

# Convergent Filter Bases

Roland Coghetto  
Rue de la Brasserie 5  
7100 La Louvière, Belgium

**Summary.** We are inspired by the work of Henri Cartan [16], Bourbaki [10] (TG. I Filtres) and Claude Wagschal [34]. We define the base of filter, image filter, convergent filter bases, limit filter and the filter base of tails (fr: *filtre des sections*).

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The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [2], [33], [20], [18], [28], [11], [12], [13], [29], [3], [37], [25], [26], [4], [17], [30], [5], [14], [23], [35], [36], [22], [31], [6], [7], [9], [19], [27], and [15].

## 1. FILTERS – SET-THEORETICAL APPROACH

From now on  $X$  denotes a non empty set,  $\mathcal{F}$  denotes a filter of  $X$ , and  $S$  denotes a family of subsets of  $X$ .

Let  $X$  be a set and  $S$  be a family of subsets of  $X$ . We say that  $S$  is upper if and only if

(Def. 1) for every subsets  $Y_1, Y_2$  of  $X$  such that  $Y_1 \in S$  and  $Y_1 \subseteq Y_2$  holds  $Y_2 \in S$ .

Let us note that there exists a  $\cap$ -closed family of subsets of  $X$  which is non empty and there exists a non empty,  $\cap$ -closed family of subsets of  $X$  which is upper.

Let  $X$  be a non empty set. Let us note that there exists a non empty, upper,  $\cap$ -closed family of subsets of  $X$  which has non empty elements.

Now we state the propositions:

- (1)  $S$  is a non empty, upper,  $\cap$ -closed family of subsets of  $X$  with non empty elements if and only if  $S$  is a filter of  $X$ .
- (2) Let us consider non empty sets  $X_1, X_2$ , a filter  $\mathcal{F}_1$  of  $X_1$ , and a filter  $\mathcal{F}_2$  of  $X_2$ . Then the set of all  $f_1 \times f_2$  where  $f_1$  is an element of  $\mathcal{F}_1$ ,  $f_2$  is an element of  $\mathcal{F}_2$  is a non empty family of subsets of  $X_1 \times X_2$ .

Let  $X$  be a non empty set. We say that  $X$  is  $\cap$ -finite closed if and only if

- (Def. 2) for every finite, non empty subset  $S_1$  of  $X$ ,  $\cap S_1 \in X$ .

One can check that there exists a non empty set which is  $\cap$ -finite closed.

Now we state the proposition:

- (3) Let us consider a non empty set  $X$ . If  $X$  is  $\cap$ -finite closed, then  $X$  is  $\cap$ -closed.

Note that every non empty set which is  $\cap$ -finite closed is also  $\cap$ -closed.

- (4) Let us consider a set  $X$ , and a family  $S$  of subsets of  $X$ . Then  $S$  is  $\cap$ -closed and  $X \in S$  if and only if  $\text{FinMeetCl}(S) \subseteq S$ .
- (5) Let us consider a non empty set  $X$ , and a non empty subset  $A$  of  $X$ . Then  $\{B, \text{ where } B \text{ is a subset of } X : A \subseteq B\}$  is a filter of  $X$ .

Let  $X$  be a non empty set. Note that every filter of  $X$  is  $\cap$ -closed.

- (6) Let us consider a set  $X$ , and a family  $B$  of subsets of  $X$ . If  $B = \{X\}$ , then  $B$  is upper.
- (7) Let us consider a non empty set  $X$ , and a filter  $\mathcal{F}'$  of  $X$ . Then  $\mathcal{F}' \neq 2^X$ .

Let  $X$  be a non empty set. The functor  $\text{Filt}(X)$  yielding a non empty set is defined by the term

- (Def. 3) the set of all  $\mathcal{F}'$  where  $\mathcal{F}'$  is a filter of  $X$ .

Let  $I$  be a non empty set and  $M$  be a  $(\text{Filt}(X))$ -valued many sorted set indexed by  $I$ . The intersection of the family of filters  $M$  yielding a filter of  $X$  is defined by the term

- (Def. 4)  $\cap \text{rng } M$ .

Let  $\mathcal{F}_1, \mathcal{F}_2$  be filters of  $X$ . We say that  $\mathcal{F}_1$  is coarser than  $\mathcal{F}_2$  if and only if

- (Def. 5)  $\mathcal{F}_1 \subseteq \mathcal{F}_2$ .

One can verify that the predicate is reflexive. We say that  $\mathcal{F}_1$  is finer than  $\mathcal{F}_2$  if and only if

- (Def. 6)  $\mathcal{F}_2 \subseteq \mathcal{F}_1$ .

Observe that the predicate is reflexive.

Now we state the propositions:

- (8) Let us consider a non empty set  $X$ , a filter  $\mathcal{F}'$  of  $X$ , and a filter  $\mathcal{F}$  of  $X$ . Suppose  $\mathcal{F} = \{X\}$ . Then  $\mathcal{F}$  is coarser than  $\mathcal{F}'$ .

- (9) Let us consider a non empty set  $X$ , a non empty set  $I$ , a  $(\text{Filt}(X))$ -valued many sorted set  $M$  indexed by  $I$ , an element  $i$  of  $I$ , and a filter  $\mathcal{F}'$  of  $X$ . Suppose  $\mathcal{F}' = M(i)$ . Then the intersection of the family of filters  $M$  is coarser than  $\mathcal{F}'$ .
- (10) Let us consider a set  $X$ , and a family  $S$  of subsets of  $X$ . Suppose  $\text{FinMeetCl}(S)$  has non empty elements. Then  $S$  has non empty elements.
- (11) Let us consider a non empty set  $X$ , a family  $G$  of subsets of  $X$ , and a filter  $\mathcal{F}'$  of  $X$ . Suppose  $G \subseteq \mathcal{F}'$ . Then
  - (i)  $\text{FinMeetCl}(G) \subseteq \mathcal{F}'$ , and
  - (ii)  $\text{FinMeetCl}(G)$  has non empty elements.

