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# Computational Survey on A Posteriori Error Estimators for the Crouzeix–Raviart Nonconforming Finite Element Method for the Stokes Problem

**Carsten Carstensen · Christian Merdon**

*Abstract* — This survey compares different strategies for guaranteed error control for the lowest-order nonconforming Crouzeix–Raviart finite element method for the Stokes equations. The upper error bound involves the minimal distance of the computed piecewise gradient $\mathrm{D}_{\mathrm{NC}}\, u_{\mathrm{CR}}$ to the gradients of Sobolev functions with exact boundary conditions. Several improved suggestions for the cheap computation of such test functions compete in five benchmark examples. This paper provides numerical evidence that guaranteed error control of the nonconforming FEM is indeed possible for the Stokes equations with overall efficiency indices between 1 to 4 in the asymptotic range.

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*Keywords:* Nonconforming Finite Element Method, Crouzeix–Raviart Finite Element Method, Adaptive Finite Element Method, A Posteriori Error Estimation.

## 1. Introduction

The a posteriori error analysis of conforming FEM is well established and contained even in textbooks [3, 8, 9, 22]. Although a unified framework is established [11, 14], much less is known about a posteriori error analysis for nonconforming lowest-order Crouzeix–Raviart finite element methods [1, 2, 4, 7, 16, 17]. This paper concerns the 2D Stokes equations: Given a right-hand side $f \in L^2(\Omega;\mathbb{R}^2)$ and Dirichlet data $u_D \in H^1(\Omega;\mathbb{R}^2)$ with $\int_{\partial\Omega} u_D \cdot \nu\, ds = 0$ (for the unit normal vector $\nu$), seek a pressure $p \in L^2_0(\Omega) := \{q \in L^2(\Omega) \mid \int_\Omega q\, dx = 0\}$ and a velocity field $u \in H^1(\Omega;\mathbb{R}^2)$ with

$$-\Delta u + \nabla p = f \quad \text{and} \quad \operatorname{div} u = 0 \text{ in } \Omega \quad \text{while} \quad u = u_D \text{ on } \partial\Omega.$$

The primal variable $u$ will be discretised with the nonconforming Crouzeix–Raviart FEM on some shape-regular triangulation $\mathcal{T}$ with discrete solution $u_{\mathrm{CR}}$. This paper discusses and

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compares different a posteriori error estimators for the error $e = u - u_{\mathrm{CR}}$ and its piecewise gradient $\mathrm{D}_{\mathrm{NC}}\, e$ in the (nonconforming) energy norm

$$|||e|||_{\mathrm{NC}}^2 := \|\mathrm{D}_{\mathrm{NC}}\, e\|_{L^2(\Omega)}^2 := \sum_{T \in \mathcal{T}} \|\mathrm{D}\, e\|_{L^2(T)}^2.$$

The decomposition from [2] allows for a split of this error into

$$|||e|||_{\mathrm{NC}}^2 \leqslant \eta^2 + \left(\|\mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v)\|_{L^2(\Omega)} + 1/c_0\, \|\mathrm{div}\, v\|_{L^2(\Omega)}\right)^2.$$

The first term on the right-hand side (with the first positive root $j_{1,1} \geqslant 3.8317$ of the first Bessel function $J_1$)

$$\eta := \|f_{\mathcal{T}}/2 \otimes (\bullet - \mathrm{mid}(\mathcal{T}))\|_{L^2(\Omega)} + 1/j_{1,1}\ \mathrm{osc}(f, \mathcal{T})$$

involves contributions of the data $f$ and is directly computable (up to quadrature errors). The second term employs *any* $v \in H^1(\Omega; \mathbb{R}^2)$ with $v = u_D$ along $\partial\Omega$, e.g., componentwise interpolation by [1] or novel interpolations on the red-refined triangulation $\mathrm{red}(\mathcal{T})$ from [15]. The constant $c_0$ depends only on the domain $\Omega$ and equals the smallest eigenvalue of some general eigenvalue problem [19], cf. Section 3 below. This paper compares several possible designs of $v$ and applies them in the five benchmark examples of Table 1.

| Section | Short name | Problem data & features | $c_0$ |
|---|---|---|---|
| 5.1 | Smooth Solution | $f \neq 0$, $u_D \neq 0$ | 0.3826 |
| 5.2 | 2nd Smooth Solution | $f \neq 0$, $u_D \neq 0$ | 0.3826 |
| 5.3 | Colliding Flow | $f \equiv 0$, $u_D \neq 0$ | 0.3826 |
| 5.4 | L-shaped Domain | $f \equiv 0$, $u_D \neq 0$, corner singularity | 0.3 |
| 5.5 | Backward Facing Step | $f \equiv 0$, $u_D \neq 0$, corner singularity | 0.3 |

**Table 1.** Benchmark examples and section references.

The remaining parts of this paper are outlined as follows. Section 2 introduces the necessary notation, preliminaries and our adaptive mesh refinement algorithm. Section 3 presents the a posteriori error analysis and derivates a sharp upper bound for the energy error with explicit constants. Section 4 explains the realisation of the guaranteed upper bounds. Section 5 gives details on the error estimator competition for the five benchmark problems from Table 1. Section 6 draws some conclusions from the theoretical and numerical results of this paper.

Throughout this paper we use standard notation for Lebesgue and Sobolev spaces and their norms: $V := H_0^1(\Omega; \mathbb{R}^2)$ is endowed with the energy norm $|||\cdot||| := \|\nabla \cdot\|_{L^2(\Omega)} = |\cdot|_{H^1(\Omega)}$. Finally $a \lesssim b$ abbreviates $a \leqslant Cb$ for some generic constant $C$ that depends only on the shape regularity of the triangulation while $a \approx b$ stands for $a \lesssim b \lesssim a$.

## 2. Notation and Preliminaries

### 2.1. Nonconforming Finite Element Spaces

Given a regular triangulation $\mathcal{T}$ of the bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$ in the sense of Ciarlet into closed triangles with edges $\mathcal{E}$, nodes $\mathcal{N}$ and free nodes $\mathcal{M}$, the midpoints of all edges are $\operatorname{mid}(\mathcal{E}) := \{\operatorname{mid}(E) \mid E \in \mathcal{E}\}$ and $\mathcal{E}(\partial\Omega)$ denotes the edges along the boundary $\partial\Omega$. The point $\operatorname{mid}(T)$ denotes the center of gravity of $T \in \mathcal{T}$ and defines the piecewise constant $L^2$ function $\operatorname{mid}(\mathcal{T}) \in P_0(\mathcal{T};\mathbb{R}^2)$ by $\operatorname{mid}(\mathcal{T})|_T = \operatorname{mid}(T)$ for all $T \in \mathcal{T}$. The set $\mathcal{E}(T)$ contains the three edges of a triangle $T \in \mathcal{T}$. With the elementwise first-order polynomials $P_1(\mathcal{T};\mathbb{R}^2)$, the nonconforming Crouzeix–Raviart finite element spaces are defined by

$$\begin{aligned}
\mathrm{CR}^1(\mathcal{T};\mathbb{R}^2) &:= \big\{v \in P_1(\mathcal{T};\mathbb{R}^2) \mid v \text{ is continuous at } \operatorname{mid}(\mathcal{E})\big\},\\
\mathrm{CR}^1_0(\mathcal{T};\mathbb{R}^2) &:= \big\{v \in \mathrm{CR}^1(\mathcal{T};\mathbb{R}^2) \mid v(\operatorname{mid}(E)) = 0 \text{ for all } E \in \mathcal{E}(\partial\Omega)\big\}.
\end{aligned}$$

The Crouzeix–Raviart finite elements form a subspace of the broken Sobolev functions

$$H^1(\mathcal{T}) := \big\{v \in L^2(\Omega) \mid v|_T \in H^1(T) \text{ for all } T \in \mathcal{T}\big\}.$$

The diameter $\operatorname{diam}(T)$ of $T \in \mathcal{T}$ is denoted by $h_T$ and $h_{\mathcal{T}}$ denotes their piecewise constant values with $h_{\mathcal{T}}|_T := h_T := \operatorname{diam}(T)$ for all $T \in \mathcal{T}$. The integral mean of a function $f \in L^2(\omega)$ (or any vector $f \in L^2(\omega;\mathbb{R}^2)$) over some open set $\omega$ is denoted by

$$f_\omega := \fint_\omega f\,dx := \int_\omega f\,dx/|\omega|.$$

The oscillations of $f \in L^2(\Omega)$ (as well as of vectors $f \in L^2(\Omega;\mathbb{R}^2)$) read

$$\operatorname{osc}(f,\mathcal{T}) := \Big(\sum_{T\in\mathcal{T}} \|h_T(f - f_T)\|^2_{L^2(T)}\Big)^{1/2} = \|h_{\mathcal{T}}(f - f_{\mathcal{T}})\|_{L^2(\Omega)}.$$

### 2.2. Crouzeix–Raviart FEM for the Stokes Equations

The discrete bilinear form reads

$$a_{\mathrm{NC}}(u_{\mathrm{CR}}, v_{\mathrm{CR}}) := \sum_{T\in\mathcal{T}} \int_T \mathrm{D}\,u_{\mathrm{CR}} : \mathrm{D}\,v_{\mathrm{CR}}\,dx$$

for all $u_{\mathrm{CR}}, v_{\mathrm{CR}} \in X_h := \mathrm{CR}^1(\mathcal{T};\mathbb{R}^2) \subseteq H^1(\mathcal{T};\mathbb{R}^2)$ with $A : B := \sum_{j,k=1,2} A_{jk}B_{jk}$ for all $2\times 2$ matrices $A, B \in \mathbb{R}^{2\times 2}$. The particular choice of

$$Y_h := P_0(\mathcal{T}) \cap L^2_0(\Omega) \quad \text{and} \quad b_{\mathrm{NC}}(v,q) := \int_\Omega q\,\operatorname{div}_{\mathrm{NC}} v\,dx,$$

with the piecewise divergence operator $\operatorname{div}_{\mathrm{NC}}$, leads to the discrete counterpart

$$Z_{\mathrm{NC}} := \big\{v_{\mathrm{CR}} \in \mathrm{CR}^1_0(\mathcal{T};\mathbb{R}^2) \mid \operatorname{div}_{\mathrm{NC}} v_{\mathrm{CR}} = 0\big\}$$

of the space of divergence-free functions

$$Z := \big\{v \in H^1_0(\Omega;\mathbb{R}^2) \mid \operatorname{div} v = 0\big\}.$$

