.

Frege explains what he calls *the first way* of forming new names as follows:

> Thus there arises
> (A) a proper name
> (1) from a proper name and a name of a first-level function of one argument, or
> (2) from a name of a first-level function and a name of a second-level function of one argument, or
> (3) from a name of a second-level function of one argument of type 2 and the name [of the second-level quantification-function];
> (B) the name of a first-level function of one argument
> (1) from a proper name and a name of a first-level function of two arguments. (p. 85)

Applying the first way, we can form proper names such as $\ulcorner \mathfrak{f}_i(\mathfrak{a}_j) \urcorner$, $\ulcorner \mathfrak{F}_i(\mathfrak{f}_j) \urcorner$, and $\ulcorner \forall f.\mathfrak{F}_i(f) \urcorner$, as well as one-place function names such as $\ulcorner \mathfrak{g}_i(\mathfrak{a}_j, \zeta) \urcorner$ and $\ulcorner \mathfrak{g}_i(\xi, \mathfrak{a}_j) \urcorner$. And obviously, Frege allows the first way to be iterated: "The names so formed may be used in the same way for the formation of further names" (p. 85).

If free variables had been allowed, the first way would have generated the entire class of well-formed formulas. But because free variables aren't allowed, there will be well-formed names that cannot be formed in the first way. We prove this as follows. The characteristic trait of the first way of forming new names is that a name fills an argument-place of a function name of the next type up. Let's call this latter function name *the main function name*. Define *the degree of incompleteness* of an expression E as follows. Choose some enumeration of the finite number n of different types of argument-places. Let the degree of incompleteness of E be the ordered n-tuple $\langle a_1, \ldots, a_n \rangle$, where a_i is the number of (not necessarily distinct) open argument-places in E of type i. On these n-tuples define a partial ordering by

$$\langle a_1, \ldots, a_n \rangle \leq \langle a'_1, \ldots, a'_n \rangle \text{ iff, for all } i,\ a_i \leq a'_i.$$

Note that the first way of forming new names always decreases the degree of incompleteness of the main function name. So the only names that can be formed in this way are those that can be formed by a sequence of steps each of which decreases the degree of incompleteness.

Let's consider some examples. Syntactically, it makes perfect sense to combine the function names $\mathfrak{f}_1$, $\mathfrak{f}_2$, and $\mathfrak{g}$ to form complex function names such as $\ulcorner \mathfrak{f}_1\mathfrak{f}_2\xi \urcorner$, $\ulcorner \mathfrak{g}(\xi, \xi) \urcorner$, $\ulcorner \mathfrak{g}(\mathfrak{f}_1\xi, \zeta) \urcorner$, $\ulcorner \mathfrak{g}(\xi, \mathfrak{f}_2\zeta) \urcorner$, and $\ulcorner \mathfrak{g}(\mathfrak{f}_1\xi, \mathfrak{f}_2\zeta) \urcorner$. But since these complex function names have the same degree of incompleteness as their main function name, they cannot be formed in the first way. There are proper names as well that cannot be formed in the first way, namely, those that arise from applying names of second-level functions to names of first-level functions that cannot be formed in the first way. One example is '$\forall x(x = x)$', the formation of which proceeds via the function name '$\xi = \xi$'.

So the first way must be supplemented with some *second way* that increases the degree of incompleteness. Frege describes this second way as follows:

> [W]e begin by forming a name in the first way, and we then exclude from it at all or some places, a proper name that is a part of it (or coincides with it entirely)—but in such a way that these places remain recognizable as argument-places of type 1. (p. 86)

