The $1/k$-Eulerian polynomials of type B

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Abstract

In this paper, we define the $1/k$-Eulerian polynomials of type B. Properties of these polynomials, including combinatorial interpretations, recurrence relations and γ-positivity are studied. In particular, we show that the $1/k$-Eulerian polynomials of type B are γ-positive when $k > 0$. Moreover, we define the $1/k$-derangement polynomials of type B, denoted $d_n^B(x;k)$. We show that the polynomials $d_n^B(x;k)$ are bi-γ-positive when $k \geqslant 1/2$. In particular, we get a symmetric decomposition of the polynomials $d_n^B(x;1/2)$ in terms of the classical derangement polynomials.

Mathematics Subject Classifications: 05A05, 05A15

1 Introduction

Throughout this paper, we always let k be a fixed positive number. Following Savage and Viswanathan [23], the $1/k$-*Eulerian polynomials* $A_n^{(k)}(x)$ are defined by

$$\sum_{n=0}^{\infty} A_n^{(k)}(x)\frac{z^n}{n!} = \left(\frac{1-x}{e^{kz(x-1)}-x}\right)^{\frac{1}{k}}. \tag{1}$$

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When $k=1$, the polynomial $A_n^{(k)}(x)$ reduces to the classical Eulerian polynomial $A_n(x)$. Savage and Viswanathan [23] showed that

$$A_n^{(k)}(x) = \sum_{\mathbf{e}\in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})},$$

where $I_{n,k} = \{\mathbf{e} \mid 0 \leqslant e_i \leqslant (i-1)k\}$ is the set of n-dimensional *k-inversion sequences* with $\mathbf{e} = (e_1, e_2, \ldots, e_n) \in \mathbb{Z}^n$ and

$$\mathrm{asc}\,(\mathbf{e}) = \#\left\{ i : 1 \leqslant i \leqslant n-1 \;\Big|\; \frac{e_i}{(i-1)k+1} < \frac{e_{i+1}}{ik+1} \right\}.$$

In the following, we first recall the other combinatorial interpretations of $A_n^{(k)}(x)$, and then we define the $1/k$-Eulerian polynomials of type B as well as the $1/k$-derangement polynomials of type B.

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n] = \{1, 2, \ldots, n\}$ and let $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$. A *descent* (resp. *ascent, excedance*) of π is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$, $\pi(i) > i$). Let $\mathrm{des}\,(\pi)$ (resp. $\mathrm{asc}\,(\pi)$, $\mathrm{exc}\,(\pi)$) denote the number of descents (resp. ascents, excedances) of π. It is well known that the statistics $\mathrm{des}\,(\pi), \mathrm{asc}\,(\pi)$ and $\mathrm{exc}\,(\pi)$ are equidistributed over $\mathfrak{S}_n$, and their common enumerative polynomial is the *Eulerian polynomial* $A_n(x)$, i.e.,

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{des}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)}.$$

In [13], Foata and Schützenberger introduced a q-analog of $A_n(x)$ defined by

$$A_n(x,q) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)},$$

where $\mathrm{cyc}\,(\pi)$ is the number of cycles of π. Brenti [6] showed that some crucial properties of Eulerian polynomials have nice q-analogues for the polynomials $A_n(x,q)$. According to [6, Proposition 7.3], we have

$$\sum_{n=0}^{\infty} A_n(x,q)\frac{z^n}{n!} = \left(\frac{1-x}{e^{z(x-1)}-x}\right)^q.$$

By comparing this with (1), one can immediately get that

$$A_n^{(k)}(x) = k^n A_n(x, 1/k) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} k^{n-\mathrm{cyc}\,(\pi)}.$$

A *left-to-right minimum* in π is an index i such that $\pi(i) < \pi(j)$ for any $j < i$ or $i = 1$. Let $\mathrm{lrmin}\,(\pi)$ denote the number of left-to-right minima of π. By using the fundamental transformation of Foata and Schützenberger [13], the pairs of statistics (exc , cyc) and (asc , lrmin) are equidistributed over $\mathfrak{S}_n$. Thus

$$\sum_{\mathbf{e}\in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} k^{n-\mathrm{lrmin}\,(\pi)}. \tag{2}$$

A bijective proof of (2) was recently given in [7]. According to [18, Theorem 2], the $1/k$-Eulerian polynomial $A_n^{(k)}(x)$ is also the longest ascent plateau polynomial of k-Stirling permutations of order n.

Let $f(x) = \sum_{i=0}^{n} f_i x^i$ be a symmetric polynomial, i.e., $f_i = f_{n-i}$ for any $0 \leqslant i \leqslant n$. Then $f(x)$ can be expanded uniquely as $f(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_k x^k (1+x)^{n-2k}$, and it is said to be *$\gamma$-positive* if $\gamma_k \geqslant 0$ for $0 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor$ (see [15, 16]). The γ-positivity of $f(x)$ implies unimodality of $f(x)$. We refer the reader to Athanasiadis's survey article [1] for details. The γ-positivity of Eulerian polynomials was first obtained by Foata and Schützenberger [13]. Subsequently, Foata and Strehl [14] proved the γ-positivity of Eulerian polynomials by using a group action. Using the theory of enriched P-partitions, Stembridge [28, Remark 4.8] showed that

$$A_n(x) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} 4^i P(n,i) x^i (1+x)^{n-1-2i},$$

where $P(n,i)$ is the number of permutations in $\mathfrak{S}_n$ with i *interior peaks*, i.e., the indices $i \in \{2, \ldots, n-1\}$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$. It should be noted that if $k \neq 1$, then the polynomial $A_n^{(k)}(x)$ is not symmetric, and so it is not γ-positive.