The nonconforming representation of the Stokes problem reads: Given $f \in L^2(\Omega;\mathbb{R}^2)$, seek $u_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T};\mathbb{R}^2)$ with $\operatorname{div}_{\mathrm{NC}} u_{\mathrm{CR}} = 0$,

$$u_{\mathrm{CR}}(\operatorname{mid}(E)) = \fint_E u_D\,ds \quad \text{for all } E \in \mathcal{E}(\partial\Omega),$$

and

$$a_{\mathrm{NC}}(u_{\mathrm{CR}}, v_{\mathrm{CR}}) = F(v_{\mathrm{CR}}) := \int_\Omega f \cdot v_{\mathrm{CR}}\, dx \quad \text{for all } v_{\mathrm{CR}} \in Z_{\mathrm{NC}}.$$

In other words, up to boundary conditions, $u_{\mathrm{CR}}$ is computed from the Riesz representation of a linear functional (given as right-hand side plus boundary modifications) in the Hilbert space $(Z_{\mathrm{NC}}, a_{\mathrm{NC}})$. The actual implementation uses unconstrained Crouzeix–Raviart elements $v_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T}; \mathbb{R}^2)$ and another Lagrange multiplier to enforce the global constraint

$$\mathrm{div}_{\mathrm{NC}}\, u_{\mathrm{CR}} = 0 \quad \text{a.e. in } \Omega.$$

## 2.3. Adaptive Mesh Refinement Algorithm

**Algorithm 2.1** (ACRFEM)**.**
**Input** Coarse regular triangulation $\mathcal{T}_0$ and $0 < \Theta \leqslant 1$.
For level $\ell = 0, 1, 2, \ldots$ until termination do

**Compute** discrete solution $u_{\mathrm{CR}}$ on $\mathcal{T}_\ell$ with $N_\ell$ degrees of freedom. For any $v_{\mathrm{xyz}}$ for xyz $\in$ {A, PMA, MAred, PMred, MP1, MP2, MP2CG5, MP1red, MP1redCG3}, compute the refinement indicators

$$\begin{aligned}\eta_{\mathrm{xyz}}(T)^2 &:= \eta(T, v_{\mathrm{xyz}})^2 \\ &:= \left\| f_{\mathcal{T}}/2 \otimes (\bullet - \mathrm{mid}(T)) \right\|^2_{L^2(T)} + 1/j_{1,1}^2 \left\| h_T (f - f_T) \right\|^2_{L^2(T)} \\ &\quad + \left\| \mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v_{\mathrm{xyz}}) \right\|^2_{L^2(T)} + 1/c_0^2 \left\| \mathrm{div}\, v_{\mathrm{xyz}} \right\|^2_{L^2(T)} \\ &\quad + (1 + 1/c_0^2) C_\gamma^2 \left\| h_{\mathcal{E}}^{3/2} \partial^2_{\mathcal{E}} (u_D - v_{\mathrm{xyz}})/\partial s^2 \right\|^2_{L^2(\partial\Omega \cap \partial T)}\end{aligned}$$

with $C_\gamma$ from an estimate for inhomogeneous Dirichlet boundary data from Section 4.4. With the tangential jump $[\mathrm{D}_{\mathrm{NC}}\, u_{\mathrm{CR}} \cdot \tau_E]_E$ for interior edges $E \in \mathcal{E} \setminus \mathcal{E}(\partial\Omega)$ and

$$[\mathrm{D}_{\mathrm{NC}}\, u_{\mathrm{CR}} \cdot \tau_E]_E := \left\| \partial u_D/\partial s - \mathrm{D}_{\mathrm{NC}}\, u_{\mathrm{CR}} \cdot \tau_E \right\|_{L^2(E)}$$

for boundary edges $E \in \mathcal{E}(\partial\Omega)$ with tangential vector $\tau_E$, the residual-based refinement indicators read

$$\eta_{\mathrm{R}}(T)^2 := |T| \|f\|^2_{L^2(T)} + |T|^{1/2} \sum_{E \in \mathcal{E}(T)} \left\| [\mathrm{D}_{\mathrm{NC}}\, u_{\mathrm{CR}} \cdot \tau_E]_E \right\|^2_{L^2(E)}.$$

Estimate

$$\begin{aligned}\eta(v_{\mathrm{xyz}})^2 := \Big( \left\| f_{\mathcal{T}}/2 \otimes (\bullet - \mathrm{mid}(\mathcal{T})) \right\|_{L^2(\Omega)} + 1/j_{1,1}\, \mathrm{osc}(f, \mathcal{T}) \Big)^2 + \Big( \left\| \mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v_{\mathrm{xyz}}) \right\|_{L^2(\Omega)} \\ + 1/c_0 \left\| \mathrm{div}\, v_{\mathrm{xyz}} \right\|_{L^2(\Omega)} + (1 + 1/c_0) C_\gamma \left\| h_{\mathcal{E}}^{3/2} \partial^2_{\mathcal{E}} (u_D - v_{\mathrm{xyz}})/\partial s^2 \right\|_{L^2(\partial\Omega)} \Big)^2.\end{aligned}$$

**Mark** a minimal subset $\mathcal{M}_\ell$ of $\mathcal{T}_\ell$ (based on one set of refinement indicators) such that

$$\Theta \sum_{T \in \mathcal{T}_\ell} \eta_{\mathrm{xyz}}(T)^2 \leqslant \sum_{T \in \mathcal{M}_\ell} \eta_{\mathrm{xyz}}(T)^2.$$

**Refine** $\mathcal{T}_\ell$ by *red*-refinement of triangles in $\mathcal{M}_\ell$ and *red-green-blue*-refinement [10, 23] of further triangles to avoid hanging nodes and compute $\mathcal{T}_{\ell+1}$.

**Output** Sequence of meshes $\mathcal{T}_0, \mathcal{T}_1, \ldots$ with respective discrete solution $u_{\mathrm{CR}}$ and residual-based error estimator

$$\eta_{\mathrm{R}}(\mathcal{T}_\ell) := \|h_{\mathcal{T}_\ell} f\|_{L^2(\Omega)} + \Big( \sum_{E \in \mathcal{E}_\ell} |E| \, \big\| [\mathrm{D}_{\mathrm{NC}}\, u_{\mathrm{CR}} \cdot \tau_E]_E \big\|^2_{L^2(E)} \Big)^{1/2}.$$

## 3. A Posteriori Error Estimation for Stokes Equations

This section is devoted to guaranteed upper bounds for the error in the nonconforming energy norm for the Stokes problem based on a decomposition from [2] with a slightly sharper upper bound. The general reliability result involves the computable term

$$\eta := \big\| f_{\mathcal{T}}/2 \otimes (\bullet - \mathrm{mid}(\mathcal{T})) \big\|_{L^2(\Omega)} + 1/j_{1,1}\ \mathrm{osc}(f, \mathcal{T}).$$

Here, $j_{1,1}$ is the first positive root of the first Bessel function $J_1$ and, for two vectors $x, y \in \mathbb{R}^2$, $x \otimes y := (x_1y_1, x_1y_2; x_2y_1, x_2y_2) \in \mathbb{R}^{2\times 2}$ denotes their dyadic product. For every triangle $T \in \mathcal{T}$ with the set of its edges $\mathcal{E}(T)$ and $s(T)^2 := \sum_{E\in\mathcal{E}(T)} |E|^2$, an elementary calculation shows

$$\big\| f_{\mathcal{T}}/2 \otimes (\bullet - \mathrm{mid}(\mathcal{T})) \big\|^2_{L^2(T)} := |f_{\mathcal{T}}/2|^2 \big\| \bullet - \mathrm{mid}(\mathcal{T}) \big\|^2_{L^2(T)} = |f_{\mathcal{T}}|^2 |T| s(T)^2/144.$$

The constant $c_0$ in the inf-sup condition

$$0 < c_0 := \inf_{q \in L^2_0(\Omega)} \ \sup_{v \in H^1(\Omega;\mathbb{R}^2)/\mathbb{R}^2} \frac{\int_\Omega q \, \mathrm{div}(v)\, dx}{\|\mathrm{D}\, v\|_{L^2(\Omega)} \|q\|_{L^2(\Omega)}}$$

depends only on the domain $\Omega$ and equals the smallest eigenvalue of some general eigenvalue problem [19]. Their values are given in Table 1.

The following a posteriori error estimate resembles the upper bound from [2, Theorem 1]

$$\begin{aligned} |||e|||_{\mathrm{NC}} \leqslant{}& \big\| f_{\mathcal{T}}/2 \otimes (\bullet - \mathrm{mid}(\mathcal{T})) \big\|_{L^2(\Omega)} + C\, \mathrm{osc}(f, \mathcal{T}) \\ &+ \big\| \mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v) \big\|_{L^2(\Omega)} + 1/c_0 \, \|\mathrm{div}\, v\|_{L^2(\Omega)} \end{aligned} \tag{3.1}$$

and gives a refined version with an explicit value for $C$.

**Theorem 3.1.** *Any $v \in H^1(\Omega; \mathbb{R}^2)$ with $v = u_D$ on $\partial\Omega$ satisfies*

$$|||e|||^2_{\mathrm{NC}} \leqslant \eta^2 + \big( \big\| \mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v) \big\|_{L^2(\Omega)} + 1/c_0 \, \|\mathrm{div}\, v\|_{L^2(\Omega)} \big)^2. \tag{3.2}$$

*Proof.* The analysis follows [2] and is repeated here for convenient reading to stress the little differences to [2, Theorem 1]. These differences concern the powers in (3.2) compared to (3.1), which lead to a sharper guaranteed upper bound, and the explicitly given constant in (3.5). The point of departure is the orthogonal split

$$\mathrm{D}_{\mathrm{NC}}\, e = \mathrm{D}\, z + y$$

into $z \in Z$ with

$$\int_\Omega \mathrm{D}\, z : \mathrm{D}\, v \, dx = \int_\Omega \mathrm{D}_{\mathrm{NC}}\, e : \mathrm{D}\, v \, dx \quad \text{for all } v \in Z,$$

and the remainder

$$y \in Y := \left\{ y \in L^2(\Omega;\mathbb{R}^{2\times 2}) \mid \int_\Omega y : \mathrm{D}\, v\, dx = 0 \text{ for all } v \in Z \right\}.$$

Orthogonality holds in the sense of

$$|||e|||_{\mathrm{NC}}^2 = |||z|||_{\mathrm{NC}}^2 + \|y\|_{L^2(\Omega)}^2 = \int_\Omega \mathrm{D}_{\mathrm{NC}}\, e : \mathrm{D}\, z\, dx + \int_\Omega \mathrm{D}_{\mathrm{NC}}\, e : y\, dx.$$

