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Flexary Operations[1]

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Summary. In this article we introduce necessary notation and definitions to prove the Euler's Partition Theorem according to H.S. Wilf's lecture notes [31]. Our aim is to create an environment which allows to formalize the theorem in a way that is as similar as possible to the original informal proof.

Euler's Partition Theorem is listed as item #45 from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at `http://www.cs.ru.nl/F.Wiedijk/100/` [30].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [6], [8], [15], [27], [13], [14], [23], [9], [10], [7], [25], [24], [3], [4], [19], [5], [22], [32], [33], [11], [21], [28], [18], and [12].

1. Auxiliary Facts about Finite Sequences Concatenation

From now on x, y denote objects, D, D_1, D_2 denote non empty sets, i, j, k, m, n denote natural numbers, f, g denote finite sequences of elements of D^*, f_1 denotes a finite sequence of elements of ${D_1}^*$, and f_2 denotes a finite sequence of elements of ${D_2}^*$.

Now we state the propositions:

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(1) Let us consider a function yielding function F, and an object a. Then $a \in \operatorname{Values} F$ if and only if there exists x and there exists y such that $x \in \operatorname{dom} F$ and $y \in \operatorname{dom}(F(x))$ and $a = F(x)(y)$.

(2) Let us consider a set D, and finite sequences f, g of elements of D^*. Then $\operatorname{Values} f \frown g = \operatorname{Values} f \cup \operatorname{Values} g$.

PROOF: Set $F = f \frown g$. $\operatorname{Values} f \subseteq \operatorname{Values} F$ by (1), [6, (26)]. $\operatorname{Values} g \subseteq \operatorname{Values} F$ by (1), [6, (28)]. $\operatorname{Values} F \subseteq \operatorname{Values} f \cup \operatorname{Values} g$ by (1), [6, (25)]. □

(3) The concatenation of $D \odot f \frown g$ = (the concatenation of $D \odot f$) $\frown$ (the concatenation of $D \odot g$).

(4) rng(the concatenation of $D \odot f$) = $\operatorname{Values} f$.

PROOF: Set D_3 = the concatenation of D. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence f of elements of D^* such that $\operatorname{len} f = \$_1$ holds $\operatorname{rng}(D_3 \odot f) = \operatorname{Values} f$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(5) If $f_1 = f_2$, then the concatenation of $D_1 \odot f_1$ = the concatenation of $D_2 \odot f_2$.

PROOF: Set C = the concatenation of D_2. Set N = the concatenation of D_1. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence f_4 of elements of ${D_1}^*$ for every finite sequence f_3 of elements of ${D_2}^*$ such that $\$_1 = \operatorname{len} f_4$ and $f_4 = f_3$ holds $N \odot f_4 = C \odot f_3$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(6) $i \in$ dom(the concatenation of $D \odot f$) if and only if there exists n and there exists k such that $n+1 \in \operatorname{dom} f$ and $k \in \operatorname{dom}(f(n+1))$ and $i = k + $ len(the concatenation of $D \odot f \restriction n$).

PROOF: Set D_3 = the concatenation of D. Define $\mathcal{P}$[natural number] $\equiv$ for every i for every finite sequence f of elements of D^* such that $\operatorname{len} f = \$_1$ holds $i \in \operatorname{dom}(D_3 \odot f)$ iff there exists n and there exists k such that $n+1 \in \operatorname{dom} f$ and $k \in \operatorname{dom}(f(n+1))$ and $i = k + \operatorname{len}(D_3 \odot f \restriction n)$. $\mathcal{P}[0]$. If $\mathcal{P}[j]$, then $\mathcal{P}[j+1]$ by [8, (19), (16)], (3), [27, (11)]. $\mathcal{P}[j]$ from [4, Sch. 2]. □

(7) Suppose $i \in$ dom(the concatenation of $D \odot f$). Then

(i) (the concatenation of $D \odot f$)(i) = (the concatenation of $D \odot f \frown g$)(i), and

(ii) (the concatenation of $D \odot f$)(i) = (the concatenation of $D \odot g \frown f$)$(i+$ len(the concatenation of $D \odot g$)).

The theorem is a consequence of (3).

(8) Suppose $k \in \operatorname{dom}(f(n+1))$. Then $f(n+1)(k)$ = (the concatenation of

$D \odot f)(k + \mathrm{len}(\text{the concatenation of } D \odot f{\restriction}n))$. The theorem is a consequence of (3).