The theorem is a consequence of (4).

Let  $X$  be a non empty set,  $\mathcal{F}'$  be a filter of  $X$ , and  $B$  be a non empty subset of  $\mathcal{F}'$ . We say that  $B$  is filter basis if and only if

- (Def. 7) for every element  $f$  of  $\mathcal{F}'$ , there exists an element  $b$  of  $B$  such that  $b \subseteq f$ .

Now we state the proposition:

- (12) Let us consider a non empty set  $X$ , a filter  $\mathcal{F}'$  of  $X$ , and a non empty subset  $B$  of  $\mathcal{F}'$ . Then  $\mathcal{F}'$  is coarser than  $B$  if and only if  $B$  is filter basis.

Let  $X$  be a non empty set and  $\mathcal{F}'$  be a filter of  $X$ . Observe that there exists a non empty subset of  $\mathcal{F}'$  which is filter basis.

A generalized basis of  $\mathcal{F}'$  is a filter basis, non empty subset of  $\mathcal{F}'$ . Now we state the proposition:

- (13) Let us consider a non empty set  $X$ . Then every filter of  $X$  is a generalized basis of  $\mathcal{F}'$ .

Let  $X$  be a set and  $B$  be a family of subsets of  $X$ . The functor  $[B]$  yielding a family of subsets of  $X$  is defined by

- (Def. 8) for every subset  $x$  of  $X$ ,  $x \in [B]$  iff there exists an element  $b$  of  $B$  such that  $b \subseteq x$ .

Now we state the propositions:

- (14) Let us consider a set  $X$ , and a family  $S$  of subsets of  $X$ . Then  $[S] = \{x, \text{ where } x \text{ is a subset of } X : \text{ there exists an element } b \text{ of } S \text{ such that } b \subseteq x\}$ .
- (15) Let us consider a set  $X$ , and an empty family  $B$  of subsets of  $X$ . Then  $[B] = 2^X$ .
- (16) Let us consider a set  $X$ , and a family  $B$  of subsets of  $X$ . If  $\emptyset \in B$ , then  $[B] = 2^X$ .

## 2. FILTERS – LATTICE-THEORETICAL APPROACH

Now we state the propositions:

- (17) Let us consider a set  $X$ , a non empty family  $B$  of subsets of  $X$ , and a subset  $L$  of  $2^X$ . If  $B = L$ , then  $[B] = \uparrow L$ .
- (18) Let us consider a set  $X$ , and a family  $B$  of subsets of  $X$ . Then  $B \subseteq [B]$ .

Let  $X$  be a set and  $B_1, B_2$  be families of subsets of  $X$ . We say that  $B_1$  and  $B_2$  are equivalent generators if and only if

- (Def. 9) for every element  $b_1$  of  $B_1$ , there exists an element  $b_2$  of  $B_2$  such that  $b_2 \subseteq b_1$  and for every element  $b_2$  of  $B_2$ , there exists an element  $b_1$  of  $B_1$  such that  $b_1 \subseteq b_2$ .

Let us note that the predicate is reflexive and symmetric.

Let us consider a set  $X$  and families  $B_1, B_2$  of subsets of  $X$ .

Let us assume that  $B_1$  and  $B_2$  are equivalent generators. Now we state the propositions:

- (19)  $[B_1] \subseteq [B_2]$ .
- (20)  $[B_1] = [B_2]$ .

Let  $X$  be a non empty set,  $\mathcal{F}'$  be a filter of  $X$ , and  $B$  be a non empty subset of  $\mathcal{F}'$ . The functor  $\# B$  yielding a non empty family of subsets of  $X$  is defined by the term

- (Def. 10)  $B$ .

Now we state the propositions:

- (21) Let us consider a non empty set  $X$ , a filter  $\mathcal{F}'$  of  $X$ , and a generalized basis  $B$  of  $\mathcal{F}'$ . Then  $\mathcal{F}' = [\# B]$ .
- (22) Let us consider a non empty set  $X$ , a filter  $\mathcal{F}'$  of  $X$ , and a family  $B$  of subsets of  $X$ . If  $\mathcal{F}' = [B]$ , then  $B$  is a generalized basis of  $\mathcal{F}'$ .
- (23) Let us consider a non empty set  $X$ , a filter  $\mathcal{F}'$  of  $X$ , a generalized basis  $B$  of  $\mathcal{F}'$ , a family  $S$  of subsets of  $X$ , and a subset  $S_1$  of  $\mathcal{F}'$ . Suppose  $S = S_1$  and  $\# B$  and  $S$  are equivalent generators. Then  $S_1$  is a generalized basis of  $\mathcal{F}'$ . The theorem is a consequence of (19), (21), and (22).
- (24) Let us consider a non empty set  $X$ , a filter  $\mathcal{F}'$  of  $X$ , and generalized bases  $\mathcal{B}_1, \mathcal{B}_2$  of  $\mathcal{F}'$ . Then  $\# \mathcal{B}_1$  and  $\# \mathcal{B}_2$  are equivalent generators. The theorem is a consequence of (21).

Let  $X$  be a set and  $B$  be a family of subsets of  $X$ . We say that  $B$  is quasi basis if and only if

- (Def. 11) for every elements  $b_1, b_2$  of  $B$ , there exists an element  $b$  of  $B$  such that  $b \subseteq b_1 \cap b_2$ .