A permutation $\pi \in \mathfrak{S}_n$ is a *derangement* if it has no fixed points, i.e., $\pi(i) \neq i$ for all $i \in [n]$. Let $\mathcal{D}_n$ be the set of derangements in $\mathfrak{S}_n$, and let $d_n(x) = \sum_{\pi\in\mathcal{D}_n} x^{\mathrm{exc}\,(\pi)}$ be the derangement polynomials. It is well known that the generating function of $d_n(x)$ is given as follows (see [4, Proposition 6]):

$$d(x,z) = \sum_{n\geqslant 0} d_n(x) \frac{z^n}{n!} = \frac{1-x}{e^{xz} - xe^z}. \tag{3}$$

Using continued fractions, Shin and Zeng [24, Theorem 11] obtained the following result.

Theorem 1. *For $n \geqslant 2$, we have*

$$\sum_{\pi\in\mathcal{D}_n} x^{\mathrm{exc}\,(\pi)} q^{\mathrm{cyc}\,(\pi)} = \sum_{i=1}^{\lfloor n/2 \rfloor} c_{n,k}(q) x^k (1+x)^{n-2k},$$

where $c_{n,k}(q) = \sum_{\pi\in\mathcal{D}_n(k)} q^{\mathrm{cyc}\,(\pi)}$ and $\mathcal{D}_n(k)$ is the subset of derangements in $\mathcal{D}_n$ with exactly k cyclic valleys and without cyclic double descents.

Let $\pm[n] = [n] \cup \{-1, \ldots, -n\}$. Let B_n be the *hyperoctahedral group* of rank n and let $w = w(1)w(2)\cdots w(n) \in B_n$. Elements of B_n are permutations of $\pm[n]$ with the property that $w(-i) = -w(i)$ for all $i \in [n]$. Let

$$\mathrm{des}_B(w) = \#\{i \in \{0, 1, \ldots, n-1\} \mid w(i) > w(i+1)\},$$

where $w(0)=0$. As usual, we denote by $\bar{i}$ the negative element $-i$. We say that $i\in[n]$ is a *weak excedance* of w if $w(i)=i$ or $w(|w(i)|)>w(i)$ (see [5, p. 431]). An *excedance* of w is an index $i\in[n]$ such that $w(|w(i)|)>w(i)$. A *fixed point* (resp. *singleton*) of w is an index $i\in[n]$ such that $w(i)=i$ (resp. $w(i)=\bar{i}$). Let $\operatorname{wexc}(w)$ (resp. $\operatorname{exc}(w)$, $\operatorname{fix}(w)$, $\operatorname{single}(\pi)$) denote the number of weak excedances (resp. excedances, fixed points, singletons) of w. By definition, we have $\operatorname{wexc}(w)=\operatorname{exc}(w)+\operatorname{fix}(w)$. According to [5, Theorem 3.15], the statistics $\operatorname{des}_B(w)$ and $\operatorname{wexc}(w)$ have the same distribution over B_n, and their common enumerative polynomial is the *Eulerian polynomial of type B*:

$$B_n(x)=\sum_{w\in B_n}x^{\operatorname{des}_B(w)}=\sum_{w\in B_n}x^{\operatorname{wexc}(w)}.$$

A *left peak* of $\pi\in\mathfrak{S}_n$ is an index $i\in[n-1]$ such that $\pi(i-1)<\pi(i)>\pi(i+1)$, where we take $\pi(0)=0$. Let $\operatorname{lpk}(\pi)$ be the number of left peaks in π. Let $Q(n,i)$ be the number of permutations in $\mathfrak{S}_n$ with i left peaks. By using the theory of enriched P-partitions, Petersen [21, Proposition 4.15] obtained the following result.

Theorem 2. *We have* $B_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}4^iQ(n,i)x^i(1+x)^{n-2i}$.

In recent years, various refinements of Theorem 2 have been studied by several authors, see [16, 25, 29] and references therein.

For $w\in B_n$, we say that w is a type B *derangement* if $\operatorname{fix}(w)=0$. Let $\mathcal{D}_n^B$ be the set of all type B derangements in B_n. Clearly, $\operatorname{wexc}(w)=\operatorname{exc}(w)$ for $w\in\mathcal{D}_n^B$. The *type B derangement polynomials* $d_n^B(x)$ are defined by

$$d_n^B(x)=\sum_{\pi\in\mathcal{D}_n^B}x^{\operatorname{exc}(\pi)},$$

which have been studied by Chen et al. [9] in a slightly different form. According to [11, Theorem 3.2], the generating function of $d_n^B(x)$ is given as follows:

$$\sum_{n=0}^{\infty}d_n^B(x)\frac{z^n}{n!}=\frac{(1-x)e^z}{e^{2xz}-xe^{2z}}. \tag{4}$$

Combining (3) and (4), we obtain $d_n^B(x)=\sum_{i=0}^{n}\binom{n}{i}2^id_i(x)$.

The *type B 1/k-Eulerian polynomials* $B_n^{(k)}(x)$ and the *type B 1/k-derangement polynomials* $d_n^B(x;k)$ are defined by using the following generating functions:

$$\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{z^n}{n!}=\left(\frac{(1-x)e^{kz(1-x)}}{1-xe^{2kz(1-x)}}\right)^{\frac{1}{k}}, \tag{5}$$

$$\sum_{n=0}^{\infty}d_n^B(x;k)\frac{z^n}{n!}=\left(\frac{(1-x)e^{kz}}{e^{2kxz}-xe^{2kz}}\right)^{\frac{1}{k}}. \tag{6}$$

In particular, $B_n^{(1)}(x) = B_n(x)$ and $d_n^B(x;1) = d_n^B(x)$. Comparing (5) with (6), we have

$$B_n^{(k)}(x) = \sum_{i=0}^{n} \binom{n}{i} d_i^B(x;k)x^{n-i}.$$

This paper is organized as follows. In the next section, we present the main results. In particular, we show that the type B $1/k$-Eulerian polynomials $B_n^{(k)}(x)$ are γ-positive when k is positive and the type B $1/k$-derangement polynomials $d_n^B(x;k)$ are bi-γ-positive when $k \geqslant 1/2$. In Sections 3 and 4, we respectively prove Theorem 4 and Theorem 9.

