Notre Dame Journal of Formal Logic
Volume 45, Number 4, 2004

Equivalence of Syllogisms

Fred Richman

Abstract We consider two categorical syllogisms, valid or invalid, to be equivalent if they can be transformed into each other by certain transformations, going back to Aristotle, that preserve validity. It is shown that two syllogisms are equivalent if and only if they have the same models. Counts are obtained for the number of syllogisms in each equivalence class. For a more natural development, using group-theoretic methods, the space of syllogisms is enlarged to include nonstandard syllogisms, and various groups of transformations on that space are studied.

> Happy families are all alike; every unhappy family is unhappy in its own way. (Leo Nikolaevich Tolstoi. *Anna Karenina*)

1 Categorical Syllogisms

Studies of categorical syllogisms typically focus on the valid ones: which syllogisms are valid, why they are valid, how the valid ones are classified, how to derive valid ones from other valid ones. As Lear [4] put it, "Our principal interest in invalid inferences is to discard them." But the valid syllogisms exist in the context of all syllogisms, just as tautologies exist in the context of all propositional forms. To understand them and the ways we manipulate them, we need to consider this context. In this paper we examine the structure on the set of *all* syllogisms induced by the traditional methods for transforming one valid syllogism into another.

Categorical syllogisms are inferences of the form $p \wedge q \Rightarrow r$ where p, q, and r are statements about pairs of classes. The statements p and q are the *premises*, the statement r is the *conclusion*. Each statement is of one of the four types:

Received May 6, 2003; accepted August 1, 2003; printed October 26, 2004
2000 Mathematics Subject Classification: Primary, 03B99
Keywords: categorical syllogism

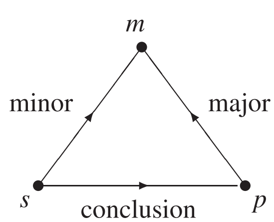
Axy	All x are y
Exy	No x are y
Ixy	Some x are y
Oxy	Some x are not y

Note that O is the negation of A and E is the negation of I. The terms x and y refer to classes. The term x is called the *subject*, y the *predicate*. A typical example of a statement is ‘all dogs are animals’. The traditional syllogism contains precisely three terms each of which occurs in two of the three statements. The term common to the two premises is called the *middle term*. The subject and predicate of the conclusion are the *subject term* and the *predicate term* of the syllogism. The premise containing the predicate term is the *major premise*, the other is the *minor premise*. The major premise is traditionally written first (so p is the major premise).

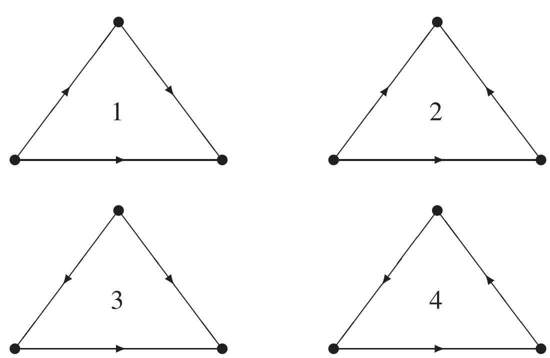
With this structure there are exactly four ways for the statements to share terms. These are the traditional *figures,* described by the following table, where s denotes the subject term, p the predicate term, and m the middle term.

Major	Minor	Conclusion	Figure
mp	sm	sp	1
pm	sm	sp	2
mp	ms	sp	3
pm	ms	sp	4

We can represent the figures by triangles, as is done in Richards [7]. The vertices of the triangles correspond to the terms, the sides to the statements. We put an arrow on each side from the subject to the predicate. The base of the triangle represents the conclusion, and we take that arrow from left to right, so the major premise is the right side of the triangle and the minor premise is the left side.



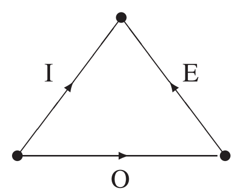
Here are the triangles for the four figures:



A *form* of a syllogism is obtained taking a figure and assigning one of the four statement types, A, E, I, and O, to each side of the triangle. This assignment is called the *mood* of the syllogism. Thus there are 256 forms, 64 in each figure. We denote the forms by symbols like EIO-2, which means the major premise has type E, the minor premise has type I, the conclusion has type O, and the figure is 2. To get a syllogism from a form, we assign a class to each vertex. For example, EIO-2 is exemplified (or instantiated) by the argument,

No dogs are cats.
Some carnivores are cats.
Therefore, some carnivores are not dogs.

This is a *valid* syllogism, an instantiation of a valid form: if we replace the terms 'dogs', 'cats', and 'carnivores', by terms denoting any classes whatsoever, the conclusion will be true if the premises are. This form, EIO-2, is traditionally called *Festino*, a mnemonic whose vowels are EIO. Its triangular representation is



Here are the traditional mnemonic names of 24 of the forms, arranged by figures:

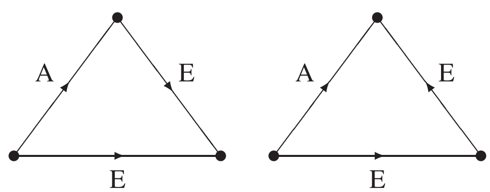
1	2	3	4
Barbara	Cesare	Darapti ∗	Bramantip ∗
Celarent	Camestres	Felapton ∗	Camenes
Darii	Festino	Disamis	Dimaris
Ferio	Baroco	Datisi	Fesapo ∗
Barbari †	Camestrop †	Bocardo	Fresison
Celaront †	Cesaro †	Ferison	Camenop †

The 15 unmarked forms are the valid ones. The 9 marked forms are valid if Axy is interpreted as 'all x are y, and there are some x'. This is known as attributing *existential import* to A. The 5 forms marked with a dagger come from other forms of the same figure by weakening the conclusion (assuming existential import). For example, the conclusion of *Barbari* is that some x are y, while the conclusion of *Barbara* is that all x are y.

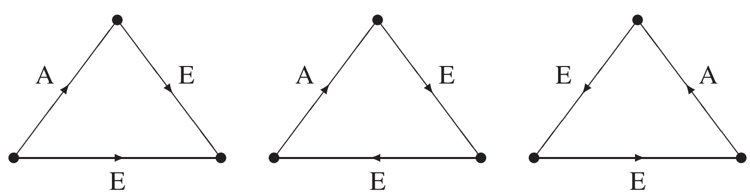
Having pointed out the distinction between forms and their instantiations, the reader is warned that we will tend to use the terms "syllogism" and "form" interchangeably.