Let's consider some examples. If $\mathfrak{a}$ and $\mathfrak{b}$ are denoting proper names, we can use the first way to form the proper names $\ulcorner \mathfrak{f}_1\mathfrak{f}_2\mathfrak{a} \urcorner$, $\ulcorner \mathfrak{g}(\mathfrak{a}, \mathfrak{a}) \urcorner$,$\ulcorner \mathfrak{g}(\mathfrak{a}, \mathfrak{f}_2\mathfrak{b}) \urcorner$, $\ulcorner \mathfrak{g}(\mathfrak{f}_1\mathfrak{a}, \mathfrak{b}) \urcorner$, and $\ulcorner \mathfrak{g}(\mathfrak{f}_1\mathfrak{a}, \mathfrak{f}_2\mathfrak{b}) \urcorner$. By applying the second way to these complex proper names we can form the function names $\ulcorner \mathfrak{f}_1\mathfrak{f}_2\xi \urcorner$, $\ulcorner \mathfrak{g}(\xi, \xi) \urcorner$, $\ulcorner \mathfrak{g}(\xi, \mathfrak{f}_2\zeta) \urcorner$, $\ulcorner \mathfrak{g}(\mathfrak{f}_1\xi, \zeta) \urcorner$, and $\ulcorner \mathfrak{g}(\mathfrak{f}_1\xi, \mathfrak{f}_2\zeta) \urcorner$, none of which could be formed in the first way.

Although Frege doesn't explicitly say so, the second way must be allowed to form new function names of *higher types* as well. Since this is just an oversight on his part, I will add this to the list of formation rules. Next, Frege writes that names formed thus far "may be used further to form denoting names in the first way or in the second." That is, he allows his two ways to be iterated. Finally, at the end of §30 he lays down the closure condition that "All correctly-formed names are formed in this manner" (p. 86). It is easily proved that an expression is a well-formed name in the sense of Frege's definition just in case it is a closed well-formed formula in the contemporary sense, based on Frege's primitives.

8 The Induction Step in §30

Frege's treatment of the induction step in §30 is for the most part very quick, which makes the task of interpreting it quite hard. But it is clear that the context criteria are supposed to play a central role. For the sentence preceding Frege's statement of the first way of forming new names states that the context criteria "can serve . . . in the extension step by step of the sphere of [denoting] names. From them it follows that every name formed out of denoting names does denote something" (p. 86). However, not much explanation is provided. For instance, after his statement of the first way, Frege simply asserts that "all names arising in this way succeed in denoting if the primitive simple names do so" (p. 85).

Since the induction step is sometimes thought to favor the auxiliary reading of the context criteria (see, for example, Heck [14], Section 4). I'll begin by examining it on this reading. On this reading, it is extremely easy to show that the first way preserves the property of denoting. For when a function name denotes, it will then form denoting names when combined with *any* denoting name of the type immediately below. The second way is not much harder. Frege argues in the second paragraph of §30 that this way too preserves the property of denoting. For simplicity, I will limit myself to the simplest case of his argument, namely, showing that $\ulcorner \mathfrak{f}_1\mathfrak{f}_2\xi \urcorner$ denotes when $\mathfrak{f}_1$ and $\mathfrak{f}_2$ do; the other cases are analogous. So let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be two denoting one-place function names, and 'Δ', an auxiliary proper name. By two applications of the first way, we form the proper name $\ulcorner \mathfrak{f}_1\mathfrak{f}_2\Delta \urcorner$, which denotes provided 'Δ' does. Then, by applying the second way, we form the function name $\ulcorner \mathfrak{f}_1\mathfrak{f}_2\xi \urcorner$. To verify that this name denotes we need to show that, if 'Δ' denotes, so does the complex proper name $\ulcorner \mathfrak{f}_1\mathfrak{f}_2\Delta \urcorner$. But this follows from the fact that this complex proper name can be formed in the first way. Finally, although Frege doesn't say anything about how the proof is supposed to go for iterations of these two ways, it is not hard to see that on the auxiliary reading this too will go through nicely.

So clearly, the auxiliary reading of the context criteria allows a very nice proof of the induction step. However, I've given a number of arguments in favor of the alternative, nonauxiliary reading. So let's investigate whether this reading too allows a proof of the induction step.

The obvious worry is that on the nonauxiliary reading, the information from the basis case will be too weak to support the induction step. Assume, for instance, we've carried out the basis case for some new function name $\mathfrak{f}$ that we want to add. All this tells us is that $\ulcorner \mathfrak{f}\mathfrak{a} \urcorner$ denotes whenever $\mathfrak{a}$ is a proper name accepted into the sphere of denoting names. So it seems we don't even know that the results of iterated applications of the first way, such as $\ulcorner \mathfrak{f}\mathfrak{f}\mathfrak{a} \urcorner$ and $\ulcorner \mathfrak{f}\mathfrak{f}\mathfrak{f}\mathfrak{a} \urcorner$, denote. But this worry underestimates the nonauxiliary reading. Recall from Section 5 how $\ulcorner \mathfrak{f}\mathfrak{a} \urcorner$ can be shown to denote: either by being equivalent to some proper name $\mathfrak{b}$ already in the sphere of denoting names, or by being such that for every one-place function name $\mathfrak{g}$ in the sphere of denoting names, $\ulcorner \mathfrak{g}(\mathfrak{f}\mathfrak{a}) \urcorner$ is equivalent to some such $\mathfrak{b}$. In either case, we have a lot of control over the situation.