2 Main results

2.1 The $1/k$-Eulerian polynomials of type B

An element is a *left-to-right maximum* of $\pi \in \mathfrak{S}_n$ if it is larger than or equal to all the elements to its left. We always assume that $\pi(1)$ is a left-to-right maximum. Let $\operatorname{lrmax}(\pi)$ be the number of left-to-right maxima of π. We can write $\pi \in \mathfrak{S}_n$ in standard cycle decomposition, where each cycle is written with its largest entry first and the cycles are written in increasing order of their largest entry. A *cycle peak* of π is an index i such that $\pi^{-1}(i) < i > \pi(i)$. Let $\operatorname{cpk}(\pi)$ be the number of cycle peaks of π. By using the *fundamental transformation* of Foata and Schützenberger [13], it is easy to verify that

$$\sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{cpk}(\pi)} y^{\operatorname{cyc}(\pi)} = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{lpk}(\pi)} y^{\operatorname{lrmax}(\pi)}.$$

In the following discussion, we always write $w \in B_n$ by using its *standard cycle decomposition*, in which each cycle is written with its largest entry last and the cycles are written in ascending order of their last entry. It should be noted that the n letters appearing in the cycle notation of $w \in B_n$ are the letters $w(1), w(2), \ldots, w(n)$. Let $\operatorname{cyc}(w)$ be the number of cycles of w.

Example 3. The signed permutation $w = \overline{3}51\overline{7}24689$ can be written as

$$(\overline{9})(\overline{3},1)(2,5)(4,\overline{7},6)(8).$$

Moreover, w has only one singleton 9, one fixed point 8, $\operatorname{cyc}(w) = 5$ and $\operatorname{exc}(w) = 3$.

We can now present the first main result of this paper.

Theorem 4. *(i) For $n \geqslant 1$, we have*

$$B_n^{(k)}(x) = \sum_{w \in B_n} x^{\operatorname{wexc}(w)} k^{n-\operatorname{cyc}(w)},$$

and the polynomials $B_n^{(k)}(x)$ satisfy the recurrence relation

$$B_{n+1}^{(k)}(x) = (1 + x + 2knx)B_n^{(k)}(x) + 2kx(1-x)\frac{d}{dx}B_n^{(k)}(x), \tag{7}$$

with the initial conditions $B_0^{(k)}(x) = 1$ *and* $B_1^{(k)}(x) = 1 + x$*;*

(ii) When $k > 0$*, the polynomial* $B_n^{(k)}(x)$ *is* γ*-positive. More precisely, we have*

$$B_n^{(k)}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \left(\sum_{j=i}^{n-1} b_{n,i,j} k^j 4^i \right) x^i (1+x)^{n-2i}, \tag{8}$$

where the numbers $b_{n,i,j}$ *satisfy the recurrence relation*

$$b_{n+1,i,j} = b_{n,i,j} + 2ib_{n,i,j-1} + (n-2i+2)b_{n,i-1,j-1}, \tag{9}$$

with $b_{1,0,0} = 1$ *and* $b_{1,i,j} = 0$ *for* $(i,j) \neq (0,0)$*;*

(iii) For $n \geqslant 1$*, we define*

$$b_n(x,q) = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=i}^{n-1} b_{n,i,j} x^i q^j.$$

Set $b_0(x,q) = 1$*. Then the generating function of* $b_n(x,q)$ *is given as follows:*

$$b(x,q,z) = \sum_{n=0}^{\infty} b_n(x,q) \frac{z^n}{n!} = \left(\frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(qz\sqrt{1-x}) - \sinh(qz\sqrt{1-x})} \right)^{\frac{1}{q}};$$

(iv) For $n \geqslant 1$*, we have*

$$b_n(x,q) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{cpk}(\pi)} q^{n - \operatorname{cyc}(\pi)}; \tag{10}$$

(v) For $n \geqslant 1$*, we have*

$$B_{n+1}^{(k)}(x) = (1+x)B_n^{(k)}(x) + x \sum_{i=0}^{n-1} \binom{n}{i} 2^{n+1-i} k^{n-i} B_i^{(k)}(x) A_{n-i}(x). \tag{11}$$

When $q = 1$, the generating function $b(x,q,z)$ reduces to the the generating function of the polynomials $\sum_{i=0}^{\lfloor n/2 \rfloor} Q(n,i)x^i$, which is due to Gessel [26, A008971]. Thus the polynomial $b_n(x,q)$ can be called the *$1/q$-left peak polynomial*. From the explicit formula of $b(x,q,z)$, it is routine to verify the following result.

Corollary 5. *For* $n \geqslant 1$*, we have*

$$\sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{cpk}(\pi)} (-1)^{n - \operatorname{cyc}(\pi)} = (1-x)^{\lfloor n/2 \rfloor}.$$

Let

$$T(n,i) = \frac{n!}{i!(n-2i)!2^i}$$

be the *Bessel number*, which is the number of involutions of $[n]$ with i pairs. In other words, the number $T(n,i)$ counts involutions of $[n]$ with $n-i$ cycles. Note that

$$b_{n,i,i} = \#\{\pi \in \mathfrak{S}_n : \operatorname{cpk}(\pi) = i, \operatorname{cyc}(\pi) = n-i\}.$$

So the following corollary is immediate.