2. Flexary Plus

From now on f denotes a complex-valued function and g, h denote complex-valued finite sequences.

Let us consider k and n. Let f, g be complex-valued functions. The functor $(f,k)+\ldots+(g,n)$ yielding a complex number is defined by

(Def. 1) (i) $h(0+1) = f(0+k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$, then $it = \sum(h{\restriction}(n -' k + 1))$, **if** $f = g$ and $k \leqslant n$,

(ii) $it = 0$, **otherwise**.

Now we state the propositions:

(9) Suppose $k \leqslant n$. Then there exists h such that

(i) $(f,k)+\ldots+(f,n) = \sum h$, and

(ii) $\mathrm{len}\, h = n -' k + 1$, and

(iii) $h(0+1) = f(0+k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$.

PROOF: Define $\mathcal{P}(\text{natural number}) = f(k + \$_1 - 1)$. Set $n_3 = n -' k + 1$. Consider p being a finite sequence such that $\mathrm{len}\, p = n_3$ and for every i such that $i \in \mathrm{dom}\, p$ holds $p(i) = \mathcal{P}(i)$ from [6, Sch. 2]. $\mathrm{rng}\, p \subseteq \mathbb{C}$. $p(1+0) = f(k+0)$ and ... and $p(1 + (n -' k)) = f(k + (n -' k))$ by [4, (11)], [26, (25)]. $\square$

(10) If $(f,k)+\ldots+(f,n) \neq 0$, then there exists i such that $k \leqslant i \leqslant n$ and $i \in \mathrm{dom}\, f$.

PROOF: Consider h such that $(f,k)+\ldots+(f,n) = \sum h$ and $\mathrm{len}\, h = n -' k + 1$ and $h(0+1) = f(0+k)$ and ... and $h(n -' k + 1) = f(n -' k + k)$. $\mathrm{rng}\, h \subseteq \{0\}$ by [26, (25)], [4, (11)]. $\square$

(11) $(f,k)+\ldots+(f,k) = f(k)$. The theorem is a consequence of (9).

(12) If $k \leqslant n+1$, then $(f,k)+\ldots+(f,(n+1)) = ((f,k)+\ldots+(f,n)) + f(n+1)$. The theorem is a consequence of (11) and (9).

(13) If $k \leqslant n$, then $(f,k)+\ldots+(f,n) = f(k) + ((f,(k+1))+\ldots+(f,n))$. The theorem is a consequence of (11) and (9).

(14) If $k \leqslant m \leqslant n$, then $((f,k)+\ldots+(f,m)) + ((f,(m+1))+\ldots+(f,n)) = (f,k)+\ldots+(f,n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv ((f,k)+\ldots+(f,m)) + ((f,(m+1))+\ldots+(f,(m+\$_1))) = (f,k)+\ldots+(f,(m+\$_1))$. $\mathcal{P}[0]$ by [4, (13)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (11)], (12). $\mathcal{P}[i]$ from [4, Sch. 2]. $\square$

(15) If $k > \operatorname{len} h$, then $(h,k)+\ldots+(h,n) = 0$. The theorem is a consequence of (9).

(16) If $n \geqslant \operatorname{len} h$, then $(h,k)+\ldots+(h,n) = (h,k)+\ldots+(h,\operatorname{len} h)$. The theorem is a consequence of (15) and (12).

(17) $(h,0)+\ldots+(h,k) = (h,1)+\ldots+(h,k)$. The theorem is a consequence of (13).

(18) $(h,1)+\ldots+(h,\operatorname{len} h) = \sum h$. The theorem is a consequence of (9).

(19) $(g \frown h,k)+\ldots+(g \frown h,n) = ((g,k)+\ldots+(g,n))+((h,(k -' \operatorname{len} g))+\ldots+(h,(n -' \operatorname{len} g)))$. The theorem is a consequence of (11), (15), (16), (17), and (14).

Let us consider n and k. Let f be a real-valued finite sequence. One can check that $(f,k)+\ldots+(f,n)$ is real.

Let f be a natural-valued finite sequence. Note that $(f,k)+\ldots+(f,n)$ is natural.

Let f be a complex-valued function. Assume $\operatorname{dom} f \cap \mathbb{N}$ is finite. The functor $(f,n)+\ldots$ yielding a complex number is defined by

(Def. 2) for every k such that for every i such that $i \in \operatorname{dom} f$ holds $i \leqslant k$ holds $it = (f,n)+\ldots+(f,k)$.