I therefore contend that on the nonauxiliary reading, the induction step looks reasonably promising; in particular, it looks promising enough for §30 not to count against this reading, as it is sometimes thought to do. This contention will be borne out in what follows, where I assemble all the pieces and examine how far Frege's proof gets before it founders. I'll proceed stage by stage.

At stage 0 Frege accepts the names of the truth-values as denoting. So both the initial set of primitives P^0 and the set $\mathcal{L}^0$ of names formed from these primitives consist of nothing but names of truth-values.

At stage 1 Frege introduces the one-place function names '$—\xi$' and '$\neg\xi$'. He carries out the basis case as shown in Section 5.[36] And the induction step goes as Frege sketches in §30. The first way of forming new names preserves the property of denoting because all names formed in this way reduce to names in $\mathcal{L}^0_0$. And the second way preserves denotation because all proper names in $\mathcal{L}^1_0$ can be formed in the first way.

At stage 2 Frege introduces the names of the identity and conditional functions. The basis case goes as discussed in Section 6, and the induction step, as sketched above for stage 1.

At stage 3 Frege introduces the names of the quantifiers. Now things get a bit more complicated. Since these names are of higher type, Frege attempts in the basis case to show that they satisfy the open-ended conditions (OE_2) and (OE_3) discussed in Section 5. As for the induction step, the first way preserves the property of denoting, because every application of the function name '$\forall x.\varphi(x)$' to a function name from $\mathcal{L}^2_1$ denotes a truth-value. To show that the second way too preserves denotation, assume that the proper name $\ulcorner \ldots \forall x \ldots \Delta \ldots \urcorner$ has been formed in the first way and that the second way is applied to "knock out" 'Δ' so as to form the function name $\ulcorner \ldots \forall x \ldots \xi \ldots \urcorner$. Here we run into a problem: to show that this function name denotes, we need to show that the proper name denotes which arises from applying this function name to the proper name $\ulcorner \forall y (\ldots \forall x \ldots y \ldots) \urcorner$, provided this latter denotes. But to find out whether this latter proper name denotes, we need to know whether our original function name does (see Heck [14], p. 448).

However, this problem can be bypassed because the names of the quantifiers are nonexpansive. So we know that adding these function names won't extend the domain. Thus, to show that $\ulcorner \ldots \forall x \ldots \xi \ldots \urcorner$ denotes, it suffices to show that the proper names that result from replacing 'ξ' with *names of truth-values* denote, which they clearly do. The same goes for the name of the second-order quantifier.

At stage 5 Frege attempts to introduce the name of his value-range function. Since this too is a name of higher type, he attempts to show that it satisfies the open-ended criterion (OE_2): he attempts to show that for any function name '$\Phi(\xi)$' we may in the future adopt as denoting, '$\grave{\epsilon}.\Phi(\epsilon)$' denotes. As shown in Section 6, Frege applies a nonauxiliary context condition to these proper names and argues correctly that everything reduces to showing that '$\grave{\epsilon}.\Phi\epsilon = \grave{\epsilon}.\Psi\epsilon$' denotes when '$\Phi\xi$' and '$\Psi\xi$' are denoting function names. And this appears to be ensured by the semantic counterpart of Basic Law V, which stipulates that '$\grave{\epsilon}.\Phi\epsilon = \grave{\epsilon}.\Psi\epsilon$' is to denote the same as '$\forall x(\Phi x \leftrightarrow \Psi x)$'.