Corollary 6. *For $0 \leqslant i \leqslant \lfloor n/2 \rfloor$, we have $b_{n,i,i} = T(n,i)$.*

It follows from (9) that

$$b_{n+1,i,n} = 2ib_{n,i,n-1} + (n-2i+2)b_{n,i-1,n-1}. \tag{12}$$

Let $\pi \in \mathfrak{S}_{n+1}$ with $\operatorname{cpk}(\pi) = i+1$ and $\operatorname{cyc}(\pi) = 1$. We write π in standard cycle decomposition. If π' is obtained from π by deleting its parentheses and the element $n+1$, then π' is a permutation in $\mathfrak{S}_n$ with i interior peaks. So we get the following corollary.

Corollary 7. *For $0 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$, the number $b_{n+1,i+1,n}$ equals the number of permutations in $\mathfrak{S}_n$ with i interior peaks.*

2.2 The $1/k$-derangement polynomials of type B

Let $p(x) = \sum_{i=0}^{d} p_i x^i$. There is a unique decomposition: $p(x) = a(x) + xb(x)$, where

$$a(x) = \frac{p(x) - x^{d+1}p(1/x)}{1-x},\ b(x) = \frac{x^d p(1/x) - p(x)}{1-x}. \tag{13}$$

It is clear that $a(x)$ and $b(x)$ are symmetric polynomials satisfying $a(x) = x^d a(\frac{1}{x})$ and $b(x) = x^{d-1} b(\frac{1}{x})$. We call the ordered pair of polynomials $(a(x), b(x))$ the *symmetric decomposition* of $p(x)$ (see [2]).

Definition 8. Let $(a(x), b(x))$ be the symmetric decomposition of $p(x)$. If $a(x)$ and $b(x)$ are both γ-positive, then we say that $p(x)$ is *bi-γ-positive.*

We say that $p(x)$ is *alternatingly increasing* if

$$p_0 \leqslant p_d \leqslant p_1 \leqslant p_{d-1} \leqslant \cdots \leqslant p_{\lfloor \frac{d+1}{2} \rfloor}.$$

As pointed out by Brändén and Solus [3], the polynomial $p(x)$ is alternatingly increasing if and only if the pair of polynomials in its symmetric decomposition are both unimodal and have nonnegative coefficients. Thus the bi-γ-positivity of $p(x)$ implies that $p(x)$ is alternatingly increasing.

We now present a counterpart of Theorem 1.

Theorem 9. *For $n \geqslant 1$, we have*

$$d_n^B(x;k) = \sum_{\pi \in \mathcal{D}_n^B} x^{\operatorname{exc}(\pi)} k^{n-\operatorname{cyc}(\pi)}. \tag{14}$$

When $k \geqslant 1/2$, the polynomials $d_n^B(x;k)$ are bi-γ-positive. More precisely, we have

$$d_n^B(x;k) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} p(n,j;k) x^j (1+x)^{n-1-2j} + \sum_{j=0}^{\lfloor n/2 \rfloor} q(n,j;k) x^j (1+x)^{n-2j}, \tag{15}$$

where the numbers $p(n,j;k)$ and $q(n,j;k)$ satisfy the following recurrence system:

$$\begin{aligned}p(n+1,j;k) &= (1+2kj)p(n,j;k)+4k(n-2j+1)p(n,j-1;k)+\\ &\quad 2knp(n-1,j-1;k)+q(n,j;k),\\ q(n+1,j;k) &= 2kjq(n,j;k)+4k(n-2j+2)q(n,j-1;k)+2knq(n-1,j-1;k)+\\ &\quad (2k-1)p(n,j-1;k),\end{aligned}$$

with the initial conditions $q(0,0;k)=1$, $q(0,j;k)=0$ for $j\neq 0$, $p(0,j;k)=0$ for any j.

For $n\geqslant 1$, we define

$$P_n(x;k) = \sum_{j=0}^{\lfloor (n-1)/2\rfloor} p(n,j;k)x^j(1+x)^{n-1-2j},$$

$$Q_n(x;k) = \sum_{j=0}^{\lfloor n/2\rfloor} q(n,j;k)x^j(1+x)^{n-2j}.$$

The first few $P_n(x;k)$ and $Q_n(x;k)$ are given as follows:

$$P_1(x;k)=1,\ P_2(x;k)=1+x,\ P_3(x;k)=1+(1+12k)x+x^2,$$

$$Q_1(x;k)=0,\ Q_2(x;k)=(4k-1)x,\ Q_3(x;k)=(8k^2-1)x(1+x).$$

Corollary 10. *The polynomials $P_n(x;k)$ and $Q_n(x;k)$ satisfy the recurrence system*

$$\begin{aligned}P_{n+1}(x;k) &= (1+(2kn-2k+1)x)P_n(x;k)+2kx(1-x)P_n'(x;k)+\\ &\quad 2knxP_{n-1}(x;k)+Q_n(x;k),\\ Q_{n+1}(x;k) &= 2knxQ_n(x;k)+2kx(1-x)Q_n'(x;k)+2knxQ_{n-1}(x;k)+\\ &\quad (2k-1)xP_n(x;k),\end{aligned}$$

with the initial conditions $P_0(x;k)=0, P_1(x;k)=1, Q_0(x;k)=1$ and $Q_1(x;k)=0$.

Proof. For $n\geqslant 1$, we define

$$p_n(x) = \sum_{j=0}^{\lfloor (n-1)/2\rfloor} p(n,j;k)x^j,\ q_n(x) = \sum_{j=0}^{\lfloor n/2\rfloor} q(n,j;k)x^j.$$