Let us consider h. One can check that the functor $(h,n)+\ldots$ yields a complex number and is defined by the term

(Def. 3) $(h,n)+\ldots+(h,\operatorname{len} h)$.

Let n be a natural number and h be a natural-valued finite sequence. Let us note that $(h,n)+\ldots$ is natural.

Now we state the propositions:

(20) Let us consider a finite, complex-valued function f. Then $f(n)+(f,(n+1))+\ldots = (f,n)+\ldots$. The theorem is a consequence of (13).

(21) $\sum h = (h,1)+\ldots$.

(22) $\sum h = h(1)+(h,2)+\ldots$. The theorem is a consequence of (18) and (20).

The scheme *TT* deals with complex-valued finite sequences f, g and natural numbers a, b and non zero natural numbers n, k and states that

(Sch. 1) $(f,a)+\ldots = (g,b)+\ldots$

provided

- for every j, $(f,(a+j\cdot n))+\ldots+(f,(a+j\cdot n+(n -' 1))) = (g,(b+j\cdot k))+\ldots+(g,(b+j\cdot k+(k -' 1)))$.

3. POWER FUNCTION

Let r be a real number and f be a real-valued function. The functor r^f yielding a real-valued function is defined by

(Def. 4) $\operatorname{dom} it = \operatorname{dom} f$ and for every x such that $x \in \operatorname{dom} f$ holds $it(x) = r^{f(x)}$.

Let n be a natural number and f be a natural-valued function. One can verify that n^f is natural-valued.

Let r be a real number and f be a real-valued finite sequence. One can check that r^f is finite sequence-like and r^f is $(\operatorname{len} f)$-element.

Let f be a one-to-one, natural-valued function. Observe that $(2+n)^f$ is one-to-one.

(23) Let us consider real numbers r, s. Then $r^{\langle s \rangle} = \langle r^s \rangle$.

(24) Let us consider a real number r, and real-valued finite sequences f, g. Then $r^{f \frown g} = r^f \frown r^g$.

PROOF: Set $f_5 = f \frown g$. Set $r_2 = r^f$. Set $r_3 = r^g$. For every i such that $1 \leqslant i \leqslant \operatorname{len} f_5$ holds $r^{f_5}(i) = (r_2 \frown r_3)(i)$ by [26, (25)], [6, (25)]. $\square$

(25) Let us consider a real-valued function f, and a function g. Then $2^f \cdot g = 2^{f \cdot g}$. PROOF: Set $h = 2^f$. Set $f_5 = f \cdot g$. $\operatorname{dom}(h \cdot g) \subseteq \operatorname{dom} 2^{f_5}$ by [9, (11)]. $\operatorname{dom} 2^{f_5} \subseteq \operatorname{dom}(h \cdot g)$ by [9, (11)]. For every x such that $x \in \operatorname{dom} 2^{f_5}$ holds $(h \cdot g)(x) = 2^{f_5}(x)$ by [9, (11), (13)]. $\square$

(26) Let us consider an increasing, natural-valued finite sequence f. If $n > 1$, then $n^f(1) + (n^f, 2) + \ldots < 2 \cdot n^{f(\operatorname{len} f)}$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequence f such that $n > 1$ and $f(\operatorname{len} f) \leqslant \$_1$ and $f \neq \emptyset$ holds $\sum n^f < 2 \cdot n^{f(\operatorname{len} f)}$. For every natural-valued finite sequence f such that $n > 1$ and $\operatorname{len} f = 1$ holds $\sum n^f < 2 \cdot n^{f(\operatorname{len} f)}$ by [26, (25)], [19, (83)], [6, (40)], [11, (73)]. $\mathcal{P}[0]$ by [26, (25)], [4, (25)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (8), (25), (13)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. $\sum n^f = n^f(1) + (n^f, 2) + \ldots$. $\square$

(27) Let us consider increasing, natural-valued finite sequences f_1, f_2. Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = n^{f_2}(1) + (n^{f_2}, 2) + \ldots$. Then $f_1 = f_2$.