Assume for now that this appearance is correct. We would then have an equivalence relation $\sim$ on the set $\{$'$\grave{\epsilon}.\Phi\epsilon$' $\mid$ '$\Phi\xi$' $\text{denotes}_1\}$ of proper value-range names, and our domain would correspond to the $\sim$-equivalence classes on this set. Moreover, on this assumption the induction step would go through exactly as intended.[37]

However, this assumption cannot be correct. For if it were, the "Russell sentence" too would have been assigned a unique denotation. Let '$R\xi$' abbreviate '$\exists f(\xi = \grave{\epsilon}.f\epsilon \wedge \neg f\xi)$' ("the property of being a value-range that doesn't satisfy its own membership criterion"). Let 'r' abbreviate '$\grave{\epsilon}.R\epsilon$'. Familiar reasoning then shows that the Russell sentence 'Rr' is provably equivalent to its own negation. So this sentence cannot have a unique denotation, contrary to what we thought we just proved.

What went wrong? The problem is that the semantic counterpart of Basic Law V fails to define an equivalence relation $\sim$ on value-range names. To determine whether '$\grave{\epsilon}.\Phi\epsilon = \grave{\epsilon}.\Psi\epsilon$' denotes the True, we need to determine what '$\forall x(\Phi x = \Psi x)$' denotes. But the variable 'x' in the latter formula ranges over value-ranges, and this range isn't well defined unless it has *already* been determined when '$\grave{\epsilon}.\Phi\epsilon = \grave{\epsilon}.\Psi\epsilon$' denotes the True and when the False. So here we have a vicious circle.

How can Frege have failed to notice this? My guess is that he was misled by the fact that the name of the first-order quantifier is nonexpansive and incorrectly inferred from this that '$\forall x(\Phi x = \Psi x)$' had been assigned a denotation. But as we've seen, sentences of the form '$\forall x.\Phi x$' haven't been assigned any denotation unless some range of quantification has been specified.

9 Concluding Remarks

Given this flaw, what remains of Frege's proof of referentiality? I have argued that Frege's mistake lies, not in his contextual account of reference or in his procedure for extending the sphere of denoting names, but in his *application* of these ideas to the case at hand.

The problem has to do with the *impredicativity* of (V). Recall from the Introduction that an abstraction principle of the form

(∗) $$\Sigma(\alpha) = \Sigma(\beta) \leftrightarrow \alpha \sim \beta$$

is said to be *predicative* if the equivalence relation $\alpha \sim \beta$ doesn't quantify over entities of the kind to which names of the form '$\Sigma(\alpha)$' purport to refer, and *impredicative*, otherwise. The problem can be analyzed as follows. We are told to assign to each instance of the schematic identity statement on the left-hand side of (∗) the same truth-value as that assigned to the corresponding instance of the right-hand side. But in order to determine the truth-value of a statement that quantifies over certain objects, we need to know *which* objects are quantified over; in particular, we need to know the criteria of identity for the objects in the range of quantification. When (∗) is impredicative, this gives rise to a problem with Frege's procedure for extending the sphere of denoting names. For then the criteria of identity associated with the objects over which the right-hand side of (∗) quantifies are precisely what we are attempting to determine. So we are going round in a circle. Note that this holds for *all* impredicative abstraction principles: not only for *inconsistent* ones such as (V), but also for *consistent* ones such as Hume's Principle

(HP) $$Nx.Fx = Nx.Gx \leftrightarrow F \approx G$$

where $F \approx G$ is the relation of equinumerosity of concepts.[38]

However, for *predicative* abstraction principles Frege's procedure is sound. For then the truth-values of the instances of the right-hand side of (∗) are determined in a way that doesn't involve the truth-values of the instances of the schematic identity statement on its left-hand side. For example, Frege's procedure is sound for the direction abstraction principle

(D) $$d(l) = d(l') \leftrightarrow l \mathbin{//} l'$$

where the relation // is parallelism of lines.[39] The procedure also works for the two-sorted version of (V), where the abstracts introduced on the left-hand side belong to a sort different from that over which the right-hand side quantifies. For this abstraction principle is predicative.[40]

These technical facts give rise to some interesting philosophical questions.

1. Is Frege's contextual account of reference philosophically defensible?
2. We've seen that Frege's procedure for extending the sphere of denoting names can, when restricted to predicative abstraction principles, ensure that the conditions of his contextual account of reference are satisfied. Are there alternative ways of ensuring this?
3. If so, do any of these alternative ways give a privileged status to abstraction principles?