Multiplying both sides of the recurrence system of the numbers $p(n,j;k)$ and $q(n,j;k)$ by x^j and summing over all j, we get the following recurrence system:

$$\begin{aligned}p_{n+1}(x) &= (1+4k(n-1)x)p_n(x)+2kx(1-4x)p_n'(x)+2knxp_{n-1}(x)+q_n(x),\\ q_{n+1}(x) &= 4knxq_n(x)+2kx(1-4x)q_n'(x)+2knxq_{n-1}(x)+(2k-1)xp_n(x),\end{aligned}$$

with the initial conditions $p_0(x) = 0$, $p_1(x) = 1$, $p_2(x) = 1$, $q_0(x) = 1$, $q_1(x) = 0$ and $q_2(x) = (4k-1)x$. For $n \geqslant 1$, we have

$$P_n(x;k) = (1+x)^{n-1} p_n\left(\frac{x}{(1+x)^2}\right),$$

$$Q_n(x;k) = (1+x)^n q_n\left(\frac{x}{(1+x)^2}\right).$$

Substituting $x \to x/(1+x)^2$ into the recurrence system of the polynomials $p_n(x)$ and $q_n(x)$ and simplifying some terms leads to the desired result. □

A *succession* of $\pi \in \mathfrak{S}_n$ is an index i such that $\pi(i+1) = \pi(i)+1$, where $i \in [n-1]$. Let $\mathfrak{S}_n^s$ denote the set of permutations in $\mathfrak{S}_n$ with no successions. We can now give the following result.

Theorem 11. *For $n \geqslant 1$, we have*

$$d_n^B(x;1/2) = \frac{1}{x} d_{n+1}(x) + d_n(x),$$

where $d_n(x)$ is the derangement polynomial. Moreover, we have

$$d_n^B(x;1/2) = \sum_{\pi \in \mathfrak{S}_{n+1}^s} x^{\operatorname{asc}(\pi)}. \tag{16}$$

Proof. Let $P_n(x) = P_n(x;1/2)$ and $Q_n(x) = Q_n(x;1/2)$. It follows from Theorem 9 that $d_n^B(x;1/2) = P_n(x) + Q_n(x)$. By using Corollary 10, we see that the polynomials $P_n(x)$ and $Q_n(x)$ satisfy the following recurrence system:

$$\begin{aligned}
P_{n+1}(x) &= (1+nx)P_n(x) + x(1-x)P_n'(x) + nxP_{n-1}(x) + Q_n(x),\\
Q_{n+1}(x) &= nxQ_n(x) + x(1-x)Q_n'(x) + nxQ_{n-1}(x),
\end{aligned}$$

with the initial conditions $P_0(x) = 0, P_1(x) = 1, Q_0(x) = 1$ and $Q_1(x) = 0$. According to [17, Eq. (3.2)], the polynomials $Q_n(x)$ satisfy the same recurrence relation and initial conditions as $d_n(x)$, so they agree. We now prove that

$$P_n(x) = \frac{1}{x} d_{n+1}(x).$$

Clearly, it holds for $n = 0, 1, 2$. Assume it holds for n. Then we get

$$\begin{aligned}
P_{n+1}(x) &= \frac{1+nx}{x} d_{n+1}(x) + \frac{x(1-x)}{x^2}(xd_{n+1}'(x) - d_{n+1}(x)) + \frac{nx}{x} d_n(x) + d_n(x)\\
&= (n+1)d_{n+1}(x) + (1-x)d_{n+1}'(x) + (n+1)d_n(x)\\
&= \frac{1}{x} d_{n+2}(x),
\end{aligned}$$

as desired. The combinatorial interpretation (16) follows immediately from [22, Eq. (3.8)]. This completes the proof. □

Let

$$S_n(x) = \sum_{\pi \in \mathfrak{S}^s_{n+1}} x^{\mathrm{asc}\,(\pi)}.$$

Combining (6) and (16), we obtain

$$\sum_{n=0}^{\infty} S_n(x) \frac{z^n}{n!} = e^z \left(\frac{1-x}{e^{xz} - xe^z} \right)^2. \tag{17}$$

Combining (1), (3), (4) and (17), we get the following result.

Theorem 12. *For $n \geqslant 0$, we have*

$$S_n(x) = \frac{1}{2^n} \sum_{i=0}^{n} \binom{n}{i} d_i^B(x) d_{n-i}^B(x),$$

$$S_n(x) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) d_{n-i}(x).$$

3 Proof of Theorem 4

In this section we complete the proof of Theorem 4 by using the theory of context-free grammars. For an alphabet V, let $\mathbb{Q}[[V]]$ be the rational commutative ring of formal power series in monomials formed from letters in V. A *context-free grammar* over V is a function $G : V \to \mathbb{Q}[[V]]$ that replaces a letter in V by an element of $\mathbb{Q}[[V]]$ (see [8, 12]). The formal derivative D_G is a linear operator defined with respect to a grammar G. In other words, D_G is the unique derivation satisfying $D_G(x) = G(x)$ for $x \in V$, and for any two formal functions u and v, we have

$$D_G(u + v) = D_G(u) + D_G(v),\ D_G(uv) = D_G(u)v + uD_G(v).$$

For a constant c, we have $D_G(c) = 0$. It follows from *Leibniz's rule* that

$$D_G^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_G^k(u) D_G^{n-k}(v). \tag{18}$$

For example, if $G = \{x \to xy, y \to y\}$, then

$$D_G(x) = xy,\ D_G(y) = y,\ D_G^2(x) = D_G(xy) = xy^2 + xy.$$

A *grammatical labeling* is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar (see [10]). Following [20, Definition 1], a *change of grammars* is a substitution method in which the original grammars are replaced with functions of other grammars.

In the following discussion, we always write $w \in B_n$ by its standard cycle decomposition. For $w \in B_n$, we say that $i \in [n]$ is an *anti-excedance* of w if $w(i) = \bar{i}$ or $w(i) > w(|w(i)|)$. Let $\mathrm{aexc}\,(w)$ be the number of anti-excedances of w. It is clear that $\mathrm{wexc}\,(w) + \mathrm{aexc}\,(w) = n$ for $w \in B_n$. The following lemma is fundamental.