PROOF: For every natural-valued finite sequence f such that $n > 1$ and $\sum n^f \leqslant 0$ holds $f = \emptyset$ by [11, (85)], [19, (83)]. Define $\mathcal{P}[\text{natural number}] \equiv$ for every increasing, natural-valued finite sequences f_1, f_2 such that $n > 1$ and $\sum n^{f_1} \leqslant \$_1$ and $\sum n^{f_1} = \sum n^{f_2}$ holds $f_1 = f_2$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by (21), (22), [4, (8)], [11, (72)]. $\mathcal{P}[i]$ from [4, Sch. 2]. $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = \sum n^{f_1}$. $n^{f_2}(1) + (n^{f_2}, 2) + \ldots = \sum n^{f_2}$. $\square$

(28) Let us consider a natural-valued function f. If $n > 1$, then $\operatorname{Coim}(n^f, n^k) = \operatorname{Coim}(f, k)$. PROOF: $\operatorname{Coim}(n^f, n^k) \subseteq \operatorname{Coim}(f, k)$ by [17, (30)]. $\square$

(29) Let us consider natural-valued functions f_1, f_2. Suppose $n > 1$. Then f_1 and f_2 are fiberwise equipotent if and only if n^{f_1} and n^{f_2} are fiberwise equipotent. PROOF: If f_1 and f_2 are fiberwise equipotent, then n^{f_1} and n^{f_2} are fiberwise equipotent by [9, (72)], [17, (30)], (28). For every object x, $\overline{\overline{\mathrm{Coim}(f_1, x)}} = \overline{\overline{\mathrm{Coim}(f_2, x)}}$ by [9, (72)], [17, (30)], (28). □

(30) Let us consider one-to-one, natural-valued finite sequences f_1, f_2. Suppose $n > 1$ and $n^{f_1}(1) + (n^{f_1}, 2) + \ldots = n^{f_2}(1) + (n^{f_2}, 2) + \ldots$. Then $\operatorname{rng} f_1 = \operatorname{rng} f_2$.
PROOF: Reconsider $F_1 = f_1$, $F_2 = f_2$ as a finite sequence of elements of $\mathbb{R}$. Set $s_1 = \mathrm{sort_a}\, F_1$. Set $s_2 = \mathrm{sort_a}\, F_2$. n^{F_1} and n^{s_1} are fiberwise equipotent. n^{F_2} and n^{s_2} are fiberwise equipotent. For every extended reals e_1, e_2 such that e_1, $e_2 \in \operatorname{dom} s_1$ and $e_1 < e_2$ holds $s_1(e_1) < s_1(e_2)$ by [16, (2)], [2, (77)]. For every extended reals e_1, e_2 such that e_1, $e_2 \in \operatorname{dom} s_2$ and $e_1 < e_2$ holds $s_2(e_1) < s_2(e_2)$ by [16, (2)], [2, (77)]. $\sum n^{s_1} = n^{s_1}(1) + (n^{s_1}, 2) + \ldots$. $\sum n^{f_1} = n^{f_1}(1) + (n^{f_1}, 2) + \ldots$. $\sum n^{s_1} = \sum n^{s_2}$. $n^{s_1}(1) + (n^{s_1}, 2) + \ldots = n^{s_2}(1) + (n^{s_2}, 2) + \ldots$ and s_1 is increasing and natural-valued. □

(31) There exists an increasing, natural-valued finite sequence f such that $n = 2^f(1) + (2^f, 2) + \ldots$.
PROOF: Set $D = \mathrm{digits}(n, 2)$. Consider d being a finite 0-sequence of $\mathbb{N}$ such that $\operatorname{dom} d = \operatorname{dom} D$ and for every natural number i such that $i \in \operatorname{dom} d$ holds $d(i) = D(i) \cdot 2^i$ and $\mathrm{value}(D, 2) = \sum d$. Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} d$, then there exists an increasing, natural-valued finite sequence f such that ($\operatorname{len} f = 0$ or $f(\operatorname{len} f) < \$_1$) and $\sum 2^f = \sum(d{\restriction}\$_1)$. $\mathcal{P}$[(0 **qua** natural number)] by [11, (72)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (13)], [29, (86)], [20, (65)], [4, (25), (23)]. $\mathcal{P}[i]$ from [4, Sch. 2]. Consider f being an increasing, natural-valued finite sequence such that $\operatorname{len} f = 0$ or $f(\operatorname{len} f) < \operatorname{len} d$ and $\sum 2^f = \sum(d{\restriction}\operatorname{len} d)$. $\sum 2^f = 2^f(1) + (2^f, 2) + \ldots$.
□

4. Value-based Function (Re)Organization

Let o be a function yielding function and x, y be objects. The functor $o_{x,y}$ yielding a set is defined by the term

(Def. 5) $o(x)(y)$.

