FORMALIZED MATHEMATICS
Vol. 23, No. 2, Pages 75–79, 2015
DOI: 10.1515/forma-2015-0007

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Morley's Trisector Theorem

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. Morley's trisector theorem states that "The points of intersection of the adjacent trisectors of the angles of any triangle are the vertices of an equilateral triangle" [10].

There are many proofs of Morley's trisector theorem [12, 16, 9, 13, 8, 20, 3, 18]. We follow the proof given by A. Letac in [15].

MSC: 51M04 03B35

Keywords: Euclidean geometry; Morley's trisector theorem; equilateral triangle

MML identifier: EUCLID11, version: 8.1.04 5.32.1237

The notation and terminology used in this paper have been introduced in the following articles: [1], [11], [7], [14], [19], [2], [4], [23], [5], [24], [21], [22], and [6].

1. Preliminaries

From now on A, B, C, D, E, F, G denote points of $\mathcal{E}^2_{\mathrm{T}}$.

Now we state the propositions:

(1) $\measuredangle(A, B, A) = 0$.

(2) $0 \leqslant \measuredangle(A, B, C) < 2 \cdot \pi$.

(3) (i) $0 \leqslant \measuredangle(A, B, C) < \pi$, or

(ii) $\measuredangle(A, B, C) = \pi$, or

(iii) $\pi < \measuredangle(A, B, C) < 2 \cdot \pi$.

The theorem is a consequence of (2).

(4) $|F - E|^{\mathbf{2}} = |A - E|^{\mathbf{2}} + |A - F|^{\mathbf{2}} - 2 \cdot |A - E| \cdot |A - F| \cdot \cos \measuredangle(E, A, F)$.

(5) If A, B, C are mutually different and $0 < \measuredangle(A, B, C) < \pi$, then $0 < \measuredangle(B, C, A) < \pi$ and $0 < \measuredangle(C, A, B) < \pi$.

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ISSN 1426–2630(Print), 1898-9934(Online)

(6) Suppose A, B, C are mutually different and $\measuredangle(A,B,C) = 0$. Then

(i) $\measuredangle(B,C,A) = 0$ and $\measuredangle(C,A,B) = \pi$, or

(ii) $\measuredangle(B,C,A) = \pi$ and $\measuredangle(C,A,B) = 0$ and $\measuredangle(A,B,C) + \measuredangle(B,C,A) + \measuredangle(C,A,B) = \pi$.

(7) Suppose A, B, C are mutually different and $\measuredangle(A,B,C) = \pi$. Then

(i) $\measuredangle(B,C,A) = 0$, and

(ii) $\measuredangle(C,A,B) = 0$, and

(iii) $\measuredangle(A,B,C) + \measuredangle(B,C,A) + \measuredangle(C,A,B) = \pi$.

(8) If A, B, C are mutually different and $\measuredangle(A,B,C) > \pi$, then $\measuredangle(A,B,C) + \measuredangle(B,C,A) + \measuredangle(C,A,B) = 5 \cdot \pi$.

Let us assume that $\measuredangle(C,B,A) < \pi$. Now we state the propositions:

(9) $0 \leqslant$ area of $\triangle(A,B,C)$. The theorem is a consequence of (2).

(10) $0 \leqslant \varnothing_{\triangleleft}(A,B,C)$. The theorem is a consequence of (9).

2. Morley's Theorem

Now we state the propositions:

(11) Suppose A, F, C form a triangle and $\measuredangle(C,F,A) < \pi$ and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $(\measuredangle(A,C,B)/3) + (\measuredangle(B,A,C)/3) + (\measuredangle(C,B,A)/3) = \pi/3$.
Then $|A - F| \cdot \sin((\pi/3) - (\measuredangle(C,B,A)/3)) = |A - C| \cdot \sin(\measuredangle(A,C,B)/3)$.

(12) Suppose A, B, C form a triangle and A, F, C form a triangle and $\measuredangle(C,F,A) < \pi$ and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $(\measuredangle(A,C,B)/3) + (\measuredangle(B,A,C)/3) + (\measuredangle(C,B,A)/3) = \pi/3$ and $\sin((\pi/3) - (\measuredangle(C,B,A)/3)) \neq 0$. Then $|A - F| = 4 \cdot \varnothing_{\triangleleft}(A,B,C) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin((\pi/3) + (\measuredangle(C,B,A)/3)) \cdot \sin(\measuredangle(A,C,B)/3)$. The theorem is a consequence of (11).

(13) Suppose C, A, B form a triangle and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$. Then $\measuredangle(E,A,F) = \measuredangle(B,A,C)/3$. Proof: $\measuredangle(E,A,F) \neq 4 \cdot \pi + (\measuredangle(B,A,C)/3)$ by [17, (5)], (2), [7, (30)]. $\measuredangle(E,A,F) \neq 2 \cdot \pi + (\measuredangle(B,A,C)/3)$ by (2), [7, (30)]. □

(14) Suppose C, A, B form a triangle and $\measuredangle(A,C,B) < \pi$ and A, F, C form a triangle and F, A, E form a triangle and E, A, B form a triangle and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$. Then $(\pi/3) + (\measuredangle(A,C,B)/3) + ((\pi/3) + (\measuredangle(C,B,A)/3)) + \measuredangle(E,A,F) = \pi$. The theorem is a consequence of (13).