To estimate $\int_\Omega \mathrm{D}_{\mathrm{NC}}\, e : \mathrm{D}\, z\, dx$, employ the nonconforming interpolation $z_{\mathrm{NC}} \in \mathrm{CR}^1(\mathcal{T};\mathbb{R}^2)$ of $z$ defined by

$$z_{\mathrm{NC}}(\mathrm{mid}(E)) := \fint_E z\, ds \quad \text{for all } E \in \mathcal{E}.$$

An integration by parts yields $z_{\mathrm{NC}} \in Z_{\mathrm{NC}}$ and

$$\int_T \mathrm{D}(z - z_{\mathrm{NC}})\, dx = 0 \quad \text{for all } T \in \mathcal{T}. \tag{3.3}$$

This allows for the following computation:

$$\begin{aligned}\int_\Omega \mathrm{D}_{\mathrm{NC}}\, e : \mathrm{D}\, z\, dx &= F(z) - a_{\mathrm{NC}}(u_{\mathrm{CR}}, z)\\ &= F(z - z_{\mathrm{NC}}) - a_{\mathrm{NC}}(u_{\mathrm{CR}}, z) + F(z_{\mathrm{NC}})\\ &= \int_\Omega f \cdot (z - z_{\mathrm{NC}})\, dx - a_{\mathrm{NC}}(u_{\mathrm{CR}}, z - z_{\mathrm{NC}}).\end{aligned}$$

Equation (3.3) and $\nabla u_{\mathrm{CR}} \in P_0(\mathcal{T};\mathbb{R}^{2\times 2})$ yield $a_{\mathrm{NC}}(u_{\mathrm{CR}}, z - z_{\mathrm{NC}}) = 0$. Hence,

$$\begin{aligned}\int_\Omega \mathrm{D}_{\mathrm{NC}}\, e : \mathrm{D}\, z\, dx &= \int_\Omega f \cdot (z - z_{\mathrm{NC}})\, dx\\ &= \int_\Omega f_{\mathcal{T}} \cdot (z - z_{\mathrm{NC}})\, dx + \int_\Omega (f - f_{\mathcal{T}}) \cdot (z - z_{\mathrm{NC}})\, dx.\end{aligned} \tag{3.4}$$

For every $T \in \mathcal{T}$, consider the function $q_{\mathcal{T}}|_T \in H(\mathrm{div}, T;\mathbb{R}^{2\times 2})$ defined by

$$q_{\mathcal{T}}(x)|_T := -f_{\mathcal{T}}/2 \otimes (x - \mathrm{mid}(T)) \quad \text{for } x \in T.$$

An integration by parts and some basic calculations show, for any $v \in H_0^1(\Omega;\mathbb{R}^2)$, that

$$\begin{aligned}\int_T q_{\mathcal{T}} : \nabla v\, dx &= \int_{\partial T} q_{\mathcal{T}} \nu \cdot v\, ds - \int_T \mathrm{div}\, q_{\mathcal{T}} \cdot v\, dx\\ &= -\sum_{E \in \mathcal{E}(\partial T)} f_{\mathcal{T}} |T|/(3|E|) \int_E v\, ds + \int_T f_{\mathcal{T}} \cdot v\, dx\\ &= \int_T f_{\mathcal{T}} \cdot (v - v_{\mathrm{NC}})\, dx.\end{aligned}$$

Moreover, the $P_0(\mathcal{T})$ orthogonality of $f - f_{\mathcal{T}}$ and a Poincaré inequality with constant $\operatorname{diam}(T)/j_{1,1}$ from [18] yield

$$\begin{aligned}\int_\Omega (f - f_{\mathcal{T}}) \cdot (z - z_{\mathrm{NC}})\, dx &= \int_\Omega (f - f_{\mathcal{T}}) \cdot (z - z_{\mathrm{NC}} - (z - z_{\mathrm{NC}})_{\mathcal{T}})\, dx \\ &\leqslant \sum_{T \in \mathcal{T}} \|f - f_{\mathcal{T}}\|_{L^2(T)} \|z - z_{\mathrm{NC}} - (z - z_{\mathrm{NC}})_{\mathcal{T}}\|_{L^2(T)} \\ &\leqslant 1/j_{1,1} \operatorname{osc}(f, \mathcal{T}) \|\mathrm{D}(z - z_{\mathrm{NC}})\|_{L^2(\Omega)}. \end{aligned} \tag{3.5}$$

Hence, (3.4) reads

$$\begin{aligned}\int_\Omega \mathrm{D_{NC}}\, e : \mathrm{D}\, z\, dx &= \int_\Omega q_{\mathcal{T}} : \nabla z\, dx + \int_\Omega (f - f_{\mathcal{T}}) \cdot (z - z_{\mathrm{NC}})\, dx \\ &\leqslant \|q_{\mathcal{T}}\|_{L^2(\Omega)} |||z|||_{\mathrm{NC}} + 1/j_{1,1} \operatorname{osc}(f, \mathcal{T}) |||z - z_{\mathrm{NC}}|||_{\mathrm{NC}}.\end{aligned}$$

Notice that (3.3) yields

$$\begin{aligned}\|\mathrm{D}(z - z_{\mathrm{NC}})\|^2_{L^2(T)} &= \int_T |\mathrm{D}\, z|^2\, dx - 2 \int_T \mathrm{D}\, z : \mathrm{D}\, z_{\mathrm{NC}}\, dx + \int_T |\mathrm{D}\, z_{\mathrm{NC}}|^2\, dx \\ &= \int_T |\mathrm{D}\, z|^2\, dx - \int_\Omega |\mathrm{D}\, z_{\mathrm{NC}}|^2\, dx \leqslant \|\mathrm{D}\, z\|^2_{L^2(T)}.\end{aligned}$$

It remains to estimate $\int_\Omega \mathrm{D_{NC}}\, e : y\, dx$. Recall from [2] that, for each $y \in Y$, there exists some

$$w \in L^2_0(\Omega) := \left\{ q \in L^2(\Omega) \mid \int_\Omega q\, dx = 0 \right\}$$

with

$$\int_\Omega y : \mathrm{D}\, v\, dx = \int_\Omega w \operatorname{div} v\, dx \quad \text{for all } v \in H^1(\Omega; \mathbb{R}^2)$$

and

$$\|w\|_{L^2(\Omega)} \leqslant 1/c_0 \, \|y\|_{L^2(\Omega)}.$$

Hence, given any $v \in H^1(\Omega; \mathbb{R}^2)$ with $u - v = 0$ on $\partial\Omega$, it holds

$$\begin{aligned}\int_\Omega \mathrm{D_{NC}}\, e : y\, dx &= \int_\Omega \mathrm{D_{NC}}(u_{\mathrm{CR}} - v) : y\, dx + \int_\Omega \mathrm{D}(v - u) : y\, dx \\ &\leqslant \big\|\mathrm{D_{NC}}(u_{\mathrm{CR}} - v)\big\|_{L^2(\Omega)} \|y\|_{L^2(\Omega)} + \int_\Omega \operatorname{div}(v - u) w\, dx \\ &\leqslant \Big(\big\|\mathrm{D_{NC}}(u_{\mathrm{CR}} - v)\big\|_{L^2(\Omega)} + 1/c_0 \, \|\operatorname{div} v\|_{L^2(\Omega)}\Big) \|y\|_{L^2(\Omega)}.\end{aligned}$$

The combination of all mentioned results concludes the proof. □

## 4. Realisations of Guaranteed Upper Bounds

The subsequent Sections 4.1–4.3 discuss nine designs for $v$ and the estimation of $|||e|||$ via Theorem 3.1 with

$$\mu(v) := \|\mathrm{D_{NC}}(u_{\mathrm{CR}} - v)\|_{L^2(\Omega)} + 1/c_0 \, \|\operatorname{div} v\|_{L^2(\Omega)}. \tag{4.1}$$

The significant difference to [15] on the Poisson problem lies in the additional divergence term which leads to a sum of $L^2$ norms and Algorithms 4.1 and 4.2.

### 4.1. Interpolation after Ainsworth

This subsection introduces the interpolation after Ainsworth [1] that designs some piecewise linear $v_{\mathrm{A}} \in H^1(\Omega;\mathbb{R}^2)$ with respect to the original triangulation $\mathcal{T}$,

$$v_{\mathrm{A}}(z) := \begin{cases} u_D(z) & \text{if } z \in \mathcal{N} \setminus \mathcal{M}, \\ \big(\sum_{T\in\mathcal{T}(z)} u_{\mathrm{CR}}|_T(z)\big)/|\mathcal{T}(z)| & \text{if } z \in \mathcal{M}. \end{cases}$$

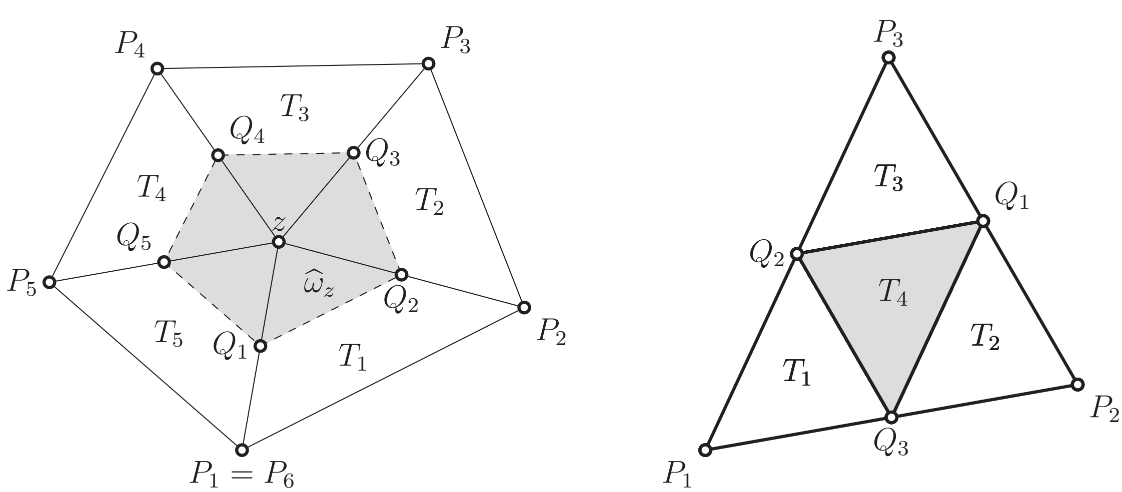
Here, the set $\mathcal{T}(z) := \{T \in \mathcal{T} \mid z \in T\}$ contains the triangles adjacent to $z \in \mathcal{N}$. The related error estimator reads

$$\eta_{\mathrm{A}}^2 := \eta^2 + \mu(v_{\mathrm{A}})^2.$$

### 4.2. Modified Interpolation Operator

This subsection introduces an improved interpolation that designs some piecewise linear $v_{\mathrm{red}} \in H^1_0(\Omega;\mathbb{R}^2)$ with respect to the red refined triangulation $\mathrm{red}(\mathcal{T})$. The red refinement connects the three edge midpoints $\mathrm{mid}(\mathcal{E}(T))$ within every triangle $T \in \mathcal{T}$ and so divides every triangle into four triangles with the same area. The nodes of $\mathrm{red}(\mathcal{T})$ consist of the original nodes $\mathcal{N}$ and the edge midpoints $\mathrm{mid}(\mathcal{E})$ of $\mathcal{T}$. At the boundary the interpolation equals the nodal interpolation of $u_D$ and on all interior edge midpoints it equals $u_{\mathrm{CR}}$;

$$v_{\mathrm{red}}(z) := \begin{cases} u_{\mathrm{CR}}(z) & \text{for } z \in \mathrm{mid}(\mathcal{E}) \setminus \mathrm{mid}(\mathcal{E}(\partial\Omega)), \\ u_D(z) & \text{for } z \in (\mathcal{N} \cup \mathrm{mid}(\mathcal{E})) \cap \partial\Omega, \\ v_z & \text{for } z \in \mathcal{M}. \end{cases} \tag{4.2}$$




(a) Interior patch

(b) Central subtriangle $T_4 = \mathrm{conv}\{\mathrm{mid}(\mathcal{E}(T))\}$ in $\mathrm{red}(T)$ for $T \in \mathcal{T}$.