I will end with some brief remarks about how this paper bears on these questions.

Concerning question 1 By clearing away certain technical difficulties, this paper gives us reason to be hopeful that the first question can be answered affirmatively. In particular, the paper makes it clear that the contextual account needn't incorporate the controversial *Grundgesetze* subsumption of the category of sentences under that of proper names. By giving up this subsumption, the contextual account can be cleaned up and made more attractive. But how is the resulting notion of reference to be understood? Dummett argues in [6] that this notion of reference will be "semantically inert" and therefore incapable of supporting a realist interpretation of the discourse in question. Given the syntactical nature of the account, Dummett's view clearly has force, and nothing I have said in this paper contradicts it.[41] But even if we haven't established a robust form of platonism, we have at least shown that the mere presence of terms purporting to denote abstract mathematical objects doesn't prevent a sentence from being true.

Concerning question 2 Alternatives to Frege's method of stepwise extending the sphere of denoting names clearly exist. One interesting alternative is what we may call *Dedekind abstraction*, which begins with a particular realization of a mathematical structure and proceeds from this to the corresponding abstract structure. The positions of this abstract structure are regarded as objects whose only properties are those they have in virtue of being positions in this structure. An example of Dedekind abstraction is Dedekind's own realization of the real numbers in set theory (as Dedekind cuts) and his postulation, based upon this, of the real numbers as sui generis mathematical objects. Clearly, Dedekind abstraction doesn't in any way privilege abstraction principles.

Concerning question 3 The question whether there are alternatives to Frege's method which give privileged status to abstraction principles is important for the assessment of neologicism, which seeks to base mathematics on abstraction principles. (See Wright [27] and Hale and Wright [13].) This paper has shown that Frege's own interest in abstraction principles, in *Grundlagen* as well as in *Grundgesetze*, was intimately connected with his contextual account of reference and in particular with his method of stepwise extending the sphere of denoting names. However, it has also been shown that this method fails to underwrite anything beyond predicative abstraction principles. But predicative abstraction principles are insufficient for the neologicists' attempt to reconstruct classical mathematics.[42] In particular, the neologicists are firmly attached to the impredicative abstraction principle (HP).

The neologicists are therefore forced to part company with Frege and look elsewhere for a method which *both* is strong enough to underwrite impredicative abstraction principles *and* privileges abstraction principles. At the very least, this calls into question the neologicists' alleged Fregean ancestry. But more seriously, it is not clear that a method of the desired kind exists at all.[43] Until such a method has been worked out, the neologicists' right to go beyond predicative abstraction principles will remain in doubt.

Notes

1. Henceforth, all references to Frege [8] will be indicated by page or section number only. Most of the relevant parts of this work are translated as Frege [10].

2. See Parsons [21], Martin [20], Resnik [24], Dummett [6], Heck [14], Weiner [26].

3. The clearest example of this view is Resnik [24]. But see also Heck [14], which defends a hybrid view, involving elements of both a contextual and a noncontextual account of reference. There is also a much more radical view which denies not only that Frege held a contextual account of reference but that he attempted any semantic or metatheoretic reasoning whatsoever; see, for example, Ricketts [25] and Weiner [26].

4. The primary example of this view is Dummett [6], especially Chapter 17. An early but rather condensed statement of this can be found in Parsons [21]. See also Martin [20].

5. I will follow Frege's convention of using lowercase Greek letters to indicate argument places, and uppercase Greek letters as unspecified names of determinate entities (more of which below). Other than that, I will mostly translate his logical formulas into modern symbolism.

6. I will follow Frege in using 'level' to characterize ontological entities. I will use 'type' to characterize syntactic entities; for instance, I will say that the name of a second-level function is of *second type*.

7. Although the 'X' used in §10 looks like a capital Latin letter, it is in fact a capital Greek letter.

8. See Heck [14], p. 8 where he claims that "Frege's talk of truth of instances formed using auxiliary names is not an approximation but an alternative to Tarski's talk of satisfaction by sequences."