Let  $X$  be a non empty set. Let us note that there exists a non empty family of subsets of  $X$  which is quasi basis and there exists a non empty, quasi basis family of subsets of  $X$  which has non empty elements.

A filter base of  $X$  is a non empty, quasi basis family of subsets of  $X$  with non empty elements. Now we state the proposition:

- (25) Let us consider a non empty set  $X$ , and a filter base  $B$  of  $X$ . Then  $[B]$  is a filter of  $X$ .

Let  $X$  be a non empty set and  $B$  be a filter base of  $X$ . The functor  $[B]$  yielding a filter of  $X$  is defined by the term

(Def. 12)  $[B]$ .

Now we state the propositions:

- (26) Let us consider a non empty set  $X$ , and filter bases  $\mathcal{B}_1, \mathcal{B}_2$  of  $X$ . Suppose  $[\mathcal{B}_1] = [\mathcal{B}_2]$ . Then  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are equivalent generators.
- (27) Let us consider a non empty set  $X$ , a filter base  $\mathcal{F}$  of  $X$ , and a filter  $\mathcal{F}'$  of  $X$ . Suppose  $\mathcal{F} \subseteq \mathcal{F}'$ . Then  $[\mathcal{F}]$  is coarser than  $\mathcal{F}'$ .
- (28) Let us consider a non empty set  $X$ , and a family  $G$  of subsets of  $X$ . Suppose  $\text{FinMeetCl}(G)$  has non empty elements. Then
- (i)  $\text{FinMeetCl}(G)$  is a filter base of  $X$ , and
  - (ii) there exists a filter  $\mathcal{F}'$  of  $X$  such that  $\text{FinMeetCl}(G) \subseteq \mathcal{F}'$ .

The theorem is a consequence of (4).

- (29) Let us consider a non empty set  $X$ , and a filter  $\mathcal{F}'$  of  $X$ . Then every generalized basis of  $\mathcal{F}'$  is a filter base of  $X$ .
- (30) Let us consider a non empty set  $X$ . Then every filter base of  $X$  is a generalized basis of  $[B]$ .
- (31) Let us consider a non empty set  $X$ , a filter  $\mathcal{F}'$  of  $X$ , a generalized basis  $B$  of  $\mathcal{F}'$ , and a subset  $L$  of  $2_{\subseteq}^X$ . If  $L = \# B$ , then  $\mathcal{F}' = \uparrow L$ . The theorem is a consequence of (21) and (17).
- (32) Let us consider a non empty set  $X$ , a filter base  $B$  of  $X$ , and a subset  $L$  of  $2_{\subseteq}^X$ . If  $L = B$ , then  $[B] = \uparrow L$ .
- (33) Let us consider a non empty set  $X$ , filters  $\mathcal{F}_1, \mathcal{F}_2$  of  $X$ , a generalized basis  $\mathcal{B}_1$  of  $\mathcal{F}_1$ , and a generalized basis  $\mathcal{B}_2$  of  $\mathcal{F}_2$ . Then  $\mathcal{F}_1$  is coarser than  $\mathcal{F}_2$  if and only if  $\mathcal{B}_1$  is coarser than  $\mathcal{B}_2$ . The theorem is a consequence of (21).
- (34) Let us consider non empty sets  $X, Y$ , a function  $f$  from  $X$  into  $Y$ , a filter  $\mathcal{F}'$  of  $X$ , and a generalized basis  $B$  of  $\mathcal{F}'$ . Then
- (i)  $f^\circ(\# B)$  is a filter base of  $Y$ , and
  - (ii)  $[f^\circ(\# B)]$  is a filter of  $Y$ , and

(iii)  $[f^\circ(\# B)] = \{M, \text{ where } M \text{ is a subset of } Y : f^{-1}(M) \in \mathcal{F}'\}$ .

PROOF: Set  $\mathcal{F} = f^\circ(\# B)$ .  $\mathcal{F}$  is a quasi basis, non empty family of subsets of  $Y$  by (29), [35, (123), (121)].  $\mathcal{F}$  has non empty elements by [35, (118)].  $[\mathcal{F}] = \{M, \text{ where } M \text{ is a subset of } Y : f^{-1}(M) \in \mathcal{F}'\}$  by [35, (143)], [12, (42)], (21), [35, (123)].  $\square$

Let  $X, Y$  be non empty sets,  $f$  be a function from  $X$  into  $Y$ , and  $\mathcal{F}'$  be a filter of  $X$ . The image of filter  $\mathcal{F}'$  under  $f$  yielding a filter of  $Y$  is defined by the term

(Def. 13)  $\{M, \text{ where } M \text{ is a subset of } Y : f^{-1}(M) \in \mathcal{F}'\}$ .

Now we state the propositions:

(35) Let us consider non empty sets  $X, Y$ , a function  $f$  from  $X$  into  $Y$ , and a filter  $\mathcal{F}'$  of  $X$ . Then

- (i)  $f^\circ \mathcal{F}'$  is a filter base of  $Y$ , and
- (ii)  $[f^\circ \mathcal{F}'] = \text{the image of filter } \mathcal{F}' \text{ under } f$ .

The theorem is a consequence of (13) and (34).

(36) Let us consider a non empty set  $X$ , and a filter base  $B$  of  $X$ . If  $B = [B]$ , then  $B$  is a filter of  $X$ .

(37) Let us consider non empty sets  $X, Y$ , a function  $f$  from  $X$  into  $Y$ , a filter  $\mathcal{F}'$  of  $X$ , and a generalized basis  $B$  of  $\mathcal{F}'$ . Then

- (i)  $f^\circ(\# B)$  is a generalized basis of the image of filter  $\mathcal{F}'$  under  $f$ , and
- (ii)  $[f^\circ(\# B)] = \text{the image of filter } \mathcal{F}' \text{ under } f$ .