**Figure 1.** Notation for red-refinements.

In this way, the interpolation $v_{\mathrm{red}}$ equals $u_{\mathrm{CR}}$ on all central subtriangles like $T_4$ in Figure 1 (b) and it remains to determine the values $v_z$ at the free nodes $z \in \mathcal{M}$. They may be chosen as in the design of $v_{\mathrm{A}}$, but we suggest to choose them locally optimal with Algorithm 4.1 by solving local problems around each node patch $\widehat{\omega}_z$ with respect to the red-refined triangulation as in Figure 1 (a) under the side condition of the fixed values at the edge midpoints $Q_j$ of the adjacent edges with the corresponding nodal basis functions $\varphi_{Q_j}^{\mathrm{red}}$ with respect to the red-refined triangulation. The value $v_z$ at $z$ remains the only degree of freedom in this local problem.

**Algorithm 4.1** (Patchwise minimisation).
**Input** $u_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T};\mathbb{R}^2)$ and $c_0 > 0$. Set $\lambda := 1$.
For $j = 1, 2, \dots$ until termination compute (a) and (b):

(a) For all $z \in \mathcal{N}(\Omega)$,

$$\begin{aligned} v_0 &:= \sum_{E\in\mathcal{E}} v_{\mathrm{red}}(\mathrm{mid}(E))\varphi_{\mathrm{mid}(E)}^{\mathrm{red}},\\ v_z &:= \operatorname*{argmin}_{w\in\mathbb{R}^2}\Big((1+\lambda)\big\|\mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v_0 - w\varphi_z^{\mathrm{red}})\big\|_{L^2(\widehat{\omega}_z)}^2 \\ &\qquad\qquad + (1+1/\lambda)/c_0^2\,\big\|\mathrm{div}(v_0 - w\varphi_z^{\mathrm{red}})\big\|_{L^2(\widehat{\omega}_z)}^2\Big). \end{aligned}$$

(b) Update

$$\begin{aligned} v_{\mathrm{red}} &:= v_0 + \sum_{z\in\mathcal{N}(\Omega)} v_z\varphi_z^{\mathrm{red}},\\ \lambda &:= \|\mathrm{div}\, v_{\mathrm{red}}\|_{L^2(\Omega)}\big/\big(c_0\big\|\mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v_{\mathrm{red}})\big\|_{L^2(\Omega)}\big),\\ \eta_{\mathrm{PMred}(j)}^2 &:= \eta^2 + \mu(v_{\mathrm{red}})^2. \end{aligned}$$

**Output** $\eta_{\mathrm{PMred}(j)}$ for $j = 1, 2, \dots$

We distinguish between the optimal version $\eta_{\mathrm{PMred}(j)}$ from Algorithm 4.1, and $\eta_{\mathrm{MAred}}$ with the suboptimal choice $v_z$ as in Section 4.1. This can be seen as a modification $v_{\mathrm{MAred}}$ of $v_{\mathrm{A}}$ at the edge midpoints.

Some numerical examples below suggest that the fixed values at the edge midpoints from $v_{\mathrm{red}}$ lead to unexpectedly large divergence terms. Indeed, even the interpolation $v_{\mathrm{A}}$ may lead to better results than $v_{\mathrm{MAred}}$ on fine meshes as displayed in Section 5.4. Of course, it is possible to describe other values at the edge midpoints and substitute (4.2) by, e.g.,

$$v_{\mathrm{red}}(z) := \begin{cases} v_{\mathrm{A}}(z) & \text{for } z \in \mathrm{mid}(\mathcal{E}) \setminus \mathrm{mid}(\mathcal{E}(\partial\Omega)),\\ u_D(z) & \text{for } z \in (\mathcal{N} \cup \mathrm{mid}(\mathcal{E})) \cap \partial\Omega,\\ v_z & \text{for } z \in \mathcal{M}. \end{cases} \tag{4.3}$$

The optimal value $v_z$ can again be computed similar to Algorithm 4.1 with $v_{\mathrm{red}}$ replaced by $v_{\mathrm{A}}$ and output $\eta_{\mathrm{PMA}(j)}$. Since $v_z = v_{\mathrm{A}}(z)$ is admissible in this optimisation, $\eta_{\mathrm{PMA}(j)}$ can only lead to better results than the interpolation $\eta_{\mathrm{A}}$.

Table 2 compares the outcome $\eta_{\mathrm{PMred}(j)}$ (resp. $\eta_{\mathrm{PMA}(j)}$) of Algorithm 4.1 for $j = 1, 2, 3, 5, 10$ with the edge values (4.2) (resp. (4.3)). There is no significant improvement for coarse meshes and only little improvement on fine meshes. Surprisingly, the design of $\eta_{\mathrm{PMA}}$ is somehow insensitive for $j \geqslant 2$ in Algorithm 4.1.

| ndof | 13 | 57 | 241 | 993 | 4033 | 16257 | 65281 | 261633 |
|---|---|---|---|---|---|---|---|---|
| $\eta_{\mathrm{A}}$ | 1817.92 | 699.646 | 276.868 | 112.429 | 46.5926 | 19.7549 | 8.59524 | 3.83932 |
| $\eta_{\mathrm{PMA}(1)}$ | 719.926 | 290.836 | 131.632 | 61.9372 | 29.1130 | 13.6895 | 6.48276 | 3.10096 |
| $\eta_{\mathrm{PMA}(2)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| $\eta_{\mathrm{PMA}(3)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| $\eta_{\mathrm{PMA}(4)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| $\eta_{\mathrm{PMA}(5)}$ | 719.926 | 290.773 | 131.611 | 61.9337 | 29.1126 | 13.6894 | 6.48274 | 3.10096 |
| $\eta_{\mathrm{MAred}}$ | 719.926 | 286.729 | 122.127 | 55.5387 | 25.5059 | 11.7695 | 5.48898 | 2.59459 |
| $\eta_{\mathrm{PMred}(1)}$ | 719.926 | 286.684 | 121.653 | 54.4621 | 24.4546 | 11.0798 | 5.10119 | 2.39005 |
| $\eta_{\mathrm{PMred}(2)}$ | 719.926 | 286.677 | 121.632 | 54.4186 | 24.4286 | 11.0678 | 5.09563 | 2.38739 |
| $\eta_{\mathrm{PMred}(3)}$ | 719.926 | 286.677 | 121.632 | 54.4183 | 24.4281 | 11.0674 | 5.09542 | 2.38728 |
| $\eta_{\mathrm{PMred}(4)}$ | 719.926 | 286.677 | 121.632 | 54.4183 | 24.4281 | 11.0674 | 5.09542 | 2.38728 |
| $\eta_{\mathrm{PMred}(5)}$ | 719.926 | 286.677 | 121.632 | 54.4183 | 24.4281 | 11.0674 | 5.09542 | 2.38728 |

**Table 2.** Error estimators $\eta_{\mathrm{A}}$, $\eta_{\mathrm{MAred}}$, $\eta_{\mathrm{PMA}(j)}$ and $\eta_{\mathrm{PMred}(j)}$ for $j = 1, 2, 3, 5, 10$ in Algorithm 4.1 and uniform mesh refinement in the example of Section 5.3 (with Dirichlet data error contribution from Section 4.4).

### 4.3. Optimal Choices

The global minimisers $v_{\mathrm{MP1}}$ in $P_1(\mathcal{T}) \cap C(\Omega)$, $v_{\mathrm{MP2}}$ in $P_2(\mathcal{T}) \cap C(\Omega)$ and $v_{\mathrm{MP1red}}$ in $P_1(\mathrm{red}(\mathcal{T})) \cap C(\Omega)$ on the red-refined triangulation $\mathrm{red}(\mathcal{T})$ of the functional $\mu$ from (4.1) are computed by Algorithm 4.2.

**Algorithm 4.2** (Global minimisation)**.**

**Input** $u_{\mathrm{CR}} \in \mathrm{CR}^1(\mathcal{T}; \mathbb{R}^2)$, $W(\mathcal{T}) \in \{P_1(\mathcal{T}) \cap C(\Omega), P_1(\mathrm{red}(\mathcal{T})) \cap C(\Omega), P_2(\mathcal{T}) \cap C(\Omega)\}$ and $c_0 > 0$. Set $\lambda := 1$.

For $j = 1, 2, \ldots$ until termination do

$$\begin{aligned} v_{W(\mathcal{T})} &:= \operatorname*{argmin}_{v \in W(\mathcal{T})} \Big( (1+\lambda) \big\| \mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v) \big\|^2_{L^2(\Omega)} + (1 + 1/\lambda) \| \operatorname{div} v \|^2_{L^2(\Omega)} / c_0^2 \Big), \qquad (4.4) \\ \eta^2_{j,W(\mathcal{T})} &:= \eta^2 + \mu(v_{W(\mathcal{T})})^2, \\ \lambda &:= \| \operatorname{div} v_{W(\mathcal{T})} \|_{L^2(\Omega)} / \Big( c_0 \big\| \mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v_{W(\mathcal{T})}) \big\|_{L^2(\Omega)} \Big). \end{aligned}$$

**Output** $\eta_{\mathrm{MP1}(j)} := \eta_{j, P_1(\mathcal{T}) \cap C(\Omega)}$, $\eta_{\mathrm{MP1red}(j)} := \eta_{j, P_1(\mathrm{red}(\mathcal{T})) \cap C(\Omega)}$, $\eta_{\mathrm{MP2}(j)} := \eta_{j, P_2(\mathcal{T}) \cap C(\Omega)}$ for $j = 1, 2, \ldots$

Table 3 displays values of $\eta_{\mathrm{MP2}(j)}$ for $j = 1, 2, \ldots, 5$ and suggests that there is a more significant improvement by Algorithm 4.2 compared to Algorithm 4.1 for the local designs. In the computational examples of Section 5, the termination of Algorithms 4.1 and 4.2 is with $j = 3$.