Let F be a function yielding function. We say that F is double one-to-one if and only if

(Def. 6) for every objects x_1, x_2, y_1, y_2 such that $x_1 \in \operatorname{dom} F$ and $y_1 \in \operatorname{dom}(F(x_1))$ and $x_2 \in \operatorname{dom} F$ and $y_2 \in \operatorname{dom}(F(x_2))$ and $F_{x_1,y_1} = F_{x_2,y_2}$ holds $x_1 = x_2$ and $y_1 = y_2$.

Let D be a set. Observe that every finite sequence of elements of D^* which is empty is also double one-to-one and there exists a function yielding function which is double one-to-one and there exists a finite sequence of elements of D^* which is double one-to-one.

Let F be a double one-to-one, function yielding function and x be an object. One can check that $F(x)$ is one-to-one.

Let F be a one-to-one function. One can check that $\langle F \rangle$ is double one-to-one.

Now we state the propositions:

(32) Let us consider a function yielding function f. Then f is double one-to-one if and only if for every x, $f(x)$ is one-to-one and for every x and y such that $x \neq y$ holds $\mathrm{rng}(f(x))$ misses $\mathrm{rng}(f(y))$.

(33) Let us consider a set D, and double one-to-one finite sequences f_1, f_2 of elements of D^*. Suppose Values f_1 misses Values f_2. Then $f_1 \frown f_2$ is double one-to-one. The theorem is a consequence of (1).

Let D be a finite set.

A double reorganization of D is a double one-to-one finite sequence of elements of D^* and is defined by

(Def. 7) Values $it = D$.

Now we state the propositions:

(34) (i) $\emptyset$ is a double reorganization of $\emptyset$, and

(ii) $\langle \emptyset \rangle$ is a double reorganization of $\emptyset$.

(35) Let us consider a finite set D, and a one-to-one, onto finite sequence F of elements of D. Then $\langle F \rangle$ is a double reorganization of D.

(36) Let us consider finite sets D_1, D_2. Suppose D_1 misses D_2. Let us consider a double reorganization o_1 of D_1, and a double reorganization o_2 of D_2. Then $o_1 \frown o_2$ is a double reorganization of $D_1 \cup D_2$. The theorem is a consequence of (33) and (2).

(37) Let us consider a finite set D, a double reorganization o of D, and a one-to-one finite sequence F. Suppose $i \in \mathrm{dom}\, o$ and $\mathrm{rng}\, F \cap D \subseteq \mathrm{rng}(o(i))$. Then $o \mathbin{+\cdot} (i, F)$ is a double reorganization of $\mathrm{rng}\, F \cup (D \setminus \mathrm{rng}(o(i)))$.

Proof: Set $r_1 = \mathrm{rng}\, F$. Set $o_3 = o(i)$. Set $r_4 = \mathrm{rng}\, o_3$. Set $o_4 = o \mathbin{+\cdot} (i, F)$. $\mathrm{rng}\, o_4 \subseteq (r_1 \cup (D \setminus r_4))^*$ by [7, (31), (32)]. o_4 is double one-to-one by [7, (32)], (1). Values $o_4 \subseteq r_1 \cup (D \setminus r_4)$ by (1), [7, (31), (32)]. $D \setminus r_4 \subseteq$ Values o_4 by (1), [7, (32)]. $r_1 \subseteq$ Values o_4. $\square$

Let D be a finite set and n be a non zero natural number. One can check that there exists a double reorganization of D which is n-element.

Let D be a finite, natural-membered set, o be a double reorganization of D, and x be an object. One can verify that $o(x)$ is natural-valued.

Now we state the propositions:

(38) Let us consider a non empty finite sequence F, and a finite function G. Suppose $\operatorname{rng} G \subseteq \operatorname{rng} F$. Then there exists a $(\operatorname{len} F)$-element double reorganization o of $\operatorname{dom} G$ such that for every n, $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\operatorname{len}(o(n))})$.