The theorem is a consequence of (34) and (30).

(38) Let us consider non empty sets  $X, Y$ , a function  $f$  from  $X$  into  $Y$ , and filter bases  $\mathcal{B}_1, \mathcal{B}_2$  of  $X$ . Suppose  $\mathcal{B}_1$  is coarser than  $\mathcal{B}_2$ . Then  $[\mathcal{B}_1]$  is coarser than  $[\mathcal{B}_2]$ . The theorem is a consequence of (30) and (33).

(39) Let us consider non empty sets  $X, Y$ , a function  $f$  from  $X$  into  $Y$ , and a filter  $\mathcal{F}'$  of  $X$ . Then  $f^\circ \mathcal{F}'$  is a filter of  $Y$  if and only if  $Y = \text{rng } f$ .

PROOF: Reconsider  $f_3 = f^\circ \mathcal{F}'$  as a filter base of  $Y$ .  $[f_3] \subseteq f_3$  by [35, (143)], [11, (76), (77)].  $\square$

(40) Let us consider a non empty set  $X$ , a non empty subset  $A$  of  $X$ , a filter  $\mathcal{F}'$  of  $A$ , and a generalized basis  $B$  of  $\mathcal{F}'$ . Then

- (i)  $(\overset{A}{\hookrightarrow})^\circ(\# B)$  is a filter base of  $X$ , and
- (ii)  $[(\overset{A}{\hookrightarrow})^\circ(\# B)]$  is a filter of  $X$ , and
- (iii)  $[(\overset{A}{\hookrightarrow})^\circ(\# B)] = \{M, \text{ where } M \text{ is a subset of } X : (\overset{A}{\hookrightarrow})^{-1}(M) \in \mathcal{F}'\}$ .

Let  $L$  be a non empty relational structure. The functor  $\text{Tails}(L)$  yielding a non empty family of subsets of  $L$  is defined by the term

(Def. 14) the set of all  $\uparrow i$  where  $i$  is an element of  $L$ .

Now we state the proposition:

(41) Let us consider a non empty, transitive, reflexive relational structure  $L$ . Suppose  $\Omega_L$  is directed. Then  $[\text{Tails}(L)]$  is a filter of  $\Omega_L$ .

PROOF:  $\text{Tails}(L)$  is non empty family of subsets of  $L$  and quasi basis and has non empty elements by [6, (22)].  $\square$

Let  $L$  be a non empty, transitive, reflexive relational structure. Assume  $\Omega_L$  is directed. The functor  $\text{TailsFilter } L$  yielding a filter of  $\Omega_L$  is defined by the term

(Def. 15)  $[\text{Tails}(L)]$ .

Now we state the proposition:

(42) Let us consider a non empty, transitive, reflexive relational structure  $L$ . Suppose  $\Omega_L$  is directed. Then  $\text{Tails}(L)$  is a generalized basis of  $\text{TailsFilter } L$ . The theorem is a consequence of (22).

Let  $L$  be a relational structure and  $x$  be a family of subsets of  $L$ . The functor  $\# x$  yielding a family of subsets of  $\Omega_L$  is defined by the term

(Def. 16)  $x$ .

Now we state the proposition:

(43) Let us consider a non empty set  $X$ , a non empty, transitive, reflexive relational structure  $L$ , and a function  $f$  from  $\Omega_L$  into  $X$ . Suppose  $\Omega_L$  is directed. Then  $f^\circ(\# \text{Tails}(L))$  is a generalized basis of the image of filter  $\text{TailsFilter } L$  under  $f$ . The theorem is a consequence of (42) and (37).

Let us consider a non empty set  $X$ , a non empty, transitive, reflexive relational structure  $L$ , a function  $f$  from  $\Omega_L$  into  $X$ , and a subset  $x$  of  $X$ . Now we state the propositions:

(44) Suppose  $\Omega_L$  is directed and  $x \in f^\circ(\# \text{Tails}(L))$ . Then there exists an element  $j$  of  $L$  such that for every element  $i$  of  $L$  such that  $i \geq j$  holds  $f(i) \in x$ .

(45) Suppose  $\Omega_L$  is directed and there exists an element  $j$  of  $L$  such that for every element  $i$  of  $L$  such that  $i \geq j$  holds  $f(i) \in x$ . Then there exists an element  $b$  of  $\text{Tails}(L)$  such that  $f^\circ b \subseteq x$ .

(46) Let us consider a non empty set  $X$ , a non empty, transitive, reflexive relational structure  $L$ , a function  $f$  from  $\Omega_L$  into  $X$ , a filter  $\mathcal{F}'$  of  $X$ , and a generalized basis  $B$  of  $\mathcal{F}'$ . Suppose  $\Omega_L$  is directed. Then  $\mathcal{F}'$  is coarser than the image of filter  $\text{TailsFilter } L$  under  $f$  if and only if  $B$  is coarser than  $f^\circ(\# \text{Tails}(L))$ . The theorem is a consequence of (43) and (33).

(47) Let us consider a non empty set  $X$ , a non empty, transitive, reflexive relational structure  $L$ , a function  $f$  from  $\Omega_L$  into  $X$ , and a filter base  $B$  of

$X$ . Suppose  $\Omega_L$  is directed. Then  $B$  is coarser than  $f^\circ(\# \text{Tails}(L))$  if and only if for every element  $b$  of  $B$ , there exists an element  $i$  of  $L$  such that for every element  $j$  of  $L$  such that  $i \leq j$  holds  $f(j) \in b$ . The theorem is a consequence of (44) and (45).