To reduce the computational costs of (4.4) one might use $v_{\mathrm{MAred}}$ as an initial guess for some iterative solver to draw near the minimiser of (4.4) for $W(\mathcal{T}) = P_1(\mathrm{red}(\mathcal{T})) \cap C(\Omega)$. We use a preconditioned conjugate gradients scheme and stop at the third iterate. The preconditioner is the diagonal of the system matrix named after Jacobi. This approximation of (4.4) in Algorithm 4.2 with $j = 3$ results in the estimator $\eta_{\mathrm{MP1redCG3}}$. Similarly, the nodal values of $v_{\mathrm{MAred}}$ define some piecewise quadratic function and hence an initial value for some PCG algorithm for the approximation of the minimiser of (4.4) for $W(\mathcal{T}) = P_2(\mathcal{T}) \cap C(\Omega)$. The truncation of the minimisation in (4.4) after five PCG iterations and $j = 3$ in Algorithm 4.2 defines the error estimator $\eta_{\mathrm{MP2CG5}}$.

| ndof | 13 | 57 | 241 | 993 | 4033 | 16257 | 65281 | 261633 |
|---|---|---|---|---|---|---|---|---|
| $\eta_{\mathrm{MP2}(1)}$ | 516.780 | 171.508 | 50.5265 | 18.2578 | 7.88212 | 3.72095 | 1.82154 | 0.903860 |
| $\eta_{\mathrm{MP2}(2)}$ | 516.747 | 169.398 | 49.1931 | 17.2531 | 7.24667 | 3.36511 | 1.63548 | 0.809433 |
| $\eta_{\mathrm{MP2}(3)}$ | 516.747 | 169.159 | 48.9606 | 17.1022 | 7.14785 | 3.30029 | 1.59558 | 0.787235 |
| $\eta_{\mathrm{MP2}(4)}$ | 516.747 | 169.144 | 48.9070 | 17.0820 | 7.13818 | 3.29419 | 1.59092 | 0.783648 |
| $\eta_{\mathrm{MP2}(5)}$ | 516.747 | 169.143 | 48.8919 | 17.0791 | 7.13742 | 3.29387 | 1.59069 | 0.783424 |

**Table 3.** Error estimator $\eta_{\mathrm{MP2}(j)}$ for $j = 1, \ldots, 5$ in Algorithm 4.2 and uniform mesh refinement in the example of Section 5.3 (with Dirichlet data error contribution from Section 4.4).

**Remark 4.1.** The Algorithms 4.1 and 4.2 approximate the minimum of $\mu(v)$ amongst $v \in W(\mathcal{T})$ by a series of quadratic minimisation problems based on the identity

$$\min_{v\in W(\mathcal{T})} \mu(v) = \min_{\lambda\in\mathbb{R}} \min_{v\in W(\mathcal{T})} \big((1+\lambda)\big\|\mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v)\big\|^2_{L^2(\Omega)} + (1+1/\lambda)\|\operatorname{div} v\|^2_{L^2(\Omega)}/c_0^2\big)$$

and alternating direction minimisation in the variable $v$ and $\lambda$.

### 4.4. Inhomogeneous Dirichlet Boundary Conditions

In case of inhomogeneous boundary conditions, the designs of $v_{\mathrm{xyz}}$ from the previous subsections do not satisfy $u - v_{\mathrm{xyz}} = 0$ on $\partial\Omega$ in general. To overcome this difficulty consider $w_D \in H^1(\Omega;\mathbb{R}^2)$ from [6,15] with

$$w_D|_{\partial\Omega} = u_D|_{\partial\Omega} - v_{\mathrm{xyz}}|_{\partial\Omega}.$$

Then, Theorem 3.1 for $v = v_{\mathrm{xyz}} - w_D$ yields $u - v \in H^1_0(\Omega)$ and

$$\begin{aligned}|||e|||^2_{\mathrm{NC}} &\leqslant \eta^2 + \big(\big\|\mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v - w_D)\big\|_{L^2(\Omega)} + 1/c_0\,\big\|\operatorname{div}(v + w_D)\big\|_{L^2(\Omega)}\big)^2 \\ &\leqslant \eta^2 + \big(\big\|\mathrm{D}_{\mathrm{NC}}(u_{\mathrm{CR}} - v)\big\|_{L^2(\Omega)} + 1/c_0\,\|\operatorname{div} v\|_{L^2(\Omega)} + (1+1/c_0)|||w_D|||\big)^2.\end{aligned}$$

In case of $v_{\mathrm{xyz}}|_{\partial\Omega} = \mathcal{I}u_D|_{\partial\Omega} := \sum_{z\in\mathcal{N}\setminus\mathcal{M}} u_D(z)\varphi_z|_{\partial\Omega}$, it holds [6,15]

$$|||w_D||| \leqslant C_\gamma\big\|h_{\mathcal{E}}^{3/2}\partial^2_{\mathcal{E}}(u_D - v_{\mathrm{xyz}})/\partial s^2\big\|_{L^2(\partial\Omega)}.$$

**Remark 4.2** ([15])**.** For right isosceles triangles numerical calculations suggest the constant $C_\gamma = 0.4980$. If $v_{\mathrm{xyz}}|_{\partial\Omega}$ is the nodal interpolation of $u_D|_{\partial\Omega}$ on the red-refined triangulation, $w_D$ can be designed on the red-refined triangulation with halved edge lengths and the constant reduces to $C_\gamma = 0.4980/2^{3/2} = 0.1761$. The same holds if $v_{\mathrm{xyz}}$ equals the $P_2$ interpolation of $u_D$ on $\mathcal{T}$ along $\partial\Omega$.

## 5. Numerical Experiments

This section discusses the five benchmark examples from Table 1.

### 5.1. Smooth Example

The first benchmark problem employs the right-hand side $f(x, y) = (4\pi^2 \sin(\pi(x - y)), 0)$ and inhomogeneous Dirichlet boundary data $u_D$ with exact solution

$$u(x, y)_j = \sin(\pi x)\cos(\pi y) - \cos(\pi x)\sin(\pi y) \quad \text{for } j = 1, 2$$

on the square domain $\Omega = (-1, 1)^2$ with $c_0 = 0.3826$ from [19].

Figure 2 shows the efficiency indices in case of uniform and adaptive mesh refinement in the range of 1 to 3. While $\eta_{\mathrm{MAred}}$ is superior to $\eta_{\mathrm{A}}$ in case of Poisson problems from [15], this example shows that this must not be the case for Stokes problems. The piecewise minimal improvement $\eta_{\mathrm{PMred}}$ closes the gap between $\eta_{\mathrm{A}}$ and $\eta_{\mathrm{MAred}}$, but barely leads to more efficient upper bounds than $\eta_{\mathrm{A}}$ at least for uniform mesh refinement. Here, $\eta_{\mathrm{PMA}}$ performs better and converges to efficiency indices close to 1.5 for uniform mesh refinement and efficiency indices below 2.5 for adaptive mesh refinement. The error estimator $\eta_{\mathrm{MP1redCG3}}$ and its optimal limit $\eta_{\mathrm{MP1red}}$ with efficiency indices between 1.25 and 1.75 only lose to $\eta_{\mathrm{MP2}}$ which allows for efficiency indices close to 1. The residual error estimator $\eta_{\mathrm{R}}$ with efficiency indices above 8 is not displayed. Figure 3 shows that none of the error estimators leads to significantly better refined meshes.

### 5.2. Second Smooth Example

The second benchmark problem from [2] employs the right-hand side $f(x, y) = (-4y, 4x)$ and inhomogeneous Dirichlet boundary data $u_D$ that matches the exact solution

$$u(x, y) = [x(1 - x)(1 - 2y), -y(1 - y)(1 - 2x)]$$

on the square domain $\Omega = (0, 1)^2$ with $c_0 = 0.3826$ from [19]. Since $u$ is smooth, the convergence rates for the energy error are optimal for uniform and adaptive mesh refinement as depicted in Figure 5.

The efficiency indices displayed in Figure 4 scatter more than in the first example. The error estimators $\eta_{\mathrm{MP1}}$ and $\eta_{\mathrm{A}}$ yield almost identical efficiency indices larger than 5 for uniform mesh refinement. The red($\mathcal{T}$)-based interpolation error estimators $\eta_{\mathrm{MAred}}$ and $\eta_{\mathrm{PMred}}$ yield efficiency indices of about 3, while $\eta_{\mathrm{MP1red}}$ and $\eta_{\mathrm{MP1redCG3}}$ allow efficiency indices around 2. Again, the most accurate error estimators are $\eta_{\mathrm{MP2}}$ and $\eta_{\mathrm{MP2CG5}}$.

### 5.3. Colliding Flow Example

The third benchmark problem employs $f(x, y) \equiv 0$ and the exact solution $u(x, y) = (20xy^4 - 4x^5, 20x^4y - 4y^5)$ on the square domain $\Omega = (-1, 1)^2$ with $c_0 = 0.3826$ from [19]. This is another smooth example with optimal convergence rates of the energy error for uniform and adaptive mesh refinement as shown in Figure 7.

The efficiency indices for adaptive mesh refinement displayed in Figure 6 are in the range between almost 1 in case of $\eta_{\mathrm{MP2}}$ and 3.5 in case of $\eta_{\mathrm{A}}$. The piecewise minimal interpolation $\eta_{\mathrm{PMred}}$ yields efficiency indices around 2.5 which is significantly better than $\eta_{\mathrm{PMA}}$ and also better than $\eta_{\mathrm{MP1}}$. The error estimator $\eta_{\mathrm{MP1redCG3}}$ is almost as efficient as the optimal $\eta_{\mathrm{MP1red}}$ with around 2.