9. As we will see in Section 4, Heck [14] holds that this similarity is deceptive.

10. Here we use the fact that Frege had auxiliary names of all syntactic categories.

11. In [14] Heck combines elements of both interpretations, which he refers to as, respectively, an *objectual* and a *substitutional* interpretation. I find this terminology somewhat misleading because the quantification involved in (what I call) a nonauxiliary interpretation needn't be substitutional but can equally well be objectual quantification over syntactic entities. In fact, this is how I will understand the nonauxiliary reading developed below.

12. For some harsh criticism of this subsumption, see Dummett [4], pp. 183–84 and 644.

13. Investigating the Context Principle as a principle about sense rather than reference, Dummett notices an analogous circularity; see [6], pp. 202–204.

14. *Grundlagen* may give the impression that Frege took it to be sufficient to ensure that *identity statements* are meaningful; see especially §62. But as we will see below, in the *Grundgesetze* proof of referentiality Frege was equally concerned to demonstrate the meaningfulness of contexts involving predicates other than that of identity. (Leaving Frege exegesis behind, one might, of course, argue that Frege *ought to* give identity statements a privileged role in the characterization of what it is for a proper name to denote. I pursue this line in my Linnebo [19].)

15. Given that Frege's sense/reference distinction post-dates *Grundlagen* by eight years, it is slightly anachronistic to talk about a predicate's *referring* rather than *being meaningful*. I indulge in this terminology because it is clearer and because nothing turns on this choice.

16. The first simplification is permissible because, whenever $\mathfrak{a}$ satisfies (CC_0), it also satisfies (CC_0'). We see this as follows. Assume that $\mathfrak{g}$ denotes$_{11}$. We need to show that $\ulcorner\mathfrak{g}(\mathfrak{a}, \zeta)\urcorner$ and $\ulcorner\mathfrak{g}(\xi, \mathfrak{a})\urcorner$ denote$_1$. Let's focus on the former case, as the latter is analogous. By (CC_1) it suffices to show that for every $\mathfrak{b}$ that denotes$_0$, $\ulcorner\mathfrak{g}(\mathfrak{a}, \mathfrak{b})\urcorner$ denotes$_0$. But if $\mathfrak{g}$ denotes$_{11}$ and $\mathfrak{b}$ denotes$_0$, then by (CC_{11}) $\ulcorner\mathfrak{g}(\xi, \mathfrak{b})\urcorner$ denotes$_1$, whence by (CC_0) it follows that $\ulcorner\mathfrak{g}(\mathfrak{a}, \mathfrak{b})\urcorner$ denotes$_0$. The second simplification is easily justified by combining (CC_{11}') and (CC_1).

17. See the footnote to §5, where Frege first introduces auxiliary names.

18. Instead of using the customary Gothic variables to range over syntactic items, I here follow Frege's slight abuse of notation and quantify into a quotational context. This allows me to follow his convention of using uppercase Greek letters to indicate that the names in question are auxiliary.

19. Following Frege we always assume that a language contains the names of the truth-values.

20. See [14]. For the claim that (CC_0) must be read in the nonauxiliary way, see p. 154; for the claim that the other context criteria must be read in the auxiliary way, see Section 4.

21. For instance, a mixed reading of the context criteria will be appealing if one modifies Frege's characterization of what it is for a proper name to denote such that only some kinds of contexts in which the proper name can occur matter. See footnote 14 and Linnebo [19].

22. We will see shortly that the property of denoting simpliciter is stable under Frege's extensions of the sphere of denoting names. This allows languages to be cumulative.

23. I here ignore the name of Frege's function $\backslash\xi$, which is introduced at the very end of the proof, after all the hard work has been completed, and which can thus be ignored for present purposes.

24. For instance, that $\ulcorner\neg\alpha\urcorner$ has been assigned the False just in case α has been assigned the True.

25. *Modeled by*, not *identified with*: for even if he had wanted to, Frege wouldn't here be allowed to help himself to sets.

26. We know that $\ulcorner\mathfrak{F}(\Phi)\urcorner$ denotes for every denoting 'Φ' because $\mathfrak{F}$ was shown to satisfy (OE_2) when it was adopted.

27. It may be objected that these names aren't really part of Frege's Begriffsschrift. I admit they don't play any role in Frege's further use of this language in *Grundgesetze*. However, when Frege characterizes his procedure for extending the sphere of denoting names, he writes that "We start from the fact that the names of the truth-values denote . . . . We then gradually widen the sphere of names to be recognized as succeeding in denoting" (p. 87). This can only mean that the names of the truth-values have been included in the sphere of denoting names.

