The ascent-plateau statistics on Stirling permutations

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Abstract

A permutation σ of the multiset $\{1,1,2,2,\ldots,n,n\}$ is called a Stirling permutation of order n if $\sigma_s > \sigma_i$ as long as $\sigma_i = \sigma_j$ and $i < s < j$. In this paper, we present a unified refinement of the ascent polynomials and the ascent-plateau polynomials of Stirling permutations. In particular, by using Foata and Strehl's group action, we prove that the pairs of statistics (left ascent-plateau, ascent) and (left ascent-plateau, plateau) are equidistributed over Stirling permutations of given order, and we show the γ-positivity of the enumerative polynomial of left ascent-plateaus, double ascents and descent-plateaus. A connection between the γ-coefficients of this enumerative polynomial and Eulerian numbers is also established.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

A *Stirling permutation* of order n is a permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ such that for each i, $1 \leqslant i \leqslant n$, all entries between the two occurrences of i are larger than i.

Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. For $1 \leqslant i \leqslant 2n$, we say that an index i is a *descent* of σ if $\sigma_i > \sigma_{i+1}$ or $i = 2n$, and we say

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that an index i is an *ascent* of σ if $\sigma_{i-1} < \sigma_i$ or $i = 1$. Hence the index $i = 1$ is always an ascent and $i = 2n$ is always a descent. A *plateau* of σ is an index i such that $\sigma_i = \sigma_{i+1}$, where $1 \leqslant i \leqslant 2n-1$. Let $\operatorname{des}(\sigma), \operatorname{asc}(\sigma)$ and $\operatorname{plat}(\sigma)$ denote the numbers of descents, ascents and plateaus of σ, respectively.

Stirling permutations were introduced by Gessel and Stanley [7], and they proved that

$$(1-x)^{2k+1}\sum_{n=0}^{\infty}\left\{{n+k \atop n}\right\}x^n = \sum_{\sigma\in\mathcal{Q}_k} x^{\operatorname{des}\sigma},$$

where $\left\{{n \atop k}\right\}$ is the *Stirling number of the second kind*, i.e., the number of ways to partition a set of n objects into k non-empty subsets. A classical result of Bóna [2] says that descents, ascents and plateaus are equidistributed, i.e.,

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{des}\sigma} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}\sigma} = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{plat}\sigma}. \tag{1}$$

This equidistributed result have been extensively studied by Janson, Kuba, Panholzer, Haglund, Visontai, Chen, Fu et al., see [5, 8, 9, 10] and references therein.

It is natural to explore multivariate extension of (1). Let us now recall some definitions.

Definition 1 ([14])**.** An occurrence of an *ascent-plateau* of $\sigma \in \mathcal{Q}_n$ is an index i such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{2, 3, \ldots, 2n-1\}$. An occurrence of a *left ascent-plateau* is an index i such that $\sigma_{i-1} < \sigma_i = \sigma_{i+1}$, where $i \in \{1, 2, \ldots, 2n-1\}$ and $\sigma_0 = 0$.

Let $\operatorname{ap}(\sigma)$ and $\operatorname{lap}(\sigma)$ be the numbers of ascent-plateaus and left ascent-plateaus of σ, respectively. For example, $\operatorname{ap}(442\mathbf{3}32115\mathbf{6}65) = 2$ and $\operatorname{lap}(\mathbf{4}42\mathbf{3}32115\mathbf{6}65) = 3$.

Define

$$M_n(x) = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{ap}(\sigma)},$$

$$N_n(x) = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{lap}(\sigma)}.$$

According to [14, Theorem 2, Theorem 3], we have

$$M(x,t) = \sum_{n\geqslant 0} M_n(x)\frac{t^n}{n!} = \sqrt{\frac{x-1}{x-e^{2t(x-1)}}},$$

$$N(x,t) = \sum_{n\geqslant 0} N_n(x)\frac{t^n}{n!} = \sqrt{\frac{1-x}{1-xe^{2t(1-x)}}}.$$

Clearly, $M_n(x) = x^n N_n\left(\frac{1}{x}\right)$. The reader is referred to [16, 17] for further properties of these polynomials.