2 Transformations and Nonstandard Syllogisms

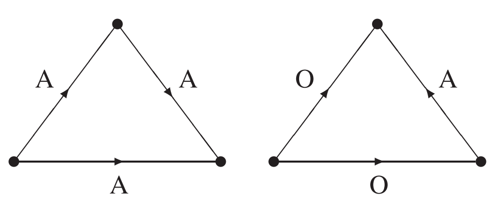
The classical transformations, used by Aristotle [1], are *conversion* and *indirect reduction* (*reductio ad impossibile*). In *conversion* the subject and predicate of an E or I statement are interchanged, and the two premises are then interchanged, if necessary, so that the major one comes first. Applying conversion to the major premise of EAE-1, *Celarent*, we get EAE-2, *Cesare*. In terms of triangles, this transformation corresponds to reversing the arrow on the right side:



Applying conversion to the conclusion of EAE-1, we get AEE-4, *Camenes*, after interchanging premises:



In *indirect reduction*, one of the premises and the conclusion are interchanged and negated, that is, we pass from $p \wedge q \Rightarrow r$ to $p \wedge \neg r \Rightarrow \neg q$ or to $\neg r \wedge q \Rightarrow \neg p$, and then interchange the two premises, if necessary. In this way we go from AAA-1, *Barbara*, to AOO-2, *Baroco* and back. Geometrically, we are flipping the triangle around one of the base vertices and changing the letters on the adjacent sides:



With conversion and indirect reduction the 15 valid forms can be obtained from just two: *Barbara* and *Celarent*.

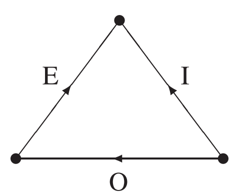
> It's quite indisputable, for
> I'll prove it with singular ease,
> You shall have it in 'Barbara' or
> 'Celarent'—whichever you please.
>
> W. S. Gilbert, *The Force of Argument*

The interchange of premises, to insure that the major premise comes first, seems a little hokey. However, no interchange is necessary if we think of a syllogism as being a major premise, a minor premise, and a conclusion, rather than a first premise, a second premise, and a conclusion. A more serious objection is that the transformations are not one-to-one, hence do not form a group. For example, both AAA-1, *Barbara*, and AOO-2, *Baroco*, are transformed to OAO-3, *Bocardo*, by applying indirect reduction to the major premise and the conclusion.

To get around this, we can enlarge the definition of a syllogism slightly. Allowing the major premise to appear in the second position, rather than just in the first, necessitates four more figures, which we denote by $1'$, $2'$, $3'$, and $4'$, in which the major premise comes second. We will call these figures *nonstandard*. For example, IEO-$2'$ is exemplified by the argument,

Some carnivores are cats.
No dogs are cats.
Therefore, some carnivores are not dogs.

In the triangle representation, we simply flip the EIO-2 triangle, *Festino*, around its vertical axis of symmetry so the premises are interchanged:



We might call this form *Festino*$'$. Note that the arrow on the base goes from right to left for nonstandard figures if we keep the convention that the statements on the sides of the triangles are read in order: right side, left side, base. Now, when we apply indirect reduction to the major premise and the conclusion of AOO-2, *Baroco*, we get AOO-$3'$, *Bocardo*$'$, rather than OAO-3, *Bocardo*, and the transformation is one-to-one.

The nonstandard figures seem to provide the right setting for indirect reduction—we get a space of 512 syllogisms with a natural group action on it. The group may be thought of as the group D_3 of symmetries of an equilateral triangle. The two flips about the vertices of the base generate this six-element group. There are 84 orbits of this group action of size six, and four of size two. The latter are $\{AAO\text{-}4, AAO\text{-}4'\}$, $\{EEI\text{-}4, EEI\text{-}4'\}$, $\{IIE\text{-}4, IIE\text{-}4'\}$, and $\{OOA\text{-}4, OOA\text{-}4'\}$. Each orbit is composed half of standard figures and half of nonstandard figures.

It is less clear that this is the right setting for conversion. We can only convert statements of type E and I. If we want a transformation on the space that corresponds, say, to conversion of the conclusion, then we have to define it also when the conclusion is not of type E or I. We could, of course, say that conversion leaves statements of type A or O unaffected, but that goes against the basic meaning of "conversion", which is to interchange subject and predicate. More natural would be to interchange subject and predicate, and change the type of the statement, if necessary, so that the resulting statement is equivalent to the original one. This would require a type of statement that "quantifies the predicate." De Morgan [3] introduced the statement types a and o in such a way that axy is equivalent to Ayx, and oxy is equivalent to Oyx. With these additional statement types, we can define conversion of the conclusion to reverse the arrow on the base and interchange the letters A and a, and the letters O and o.

A third transformation, not used systematically by Aristotle, is *obversion*. The *obverse* of the statement Axy is $Ex\bar{y}$; the obverse of the statement Ixy is $Ox\bar{y}$. Here $\bar{x}$ denotes the complement of the class denoted by x. Each statement is equivalent to its obverse—to say that all dogs are animals is to say that no dogs are nonanimals. We can apply obversion to a syllogism provided that some term appears only as a predicate, that is, in every figure except the fourth. If we apply it to the predicate of *Celarent* we get *Barbara*, and vice versa.

As with conversion, we would like to obvert at any term in a syllogism not just at those that are double predicates. So, for example, we would like a sentence type that is equivalent to $A\bar{x}y$, like Exy is equivalent to $Ax\bar{y}$. De Morgan [3] called this type e. He considered nonstandard types a, e, i, and o defined by

$$axy = A\bar{x}\bar{y} = E\bar{x}y$$

$$exy = E\bar{x}\bar{y} = A\bar{x}y$$

$$ixy = I\bar{x}\bar{y} = O\bar{x}y$$

$$oxy = O\bar{x}\bar{y} = I\bar{x}y$$

(Boole [2] also considered these as four of "the eight fundamental types of propositions.") Note that $axy = Ayx$ and $oxy = Oyx$, so adding the types a and o enables arbitrary arrow reversals (there never was a problem reversing arrows with E and I statements). Thus any syllogism with a standard figure can be put into any standard figure by arrow reversals on the top sides. For example, the second figure version of *Barbara* is aAA-2, which comes from *Barbara* by reversing the right arrow. Forms with nonstandard conclusion types, such as Aaa-2, do not correspond so directly to standard syllogisms. No form containing e or i corresponds to a standard syllogism just by arrow reversals and interchanging premises.