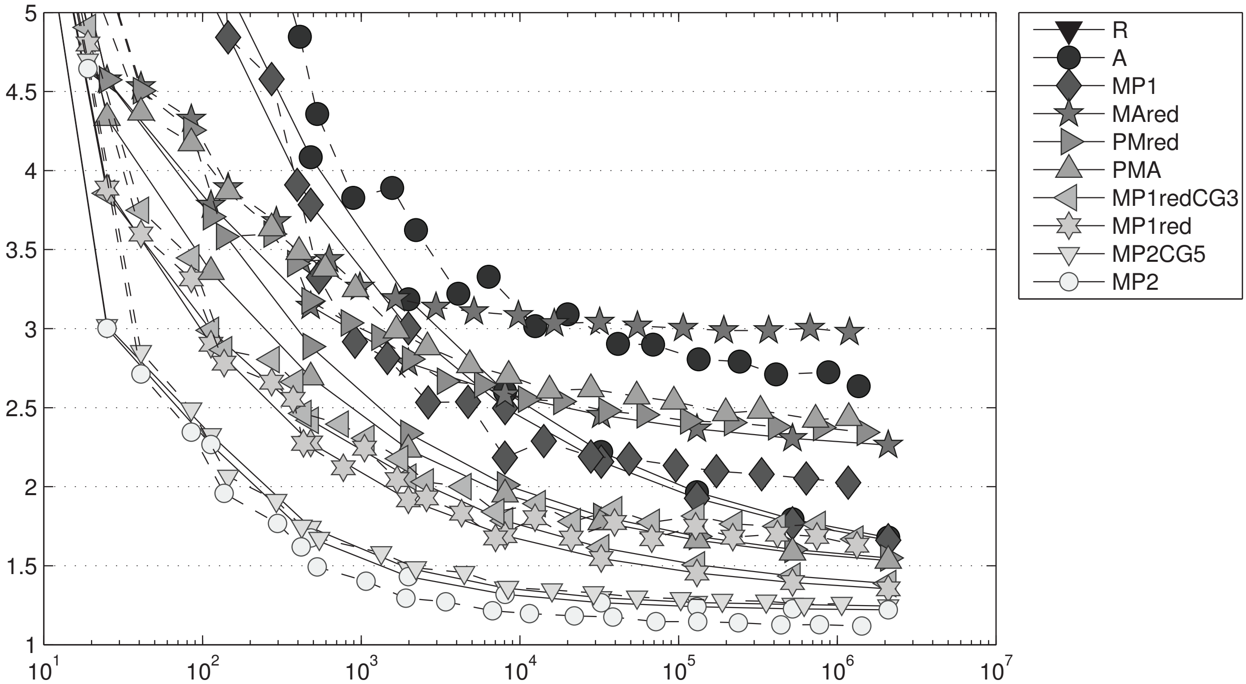



**Figure 2.** History of efficiency indices $\eta_{\mathrm{xyz}}/|||e|||_{\mathrm{NC}}$ of various a posteriori error estimators $\eta_{\mathrm{xyz}}$ labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Section 5.1.

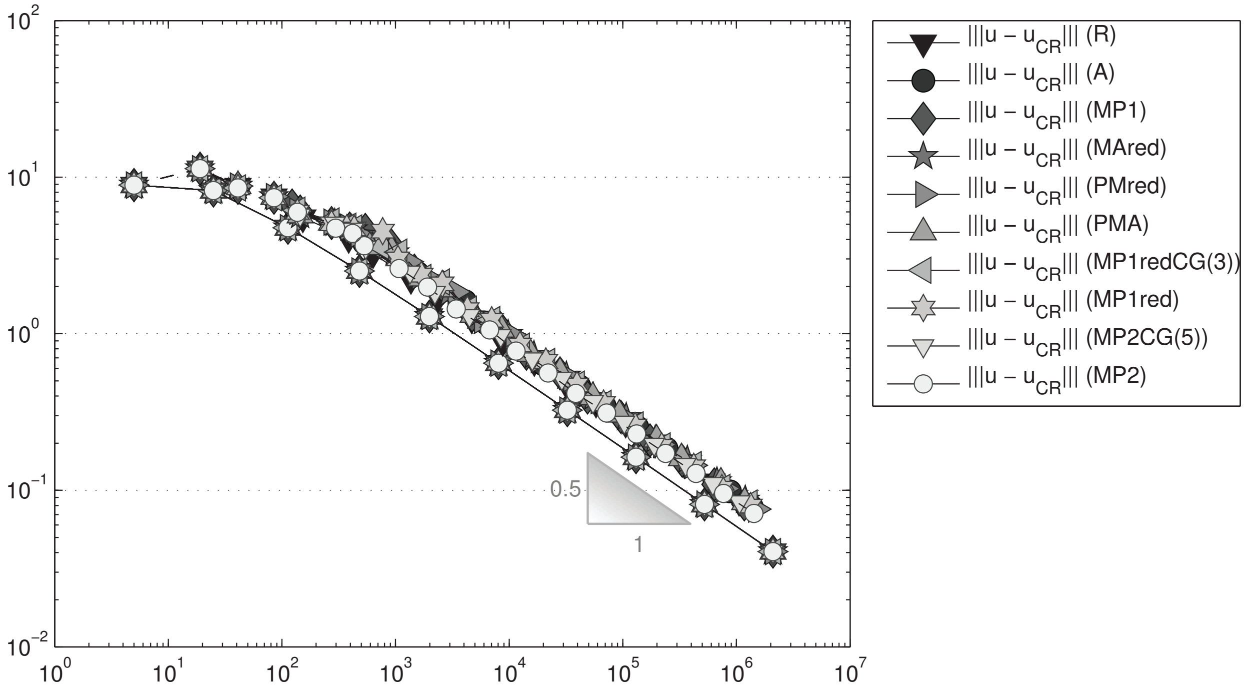



**Figure 3.** Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Section 5.1.

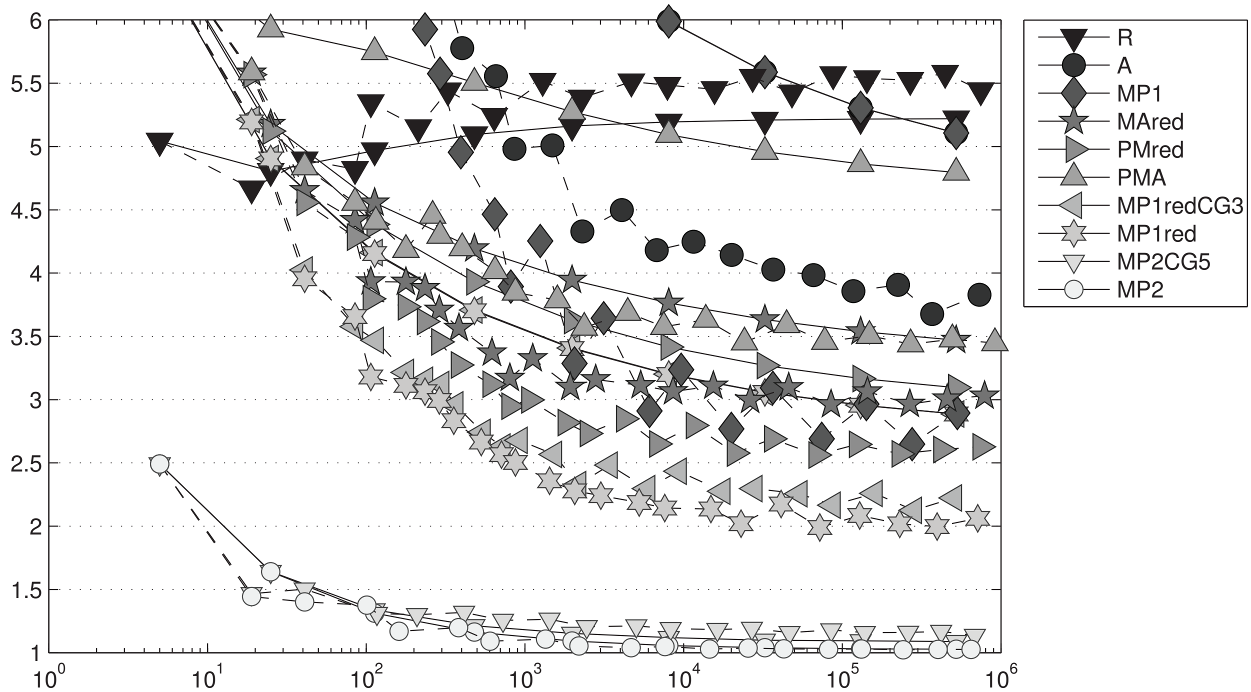



**Figure 4.** History of efficiency indices $\eta_{\mathrm{xyz}}/|||e|||_{\mathrm{NC}}$ of various a posteriori error estimators $\eta_{\mathrm{xyz}}$ labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Section 5.2.