Lemma 13. *Let $G = \{I \to I(x+y),\ x \to 2kxy,\ y \to 2kxy\}$. We have*

$$D_G^n(I) = I \sum_{w \in B_n} x^{\mathrm{wexc}\,(w)} y^{\mathrm{aexc}\,(w)} k^{n-\mathrm{cyc}\,(w)}. \tag{19}$$

Proof. We first introduce a grammatical labeling of $w \in B_n$ as follows:

(L_1) If $w(i) = i$, then put a superscript label x right after i, i.e., (i^x);

(L_2) If $w(i) = \overline{i}$, then put a superscript label y right after $\overline{i}$, i.e., $(\overline{i}^y)$;

(L_3) If $w(i) < w(|w(i)|)$, then put a superscript label x right after $w(i)$;

(L_4) If $w(i) > w(|w(i)|)$, then put a superscript label y right after $w(i)$;

(L_5) Put a subscript label k just before every element of w except the first element in each cycle;

(L_6) Put a subscript label I right after w.

The weight of w is the product of its labels. Note that the weight of w is given by

$$I x^{\mathrm{wexc}\,(w)} y^{\mathrm{aexc}\,(w)} k^{n-\mathrm{cyc}\,(w)}.$$

Every permutation in B_n can be obtained from a permutation in B_{n-1} by inserting n or $\overline{n}$. For $n = 1$, we have $B_1 = \{(1^x)_I, (\overline{1}^y)_I\}$. Note that $D(I) = I(x+y)$. Then the sum of weights of the elements in B_1 is given by $D(I)$. Hence the result holds for $n = 1$. We proceed by induction on n. Suppose that we get all labeled permutations in $w \in B_{n-1}$, where $n \geqslant 2$. Let $\widetilde{w}$ be obtained from $w \in B_{n-1}$ by inserting n or $\overline{n}$. When the inserted n or $\overline{n}$ forms a new cycle, the insertion corresponds to the substitution rule $I \to I(x+y)$. Recall that each cycle of a signed permutation is written with its largest entry last and the cycles are written in ascending order of their last entry. Now we insert n or $\overline{n}$ right after $w(i)$. If i is a weak excedance of w, then the changes of labeling are illustrated as follows:

$$\cdots(i^x)\cdots \mapsto \cdots(i^x_k n^y); \quad \cdots(i^x)\cdots \mapsto \cdots(\overline{n}^x_k i^y)\cdots;$$

$$\cdots(\cdots w(i)^x_k w(|w(i)|)\cdots)\cdots \mapsto \cdots(w(|w(i)|)_k \cdots w(i)^x_k n^y);$$

$$\cdots(\cdots w(i)^x_k w(|w(i)|)\cdots)\cdots \mapsto \cdots(\cdots w(i)^y_k \overline{n}^x_k w(|w(i)|)\cdots)\cdots;$$

If i is an anti-excedance of w, then the changes of labeling are illustrated as follows:

$$\cdots(\overline{i}^y)\cdots \mapsto \cdots(\overline{i}^x_k n^y); \quad \cdots(\overline{i}^y)\cdots \mapsto \cdots(\overline{n}^x_k \overline{i}^y)\cdots;$$

$$\cdots(\cdots w(i)^y_k w(|w(i)|)\cdots)\cdots \mapsto \cdots(w(|w(i)|)_k \cdots w(i)^x_k n^y);$$

$$\cdots(\cdots w(i)^y_k w(|w(i)|)\cdots)\cdots \mapsto \cdots(\cdots w(i)^y_k \overline{n}^x_k w(|w(i)|)\cdots)\cdots;$$

In each case, the insertion of n or $\overline{n}$ corresponds to one substitution rule in G. By induction, it is routine to check that the action of D_G on elements of B_{n-1} generates all elements of B_n. This completes the proof. □

Let

$$F_n(x,y;k) = \sum_{w\in B_n} x^{\operatorname{wexc}(w)} y^{\operatorname{aexc}(w)} k^{n-\operatorname{cyc}(w)}.$$

Lemma 14. *We have*

$$F(x,y,z;k) = \sum_{n=0}^{\infty} F_n(x,y;k)\frac{z^n}{n!} = \left(\frac{(y-x)e^{kz(y-x)}}{y-xe^{2kz(y-x)}}\right)^{\frac{1}{k}}.$$

Proof. From Lemma 13, we obtain $D_G^n(I) = IF_n(x,y;k)$. By using

$$D_G^{n+1}(I) = D_G(IF_n(x,y;k)),$$

we get that the polynomials $F_n(x,y;k)$ satisfy the following recurrence relation

$$F_{n+1}(x,y;k) = (x+y)F_n(x,y;k) + 2kxy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)F_n(x,y;k), \tag{20}$$

with the initial conditions $F_0(x,y;k) = 1$ and $F_1(x,y;k) = x+y$. By rewriting (20) in terms of the generating function $F := F(x,y,z;k)$, we have

$$\frac{\partial}{\partial z}F = (x+y)F + 2kxy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)F. \tag{21}$$

It is routine to check that the generating function

$$\widetilde{F} := \widetilde{F}(x,y,z;k) = \left(\frac{(y-x)e^{kz(y-x)}}{y-xe^{2kz(y-x)}}\right)^{\frac{1}{k}}$$

satisfies (21). Also, this generating function gives $\widetilde{F}(x,y,0;k) = 1$. Hence $F = \widetilde{F}$. □

Proof of Theorem 4. We divide our proof into five parts.

(i) Comparing (5) with Lemma 14, we obtain

$$F_n(x,y;k) = y^n B_n^{(k)}\left(\frac{x}{y}\right). \tag{22}$$

Therefore,

$$B_n^{(k)}(x) = \sum_{w\in B_n} x^{\operatorname{wexc}(w)} k^{n-\operatorname{cyc}(w)}.$$

Combining (20) and (22), it is routine to verify (7).