Let  $X$  be a non empty set and  $s$  be a sequence of  $X$ . The elementary filter of  $s$  yielding a filter of  $X$  is defined by the term

(Def. 17) the image of filter  $\text{FrechetFilter}(\mathbb{N})$  under  $s$ .

Now we state the propositions:

(48) There exists a sequence  $\mathcal{F}'$  of  $2^{\mathbb{N}}$  such that for every element  $x$  of  $\mathbb{N}$ ,  $\mathcal{F}'(x) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : x \leq y\}$ .

PROOF: Define  $\mathcal{F}(\text{object}) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : \text{there exists an element } x_0 \text{ of } \mathbb{N} \text{ such that } x_0 = \$_1 \text{ and } x_0 \leq y\}$ . There exists a function  $f$  from  $\mathbb{N}$  into  $2^{\mathbb{N}}$  such that for every object  $x$  such that  $x \in \mathbb{N}$  holds  $f(x) = \mathcal{F}(x)$  from [12, Sch. 2]. Consider  $\mathcal{F}'$  being a function from  $\mathbb{N}$  into  $2^{\mathbb{N}}$  such that for every object  $x$  such that  $x \in \mathbb{N}$  holds  $\mathcal{F}'(x) = \mathcal{F}(x)$ . For every element  $x$  of  $\mathbb{N}$ ,  $\mathcal{F}'(x) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : x \leq y\}$ .  $\square$

(49) Let us consider a natural number  $n$ . Then  $\mathbb{N} \setminus \{t, \text{ where } t \text{ is an element of } \mathbb{N} : n \leq t\}$  is finite.

PROOF:  $\mathbb{N} \setminus \{t, \text{ where } t \text{ is an element of } \mathbb{N} : n \leq t\} \subseteq n + 1$  by [8, (3), (5)], [32, (4)].  $\square$

(50) Let us consider an element  $p$  of the ordered  $\mathbb{N}$ . Then  $\{x, \text{ where } x \text{ is an element of } \mathbb{N} : \text{there exists an element } p_0 \text{ of } \mathbb{N} \text{ such that } p = p_0 \text{ and } p_0 \leq x\} = \uparrow p$ .

PROOF: For every element  $p$  of the carrier of the ordered  $\mathbb{N}$ ,  $\{x, \text{ where } x \text{ is an element of the carrier of the ordered } \mathbb{N} : p \leq x\} = \uparrow p$  by [6, (18)].  $\square$

Observe that  $\Omega_{\text{the ordered } \mathbb{N}}$  is directed and the ordered  $\mathbb{N}$  is reflexive.

Now we state the proposition:

(51) Let us consider a denumerable set  $X$ . Then  $\text{FrechetFilter}(X) =$  the set of all  $X \setminus A$  where  $A$  is a finite subset of  $X$ .

Let us consider a sequence  $\mathcal{F}'$  of  $2^{\mathbb{N}}$ .

Let us assume that for every element  $x$  of  $\mathbb{N}$ ,  $\mathcal{F}'(x) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : x \leq y\}$ . Now we state the propositions:

(52)  $\text{rng } \mathcal{F}'$  is a generalized basis of  $\text{FrechetFilter}(\mathbb{N})$ .

PROOF:  $\text{FrechetFilter}(\mathbb{N}) =$  the set of all  $\mathbb{N} \setminus A$  where  $A$  is a finite subset of  $\mathbb{N}$ . For every object  $t$  such that  $t \in \text{rng } \mathcal{F}'$  holds  $t \in \text{FrechetFilter}(\mathbb{N})$ . Reconsider  $\mathcal{F}_1 = \text{rng } \mathcal{F}'$  as a non empty subset of  $\text{FrechetFilter}(\mathbb{N})$ .  $\mathcal{F}_1$  is filter basis by [21, (2)], [4, (44)], [11, (3)].  $\square$



(53)  $\#Tails(\text{the ordered } \mathbb{N}) = \text{rng } \mathcal{F}'$ . The theorem is a consequence of (50).

Now we state the proposition:

(54) (i)  $\#Tails(\text{the ordered } \mathbb{N})$  is a generalized basis of  $\text{FrechetFilter}(\mathbb{N})$ ,  
and

(ii)  $TailsFilter \text{ the ordered } \mathbb{N} = \text{FrechetFilter}(\mathbb{N})$ .

The theorem is a consequence of (48), (53), (52), and (21).

The base of Frechet filter yielding a filter base of  $\mathbb{N}$  is defined by the term

(Def. 18)  $\#Tails(\text{the ordered } \mathbb{N})$ .

Now we state the propositions:

(55)  $\mathbb{N} \in$  the base of Frechet filter.

(56) The base of Frechet filter is a generalized basis of  $\text{FrechetFilter}(\mathbb{N})$ .

(57) Let us consider a non empty set  $X$ , filters  $\mathcal{F}_1, \mathcal{F}_2$  of  $X$ , and a filter  $\mathcal{F}'$  of  $X$ . Suppose  $\mathcal{F}'$  is finer than  $\mathcal{F}_1$  and  $\mathcal{F}'$  is finer than  $\mathcal{F}_2$ . Let us consider an element  $M_1$  of  $\mathcal{F}_1$ , and an element  $M_2$  of  $\mathcal{F}_2$ . Then  $M_1 \cap M_2$  is not empty.