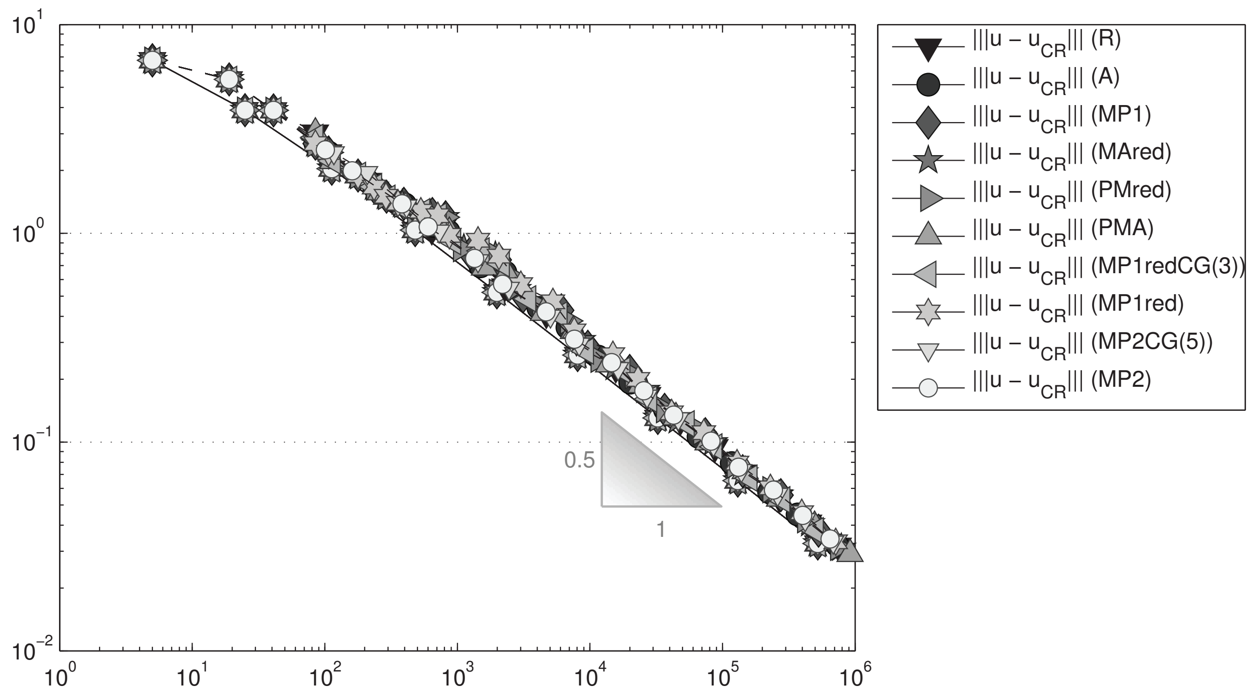



**Figure 5.** Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Section 5.2.

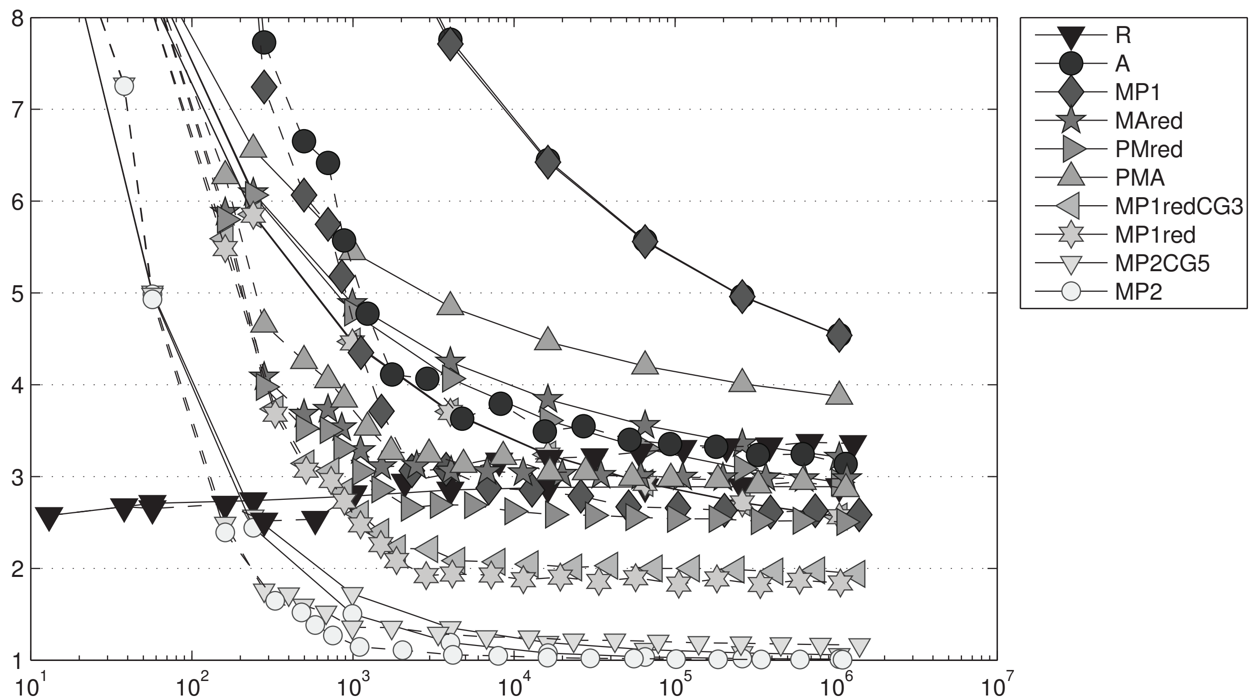



**Figure 6.** History of efficiency indices $\eta_{\mathrm{xyz}}/|||e|||_{\mathrm{NC}}$ of various a posteriori error estimators $\eta_{\mathrm{xyz}}$ labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Section 5.3.

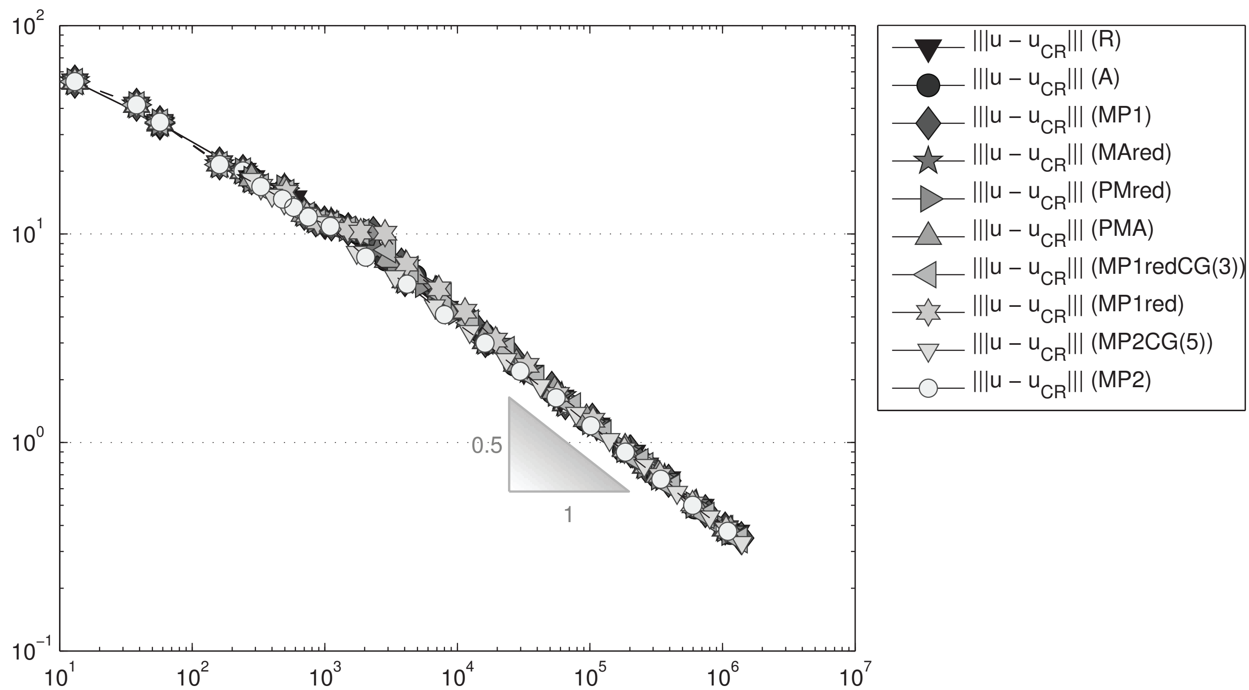



**Figure 7.** Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Section 5.3.

### 5.4. Example on L-shaped Domain

The fourth benchmark problem employs $f(x,y) \equiv 0$ and $u_D$ matching the exact solution

$$u(r,\varphi) = r^\alpha \begin{pmatrix} (\alpha+1)\sin(\varphi)\psi(\varphi) + \cos(\varphi)\psi'(\varphi) \\ -(\alpha+1)\cos(\varphi)\psi(\varphi) + \sin(\varphi)\psi'(\varphi) \end{pmatrix}^T$$

on the L-shaped domain $\Omega = (-1,1)^2 \setminus ((0,1)\times(-1,0))$ with

$$\begin{aligned}\psi(\varphi) = 1/(\alpha+1)\, &\sin((\alpha+1)\varphi)\cos(\alpha\omega) - \cos((\alpha+1)\varphi) \\ &+ 1/(\alpha-1)\,\sin((\alpha-1)\varphi)\cos(\alpha\omega) + \cos((\alpha-1)\varphi)\end{aligned}$$

for $\alpha = 856399/1572864 \approx 0.54$, $\omega = 3\pi/2$ from [21]. For the estimator we set $c_0 = 0.3$ from [19].

Figure 9 shows the convergence history of the energy error for uniform and adaptive mesh refinement. The singularity reduces the convergence speed for uniform mesh refinement significantly. The adaptive mesh refinement algorithm from Section 2.3 leads to the optimal convergence speed, independently of the chosen refinement indicators. This is also true for all other examples so far.

The efficiency indices are displayed in Figure 8 and appear similar to the examples before in the range of 1 to 4.

### 5.5. Backward Facing Step Example

The last example employs the backstep domain $\Omega = ((-2,8)\times(-1,1)) \setminus ((-2,0)\times(-1,0))$, the right-hand side $f \equiv 0$ and the inhomogeneous boundary data

$$u_D(x,y) = \begin{cases} (-y(y-1)/10, 0) & \text{at } x = -2, \\ (-(y^2-1)/80, 0) & \text{at } x = 8. \end{cases}$$

There is no known reference solution, but the example is well-understood [5, 13]. The error estimators are displayed in Figure 10 for the energy error. Again, optimal minimisation leads to significantly smaller bounds than the local interpolation designs. The employed value $c_0 = 0.3$ might not be a lower bound for the inf-sup constant and so the computed error estimates may not be guaranteed upper bounds.

## 6. Conclusions

The theoretical and practical results of this paper support the following observations.

### 6.1. Explicit Error Estimator Sufficient for Effective Mesh Design

The adaptive mesh refinement may be steered by simple $\eta_{\mathrm{R}}$-based marking. It does not appear to be favourable to spend more computational time for more laborious refinement rules.

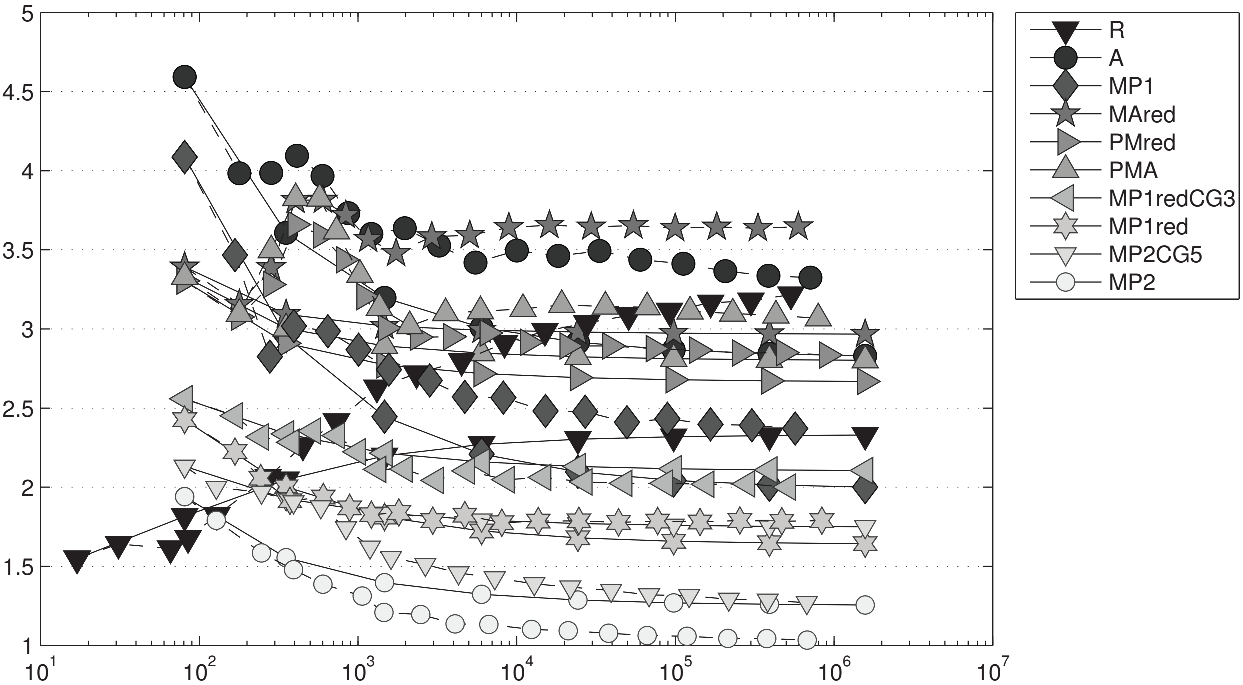



**Figure 8.** History of efficiency indices $\eta_{\mathrm{xyz}}/|||e|||_{\mathrm{NC}}$ of various a posteriori error estimators $\eta_{\mathrm{xyz}}$ labelled xyz in the figure as functions of the number of unknowns on adaptive (dashed lines) and uniform meshes (solid lines) in Section 5.4.