While a and o can be viewed as simply quantifying the predicate, and thus do not differ essentially from A and O, the statement types e and i are essentially different. Unlike the other six types, they involve unbounded quantification. For example, the i statement 'there is something that is neither a cat nor a dog', requires considering objects outside the two classes in question.

With De Morgan's extended set of 8 statement types, we can construct $8^3 = 512$ forms, each in the second figure. The $6 \cdot 6 \cdot 4 = 144$ forms not containing e or i and having a standard conclusion type correspond to the standard syllogisms. Because of the phenomenon that EAA-1 and EAA-2 are associated with the same second figure form, the number 144 undercounts the number of standard syllogisms. The undercount is

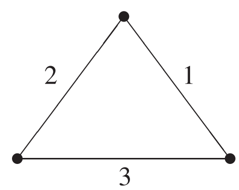
$$3(2 \cdot 2 \cdot 4) + 2(4 \cdot 2 \cdot 4) = 112.$$

The $2 \cdot 2 \cdot 4$ in the first term counts those forms that contain I or E in both premises; the second term counts those forms that contain an I or E in one premise.

3 The Syllogism Group

If we include the four nonstandard figures and the four nonstandard statement types, we get a space of $8^4 = 4096$ syllogistic forms. This is convenient for theoretical purposes because it is closed under obversion at each vertex, conversion at each side, and indirect reduction at each base vertex. So it is acted on naturally by the group G generated by obversions, conversions, and indirect reductions. Note that each of these generators of G has order two. We denote the identity of G by 1.

What is the structure of G as an abstract group (as opposed to a permutation group)? To fix the notation, number the sides of a triangle as follows,



using the same numbers for the opposite vertices. The conclusion goes on side 3, and if the arrow on side 3 goes from left to right, then the premise on side 1 is the major premise. Of the 4096 syllogisms, 512 involve only standard types and 256 of those have the major premise on side 1, that is, the arrow on side 3 goes from left to right. These are the 256 traditional forms. The generators of G are the following:

Conversion c_i reverses the arrow on side i and changes the case of the letters A, a, O, and o on that side.

Obversion o_i changes the letters on each side adjacent to vertex i. With the arrow toward the vertex, the permutation is $(AE)(IO)(ae)(io)$. With the arrow away from the vertex, the permutation is $(Ae)(Io)(Ea)(Oi)$.

Indirect reduction τ_i flips the triangle around vertex i and negates the letters on the adjacent sides. Here $i = 1, 2$. For τ_3 we don't want to negate the letters, and τ_3 is not an indirect reduction but simply the interchange of premises. In any event $\tau_3 = \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2$.

Indirect reduction (*reductio ad impossibile*) does not consider the internal structure of the statements p, q, and r. It is part of propositional calculus rather than predicate calculus. Starting from the syllogism $p \wedge q \Rightarrow r$, the operation τ_1 takes it to $p \wedge \neg r \Rightarrow \neg q$, the operation τ_2 takes it to $\neg r \wedge q \Rightarrow \neg p$, and the operation τ_3 takes it to $q \wedge p \Rightarrow r$.

What are the relations among the generators of G? Note that changing the case of A, O, a, and o commutes with negating. The following relations hold:

$$o_i^2 = c_i^2 = \tau_i^2 = 1$$
$$(\tau_1\tau_2)^3 = 1$$
$$c_i c_j = c_j c_i \qquad o_i o_j = o_j o_i \qquad c_i o_j = o_j c_i$$
$$c_i \tau_i = \tau_i c_i \qquad o_i \tau_i = \tau_i o_i$$
$$c_3 = \tau_1 c_2 \tau_1 = \tau_2 c_1 \tau_2 \qquad o_3 = \tau_1 o_2 \tau_1 = \tau_2 o_1 \tau_2$$

The τ_i generate a subgroup T isomorphic to S_3, the symmetric group on three letters. The o_i and the c_i generate commuting eight-element Abelian normal subgroups O and C of type $(2, 2, 2)$. Each element in G can be written uniquely as a product τoc with $\tau \in T$, $o \in O$, and $c \in C$, so G has order $6 \cdot 8 \cdot 8 = 384 = 2^7 3$.

The groups OT and CT are isomorphic. The group C is a normal subgroup of order 8 in CT. This subgroup is the kernel of the map from CT onto the group of permutations of the terms. The subgroup CT can be thought of as the group of symmetries of the cube by associating the sides of the triangles with the three principal axes of symmetry of the cube. The case changes and negations can be ignored because the relations are right. The order of CT is $48 = 2^4 3$ (this result appears in [7]). The same analysis applies to the subgroup OT.

The subgroup OC, which is Abelian of type $(2, 2, 2, 2, 2, 2)$, is the kernel of the map from G onto the group of permutations of the three terms, the symmetries of the (undecorated) triangle. Clearly G is a semidirect product of OC by T, the latter subgroup mapping isomorphically onto the symmetries of the triangle.

The group G has exactly three Sylow 2-subgroups: $\langle OC, \tau_1 \rangle$, $\langle OC, \tau_2 \rangle$, and $\langle OC, \tau_3 \rangle$. Thus OC, their intersection, is a natural subgroup of G. Conjugation by $c_1c_2c_3$ or $o_1o_2o_3$ fixes τ_1 and τ_2, while conjugation by τ_1 or τ_2 interchanges $\tau_2\tau_1$ and $\tau_1\tau_2$. These elements generate a 24-element subgroup fixing the three-element subgroup $\langle \tau_1\tau_2 \rangle$ under conjugation. It is not hard to show that this subgroup is the normalizer of $\langle \tau_1\tau_2 \rangle$, that is, it contains all elements that fix $\langle \tau_1\tau_2 \rangle$ under conjugation. Thus there are $384/24 = 16$ conjugates of $\langle \tau_1\tau_2 \rangle$, which constitute the Sylow 3-subgroups of G.

4 Equivalence of Syllogistic Forms

We have a syntactic notion of when two syllogisms are equivalent: when they can be transformed into each other by an element of G. We will show that the syntactic notion corresponds to a semantic notion. This will not only endow the syntactic notion with meaning, but will provide a convenient method for showing that two syllogisms are inequivalent. The semantic notion comes from considering instantiations.