(58) Let us consider a non empty set  $X$ , and filters  $\mathcal{F}_1, \mathcal{F}_2$  of  $X$ . Suppose for every element  $M_1$  of  $\mathcal{F}_1$  for every element  $M_2$  of  $\mathcal{F}_2$ ,  $M_1 \cap M_2$  is not empty. Then there exists a filter  $\mathcal{F}'$  of  $X$  such that

(i)  $\mathcal{F}'$  is finer than  $\mathcal{F}_1$ , and

(ii)  $\mathcal{F}'$  is finer than  $\mathcal{F}_2$ .

Let  $X$  be a set and  $x$  be a subset of  $X$ . The functor  $\text{SubsetToBooleSubset } x$  yielding an element of  $2_{\subseteq}^X$  is defined by the term

(Def. 19)  $x$ .

Now we state the propositions:

(59) Let us consider an infinite set  $X$ . Then  $X \in$  the set of all  $X \setminus A$  where  $A$  is a finite subset of  $X$ .

(60) Let us consider a set  $X$ , and a subset  $A$  of  $X$ . Then  $\{B, \text{ where } B \text{ is an element of } 2_{\subseteq}^X : A \subseteq B\} = \{B, \text{ where } B \text{ is a subset of } X : A \subseteq B\}$ .

(61) Let us consider a set  $X$ , and an element  $a$  of  $2_{\subseteq}^X$ . Then  $\uparrow a = \{Y, \text{ where } Y \text{ is a subset of } X : a \subseteq Y\}$ .

(62) Let us consider a set  $X$ , and a subset  $A$  of  $X$ . Then  $\{B, \text{ where } B \text{ is an element of } 2_{\subseteq}^X : A \subseteq B\} = \uparrow \text{SubsetToBooleSubset } A$ . The theorem is a consequence of (60).

(63) Let us consider a non empty set  $X$ , and a filter  $\mathcal{F}'$  of  $X$ . Then  $\bigcup \mathcal{F}' = X$ .

(64) Let us consider an infinite set  $X$ . Then the set of all  $X \setminus A$  where  $A$  is a finite subset of  $X$  is a filter of  $X$ . The theorem is a consequence of (59).

Let us consider a set  $X$ . Now we state the propositions:

(65)  $2^X$  is a filter of  $2_{\subseteq}^X$ .

(66)  $\{X\}$  is a filter of  $2_{\subseteq}^X$ .

(67) Let us consider a non empty set  $X$ . Then  $\{X\}$  is a filter of  $X$ .

Let us consider an element  $A$  of  $2_{\subseteq}^X$ . Now we state the propositions:

(68)  $\{Y, \text{ where } Y \text{ is a subset of } X : A \subseteq Y\}$  is a filter of  $2_{\subseteq}^X$ .

(69)  $\{B, \text{ where } B \text{ is an element of } 2_{\subseteq}^X : A \subseteq B\}$  is a filter of  $2_{\subseteq}^X$ . The theorem is a consequence of (60) and (68).

Now we state the proposition:

(70) Let us consider a non empty set  $X$ , and a non empty subset  $B$  of  $2_{\subseteq}^X$ . Then for every elements  $x, y$  of  $B$ , there exists an element  $z$  of  $B$  such that  $z \subseteq x \cap y$  if and only if  $B$  is filtered.

PROOF: For every elements  $x, y$  of  $B$ , there exists an element  $z$  of  $B$  such that  $z \subseteq x \cap y$  by [19, (2)].  $\square$

Let us consider a non empty set  $X$  and a non empty subset  $\mathcal{F}'$  of the lattice of subsets of  $X$ . Now we state the propositions:

(71)  $\mathcal{F}'$  is a filter of the lattice of subsets of  $X$  if and only if for every elements  $p, q$  of  $\mathcal{F}'$ ,  $p \cap q \in \mathcal{F}'$  and for every element  $p$  of  $\mathcal{F}'$  and for every element  $q$  of the lattice of subsets of  $X$  such that  $p \subseteq q$  holds  $q \in \mathcal{F}'$ .

(72)  $\mathcal{F}'$  is a filter of the lattice of subsets of  $X$  if and only if for every subsets  $Y_1, Y_2$  of  $X$ , if  $Y_1, Y_2 \in \mathcal{F}'$ , then  $Y_1 \cap Y_2 \in \mathcal{F}'$  and if  $Y_1 \in \mathcal{F}'$  and  $Y_1 \subseteq Y_2$ , then  $Y_2 \in \mathcal{F}'$ . The theorem is a consequence of (71).

Now we state the propositions:

(73) Let us consider a non empty set  $X$ , and a non empty family  $\mathcal{F}$  of subsets of  $X$ . Suppose  $\mathcal{F}$  is a filter of the lattice of subsets of  $X$ . Then  $\mathcal{F}$  is a filter of  $2_{\subseteq}^X$ . The theorem is a consequence of (71).

(74) Let us consider a non empty set  $X$ . Then every filter of  $2_{\subseteq}^X$  is a filter of the lattice of subsets of  $X$ . The theorem is a consequence of (72).

(75) Let us consider a non empty set  $X$ , and a non empty subset  $\mathcal{F}'$  of the lattice of subsets of  $X$ . Then  $\mathcal{F}'$  is filter of the lattice of subsets of  $X$  and has non empty elements if and only if  $\mathcal{F}'$  is a filter of  $X$ . The theorem is a consequence of (72).

(76) Let us consider a non empty set  $X$ . Then every proper filter of  $2_{\subseteq}^X$  is a filter of  $X$ .

PROOF:  $\mathcal{F}'$  has non empty elements by [19, (18)], [7, (4)].  $\square$

(77) Let us consider a non empty topological space  $T$ , and a point  $x$  of  $T$ . Then the neighborhood system of  $x$  is a filter of the carrier of  $T$ .