Let

$$C_n(x) = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{asc}(\sigma)}.$$

The polynomials $C_n(x)$ and $N_n(x)$, respectively, satisfy the following recurrence relation

$$C_{n+1}(x) = (2n+1)xC_n(x) + x(1-x)C'_n(x),$$

$$N_{n+1}(x) = (2n+1)xN_n(x) + 2x(1-x)N'_n(x),$$

with the initial conditions $C_0(x) = N_0(x) = 1$ (see [2, 7, 13] for instance). The purpose of this paper is to present a unified refinement of the polynomials $C_n(x)$ and $N_n(x)$. In the sequel, we always assume that Stirling permutations are prepended by 0. That is, we identify an n-Stirling permutation $\sigma_1\sigma_2\cdots\sigma_{2n}$ with the word $\sigma_0\sigma_1\sigma_2\cdots\sigma_{2n}$, where $\sigma_0 = 0$.

In this paper, we introduce the following definition.

Definition 2. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. For $1 \leqslant i \leqslant 2n-1$, a *double ascent* of σ is an index i such that $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$, a *descent-plateau* of σ is an index i such that $\sigma_{i-1} > \sigma_i = \sigma_{i+1}$.

Let $\mathrm{dasc}\,(\sigma)$ and $\mathrm{dp}\,(\sigma)$ denote the numbers of double ascents and descent-plateaus of σ, respectively. For example, $\mathrm{dasc}\,(\mathbf{2}44332115665) = 2$ and $\mathrm{dp}\,(244\mathbf{3}32\mathbf{1}15665) = 2$. It is clear that

$$\mathrm{asc}\,(\sigma) = \mathrm{lap}\,(\sigma) + \mathrm{dasc}\,(\sigma), \mathrm{plat}\,(\sigma) = \mathrm{lap}\,(\sigma) + \mathrm{dp}\,(\sigma). \tag{2}$$

It is natural to consider the polynomials $P_n(x,y,z)$ defined by

$$P_n(x,y,z) = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)}y^{\mathrm{dasc}\,(\sigma)}z^{\mathrm{dp}\,(\sigma)} = \sum_{i,j,k} p_n(i,j,k)x^iy^jz^k,$$

where $1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-1, 0 \leqslant k \leqslant n-1$. In particular,

$$P_n(x,x,1) = P_n(x,1,x) = C_n(x),$$

$$P_n(x,1,1) = N_n(x).$$

As a continuation of [2], the main result of this paper says that the pairs of statistics $(\mathrm{lap}\,(\sigma), \mathrm{asc}\,(\sigma))$ and $(\mathrm{lap}\,(\sigma), \mathrm{plat}\,(\sigma))$ are equidistributed. The main tools of the proofs are the grammatical technique and a variation of the Foata and Strehl's group action.

2 Main results

Context-free grammars can be used to study various exponential structures (see [4, 5, 15, 17] for instance). For an alphabet A, let $\mathbb{Q}((A))$ be the ring of formal Laurent series formed from letters in A. Following [4], a context-free grammar over A is a function $G : A \to \mathbb{Q}((A))$ that replaces a letter in A by an element of $\mathbb{Q}((A))$. The formal derivative D is a linear operator defined with respect to a context-free grammar G. According to [5], an advantage of the grammatical description of a combinatorial sequence is that a recursion of its generating function can be provided by attaching a labelling of the combinatorial object in accordance with the replacement rules of the grammar.

2.1 A grammatical labeling of Stirling permutations

The first few of the polynomials $P_n(x,y,z)$ are given as follows:

$$P_1(x,y,z) = x,$$
$$P_2(x,y,z) = xy + xz + x^2,$$
$$P_3(x,y,z) = x(y^2+z^2) + 4x^2(y+z) + 2xyz + 2x^2 + x^3.$$

Theorem 3. *Let* $A = \{x,y,z,p,q\}$ *and*

$$G = \{x \to xzq, y \to yzp, z \to xyz, p \to xyz, q \to xyz\}. \tag{3}$$

Then

$$D^n(z) = z \sum_{i,j,k} p_n(i,j,k)(xy)^i q^j p^k z^{2n-2i-j-k},$$

where $1 \leqslant i \leqslant n, 0 \leqslant j \leqslant n-1, 0 \leqslant k \leqslant n-1$ *and* $2i+j+k \leqslant 2n$*. Set* $P_n = P_n(x,y,z)$*. Then the polynomials* $P_n(x,y,z)$ *satisfy the recurrence relation*

$$P_{n+1} = (2n+1)xP_n + (xy+xz-2x^2)\frac{\partial}{\partial x}P_n + x(1-y)\frac{\partial}{\partial y}P_n + x(1-z)\frac{\partial}{\partial z}P_n, \tag{4}$$

with the initial condition $P_0(x,y,z) = 1$.