(ii) We now consider a change of the grammar G given in Lemma 13. Setting $u = x+y$ and $v = xy$, we get $D_G(I) = Iu,\ D_G(u) = 4kv$ and $D_G(v) = 2kuv$. We define

$$G_1 = \{I \to Iu,\ u \to 4kv,\ v \to 2kuv\}. \tag{23}$$

Note that $D_{G_1}(I) = Iu$, $D_{G_1}^2(I) = I(u^2+4kv)$, $D_{G_1}^3(I) = I(u^3+(12k+8k^2)uv)$ and $D_{G_1}^4(I) = I(u^4+(24k+32k^2+16k^3)u^2v+(48k^2+32k^3)v^2)$. By induction, it is routine to verify that there exist nonnegative integers $b_{n,i,j}$ such that

$$D_{G_1}^n(I) = I\sum_{i=0}^{\lfloor n/2\rfloor}\sum_{j=i}^{n-1} b_{n,i,j}k^j4^iv^iu^{n-2i}. \tag{24}$$

It should be noted that $b_{n,i,j}=0$ if i and j are outside the bounds given in (24). Note that

$$\begin{aligned} D_{G_1}^{n+1}(I) &= D_{G_1}\left(I\sum_{i=0}^{\lfloor n/2\rfloor}\sum_{j=i}^{n-1} b_{n,i,j}k^j4^iv^iu^{n-2i}\right) \\ &= \sum_{i,j} b_{n,i,j}k^j4^iv^i\left(u^{n-2i+1}+2kiu^{n-2i+1}+4k(n-2i)vu^{n-2i-1}\right). \end{aligned}$$

Equating the coefficients of $k^j4^iv^iu^{n+1-2i}$ in both sides of the above expression, we get the recurrence relation (9). Since $D_{G_1}(I) = Iu$, we see that $b_{1,0,0}=1$ and $b_{1,i,j}=0$ if $(i,j)\neq(0,0)$. Taking $u=x+y,\ v=xy$ in (24) and then setting $y=1$, we get (8).

(*iii*) Multiplying both sides of (9) by x^iq^j and summing over all i and j, we obtain

$$b_{n+1}(x,q) = (1+nqx)b_n(x,q)+2qx(1-x)\frac{\partial}{\partial x}b_n(x,q). \tag{25}$$

In particular, $b_1(x,q)=1$ and $b_2(x,q)=1+qx$. Set $b=b(x,q,z)$. By rewriting (25) in terms of b, we have

$$(1-qxz)\frac{\partial b}{\partial z} = b+2qx(1-x)\frac{\partial b}{\partial x}. \tag{26}$$

It is routine to verify that

$$\widetilde{b}(x,q,z) = \left(\frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(qz\sqrt{1-x})-\sinh(qz\sqrt{1-x})}\right)^{\frac{1}{q}}$$

satisfies (26). Also, this generating function gives $\widetilde{b}(x,q,0)=1$ and $\widetilde{b}(0,q,z)=e^z$. Hence $\widetilde{b}(x,q,z)=b(x,q,z)$.

(*iv*) Assume that (10) holds for n. Let $\pi\in\mathfrak{S}_n$, and let π_i be an element of $\mathfrak{S}_{n+1}$ obtained from π by inserting the entry $n+1$ right after i if $i\in[n]$ or as a new cycle $(n+1)$ if $i=n+1$. It is clear that

$$\operatorname{cyc}(\pi_i) = \begin{cases} \operatorname{cyc}(\pi), & \text{if } i\in[n]; \\ \operatorname{cyc}(\pi)+1, & \text{if } i=n+1. \end{cases}$$

Therefore, we have

$$\begin{aligned}
&b_{n+1}(x,q)\\
&=\sum_{i=1}^{n+1}\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{cpk}\,(\pi_i)}q^{n+1-\mathrm{cyc}\,(\pi_i)}\\
&=\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{cpk}\,(\pi)}q^{n-\mathrm{cyc}\,(\pi)}+\sum_{i=1}^{n}\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{cpk}\,(\pi_i)}q^{n+1-\mathrm{cyc}\,(\pi)}\\
&=b_n(x,q)+\sum_{\pi\in\mathfrak{S}_n}\left(2\mathrm{cpk}\,(\pi)x^{\mathrm{cpk}\,(\pi)}+(n-2\mathrm{cpk}\,(\pi))x^{\mathrm{cpk}\,(\pi)+1}\right)q^{n+1-\mathrm{cyc}\,(\pi)}\\
&=b_n(x,q)+nqxb_n(x,q)+2q(1-x)\sum_{\pi\in\mathfrak{S}_n}\mathrm{cpk}\,(\pi)x^{\mathrm{cpk}\,(\pi)}q^{n-\mathrm{cyc}\,(\pi)},
\end{aligned}$$

and (25) follows. Thus (10) holds for $n+1$.

(v) Let G be the grammar given in Lemma 13. It follows from (19) that

$$D_G^n(I)=I\sum_{w\in B_n} x^{\mathrm{wexc}\,(w)}y^{n-\mathrm{wexc}\,(w)}k^{n-\mathrm{cyc}\,(w)}.$$

Setting $y=1$, we get $D_G^n(I)|_{y=1}=IB_n^{(k)}(x)$. Dumont [12] discovered that if

$$G_2=\{x\to xy, y\to xy\},$$

then we have

$$D_{G_2}^n(x)=x\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)}y^{n-\mathrm{exc}\,(\pi)}=xy^nA_n\left(\frac{x}{y}\right). \tag{27}$$

By using (27), it is easy to verify that for $n\geqslant 1$, we have

$$D_G^n(x+y)=2^{n+1}k^nx\sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)}y^{n-\mathrm{exc}\,(\pi)}=2^{n+1}k^nxy^nA_n\left(\frac{x}{y}\right).$$

It follows from Leibniz's rule (18) that for $n\geqslant 1$, we have

$$\begin{aligned}
D_G^{n+1}(I)&=\sum_{i=0}^{n}\binom{n}{i}D_G^i(I)D_G^{n-i}(x+y)\\
&=(x+y)D_G^n(I)+\sum_{i=0}^{n-1}\binom{n}{i}D_G^i(I)D_G^{n-i}(x+y).
\end{aligned}$$

Setting $y=1$ in both sides of the above expression, we immediately get (11). □

4 Proof of Theorem 9

In this section we complete the proof of Theorem 9. A grammatical interpretation of the polynomial $d_n^B(x)$ was given by [19, Theorem 11]. We now give a refinement of Lemma 13.