It will be convenient to think of a syllogism as an implication rather than an inference. That is, we think of $p \wedge q \Rightarrow r$ as a statement that is either true or false. (Łukasiewicz [5] claims that Aristotle himself thought of syllogisms as implications rather than inferences.) When we say that a syllogism is *true*, we are saying that the statement of the implication is true. A syllogism is *valid*, on the other hand, if its form is valid. A syllogistic form, like EIO-2, gives rise to a function from ordered triples of sets to truth values—a *ternary relation* among sets. If α is a form we denote its ternary relation by R_α. By $R_\alpha(x, y, z)$ we mean the truth value of the syllogism that instantiates α when x is the class referred to by the subject term, y by the middle term, and z by the predicate term. The form α is *valid* if $R_\alpha(x, y, z)$ is true for all sets x, y, and z.

We will consider three increasingly weaker notions of equivalence of syllogistic forms α and β. The third one will be our definition of *equivalence*.

Definition 1 $R_\alpha = R_\beta$.

Definition 2 $R_\alpha = R_\beta \circ \pi$, where π is a permutation of the variables.

Definition 3 $R_\alpha = R_\beta \circ \pi$, where π permutes the variables and may also complement some of them.

Definition 1, which we will ignore, gives a fairly restrictive equivalence although it is nontrivial. The forms EAI-1 and EAI-2 are equivalent because of the symmetry of E. That is, the two implications

$$Eyz \wedge Axy \Rightarrow Ixz \text{ and}$$
$$Ezy \wedge Ayz \Rightarrow Ixz$$

are logically equivalent. The forms $\alpha = EAI$-1 and $\beta = AEI$-4 are not equivalent because $R_\alpha(\varnothing, \varnothing, \{0\})$ is false and $R_\alpha(\varnothing, \varnothing, \{0\})$ is true, whereas we might expect equivalence because the symmetry of I results in the logical equivalence of the two implications

$$Eyz \wedge Axy \Rightarrow Ixz \text{ and}$$
$$Axy \wedge Eyz \Rightarrow Izx.$$

For these two forms to be equivalent, we must allow a permutation of the inputs into the ternary relations. That is, we must pass to Definition 2.

Definition 2 respects *conversion* and *indirect reduction*. By indirect reduction we go from $\alpha = AAA$-1 to $\beta = AOO$-2, and we see that $R_\alpha(x, y, z) = R_\beta(x, z, y)$. Definition 3 respects obversion in addition to conversion and indirect reduction. In Definition 2 we act on the ternary relations by permuting the inputs—a six-element group isomorphic to the symmetry group of an equilateral triangle. In Definition 3, where we also allow the inputs to be complemented, we act on the relations with a 48-element group isomorphic to the symmetry group of a cube: we can think of the inputs as the (directed) major axes of the cube with complementation occurring if the directions get reversed.

Clearly any form equivalent to a valid form is also valid. More generally, if α is equivalent to β, and S is any set, then the number of triples of subsets of S making R_α true is equal to the number of triples of subsets of S making R_β true.

5 Orbits of the Syllogism Group G

An *orbit* of G is a set of the form $\{g\alpha : g \in G\}$ for some syllogism α. Two syllogisms are in the same orbit of G if some element of G transforms the one into the other. If two syllogisms are in the same orbit of G, then we can transform one into the other by a sequence of conversions, obversions, and indirect reductions.

Question 5.1 *If two standard syllogisms are in the same orbit of G, can we transform one into the other by a sequence of conversions, obversions, and indirect reductions* without leaving the subspace of standard syllogisms*?*

We will also want to answer the corresponding question with G replaced by CT, the group of classical transformations. The point is that the syllogism space was enlarged for convenience, not so that a standard syllogism could be transformed into more standard syllogisms than it could before.

We need to clarify this question a bit. Even the classical transformations take you out of the standard space of 256 forms, but the identification of α with $\tau_3\alpha$, where the major and minor premises are interchanged, is so strong that they are thought of as the same syllogism. Indeed they *are* the same syllogism if we think of a syllogism as consisting of a major premise, a minor premise, and a conclusion, rather than a first premise, a second premise, and a conclusion, as we have been doing. For example, if we convert the conclusion of *Celarent*, EAE-1, then we get EAE-4′, a nonstandard form. But simply writing the major premise first, we get AEE-4, *Camenes*. So, to stay within the subspace of standard syllogisms, we really need only stay within the space of 512 forms, that include nonstandard figures but only standard statement types, in which we allow the minor premise to come first.

The main reason for allowing nonstandard figures, and enlarging the standard space to 512, is that the transformations generated by indirect reduction form a group on that space—each transformation has an inverse. If we restrict ourselves to standard figures by writing the major premise first in the transformed syllogism, then τ_2 takes both AAA-1, *Barbara*, and AOO-2, *Baroco*, to OAO-3, *Bocardo*. As far as the computation of orbits is concerned, that doesn't really matter; we can operate just as well with a monoid of functions that are not one-to-one. What is important is that the functions be *locally invertible*: if $f(\alpha) = \beta$, then there exists g so that $g(\beta) = \alpha$. Then we can still talk about orbits.

If two syllogisms are in the same orbit of G, then they are equivalent in the sense of Section 4. Our second question is about the converse.

Question 5.2 *If two syllogisms are equivalent, are they in the same orbit of G?*

To answer both questions, we first show that every orbit of G contains a standard syllogism. Then we consider the (not necessarily invertible) transformations o_i^*, c_i^*, and τ_i^*, taking standard syllogisms to standard syllogisms, defined as follows. For $i = 1, 2, 3$, if $o_i(\alpha)$ is standard, then $o_i^*(\alpha) = o_i(\alpha)$, otherwise $o_i^*(\alpha) = \alpha$, and the same for $c_i^*(\alpha)$. For $i = 1, 2$, if $\tau_i(\alpha)$ is standard, then $\tau_i^*(\alpha) = \tau_i(\alpha)$, otherwise $\tau_i^*(\alpha) = \tau_3\tau_i(\alpha)$. These are the traditional obversions, conversions, and indirect reductions, extended in a harmless manner to standard syllogisms where they don't apply. Let G^* be the monoid generated by these functions. To show that G^* is locally invertible, it suffices to show that the generators are. Clearly o_i^* and c_i^* are their own inverses, and, for i and j distinct elements of $\{1, 2\}$, if $\tau_i(\alpha)$ is nonstandard, then $\tau_j^*\tau_i^*(\alpha) = \alpha$, while if $\tau_i(\alpha)$ is standard, then $\tau_i^*\tau_i^*(\alpha) = \alpha$. We will calculate the orbits of G^* and verify that elements of different orbits are inequivalent. This last step is computer assisted.