Proof. Now we give a labeling of $\sigma \in \mathcal{Q}_n$ as follows:

(L_1) If i is a left ascent-plateau, then put a superscript label y immediately before σ_i and a superscript label x right after σ_i;

(L_2) If i is a double ascent, then put a superscript label q immediately before σ_i;

(L_3) If i is a descent-plateau, then put a superscript label p right after σ_i;

(L_4) The rest positions in σ are labeled by superscript labels z. It should be noted that the labels z mark the descent positions of σ.

The weight of σ is defined by

$$w(\sigma) = z(xy)^{\operatorname{lap}(\sigma)} q^{\operatorname{dasc}(\sigma)} p^{\operatorname{dp}(\sigma)} z^{2n-2\operatorname{lap}(\sigma)-\operatorname{dasc}(\sigma)-\operatorname{dp}(\sigma)}.$$

For example, the labeling of 552442998813316776 is as follows:

$${}^y5^x5^z2^y4^x4^z2^y9^x9^z8^p8^z1^y3^x3^z1^q6^y7^x7^z6^z.$$

We proceed by induction on n. Note that $\mathcal{Q}_1 = \{{}^y1^x1^z\}$ and

$$\mathcal{Q}_2 = \{{}^y1^x1^y2^x2^z,\ {}^q1^y2^x2^z1^z,\ {}^y2^x2^z1^p1^z\}.$$

Thus the weight of ${}^y1^x1^z$ is given by $D(z)$ and the sum of weights of elements in $\mathcal{Q}_2$ is given by $D^2(z)$, since $D(z) = xyz$ and $D^2(x) = z(xyqz + xypz + x^2y^2)$.

Assume that the result holds for $n = m-1$, where $m \geqslant 3$. Let σ be an element counted by $p_{m-1}(i,j,k)$, and let σ' be an element of $\mathcal{Q}_m$ obtained by inserting the pair mm into σ. We distinguish five cases:

(c_1) If the pair mm is inserted at a position with label x, then the change of labeling is illustrated as follows:

$$\cdots \sigma_{\ell-1}^{y}\sigma_{\ell}^{x}\sigma_{\ell+1}\cdots \mapsto \cdots \sigma_{\ell-1}^{q}\sigma_{\ell}^{y}m^{x}m^{z}\sigma_{\ell+1}\cdots .$$

In this case, the insertion corresponds to the rule $x \mapsto xzq$ and produces i permutations in $\mathcal{Q}_m$ with i left ascent-plateaus, $j+1$ double ascents and k descent-plateaus;

(c_2) If the pair mm is inserted at a position with label y, then the change of labeling is illustrated as follows:

$$\cdots \sigma_{\ell-1}^{y}\sigma_{\ell}^{x}\sigma_{\ell+1}\cdots \mapsto \cdots \sigma_{\ell-1}^{y}m^{x}m^{z}\sigma_{\ell}^{p}\sigma_{\ell+1}\cdots .$$

In this case, the insertion corresponds to the rule $y \mapsto yzp$ and produces i permutations in $\mathcal{Q}_m$ with i left ascent-plateaus, j double ascents and $k+1$ descent-plateaus;

(c_3) If the pair mm is inserted at a position with label z, then the change of labeling is illustrated as follows:

$$\cdots \sigma_{\ell}^{z}\sigma_{\ell+1}\cdots \mapsto \cdots \sigma_{\ell}^{y}m^{x}m^{z}\sigma_{\ell+1}\cdots .$$

In this case, the insertion corresponds to the rule $z \mapsto xyz$ and produces $2m-2-2i-j-k$ permutations in $\mathcal{Q}_m$ with $i+1$ left ascent-plateaus, j double ascents and k descent-plateaus;

(c_4) If the pair mm is inserted at a position with label q, then the change of labeling is illustrated as follows:

$$\cdots \sigma_{\ell}^{q}\sigma_{\ell+1}\cdots \mapsto \cdots \sigma_{\ell}^{y}m^{x}m^{z}\sigma_{\ell+1}\cdots .$$

In this case, the insertion corresponds to the rule $q \mapsto xyz$ and produces j permutations in $\mathcal{Q}_m$ with $i+1$ left ascent-plateaus, $j-1$ double ascents and k descent-plateaus;

(c_5) If the pair mm is inserted at a position with label p, then the change of labeling is illustrated as follows:

$$\cdots \sigma_{\ell}^{p}\sigma_{\ell+1}\cdots \mapsto \cdots \sigma_{\ell}^{y}m^{x}m^{z}\sigma_{\ell+1}\cdots .$$

In this case, the insertion corresponds to the rule $p \mapsto xyz$ and produces k permutations in $\mathcal{Q}_m$ with $i+1$ left ascent-plateaus, j double ascents and $k-1$ descent-plateaus.