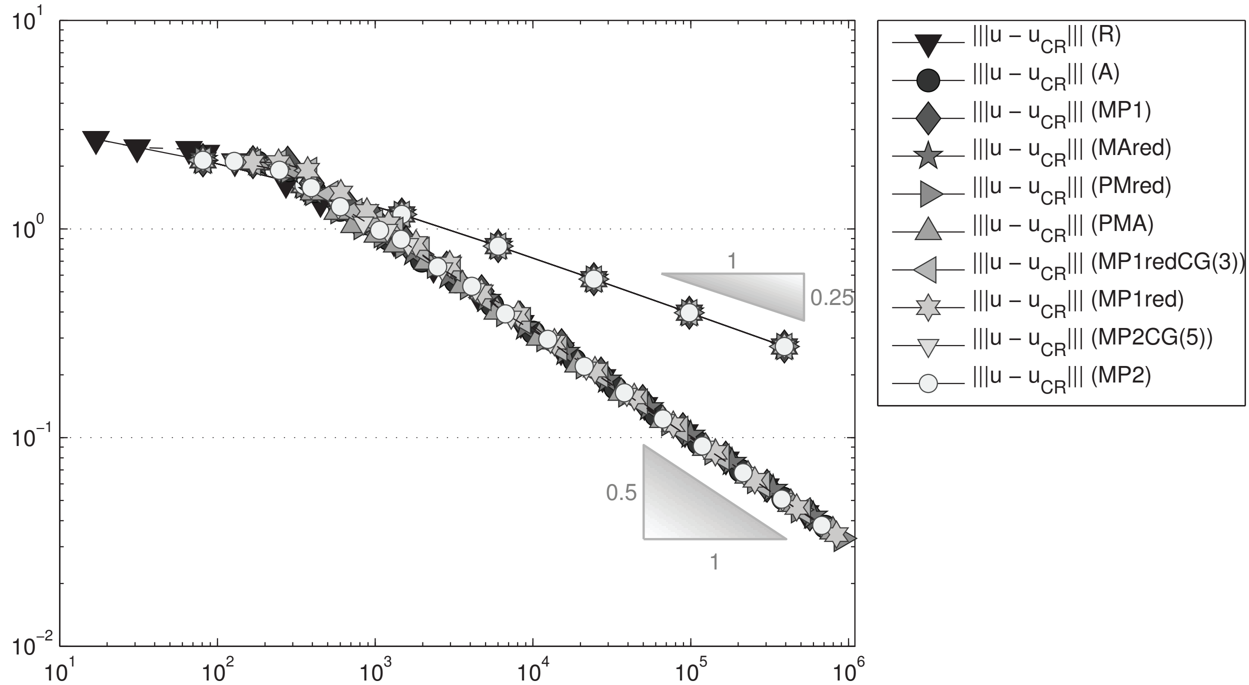



**Figure 9.** Convergence history of the energy error for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Section 5.4.

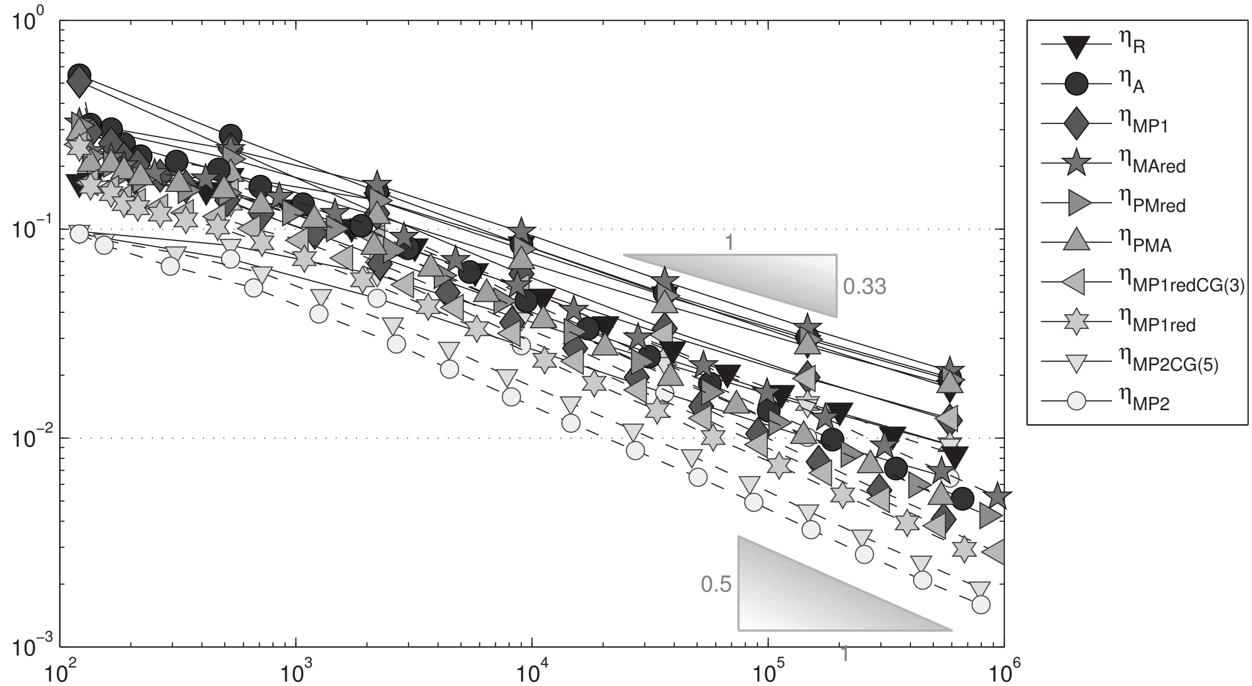



**Figure 10.** Convergence history of the error estimators for uniform (solid lines) and adaptive (dashed lines) mesh refinements in Section 5.5.

### 6.2. No Explicit Error Estimation for Reliable Error Control

The explicit residual-based error estimator $\eta_{\mathrm{R}}$ is no guaranteed upper bound for the exact error, since sharp reliability constants are unknown or hard to calculate. The shown efficiency indices for $\eta_{\mathrm{R}}$ are therefore not comparable with those of the other guaranteed error bounds. However, from related situations on the Poisson problem, we expect that the displayed $\eta_{\mathrm{R}}$ from Section 2.3 with $C_{\mathrm{rel}} = 1$ is an optimistic approximation [12,20]. Therefore, the values of $\eta_{\mathrm{R}}$ are plotted in all the figures and display that those explicit residual-based error estimates are less competitive.

### 6.3. Accurate and Cheap Error Control via $\boldsymbol{\eta_{\mathrm{PMred}}}$

The experience for Poisson problems in [15] is that the modification $v_{\mathrm{red}}$ of $v_{\mathrm{A}}$ is superior in all benchmark examples. This is not true for Stokes problems, since $\eta_{\mathrm{A}}$ sometimes is more accurate than $\eta_{\mathrm{MAred}}$. However, the associated piecewise minimal error estimator $\eta_{\mathrm{PMred}}$ is better than $\eta_{\mathrm{PMA}}$ in all benchmark examples with efficiency indices between 2 and 3.

### 6.4. Performance of Algorithms 4.1 and 4.2

Since the error estimators of Theorem 3.1, as well as that of [2], involve the sum of norms and not the sum of their squares, some alternating direction minimisation in the variable $v$ and $\lambda$ is suggested in Algorithms 4.1 and 4.2. The numerical experiments reported in Table 2 suggest that the value $\lambda = 1$ is already a good approximation. Hence, an expensive outer loop over various $j$ does not appear to be necessary in Algorithm 4.1.

### 6.5. More Accurate Error Control via $\eta_{\mathrm{MP2}}$ or $\eta_{\mathrm{MP1red}}$

Global Minimisation on the red-refined triangulation red($\mathcal{T}$) leads to the error estimator $\eta_{\mathrm{MP1red}}$ with efficiency indices between 1.5 and 2. In most benchmark examples, its approximation $\eta_{\mathrm{MP1redCG3}}$ leads to only slightly less accurate results. However, the optimisation with piecewise quadratic polynomials $\eta_{\mathrm{MP2}}$ allows the best error control with efficiency indices below 1.5, often close to 1. The error estimator $\eta_{\mathrm{MP2CG5}}$ is a very good approximation towards $\eta_{\mathrm{MP2}}$ and even yields better efficiency indices than $\eta_{\mathrm{MP1red}}$.

### 6.6. Suggested Approximation of $\eta_{\mathrm{MP1red}}$ with $\eta_{\mathrm{MP1redCG3}}$ or $\eta_{\mathrm{MP2}}$ with $\eta_{\mathrm{MP2CG5}}$

The PCG approximation $\eta_{\mathrm{MP1redCG3}}$ of $\eta_{\mathrm{MP1red}}$ is computed by three iterations of some conjugate gradient scheme with initial value $v_{\mathrm{MAred}}$ in direction of the minimiser of the sum of squares. So $\lambda$ is set to 1 and there is no outer loop of the minimisation as discussed in Section 6.4. The error estimator $\eta_{\mathrm{MP2CG5}}$ also uses the nodal values of $v_{\mathrm{MAred}}$ as coefficients for the $P_2$ ansatz functions on $\mathcal{T}$ and performs five PCG iterations to draw near $\eta_{\mathrm{MP2}}$ with $\lambda = 1$.

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