To see that every orbit of G contains a standard syllogism, start with a syllogism α. By passing to $\tau_3(\alpha)$, if necessary, we may assume that the figure is standard. By applying conversions, we may assume that the only nonstandard statement types that appear in α are i and e. If α has two nonstandard statement types, then by applying conversion we can point their arrows away from each other and obvert the subjects, giving standard types. If there are three nonstandard types, we can reduce to one nonstandard type in the same way. If there is exactly one nonstandard type, we can bring it to the conclusion with τ_1 or τ_2. If one of the top edges points to the base, we can convert the nonstandard sentence so that it points away and obvert. So we need only consider Figure 2. These forms are transformed to a standard form by the sequence of transformations o_2, o_3, c_1, c_3, τ_3. For example, AAe-2 is transformed as follows:

$$AAe\text{-}2, \quad eAa\text{-}2, \quad aEa\text{-}2, \quad AEa\text{-}1, \quad AEA\text{-}4', \quad EAA\text{-}4\,.$$

It is interesting to note that the much maligned Figure 4 is the only standard figure that appears with standard statement types in each orbit of G. Indeed, AAO-4 is the unique standard form in its orbit.

The *full negation symmetry* on the space of syllogisms, or standard syllogisms, replaces each statement type by its negation and leaves the figures alone. Elements of G and G^* commute with the full negation symmetry: It's easy to see that if the form α is transformed to β by one of the generating transformations, then its full negation $\bar{\alpha}$ is transformed to $\bar{\beta}$ by that same transformation. Moreover, a standard form α cannot be equivalent to its full negation $\bar{\alpha}$. If there is a triple of classes that makes α false, then $\bar{\alpha}$ is true for that triple. Otherwise, α is a valid syllogism and $\bar{\alpha}$ cannot also be valid because of the rules "there can be at most one negative premise" and "if one premise is negative, the conclusion must be negative" (see the rules for valid antilogisms below: there must be two positive statements and one negative statement in a valid antilogism). So the orbits come in pairs, one orbit being the full negation of the other.

The monoid G^* has 20 orbits: four each with 9 and 15 points, and two each with 1, 3, 7, 18, 21 and 30 points. *Barbara*, AAA-1, is in a 15-orbit consisting of the valid syllogisms; *Barbari*, AAI-1, is in a 9-orbit consisting of those syllogisms that are true for all nonempty classes (these are valid syllogisms if we assume *existential import*: Axy entails there exist xs).

Here are the 20 orbits of G^*, paired by the full negation symmetry, with the sizes of the orbits given on the left, and the first syllogism in the orbits on the right.

1	AAO-4	OOA-4
3	AAA-4	AOA-4
7	AAI-2	IIE-1
9	AAE-3	AIA-2
9	AAI-1	IOA-1
15	AAA-1	AOA-3
15	AAO-1	IIA-1
18	AAE-1	AOE-3
21	AAE-2	AIA-1
30	AAA-2	AIE-1

It remains to verify that no two of the syllogisms listed in the table are equivalent. This is a different kind of calculation—a semantic rather than a syntactic one. We choose a finite universe, say $\{1, 2, \ldots, n\}$, and look at all triples of subsets of it. Fortunately we can get by with a small value of n. However, there are 48 ways to hook up two syllogisms by permuting and complementing inputs in order to test for equivalence.

There is a test for inequivalence that avoids head-to-head comparison of syllogisms, but it doesn't quite do the job. If two syllogisms are equivalent, then they will be false on the same number of triples of subsets. Here are those counts for $n = 3$.

Number of triples of subsets that make a syllogism false				
AAO-4	8		OOA-4	48
AAA-4	56		AOA-4	96
AAI-2	64		IIE-1	175
AAE-3	98		AIA-2	115
AAI-1	27		IOA-1	90
AAA-1	0		AOA-3	30
AAO-1	64		IIA-1	114
AAE-1	37		AOE-3	54
AAE-2	61		AIA-1	30
AAA-2	61		AIE-1	91

We still need to compare AAA-2 with AAE-2, AOA-3 with AIA-1, and AAI-2 with AAO-1. One might hope that a bigger universe would distinguish these, but this is not the case, as we now demonstrate.

It is not hard to see that the counts will always be 4^n for AAI-2 and AAO-1. The syllogism AAI-2 is false only if m contains the disjoint sets s and p. This happens 4^n times. The syllogism AAO-1 is false only if s contains m and m contains p, which also happens 4^n times. The correspondence is $s = m$, $m = s \cup p$, and $p = p$. Conversely, $m = s$, $s = m \setminus p$, and $p = p$.

To see that the counts for AAA-2 and AAE-2 are always the same, look at the correspondence $s = s$, $m = m$, and $p = m \setminus p$—a sort of relative obversion. The count, which is $5^n - 4^n$, is gotten by choosing 4 disjoint sets, which can be done in 5^n ways, then subtracting the 4^n ways that this is done with the first set empty.

What about AIA-1 and AOA-3? The first is false if p contains m, and s intersects m and is not contained in p. The second is false if p contains m, and s does not contain m and is not contained in p. In short, both conditions say p contains m but not s. The first says also that m intersects s, the second that m intersects $p - s$. So letting $s = (s \setminus p) \cup (p \setminus s)$, gives a correspondence. By exclusion-inclusion the count is $6^n - 2 \cdot 5^n + 4^n$.

To show that none of these three pairs are equivalent, a program was written to try all 48 possibilities on each of the three pairs. They all failed. So we have answered the two questions in the affirmative.

Here are the counts for a few other syllogisms. The syllogism AAO-4 is false only if $s = p = m$. This happens 2^n times, once for each subset of the universe. The syllogism AAE-3 is false only if m is contained in s and p, and s intersects p. The count is $5^n - 3^n$. The syllogism AAI-1, *Barbari*, is false only if p is empty and m is contained in s. This happens 3^n times. (Because *Barbari* is true for all nonempty classes, it is sometimes considered valid. The only reason that it was not part of the standard list of 19 valid syllogisms, or 14 excluding Figure 4, was because it was thought that one should always use the stronger *Barbara* instead.)