PROOF: Set $D = \operatorname{dom} G$. Set $d =$ the one-to-one , onto finite sequence of elements of D. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \overline{\overline{G}}$, then there exists a $(\operatorname{len} F)$-element double reorganization o of $d^\circ(\operatorname{Seg} \$_1)$ such that for every k, $F(k) = G(o_{k,1})$ and ... and $F(k) = G(o_{k,\operatorname{len}(o(k))})$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (13)], [26, (29)], [4, (11)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

(39) Let us consider a non empty finite sequence F, and a finite sequence G. Suppose $\operatorname{rng} G \subseteq \operatorname{rng} F$. Then there exists a $(\operatorname{len} F)$-element double reorganization o of $\operatorname{dom} G$ such that for every n, $o(n)$ is increasing and $F(n) = G(o_{n,1})$ and ... and $F(n) = G(o_{n,\operatorname{len}(o(n))})$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} G$, then there exists a $(\operatorname{len} F)$-element double reorganization o of $\operatorname{Seg} \$_1$ such that for every k, $o(k)$ is increasing and $F(k) = G(o_{k,1})$ and ... and $F(k) = G(o_{k,\operatorname{len}(o(k))})$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (13)], [26, (29)], [4, (11)], [26, (25)]. $\mathcal{P}[i]$ from [4, Sch. 2]. □

Let f be a finite function, o be a double reorganization of $\operatorname{dom} f$, and x be an object. One can check that $f \cdot o(x)$ is finite sequence-like and there exists a finite sequence which is complex-functions-valued and finite sequence-yielding.

Let f be a function yielding function and g be a function. We introduce $g \odot f$ as a synonym of $[g, f]$.

One can check that $g \odot f$ is function yielding.

Let f be a $((\operatorname{dom} g)^*)$-valued finite sequence. One can check that $g \odot f$ is finite sequence-yielding.

Let x be an object. Let us note that $(g \odot f)(x)$ is $(\operatorname{len}(f(x)))$-element.

Let f be a function yielding finite sequence. One can verify that $g \odot f$ is finite sequence-like and $g \odot f$ is $(\operatorname{len} f)$-element.

Let f be a function yielding function and g be a complex-valued function. One can check that $g \odot f$ is complex-functions-valued.

Let g be a natural-valued function. One can check that $g \odot f$ is natural-functions-valued.

Let us consider a function yielding function f and a function g. Now we state the propositions:

(40) $\operatorname{Values} g \odot f = g^\circ(\operatorname{Values} f)$.

PROOF: Set $g_3 = g \odot f$. $\operatorname{Values} g_3 \subseteq g^\circ(\operatorname{Values} f)$ by (1), [9, (11), (12)]. Consider b being an object such that $b \in \operatorname{dom} g$ and $b \in \operatorname{Values} f$ and

$g(b) = a$. Consider x, y being objects such that $x \in \operatorname{dom} f$ and $y \in \operatorname{dom}(f(x))$ and $b = f(x)(y)$. □

(41) $(g \odot f)(x) = g \cdot f(x)$.

Now we state the proposition:

(42) Let us consider a function yielding function f, a finite sequence g, and objects x, y. Then $(g \odot f)_{x,y} = g(f_{x,y})$. The theorem is a consequence of (41).

Let f be a complex-functions-valued, finite sequence-yielding function. The functor $\sum f$ yielding a complex-valued function is defined by

(Def. 8) $\operatorname{dom} it = \operatorname{dom} f$ and for every set x, $it(x) = \sum(f(x))$.

Let f be a complex-functions-valued, finite sequence-yielding finite sequence. One can verify that $\sum f$ is finite sequence-like and $\sum f$ is $(\operatorname{len} f)$-element.

Let f be a natural-functions-valued, finite sequence-yielding function. One can verify that $\sum f$ is natural-valued.

Let f, g be complex-functions-valued finite sequences. One can check that $f \frown g$ is complex-functions-valued.

Let f, g be extended real-valued finite sequences. One can verify that $f \frown g$ is extended real-valued.

Let f be a complex-functions-valued function and X be a set. One can check that $f{\restriction}X$ is complex-functions-valued.

Let f be a finite sequence-yielding function. One can check that $f{\restriction}X$ is finite sequence-yielding.

Let F be a complex-valued function. One can check that $\langle F \rangle$ is complex-functions-valued.

Let us consider finite sequences f, g. Now we state the propositions:

(43) If $f \frown g$ is finite sequence-yielding, then f is finite sequence-yielding and g is finite sequence-yielding.