By induction, we see that grammar (3) generates all elements in $\mathcal{Q}_m$. Combining the above five cases, we see that

$$\begin{aligned}p_{n+1}(i,j,k) =& ip_n(i,j-1,k) + ip_n(i,j,k-1) + (j+1)p_n(i-1,j+1,k) + \\ & (k+1)p_n(i-1,j,k+1) + (2n+3-2i-j-k)p_n(i-1,j,k).\end{aligned}$$

Multiplying both sides of the above recurrence relation by $x^i y^j z^k$ for all i,j,k, we get (4). □

2.2 Equidistributed statistics

Let $i \in [2n]$ and let $\sigma = \sigma_1\sigma_2 \dots \sigma_{2n} \in \mathcal{Q}_n$. We define an action φ_i on $\mathcal{Q}_n$ as follows:

- If i is a double ascent, then $\varphi_i(\sigma)$ is obtained by moving σ_i to the right of the second σ_i, which forms a new pleateau $\sigma_i\sigma_i$;
- If i is a descent-plateau, then $\varphi_i(\sigma)$ is obtained by moving σ_i to the right of σ_k, where $k = \max\{j \in \{0, 1, 2, \dots, i-1\} : \sigma_j < \sigma_i\}$.

For instance, if $\sigma = 2447887332115665$, then

$$\varphi_1(\sigma) = 4478873322115665,\ \varphi_4(\sigma) = 2448877332115665,$$

and $\varphi_9(\varphi_1(\sigma)) = \varphi_6(\varphi_4(\sigma)) = \sigma$. In recent years, the Foata and Strehl's group actions have been extensively studied (see [3, 11] for instance). We introduce the Foata-Strehl action on Stirling permutations by

$$\varphi_i'(\sigma) = \begin{cases} \varphi_i(\sigma), & \text{if } i \text{ is a double ascent or descent-plateau;} \\ \sigma, & \text{otherwise.} \end{cases}$$

It is clear that φ_i' are involutions and that they commute. Hence, for any subset $S \subseteq [2n]$, we may define the function $\varphi_S' : \mathcal{Q}_n \mapsto \mathcal{Q}_n$ by $\varphi_S'(\sigma) = \prod_{i \in S} \varphi_i'(\sigma)$. Then the group $\mathbb{Z}_2^{2n}$ acts on $\mathcal{Q}_n$ via the function φ_S', where $S \subseteq [2n]$.

The main result of this paper is given as follows, which is also implied by (4).

Theorem 4. *For any $n \geqslant 1$, we have*

$$P_n(x, y, z) = P_n(x, z, y). \tag{5}$$

Furthermore,

$$\sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{asc}\,(\sigma)} = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{plat}\,(\sigma)}. \tag{6}$$

Proof. For any $\sigma \in \mathcal{Q}_n$, we define

$$\begin{aligned} \mathrm{Dasc}\,(\sigma) &= \{i \in [2n-1] : \sigma_{i-1} < \sigma_i < \sigma_{i+1}\}, \\ \mathrm{DP}\,(\sigma) &= \{i \in [2n-1] : \sigma_{i-1} > \sigma_i = \sigma_{i+1}\}, \\ \mathrm{LAP}\,(\sigma) &= \{i \in [2n-1] : \sigma_{i-1} < \sigma_i = \sigma_{i+1}\}. \end{aligned}$$

Let $S = S(\sigma) = \mathrm{Dasc}\,(\sigma) \cup \mathrm{DP}\,(\sigma)$. Note that

$$\mathrm{Dasc}\,(\varphi_S'(\sigma)) = \mathrm{DP}\,(\sigma),\ \mathrm{DP}\,(\varphi_S'(\sigma)) = \mathrm{Dasc}\,(\sigma) \text{ and } \mathrm{LAP}\,(\varphi_S'(\sigma)) = \mathrm{LAP}\,(\sigma).$$

Therefore,

$$P_n(x, y, z) \quad = \quad \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)} z^{\mathrm{dp}\,(\sigma)}$$

$$
\begin{aligned}
&= \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\varphi'_S(\sigma))} y^{\mathrm{dp}\,(\varphi'_S(\sigma))} z^{\mathrm{dasc}\,(\varphi'_S(\sigma))}\\
&= \sum_{\sigma'\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma')} z^{\mathrm{dasc}\,(\sigma')} y^{\mathrm{dp}\,(\sigma')}\\
&= P_n(x,z,y).
\end{aligned}
$$