It turns out that if two orbits of G contain the same number of standard syllogisms, then they have the same size. This was ascertained by a program that computed the orbits. The following table shows the number of standard syllogisms for each of the 20 orbits of G, the corresponding size of the orbit, and the number of orbits containing that number of standard syllogisms.

standard syllogisms	1	3	7	9	15	18	21	30	256
syllogisms	64	192	64	192	192	384	192	384	4096
number of orbits	2	2	2	4	4	2	2	2	20

Note that the size of four of the orbits is 384, the order of G. Of course, all of the sizes divide the order of G, the quotient being the number of symmetries of a syllogism in that orbit. For example, the syllogism AAO-4 is left invariant by the six-element subgroup T, so the size of its orbit is $384/6 = 64$.

6 Orbits of Subgroups of *G*

By restricting the allowable transformations, we get different notions of equivalence of syllogisms. The equivalence classes are the orbits of various subgroups of G. In this section we study those orbits and the corresponding orbits in the space of 256 standard syllogisms.

How many orbits are there in the space of 256 standard syllogisms if we just allow conversion? Let $c_i^*(\alpha) = c_i(\alpha)$, if $c_i(\alpha)$ is standard, and $c_i^*(\alpha) = \alpha$ otherwise (that is, if you are trying to convert an A or an O). This generates a group, C^*, of order 8. So the sizes of the orbits can only be 1, 2, 4, or 8. There are 114 orbits which break down as follows:

size of orbit	1	2	4	8
number of orbits	40	44	28	2

Of the 1-orbits (fixed points) 32 are boring: those syllogisms in which all sentence types are A or O, so conversion doesn't really ever apply. The 8 interesting fixed points are XXY-m where $X \in \{A, O\}$, $Y \in \{I, E\}$, and $m \in \{2, 3\}$. To construct an element of a 2-orbit, there are 3 choices for where the I or E goes, then 2^3 choices for the mood (the other sides must have A or O) and 4 choices for the figure. Finally you must subtract the 8 interesting fixed points, resulting $3 \cdot 8 \cdot 4 - 8 = 88$ elements in 2-orbits. There are four 4-orbits with just Es and Is, containing one syllogism from each figure: EEI-n, IIE-n, III-n, and EEE-n. In addition, there are $3 \cdot 2^3 = 24$ moods in which exactly one statement is of type A or O, and each of these gives a 4-orbit. One 8-orbit is IEI-1, IEI-2, IEI-3, IEI-4, EII-1, EII-2, EII-3, EII-4. The other 8-orbit is gotten by interchanging Es and Is.

As for the orbits of C itself, in the space of all 4096 syllogisms, each contains 8 elements and there are 512 of them. Clearly each C^* orbit is contained in a C orbit, and no C orbit contains more than one C^* orbit.

The indirect-reduction submonoid T^*, which acts on the space of standard syllogisms, has 84 orbits of 3, and 4 fixed points: AAO-4, EEI-4, IIE-4, and OOA-4. Twenty of the 3-orbits involve only fourth figure syllogisms, the other 64 contain one syllogism from each of the other three figures. These 64 orbits are what make G^* fail to be a group. Each orbit looks like

$$XYZ\text{-}1 \xleftrightarrow{\tau_1^*} X\bar{Z}\bar{Y}\text{-}2 \underset{\tau_1^*}{\overset{\tau_2^*}{\leftrightarrows}} \bar{Z}Y\bar{X}\text{-}3 \xleftrightarrow{\tau_2^*} XYZ\text{-}1,$$

while the fourth-figure orbits look like

$$XYZ\text{-}4 \underset{\tau_1^*}{\overset{\tau_2^*}{\leftrightarrows}} \bar{Z}X\bar{Y}\text{-}4 \underset{\tau_1^*}{\overset{\tau_2^*}{\leftrightarrows}} Y\bar{Z}\bar{X}\text{-}4 \underset{\tau_1^*}{\overset{\tau_2^*}{\leftrightarrows}} XYZ\text{-}4.$$

(In Patzig [6], for example, it is noted that the 24 valid forms group into triples by indirect reduction.)

The orbits of the indirect-reduction subgroup T, in the big space, have size 6 except for the eight 2-orbits

AAO-4	EEI-4	IIE-4	OOA-4	aao-4	eei-4	iie-4	ooa-4
AAO-4′	EEI-4′	IIE-4′	OOA-4′	aao-4′	eei-4′	iie-4′	ooa-4′

That gives $(4096 - 16)/3 = 1360$ orbits of size 6. (It's easier to see this in terms of antilogisms.)

Using both indirect reduction and conversion we get 44 orbits. The only fixed points are AAO-4 and OOA-4. There are two 1-orbits, eighteen 3-orbits, two 4-orbits, twelve 6-orbits, and ten 12-orbits.

$$2 \cdot 1 + 18 \cdot 3 + 2 \cdot 4 + 12 \cdot 6 + 10 \cdot 12 = 256,$$

for a total of $2 + 18 + 2 + 12 + 10 = 44$ orbits. Of the 104 orbits of CT in the big space, 44 contain standard syllogisms.

Standard syllogisms	Orbit size	Orbits
0	8	2
0	24	18
0	48	40
1	16	2
3	24	8
3	48	10
4	8	2
6	48	12
12	24	2
12	48	8

Barbara is in an orbit of size 48 containing the three standard syllogisms, *Barbara*, *Baroco*, and *Bocardo*. *Celarent* is in an orbit of size 48 containing the remaining twelve valid standard syllogisms.

Using just obversion we get 160 orbits: 64 fixed points (Figure 4) and 96 orbits of 2. Figures 1, 2, and 3 each admit a unique obversion.

Using indirect reduction and obversion we get 56 orbits: four fixed points (same as for indirect reduction alone), thirty two 6-orbits, and twenty 3-orbits. The 3-orbits contain the rest of the Figure 4 forms.

Using conversion and obversion we get 42 orbits: eight fixed points (Figure 4 with As and Os), four 3-orbits, four 5-orbits, ten 6-orbits, four 7-orbits, ten 10-orbits, and two 14-orbits.