(44) If $f \frown g$ is complex-functions-valued, then f is complex-functions-valued and g is complex-functions-valued.

Now we state the propositions:

(45) Let us consider a complex-valued finite sequence f. Then $\sum \langle f \rangle = \langle \sum f \rangle$.

(46) Let us consider complex-functions-valued, finite sequence-yielding finite sequences f, g. Then $\sum(f \frown g) = \sum f \frown \sum g$.

PROOF: For every i such that $1 \leqslant i \leqslant \operatorname{len} f + \operatorname{len} g$ holds $(\sum(f \frown g))(i) = (\sum f \frown \sum g)(i)$ by [26, (25)], [6, (25)]. □

(47) Let us consider a complex-valued finite sequence f, and a double reorganization o of $\operatorname{dom} f$. Then $\sum f = \sum \sum (f \odot o)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every complex-valued finite sequence f for every double reorganization o of $\operatorname{dom} f$ such that $\operatorname{len} f = \$_1$ holds $\sum f = \sum\sum(f \odot o)$. $\mathcal{P}[0]$ by [26, (29)], [11, (72)], [23, (11)], [11, (81)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [4, (11)], [26, (25)], (1), [12, (116)]. $\mathcal{P}[i]$ from [4, Sch. 2]. $\square$

Let us note that $\mathbb{N}^*$ is natural-functions-membered and $\mathbb{C}^*$ is complex-functions-membered.

Now we state the proposition:

(48) Let us consider a finite sequence f of elements of $\mathbb{C}^*$. Then $\sum$(the concatenation of $\mathbb{C} \odot f) = \sum\sum f$.

Proof: Set C = the concatenation of $\mathbb{C}$. Define $\mathcal{P}$[natural number] $\equiv$ for every finite sequence f of elements of $\mathbb{C}^*$ such that $\operatorname{len} f = \$_1$ holds $\sum(C \odot f) = \sum\sum f$. $\mathcal{P}[0]$. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [8, (19), (16)], (46), (45). $\mathcal{P}[i]$ from [4, Sch. 2]. $\square$

Let f be a finite function.

A valued reorganization of f is a double reorganization of $\operatorname{dom} f$ and is defined by

(Def. 9) for every n, there exists x such that $x = f(it_{n,1})$ and ... and $x = f(it_{n,\operatorname{len}(it(n))})$ and for every natural numbers n_1, n_2, i_1, i_2 such that $i_1 \in \operatorname{dom}(it(n_1))$ and $i_2 \in \operatorname{dom}(it(n_2))$ and $f(it_{n_1,i_1}) = f(it_{n_2,i_2})$ holds $n_1 = n_2$.

Now we state the propositions:

(49) Let us consider a finite function f, and a valued reorganization o of f. Then

(i) $\operatorname{rng}((f \odot o)(n)) = \emptyset$, or

(ii) $\operatorname{rng}((f \odot o)(n)) = \{f(o_{n,1})\}$ and $1 \in \operatorname{dom}(o(n))$.

Proof: Consider y such that $y \in \operatorname{rng}((f \odot o)(n))$. Consider x such that $x \in \operatorname{dom}((f \odot o)(n))$ and $(f \odot o)(n)(x) = y$. $n \in \operatorname{dom}(f \odot o)$. Consider w being an object such that $w = f(o_{n,1})$ and ... and $w = f(o_{n,\operatorname{len}(o(n))})$. $\operatorname{rng}((f \odot o)(n)) \subseteq \{f(o_{n,1})\}$ by [9, (11), (12)], [26, (25)]. $\square$

(50) Let us consider a finite sequence f, and valued reorganizations o_1, o_2 of f. Suppose $\operatorname{rng}((f \odot o_1)(i)) = \operatorname{rng}((f \odot o_2)(i))$. Then $\operatorname{rng}(o_1(i)) = \operatorname{rng}(o_2(i))$.

(51) Let us consider a finite sequence f, a complex-valued finite sequence g, and double reorganizations o_1, o_2 of $\operatorname{dom} g$. Suppose o_1 is a valued reorganization of f and o_2 is a valued reorganization of f and $\operatorname{rng}((f \odot o_1)(i)) = \operatorname{rng}((f \odot o_2)(i))$. Then $(\sum(g \odot o_1))(i) = (\sum(g \odot o_2))(i)$. The theorem is a consequence of (41).

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