Combining (2) and (5), we see that $P_n(xy,y,1)=P_n(xy,1,y)$. This completes the proof. □

Theorem 5. *For $n\geqslant 1$, we have*

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)} z^{\mathrm{dp}\,(\sigma)} = \sum_{\substack{1\leqslant i\leqslant n\\ 0\leqslant j\leqslant n-1}} \gamma_{n,i,j}x^i(y+z)^j,$$

where

$$\gamma_{n,i,j}=\#\{\sigma\in\mathcal{Q}_n:\mathrm{lap}\,(\sigma)=i,\mathrm{dasc}\,(\sigma)=j,\mathrm{dp}\,(\sigma)=0\}.$$

Proof. Let $\mathrm{NDP}_n=\{\sigma\in\mathcal{Q}_n:\mathrm{dp}\,(\sigma)=0\}$, and let

$$\mathrm{NDP}_{n,i,j}=\{\sigma\in\mathcal{Q}_n:\mathrm{lap}\,(\sigma)=i,\mathrm{dasc}\,(\sigma)=j,\mathrm{dp}\,(\sigma)=0\}.$$

For any $\sigma\in\mathrm{NDP}_{n,i,j}$, let $[\sigma]=\{\varphi'_S(\sigma)\mid S\subseteq\mathrm{Dasc}\,(\sigma)\}$. For any $\sigma'\in[\sigma]$, suppose that $\sigma'=\varphi'_S(\sigma)$ for some $S\subseteq\mathrm{Dasc}\,(\sigma)$. Then

$$\mathrm{lap}\,(\sigma')=\mathrm{lap}\,(\sigma),\mathrm{dasc}\,(\sigma')=\mathrm{dasc}\,(\sigma)-|S| \text{ and } \mathrm{dp}\,(\sigma')=|S|.$$

Moreover, $\{[\sigma]\mid\sigma\in\mathrm{NDP}_n\}$ form a partition of $\mathcal{Q}_n$. Hence,

$$
\begin{aligned}
&\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)} z^{\mathrm{dp}\,(\sigma)}\\
&= \sum_{\sigma\in\mathrm{NDP}_n}\sum_{\sigma'\in[\sigma]} x^{\mathrm{lap}\,(\sigma')} y^{\mathrm{dasc}\,(\sigma')} z^{\mathrm{dp}\,(\sigma')}\\
&= \sum_{\sigma\in\mathrm{NDP}_n}\sum_{S\subseteq\mathrm{Dasc}\,(\sigma)} x^{\mathrm{lap}\,(\varphi'_S(\sigma))} y^{\mathrm{dasc}\,(\varphi'_S(\sigma))} z^{\mathrm{dp}\,(\varphi'_S(\sigma))}\\
&= \sum_{\sigma\in\mathrm{NDP}_n}\sum_{S\subseteq\mathrm{Dasc}\,(\sigma)} x^{\mathrm{lap}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)-|S|} z^{|S|}\\
&= \sum_{\sigma\in\mathrm{NDP}_n} x^{\mathrm{lap}\,(\sigma)} \sum_{S\subseteq\mathrm{Dasc}\,(\sigma)} y^{\mathrm{dasc}\,(\sigma)-|S|} z^{|S|}\\
&= \sum_{\sigma\in\mathrm{NDP}_n} x^{\mathrm{lap}\,(\sigma)}(y+z)^{\mathrm{dasc}\,(\sigma)}\\
&= \sum_{i,j}\gamma_{n,i,j}x^i(y+z)^j.
\end{aligned}
$$

□

Taking $y = z = 1$ in Theorem 5, we have

$$N_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\mathrm{lap}\,(\sigma)} = \sum_{i=1}^{n} \left(\sum_{j=0}^{n-1} 2^j \gamma_{n,i,j} \right) x^i.$$

Let $N_n(x) = \sum_{k=1}^{n} N(n,k)x^k$. According to [13, Eq. (24)], we have

$$N_n(x) = \sum_{k=1}^{n} 2^{n-2k} \binom{2k}{k} k! \left\{ {n \atop k} \right\} x^k (1-x)^{n-k}.$$

Thus, for $n \geqslant 1$, we have

$$\sum_{j=0}^{n-1} 2^j \gamma_{n,i,j} = \sum_{j=1}^{i} (-1)^{i-j} 2^{n-2j} \binom{2j}{j} \binom{n-j}{i-j} j! \left\{ {n \atop j} \right\}.$$

Theorem 6. *Let $A = \{u, v, w\}$ and $G = \{u \to uvw, v \to 2uw, w \to uw\}$. Then*

$$D^n(w) = \sum_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant n-1}} \gamma_{n,i,j} u^i v^j w^{2n+1-2i-j}. \tag{7}$$

Furthermore, the numbers $\gamma_{n,i,j}$ satisfy the recurrence relation

$$\gamma_{n+1,i,j} = i\gamma_{n,i,j-1} + 2(j+1)\gamma_{n,i-1,j+1} + (2n+3-2i-j)\gamma_{n,i-1,j}, \tag{8}$$

with the initial conditions $\gamma_{1,1,0} = 1$ and $\gamma_{1,i,j} = 0$ for $i > 1$ and $j \geqslant 0$.