28. My view that the domain is restricted because the syntax is, rather than the other way round, enjoys considerable textual support as well. In the original German, the passage to which Heck calls attention reads "andere Gegenstände kennen wir hier noch nicht" ([8], p. 48). A better English rendering would have been: we do not yet know other objects. The epistemological ring of the word 'know' suggests that Frege's primary concern was with our linguistic access to objects rather than with the objects themselves. Indeed, Frege's choice of verb, "kennen", harks back to the verb "anerkennen" ("recognize") in the preceding sentence, where Frege talks about "the sphere of *names* to be recognized as succeeding in denoting" (p. 87, my emphasis). And two paragraphs below, another cognate of the work 'kennen'—'bekannt'—is attached to *names* in Frege's system. See [8], p. 49.

29. In connection with this proof, Frege's rehearses his earlier stipulation of when '$\forall x.\Phi x$' is to denote the True, namely, just in case '$\Phi\Delta$' denotes the True for 'Δ' as argument, "whatever 'Δ' denotes" (p. 88). So the name 'Δ' is auxiliary.

30. I leave out the name of Frege's final function $\backslash\xi$, which maps a one-member value-range onto its unique member and otherwise behaves like the identity mapping. Clearly, if the name of the value-range function denotes, then so does this function name.

31. In the original German, "gerechte". Furth translates this as "fair".

32. In fact, the only background names he considers are the *primitive* function names already in the sphere of denoting names. Presumably, the complex function names are supposed to be taken care of by the induction step.

33. Frege's negation and conditional functions are such that

$$\neg\xi = \neg(\text{—}\xi) \text{ and } \xi \to \zeta = (\text{—}\xi) \to (\text{—}\zeta).$$

34. Following §30 I consider only this single function name of third type.

35. This notion of completeness must not be confused with that which according to Frege distinguishes objects from concepts.

36. To review, he shows that for every $\mathfrak{a} \in \mathcal{L}_0^0$, $\ulcorner$—$\mathfrak{a}\urcorner$ reduces to some $\mathfrak{b} \in \mathcal{L}_0^0$; and likewise for '$\neg\xi$'.

37. I won't give any details. But as before, it suffices to show that every proper name in $\mathcal{L}^5$ denotes$_0^5$. Furthermore, using Frege's reduction, we can assume that value-range names occur only in argument-places of the name of the identity function.

38. (HP) is impredicative because its right-hand side quantifies over object in general, and the numbers are supposed to be objects.

39. This is Frege's own example when he first introduces abstraction principles; see [9], §64.

40. In fact, by successively adding new abstraction principles of this sort, Frege's procedure can be applied to simple type theory developed to any finite level.

41. However, in Linnebo [19] I develop Fregean ideas about reference in a somewhat different way, which I argue does secure genuine, "semantically active" reference for mathematical terms.

42. Most strikingly, predicative abstraction principles fail to guarantee the existence of a sufficient number of mathematical objects. Let (∗) be a predicative abstraction principle. Then clearly, if there are κ objects in the first-order domain over which the right-hand side of (∗) quantifies, then (∗) can at most establish the existence of 2^κ abstracts.

    The problem is particularly acute in the case of (HP). The predicative counterpart of (HP) is the two-sorted principle (HP2S) which looks syntactically like (HP) but where the numbers and the objects numbered belong to different logical sorts. What is known as *Frege's Theorem* says that in full second-order logic, (HP) and suitable definitions imply all the axioms of second-order Peano Arithmetic. (For a nice exposition, see Boolos [1].) The counterpart of Frege's Theorem for (HP2S) says that (HP2S) and the same definitions imply all the axioms of Peano Arithmetic *except* the Successor Axiom (which says that every natural number has a successor). In fact, there are models of (HP2S) and said definitions where the only numbers are 0 and 1. For a discussion of predicative Frege Arithmetic, see Linnebo [18].

43. Most promising is a method for "recarving contents." See Hale [12]. But there are serious doubts whether this attempt will succeed. See Dummett [7], Potter and Smiley [22], Potter and Smiley [23].

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