Let  $T$  be a non empty topological space and  $\mathcal{F}'$  be a proper filter of  $2_{\subseteq}^{\Omega T}$ . The functor  $\text{BooleanFilterToFilter}(\mathcal{F}')$  yielding a filter of the carrier of  $T$  is defined by the term

(Def. 20)  $\mathcal{F}'$ .

Let  $\mathcal{F}_1$  be a filter of the carrier of  $T$  and  $\mathcal{F}_2$  be a proper filter of  $2_{\subseteq}^{\Omega T}$ . We say that  $\mathcal{F}_1$  is finer than  $\mathcal{F}_2$  if and only if

(Def. 21)  $\text{BooleanFilterToFilter}(\mathcal{F}_2) \subseteq \mathcal{F}_1$ .

### 3. LIMIT OF A FILTER

Let  $T$  be a non empty topological space and  $\mathcal{F}'$  be a filter of the carrier of  $T$ . The functor  $\text{LimFilter}(\mathcal{F}')$  yielding a subset of  $T$  is defined by the term

(Def. 22)  $\{x, \text{ where } x \text{ is a point of } T : \mathcal{F}' \text{ is finer than the neighborhood system of } x\}$ .

Let  $B$  be a filter base of the carrier of  $T$ . The functor  $\text{Lim } B$  yielding a subset of  $T$  is defined by the term

(Def. 23)  $\text{LimFilter}(B)$ .

Now we state the proposition:

(78) Let us consider a non empty topological space  $T$ , and a filter  $\mathcal{F}'$  of the carrier of  $T$ . Then there exists a proper filter  $\mathcal{F}_1$  of  $2_{\subseteq}^{\alpha}$  such that  $\mathcal{F}' = \mathcal{F}_1$ , where  $\alpha$  is the carrier of  $T$ . The theorem is a consequence of (73) and (75).

Let  $T$  be a non empty topological space and  $\mathcal{F}'$  be a filter of the carrier of  $T$ . The functor  $\text{FilterToBooleanFilter}(\mathcal{F}', T)$  yielding a proper filter of  $2_{\subseteq}^{\Omega T}$  is defined by the term

(Def. 24)  $\mathcal{F}'$ .

Let us consider a non empty topological space  $T$ , a point  $x$  of  $T$ , and a filter  $\mathcal{F}'$  of the carrier of  $T$ . Now we state the propositions:

(79)  $x$  is a convergence point of  $\mathcal{F}'$  and  $T$  if and only if  $x$  is a convergence point of  $\text{FilterToBooleanFilter}(\mathcal{F}', T)$  and  $T$ .

(80)  $x$  is a convergence point of  $\mathcal{F}'$  and  $T$  if and only if  $x \in \text{LimFilter}(\mathcal{F}')$ . The theorem is a consequence of (78).

Let  $T$  be a non empty topological space and  $\mathcal{F}'$  be a filter of  $2_{\subseteq}^{\Omega T}$ . The functor  $\text{LimFilterB}(\mathcal{F}')$  yielding a subset of  $T$  is defined by the term

(Def. 25)  $\{x, \text{ where } x \text{ is a point of } T : \text{ the neighborhood system of } x \subseteq \mathcal{F}'\}$ .

Let us consider a non empty topological space  $T$  and a filter  $\mathcal{F}'$  of the carrier of  $T$ . Now we state the propositions:

(81)  $\text{LimFilter}(\mathcal{F}') = \text{LimFilterB}(\text{FilterToBooleanFilter}(\mathcal{F}', T))$ .

(82)  $\text{Lim}(\text{the net of } \text{FilterToBooleanFilter}(\mathcal{F}', T)) = \text{LimFilter}(\mathcal{F}')$ .

(83) Let us consider a Hausdorff, non empty topological space  $T$ , a filter  $\mathcal{F}'$  of the carrier of  $T$ , and points  $p, q$  of  $T$ . If  $p, q \in \text{LimFilter}(\mathcal{F}')$ , then  $p = q$ .

Let  $T$  be a Hausdorff, non empty topological space and  $\mathcal{F}'$  be a filter of the carrier of  $T$ . Note that  $\text{LimFilter}(\mathcal{F}')$  is trivial.

Let  $X$  be a non empty set,  $T$  be a non empty topological space,  $f$  be a function from  $X$  into the carrier of  $T$ , and  $\mathcal{F}'$  be a filter of  $X$ . The functor  $\text{lim}_{\mathcal{F}'} f$  yielding a subset of  $\Omega_T$  is defined by the term

(Def. 26)  $\text{LimFilter}(\text{the image of filter } \mathcal{F}' \text{ under } f)$ .

Let  $L$  be a non empty, transitive, reflexive relational structure and  $f$  be a function from  $\Omega_L$  into the carrier of  $T$ . The functor  $\text{LimF}(f)$  yielding a subset of  $\Omega_T$  is defined by the term

(Def. 27)  $\text{LimFilter}(\text{the image of filter } \text{TailsFilter } L \text{ under } f)$ .

Now we state the proposition:

(84) Let us consider a non empty topological space  $T$ , a non empty, transitive, reflexive relational structure  $L$ , a function  $f$  from  $\Omega_L$  into the carrier of  $T$ , a point  $x$  of  $T$ , and a generalized basis  $B$  of  $\text{BooleanFilterToFilter}(\text{the neighborhood system of } x)$ . Suppose  $\Omega_L$  is directed. Then  $x \in \text{LimF}(f)$  if and only if for every element  $b$  of  $B$ , there exists an element  $i$  of  $L$  such that for every element  $j$  of  $L$  such that  $i \leq j$  holds  $f(j) \in b$ . The theorem is a consequence of (46), (29), and (47).