Proof. From the grammar (3), we see that

$$D(xy) = xyz(p+q),$$
$$D(p+q) = 2xyz,$$
$$D(z) = xyz.$$

Set $u = xy, v = p+q$ and $w = z$. Then $D(u) = uvw, D(v) = 2uw$ and $D(w) = uw$. Combining Theorem 3 and Theorem 5, we get (7). Since $D^{n+1}(w) = D(D^n(w))$, we obtain that

$$\begin{aligned} D^{n+1}(w) &= D\left(\sum_{i,j} \gamma_{n,i,j} u^i v^j w^{2n+1-2i-j} \right) \\ &= \sum_{i,j} i\gamma_{n,i,j} u^i v^{j+1} w^{2n+2-2i-j} + 2\sum_{i,j} j\gamma_{n,i,j} u^{i+1} v^{j-1} w^{2n+2-2i-j} + \\ &\sum_{i,j} (2n+1-2i-j)\gamma_{n,i,j} u^{i+1} v^j w^{2n+1-2i-j}. \end{aligned}$$

Equating the coefficients of $u^i v^j w^{2n+1-2i-j}$ on both sides of the above equation, we obtain (8). □

Let $G_n(x,y)=\sum_{i,j}\gamma_{n,i,j}x^iy^j$. Multiplying both sides of the recurrence relation (8) by x^iy^j for all i,j, we get that

$$G_{n+1}(x,y)=(2n+1)xG_n(x,y)+(xy-2x^2)\frac{\partial}{\partial x}G_n(x,y)+(2x-xy)\frac{\partial}{\partial y}G_n(x,y). \quad (9)$$

The first few of the polynomials $G_n(x,y)$ are given as follows:

$$G_0(x,y)=1, G_1(x,y)=x, G_2(x,y)=xy+x^2, G_3(x,y)=xy^2+4x^2y+2x^2+x^3.$$

2.3 Connection with Eulerian numbers

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n]=\{1,2,\ldots,n\}$. Let $\pi=\pi(1)\pi(2)\ldots\pi(n)\in\mathfrak{S}_n$. A *descent* of π is an index $i\in[n-1]$ such that $\pi(i)>\pi(i+1)$. Let $\mathrm{des}\,(\pi)$ be the number of descents of π. The classical Eulerian polynomials are defined by

$$A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{des}\,(\pi)}.$$

Let $\#C$ denote the cardinality of a set C. The *Eulerian numbers* are defined by

$$\left\langle {n \atop k} \right\rangle=\#\{\pi\in\mathfrak{S}_n:\mathrm{des}\,(\pi)=k\}.$$

Recall that the Eulerian numbers satisfy the recurrence relation

$$\left\langle {n+1 \atop k} \right\rangle=(k+1)\left\langle {n \atop k} \right\rangle+(n+1-k)\left\langle {n \atop k-1} \right\rangle,$$

with the initial conditions $\left\langle {1 \atop 0} \right\rangle=1$ and $\left\langle {1 \atop k} \right\rangle=0$ for $k\geqslant 1$. We can now present the following result.

Theorem 7. *For $n\geqslant 1$ and $0\leqslant k\leqslant n-1$, we have*

$$\gamma_{n,n-k,k}=\left\langle {n \atop k} \right\rangle.$$

Proof. Set $a(n,k)=\gamma_{n,n-k,k}$. Then $a(n,k-1)=\gamma_{n,n-k+1,k-1}$. Using (8), it is easy to verify that $\gamma_{n,i,j}=0$ for $i+j>n$. Hence $\gamma_{n,n-k,k+1}=0$. Therefore, the numbers $a(n,k)$ satisfy the recurrence relation

$$a(n+1,k)=(k+1)a(n,k)+(n+1-k)a(n,k-1).$$

Since the numbers $a(n,k)$ and $\left\langle {n \atop k} \right\rangle$ satisfy the same recurrence relation and initial conditions, they agree. □