7 Antilogisms

An antilogism is just like a syllogism except that an antilogism says $\neg(p \wedge q \wedge r)$ rather than $p \wedge q \Rightarrow r$. Antilogisms are represented by the same kind of triangles as syllogisms are. As $\neg(p \wedge q \wedge r)$ is equivalent to $p \wedge q \Rightarrow \neg r$, antilogisms are just another way of looking at syllogisms. Because of the complete symmetry of the statements p, q, and r in $\neg(p \wedge q \wedge r)$, we view any permutation of those statements to be the same antilogism. In terms of triangles, any of the six placements of a triangle with given arrows and statement types on the sides, is considered the same antilogism—the antilogism is the decorated triangle itself, independent of its placement.

We say that the syllogism $p \wedge q \Rightarrow r$ *belongs to* the antilogism $\neg(p \wedge q \wedge \neg r)$, or that the antilogism $\neg(p \wedge q \wedge r)$ *contains* the syllogism $p \wedge q \Rightarrow \neg r$. One virtue of considering antilogisms is that one antilogism contains many syllogisms. The antilogism $\neg(p \wedge q \wedge r)$ contains the three syllogisms $p \wedge q \Rightarrow \neg r$, $p \wedge r \Rightarrow \neg q$, and $q \wedge r \Rightarrow \neg p$, where we might have to interchange premises to standardize the syllogism. Thus, indirect reduction is built into the notion of an antilogism.

There are only two antilogism figures—cyclic and acyclic—depending on whether the arrows are all head to tail or not. We call the cyclic one *Figure 4* because it contains the Figure 4 syllogisms. The acyclic antilogism, which contains the syllogisms in Figures 1, 2, and 3, we call *Figure 1*. Each standard antilogism contains exactly 3 syllogisms, except for the four antilogisms of the form XXX-4, which contain one syllogism apiece and are not valid.

We have seen that there are $84 + 4$ orbits of T^* consisting of four fixed points, twenty 3-orbits involving only Figure 4 syllogisms, and sixty-four 3-orbits containing one syllogism from each of the other three figures. The 88 orbits correspond to the 88 distinct standard antilogisms. There are $64 = 4^3$ antilogisms in Figure 1 and $24 = 4 + 12 + 8$ in Figure 4: The $4 = \binom{4}{1}$ use one of the letters A, E, I, and O, the $12 = 4 \cdot 3$ use two, and the $8 = 2\binom{4}{1}$ use three. If we also allow obversion, the 64 antilogisms in Figure 1 pair up to give $32 + 24 = 56$ orbits.

Rules for testing the validity of (standard) antilogisms are simpler than those for syllogisms. A term is *distributed* in a statement if decreasing its class cannot change the statement from true to false; so x is distributed in Axy and Exy while y is distributed in Exy and Oxy. The following rules suffice to distinguish the 5 valid antilogisms (that contain the 15 valid syllogisms).

1. Each term is distributed in some statement.
2. There are two positive statements and one negative statement.
3. There are two universal statements and one particular statements (or just not three universal statements).

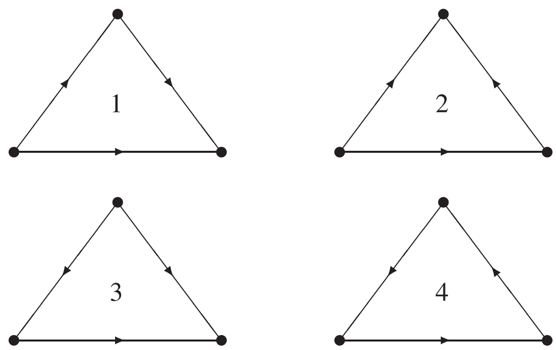
If a term is not distributed, take proper nonempty sets that make the opposite side true (one-element sets suffice), then take the third set to be bigger than each of those two sets. Three positive statements fail with the same one-element set at each vertex. Two negative statements fail by putting the same one-element set on the vertices of the positive statement, and a different one-element set (the complement) on the third vertex. Three negative statements fail by putting three different one-element sets on the vertices. Three universal statements fail by putting the empty set on each vertex (AAE-1, which contains *Barbari*, and AAE-4, which contains *Bramantip*, and EAA-1, which contains *Darapti*, are all this rules out, of those that pass the first two tests). If we eliminate the third rule, we get the 8 antilogisms that are valid assuming existential import.

Alternatively, the following two rules suffice for the 5 valid antilogisms:

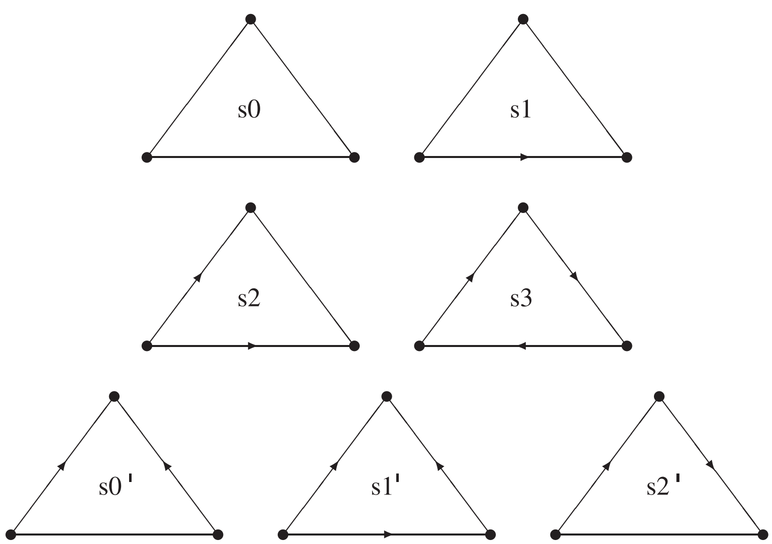
1. Each term is distributed exactly once.
2. There are two positive statements and one negative statement.

8 Symmetrized Antilogism Figures

We have seen that there are four ways to put arrows on the sides of a triangle corresponding to the four traditional figures of a syllogism.



By a *symmetrized figure* we mean a triangle that has arrows on some (possibly all or none) of its sides. The idea is that the sides with arrows represent statements of type A or O, and those without arrows represent statements of type E or I. By considering antilogisms with symmetrized figures, we build both conversion and indirect reduction into the picture. There are seven symmetrized antilogism figures, which may be described as: no arrows, one arrow, two arrows out, Figure 4, two arrows in, Figure 1, two arrows along:



To count the number of equivalence classes of antilogisms under indirect reduction and conversion, we simply count the number of distinct ways we can assign letters to these figures.