(15) If A, C, B form a triangle, then $\sin((\pi/3) - (\measuredangle(A,C,B)/3)) \neq 0$. The theorem is a consequence of (2).

(16) Suppose A, B, C form a triangle and A, B, E form a triangle and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and A, F, C form a triangle and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $\measuredangle(A,C,B) < \pi$. Then $|F - E| = 4 \cdot \varnothing_{\triangleleft}(A,B,C) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3)$.
PROOF: $\sin((\pi/3) - (\measuredangle(A,C,B)/3)) \neq 0$. $\sin((\pi/3) - (\measuredangle(C,B,A)/3)) \neq 0$. $0 < \measuredangle(A,C,B)$. $\measuredangle(C,B,A) < \pi$. $0 < \measuredangle(A,C,B) < \pi$ and A, C, B are mutually different. $\measuredangle(B,A,C) < \pi$. $0 < \measuredangle(B,A,E) < \pi$. $\measuredangle(A,E,B) < \pi$. $0 < \measuredangle(F,A,C) < \pi$. $\measuredangle(C,F,A) < \pi$. F, A, E form a triangle by [19, (4)], (5), [17, (5)], [7, (31)]. $|A - F| = \varnothing_{\triangleleft}(A,B,C) \cdot 4 \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin((\pi/3) + (\measuredangle(C,B,A)/3)) \cdot \sin(\measuredangle(A,C,B)/3)$. $(\pi/3) + (\measuredangle(A,C,B)/3) + ((\pi/3) + (\measuredangle(C,B,A)/3)) + \measuredangle(E,A,F) = \pi$. $|F - E|^{\mathbf{2}} = |A - E|^{\mathbf{2}} + |A - F|^{\mathbf{2}} - 2 \cdot |A - E| \cdot |A - F| \cdot \cos\measuredangle(E,A,F)$. $\square$

(17) Suppose A, B, C form a triangle and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$. Then A, B, E form a triangle. The theorem is a consequence of (1) and (2).

(18) Suppose A, B, C form a triangle and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$. Then A, F, C form a triangle. The theorem is a consequence of (1) and (2).

(19) Suppose A, B, C form a triangle and $\measuredangle(C,B,G) = \measuredangle(C,B,A)/3$ and $\measuredangle(G,C,B) = \measuredangle(A,C,B)/3$. Then C, G, B form a triangle. The theorem is a consequence of (1) and (2).

Let us assume that A, B, C form a triangle and $\measuredangle(A,C,B) < \pi$ and $\measuredangle(E,B,A) = \measuredangle(C,B,A)/3$ and $\measuredangle(B,A,E) = \measuredangle(B,A,C)/3$ and $\measuredangle(A,C,F) = \measuredangle(A,C,B)/3$ and $\measuredangle(F,A,C) = \measuredangle(B,A,C)/3$ and $\measuredangle(C,B,G) = \measuredangle(C,B,A)/3$ and $\measuredangle(G,C,B) = \measuredangle(A,C,B)/3$. Now we state the propositions:

(20) (i) $|F - E| = 4 \cdot \varnothing_{\triangleleft}(A,B,C) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3)$, and

(ii) $|G - F| = 4 \cdot \varnothing_{\triangleleft}(C,A,B) \cdot \sin(\measuredangle(C,B,A)/3) \cdot \sin(\measuredangle(B,A,C)/3) \cdot \sin(\measuredangle(A,C,B)/3)$, and

(iii) $|E - G| = 4 \cdot \varnothing_{\triangleleft}(B,C,A) \cdot \sin(\measuredangle(B,A,C)/3) \cdot \sin(\measuredangle(A,C,B)/3) \cdot \sin(\measuredangle(C,B,A)/3)$.

The theorem is a consequence of (17), (18), (19), (2), (5), and (16).

(21) (i) $|F - E| = |G - F|$, and

(ii) $|F - E| = |E - G|$, and

(iii) $|G - F| = |E - G|$.

The theorem is a consequence of (20).

(22) Morley's trisector theorem:

Suppose A, B, C form a triangle and $\measuredangle(A,B,C) < \pi$ and $\measuredangle(E,C,A) = \measuredangle(B,C,A)/3$ and $\measuredangle(C,A,E) = \measuredangle(C,A,B)/3$ and $\measuredangle(A,B,F) = \measuredangle(A,B,C)/3$ and $\measuredangle(F,A,B) = \measuredangle(C,A,B)/3$ and $\measuredangle(B,C,G) = \measuredangle(B,C,A)/3$ and $\measuredangle(G,B,C) = \measuredangle(A,B,C)/3$. Then

(i) $|F-E| = |G-F|$, and

(ii) $|F-E| = |E-G|$, and

(iii) $|G-F| = |E-G|$.

The theorem is a consequence of (21).

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Received March 26, 2015