A bijective proof of Theorem 7:

Proof. Let $\sigma \in \mathcal{Q}_n$. Note that every element of $[n]$ appears exactly two times in σ. Let $\alpha(\sigma)$ be the permutation of $\mathfrak{S}_n$ obtained from σ by deleting each first occurrence of $i \in [n]$. Then α is a map from $\mathcal{Q}_n$ to $\mathfrak{S}_n$. For example, $\alpha(\mathbf{34}4\mathbf{35}5\mathbf{6}6\mathbf{12}21) = 435621$. Let

$$\mathcal{D}_n = \{\sigma \in \mathcal{Q}_n : \mathrm{lap}\,(\sigma) = i, \mathrm{dasc}\,(\sigma) = n - i, \mathrm{dp}\,(\sigma) = 0\}.$$

It is clear that $\mathrm{lap}\,(\sigma) + \mathrm{dasc}\,(\sigma) = n$ implies $\mathrm{dp}\,(\sigma) = 0$ for $\sigma \in \mathcal{Q}_n$.

Let x be a given element in $[n]$. For any $\sigma \in \mathcal{Q}_n$, we define an action β_x on $\mathcal{Q}_n$ as follows:

- Read σ from left to right and let i be the first index such that $\sigma_i = x$;
- Move σ_i to the right of σ_k, where $k = \max\{j \in \{0, 1, 2, \ldots, i-1\} : \sigma_j < \sigma_i\}$, where $\sigma_0 = 0$.

For example, if $\sigma = 3443578876652211$, then

$$\beta_1(\sigma) = \mathbf{1}344357887665221,\ \beta_2(\sigma) = \mathbf{2}344357887665211,\ \beta_6(\sigma) = 34435\mathbf{6}7887652211.$$

It is clear that $\beta_x(\beta_y(\sigma)) = \beta_y(\beta_x(\sigma))$ for any $x, y \in [n]$. For any $S \subseteq [n]$, let $\beta_S : \mathcal{Q}_n \mapsto \mathcal{Q}_n$ be a function defined by

$$\beta_S(\sigma) = \prod_{x \in S} \beta_x(\sigma).$$

Clearly, $\beta_{[n]}(\sigma) \in \mathcal{D}_n$ and $\alpha(\sigma) = \alpha(\beta_{[n]}(\sigma))$. Moreover, $\beta_{[n]}(\sigma) = \sigma$ if $\sigma \in \mathcal{D}_n$.

Let $\alpha|_{\mathcal{D}_n}$ denote the restriction of the map α on the set $\mathcal{D}_n$. Then $\alpha|_{\mathcal{D}_n}$ is a map from $\mathcal{D}_n$ to $\mathfrak{S}_n$. Let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. The inverse $\alpha|_{\mathcal{D}_n}^{-1}$ is defined as follows:

- let $\sigma = \sigma_1\sigma_2\ldots\sigma_{2n}$ be the Stirling permutation such that $\sigma_{2i-1} = \sigma_{2i} = \pi(i)$ for each $i = 1, 2, \ldots, n$;
- let $S(\pi) = \{\pi_i : \pi_{i-1} > \pi_i, 2 \leqslant i \leqslant n\}$;
- let $\alpha|_{\mathcal{D}_n}^{-1}(\pi) = \beta_{S(\pi)}(\sigma)$.

Note that

$$\mathrm{lap}\,(\alpha|_{\mathcal{D}_n}^{-1}(\pi)) + \mathrm{dasc}\,(\alpha|_{\mathcal{D}_n}^{-1}(\pi)) = n \text{ and } \mathrm{dasc}\,(\alpha|_{\mathcal{D}_n}^{-1}(\pi)) = \mathrm{des}\,(\pi).$$

Then $\alpha|_{\mathcal{D}_n}$ is a bijection from $\mathcal{D}_n$ to $\mathfrak{S}_n$. This completes the proof. □

Example 8. The bijection between $\mathfrak{S}_3$ and $\mathcal{D}_3$ is demonstrated as follows:

$$\begin{aligned}
123 &\leftrightarrow 112233\ (S = \varnothing) \leftrightarrow \beta_S(112233) = 112233;\\
132 &\leftrightarrow 113322\ (S = \{2\}) \leftrightarrow \beta_S(113322) = 112332;\\
213 &\leftrightarrow 221133\ (S = \{1\}) \leftrightarrow \beta_S(221133) = 122133;\\
231 &\leftrightarrow 223311\ (S = \{1\}) \leftrightarrow \beta_S(223311) = 122331;\\
312 &\leftrightarrow 331122\ (S = \{1\}) \leftrightarrow \beta_S(331122) = 133122;\\
321 &\leftrightarrow 332211\ (S = \{1, 2\}) \leftrightarrow \beta_S(332211) = 123321.
\end{aligned}$$