1. For $s0$ the letters must be I and E and it only matters how many of each. So there are four classes: EEE, EEI, EII, and III.

2. For $s1$ we have two symmetric types around an asymmetric type. There are eight ways to do this.
3. For $s2$ we must pick two from $\{A, O\}$ and one from $\{E, I\}$, which can be done in six ways.
4. For $s3$, the letters must be A and O, and it only matters how many of each. So there are four classes: AAA, AAO, AOO, and OOO.
5. For $s0'$ we must pick two from $\{A, O\}$ and one from $\{E, I\}$, which can be done in six ways.
6. For $s1'$ we pick an ordered triple from $\{A, O\}$, so there are eight classes.
7. For $s2'$ we pick an ordered pair from $\{A, O\}$ and an element of $\{E, I\}$, so there are eight classes.

$s0$	$s1$	$s2$	$s3$	$s0'$	$s1'$	$s2'$	total
4	8	6	4	6	8	8	44

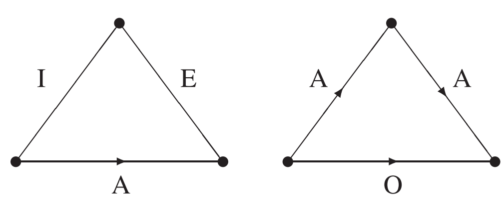
We can also count and classify the 88 antilogisms from this:

$s0$	$s1$	$s2$	$s3$	$s0'$	$s1'$	$s2'$	total
$2+2+4+2$	$4 \cdot 8$	$2 \cdot 4$	4	$2 \cdot 4$	$2 \cdot 4$	$2 \cdot 4 \cdot 4$	88

Six of the seven figures are paired by obversion. Each of the primed figures admits exactly one obversion, which takes it to the corresponding unprimed figure, and any obversion of the unprimed figure is either itself or the corresponding primed figure. We can count the equivalence classes of antilogisms under obversion, indirect reduction, and conversion, by seeing in how many distinct ways we can assign letters to the unprimed figures. For $s0$, $s2$, and $s3$ the analysis is the same as above. For $s1$, there are two pairs that are equivalent under obversion: AII and OEI, and AIE and OEE. (We list the asymmetric type first, and the type it points to second.) The obversion changes AI to OE. So there are six classes.

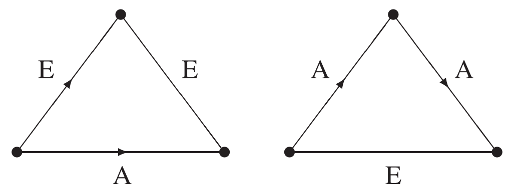
$s0$	$s1$	$s2$	$s3$	total
4	6	6	4	20

The valid syllogisms come from the class AEI-$s1$. This splits into two classes in the absence of obversion: AEI-$s1$ and AAO-1, the latter's symmetrized figure being $s1'$.



The first class consists of the twelve syllogisms *Camenes*, *Camestres*, *Celarent*, *Cesare*, *Darii*, *Datisi*, *Dimaris*, *Disamis*, *Ferio*, *Ferison*, *Festino*, and *Fresison*; the second consists of *Barbara*, *Baroco*, and *Bocardo*.

The pseudovalid syllogisms, those true for nonempty sets but not in general, are in the class AAE-$s2$. This splits into two classes in the absence of obversion: AAE-$s2$ and AAE-$s2'$.



The first class consists of *Celaront*, *Cesaro*, and *Darapti*; the second consists of the six syllogisms *Barbari*, *Bramantip*, *Camenop*, *Camestrop*, *Felapton*, and *Fesapo*.

The classification of valid and pseudovalid syllogisms into antilogism classes appears in [7].

9 Recapitulation and Questions

Two definitions of equivalence of syllogisms, one syntactic and one semantic, were given and shown to be equivalent. The standard space of 256 syllogisms was extended to a space of 4096 syllogisms acted upon by a group G of transformations that come from the classical transformations used to generate all valid syllogisms from *Barbara*. The number of orbits of G of each size were computed and the structure of G as an abstract group was ascertained. Antilogisms were suggested as a more efficient way to analyze equivalence of syllogisms.

What happens if you try to do everything with existential import? In that case, the space of syllogisms has a more complicated structure. Not only is there an equivalence relation, there is a partial order given by implication: For example, XYA-n implies XYI-n. How easy is it to describe this partial order, and how interesting is it? Actually, I haven't checked whether there are noninvertible implications even without existential import.

What happens if you try to do everything in the context of intuitionistic logic? Neither indirect reduction nor obversion is guaranteed, a priori, to produce equivalent syllogisms, yet the classically valid syllogisms are intuitionistically valid. What are the proper definitions of syntactic and semantic equivalence?

References

[1] Aristotle, *Prior Analytics*, Hackett Publishing Co., Indianapolis, 1989. Zbl 0726.01003. MR 91g:03001. 218

[2] Boole, G., *An Investigation of the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probabilities*, Dover Publications Inc., New York, 1957. MR 19,1d. 220

[3] De Morgan, A., "On the structure of the syllogism, *Transactions of the Cambridge Philosophical Society*. viii (1846)," pp. 379–408 in *"On the syllogism" and Other Logical Writings*, edited by P. Heath, Yale University Press, New Haven, 1966. MR 37:6147. 219, 220

[4] Lear, J., *Aristotle and Logical Theory*, Cambridge University Press, Cambridge, 1980. Zbl 0571.03002. MR 84h:03015. 215

[5] Łukasiewicz, J., *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford, 1951. Zbl 0043.24601. MR 14:713a. 222

[6] Patzig, G., "Syllogistic," pp. 890–98 in *Encyclopaedia Britannica*, Encyclopaedia Britannica, Chicago, 15th edition, 1979. 228

[7] Richards, D., "Categorical Syllogisms," Master's thesis, Florida Atlantic University, Boca Raton, 2000. 216, 222, 232, 233

Acknowledgments

I would like to thank my student Dawne Richards for inspiring this paper. She insisted on writing a Master's thesis in logic although she was not required to. Many of the ideas developed here are mentioned in her thesis [7].

Department of Mathematics
Florida Atlantic University
Boca Raton FL 33431
richman@fau.edu