Let  $T$  be a non empty topological space and  $s$  be a sequence of  $T$ . The functor  $\text{LimF}(s)$  yielding a subset of  $T$  is defined by the term

(Def. 28)  $\text{LimFilter}(\text{the elementary filter of } s)$ .

Now we state the proposition:

(85) Let us consider a non empty topological space  $T$ , and a sequence  $s$  of  $T$ . Then  $\text{lim}_{\text{FrechetFilter}(\mathbb{N})} s = \text{LimF}(s)$ .

Let us consider a non empty topological space  $T$  and a point  $x$  of  $T$ .

(86) The neighborhood system of  $x$  is a filter base of  $\Omega_T$ . The theorem is a consequence of (76), (13), and (29).

(87) Every generalized basis of  $\text{BooleanFilterToFilter}(\text{the neighborhood system of } x)$  is a filter base of  $\Omega_T$ .

(88) Let us consider a non empty set  $X$ , a sequence  $s$  of  $X$ , and a filter base  $B$  of  $X$ . Then  $B$  is coarser than  $s^\circ$  (the base of Frechet filter) if and only if for every element  $b$  of  $B$ , there exists an element  $i$  of the ordered  $\mathbb{N}$  such that for every element  $j$  of the ordered  $\mathbb{N}$  such that  $i \leq j$  holds  $s(j) \in b$ .

- (89) Let us consider a non empty topological space  $T$ , a sequence  $s$  of  $T$ , a point  $x$  of  $T$ , and a generalized basis  $B$  of `BooleanFilterToFilter`(the neighborhood system of  $x$ ). Then  $x \in \lim_{\text{FrechetFilter}(\mathbb{N})} s$  if and only if  $B$  is coarser than  $s^\circ$ (the base of Frechet filter). The theorem is a consequence of (46) and (54).
- (90) Let us consider a non empty topological space  $T$ , a sequence  $s$  of  $\Omega_T$ , a point  $x$  of  $T$ , and a generalized basis  $B$  of `BooleanFilterToFilter`(the neighborhood system of  $x$ ). Then  $B$  is coarser than  $s^\circ$ (the base of Frechet filter) if and only if for every element  $b$  of  $B$ , there exists an element  $i$  of the ordered  $\mathbb{N}$  such that for every element  $j$  of the ordered  $\mathbb{N}$  such that  $i \leq j$  holds  $s(j) \in b$ . The theorem is a consequence of (29) and (47).

Let us consider a non empty topological space  $T$ , a sequence  $s$  of the carrier of  $T$ , a point  $x$  of  $T$ , and a generalized basis  $B$  of `BooleanFilterToFilter`(the neighborhood system of  $x$ ).

- (91)  $x \in \lim_{\text{FrechetFilter}(\mathbb{N})} s$  if and only if for every element  $b$  of  $B$ , there exists an element  $i$  of the ordered  $\mathbb{N}$  such that for every element  $j$  of the ordered  $\mathbb{N}$  such that  $i \leq j$  holds  $s(j) \in b$ . The theorem is a consequence of (89) and (90).
- (92)  $x \in \text{LimF}(s)$  if and only if for every element  $b$  of  $B$ , there exists an element  $i$  of the ordered  $\mathbb{N}$  such that for every element  $j$  of the ordered  $\mathbb{N}$  such that  $i \leq j$  holds  $s(j) \in b$ . The theorem is a consequence of (91).

#### 4. NETS

Let  $L$  be a 1-sorted structure and  $s$  be a sequence of the carrier of  $L$ . The net of  $s$  yielding a non empty, strict net structure over  $L$  is defined by the term

(Def. 29)  $\langle \mathbb{N}, \leq_{\mathbb{N}}, s \rangle$ .

Let  $L$  be a non empty 1-sorted structure. Let us note that the net of  $s$  is non empty.

Now we state the proposition:

- (93) Let us consider a non empty 1-sorted structure  $L$ , a set  $B$ , and a sequence  $s$  of the carrier of  $L$ . Then the net of  $s$  is eventually in  $B$  if and only if there exists an element  $i$  of the net of  $s$  such that for every element  $j$  of the net of  $s$  such that  $i \leq j$  holds (the net of  $s$ )( $j$ )  $\in B$ .

Let us consider a non empty topological space  $T$ , a sequence  $s$  of the carrier of  $T$ , a point  $x$  of  $T$ , and a generalized basis  $B$  of `BooleanFilterToFilter`(the neighborhood system of  $x$ ). Now we state the propositions:

- (94) for every element  $b$  of  $B$ , there exists an element  $i$  of the ordered  $\mathbb{N}$  such that for every element  $j$  of the ordered  $\mathbb{N}$  such that  $i \leq j$  holds  $s(j) \in b$  if

and only if for every element  $b$  of  $B$ , there exists an element  $i$  of the net of  $s$  such that for every element  $j$  of the net of  $s$  such that  $i \leq j$  holds (the net of  $s$ )( $j$ )  $\in b$ .

- (95)  $x \in \text{LimF}(s)$  if and only if for every element  $b$  of  $B$ , the net of  $s$  is eventually in  $b$ . The theorem is a consequence of (92), (94), and (93).
- (96)  $x \in \text{LimF}(s)$  if and only if for every element  $b$  of  $B$ , there exists an element  $i$  of  $\mathbb{N}$  such that for every element  $j$  of  $\mathbb{N}$  such that  $i \leq j$  holds  $s(j) \in b$ . The theorem is a consequence of (91).
- (97)  $x \in \text{LimF}(s)$  if and only if for every element  $b$  of  $B$ , there exists a natural number  $i$  such that for every natural number  $j$  such that  $i \leq j$  holds  $s(j) \in b$ . The theorem is a consequence of (96).

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