3 Concluding remarks

In this paper, we introduce several variants of the ascent-plateau statistic on Stirling permutations. Here we provide another variant. Recall that the *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. As usual, we always identify a signed permutation $\pi = \pi(1)\cdots\pi(n)$ with the word $\pi(0)\pi(1)\cdots\pi(n)$, where $\pi(0) = 0$. For each $\pi \in B_n$, we define

$$\begin{aligned}
\operatorname{des}_A(\pi) &= \#\{i \in [n-1] : \pi(i) > \pi(i+1)\}, \\
\operatorname{des}_B(\pi) &= \#\{i \in \{0, 1, 2 \ldots, n-1\} : \pi(i) > \pi(i+1)\}.
\end{aligned}$$

Following [1], the number of *flag descents* of $\pi \in B_n$ is defined by

$$\operatorname{fdes}(\pi) = \begin{cases} 2\operatorname{des}_A(\pi) + 1, & \text{if } \pi(1) < 0; \\ 2\operatorname{des}_A(\pi), & \text{otherwise.} \end{cases}$$

Note that $\operatorname{fdes}(\pi) = \operatorname{des}_A(\pi) + \operatorname{des}_B(\pi)$. In the same way, it is natural to introduce the definition of the number of flag ascent-plateaus. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{2n} \in \mathcal{Q}_n$. The number of flag ascent-plateaus of σ is defined by

$$\operatorname{fap}(\sigma) = \begin{cases} 2\operatorname{ap}(\sigma) + 1, & \text{if } \sigma_1 = \sigma_2; \\ 2\operatorname{ap}(\sigma), & \text{otherwise.} \end{cases}$$

Clearly, $\operatorname{fap}(\sigma) = \operatorname{ap}(\sigma) + \operatorname{lap}(\sigma)$. A grammatical labeling of $\sigma \in \mathcal{Q}_n$ is given as follows:

(L_1) If $i \in \{2, 3, \ldots, 2n-1\}$ is an ascent-plateau, then put a superscript label y immediately before σ_i and a superscript label y right after σ_i;

(L_2) If $\sigma_1 = \sigma_2$, then put a superscript label y immediately before σ_1 and a superscript x right after σ_1;

(L_3) If $\sigma_1 < \sigma_2$, then put a superscript label x immediately before σ_1;

(L_4) The rest of positions in σ are labeled by a superscript label z.

Note that the weight of σ is given by $w(\sigma) = xy^{\operatorname{fap}(\sigma)}z^{2n-\operatorname{fap}(\sigma)}$. It is routine to check that if $G = \{x \to xyz, y \to yz^2, z \to y^2z\}$, then

$$D^n(x) = x \sum_{\sigma \in \mathcal{Q}_n} y^{\operatorname{fap}(\sigma)} z^{2n-\operatorname{fap}(\sigma)}.$$

Combining [12, Theorem 10], it is not hard to verify that

$$\sum_{\pi \in B_n} x^{\operatorname{fdes}(\pi)+1} = \sum_{k=0}^{n} \binom{n}{k} \sum_{\sigma \in \mathcal{Q}_k} x^{\operatorname{fap}(\sigma)} \sum_{\rho \in \mathcal{Q}_{n-k}} x^{2\operatorname{lap}(\rho)},$$

$$\sum_{\pi\in B_n} x^{\mathrm{fdes}\,(\pi)} = \sum_{k=0}^{n} \binom{n}{k} \sum_{\sigma\in\mathcal{Q}_k} x^{\mathrm{fap}\,(\sigma)} \sum_{\rho\in\mathcal{Q}_{n-k}} x^{2\mathrm{ap}\,(\rho)}.$$

In [18], Park studied the (p,q)-analogue of the descent polynomials of Stirling permutations:

$$C_n(x,p,q) = \sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{des}\,(\sigma)} p^{\mathrm{inv}\,(\sigma)} q^{\mathrm{maj}\,(\sigma)}.$$

It would be interesting to provide a unified refinement of $C_n(x,p,q)$ and the following polynomials:

$$\sum_{\sigma\in\mathcal{Q}_n} x^{\mathrm{ap}\,(\sigma)} y^{\mathrm{lap}\,(\sigma)} p^{\mathrm{inv}\,(\sigma)} q^{\mathrm{maj}\,(\sigma)}.$$

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