Lemma 15. *If* $G_4 = \{I \to I(y+u), x \to 2kxy, y \to 2kxy, u \to 2kxy\}$*, then*

$$D_{G_4}^n(I) = I \sum_{w \in B_n} x^{\mathrm{exc}\,(w)} y^{\mathrm{aexc}\,(w)} u^{\mathrm{fix}\,(w)} k^{n-\mathrm{cyc}\,(w)}. \tag{28}$$

Proof. We now introduce a grammatical labeling of $w \in B_n$ as follows:

(L_1) If $w(i) < w(|w(i)|)$, then put a superscript label x right after $w(i)$;

(L_2) If $w(i) > w(|w(i)|)$ or $w(i) = \bar{i}$, then put a superscript label y right after $w(i)$;

(L_3) If $w(i) = i$, then put a superscript label u right after $w(i)$, i.e., (i^u);

(L_4) Put a subscript label I right after w;

(L_5) Put a subscript label k just before every element of w except the first element in each cycle.

Then the weight of w is given by

$$Ix^{\mathrm{exc}\,(w)} y^{\mathrm{aexc}\,(w)} u^{\mathrm{fix}\,(w)} k^{n-\mathrm{cyc}\,(w)}.$$

For $n = 1$, we have $B_1 = \{(1^u)_I, (\bar{1}^y)_I\}$. Note that $D_{G_4}(I) = I(y+u)$. Thus the sum of weights of the elements in B_1 is given by $D_{G_4}(I)$. Hence the result holds for $n = 1$. We proceed by induction on n. Suppose that we get all labeled permutations in B_{n-1}, where $n \geqslant 2$. Let $\widetilde{w}$ be obtained from $w \in B_{n-1}$ by inserting n or $\overline{n}$. When the inserted n or $\overline{n}$ forms a new cycle, the insertion corresponds to the substitution rule $I \to I(y+u)$. If i is a weak excedance of w, then the changes of labeling are illustrated as follows:

$$\cdots(i^u)\cdots \mapsto \cdots(i_k^x n^y); \quad \cdots(i^u)\cdots \mapsto \cdots(\overline{n}_k^x i^y)\cdots;$$

$$\cdots(\cdots w(i)_k^x w(|w(i)|)\cdots)\cdots \mapsto \cdots(w(|w(i)|)_k \cdots w(i)_k^x n^y);$$
$$\cdots(\cdots w(i)_k^x w(|w(i)|)\cdots)\cdots \mapsto \cdots(\cdots w(i)_k^y \overline{n}_k^x w(|w(i)|)\cdots)\cdots;$$

If i is an anti-excedance of w, then the changes of labeling are illustrated as follows:

$$\cdots(\bar{i}^y)\cdots \mapsto \cdots(\bar{i}_k^x n^y); \quad \cdots(\bar{i}^y)\cdots \mapsto \cdots(\overline{n}_k^x \bar{i}^y)\cdots;$$

$$\cdots(\cdots w(i)_k^y w(|w(i)|)\cdots)\cdots \mapsto \cdots(w(|w(i)|)_k \cdots w(i)_k^x n^y);$$
$$\cdots(\cdots w(i)_k^y w(|w(i)|)\cdots)\cdots \mapsto \cdots(\cdots w(i)_k^y \overline{n}_k^x w(|w(i)|)\cdots)\cdots.$$

In each case, the insertion of n or $\overline{n}$ corresponds to one substitution rule in G_4. By induction, it is routine to check that the action of D_{G_4} on elements of B_{n-1} generates all elements of B_n. This completes the proof. ☐

We define

$$H_n(x,y,u;k)=\sum_{w\in B_n} x^{\mathrm{exc}\,(w)}y^{\mathrm{aexc}\,(w)}u^{\mathrm{fix}\,(w)}k^{n-\mathrm{cyc}\,(w)},$$

$$H:=H(x,y,u,z;k)=\sum_{n=0}^{\infty}H_n(x,y,u;k)\frac{z^n}{n!}.$$

Lemma 16. *We have*

$$H(x,y,u,z;k)=\left(\frac{(y-x)e^{kz(y+u-2x)}}{y-xe^{2kz(y-x)}}\right)^{\frac{1}{k}}. \tag{29}$$

Proof. Since $D_{G_4}^{n+1}(I)=D_{G_4}(IH_n(x,y,u;k))$, it follows that

$$D_{G_4}^{n+1}(I)=I(y+u)H_n(x,y,u;k)+2kxyI\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial u}\right)H_n(x,y,u;k).$$

Thus $H_{n+1}(x,y,u;k)=(y+u)H_n(x,y,u;k)+2kxy\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial u}\right)H_n(x,y,u;k)$. By rewriting this recurrence relation in terms of the generating function H, we have

$$\frac{\partial}{\partial z}H=(y+u)H+2kxy\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial u}\right)H. \tag{30}$$

It is routine to check that the generating function

$$\widetilde{H}(x,y,u,z;k)=\left(\frac{(y-x)e^{kz(y+u-2x)}}{y-xe^{2kz(y-x)}}\right)^{\frac{1}{k}}$$

satisfies (30). Note that $\widetilde{H}(x,y,u,0;k)=1$, $\widetilde{H}(0,y,u,z;k)=e^{u+y}$ and $\widetilde{H}(x,0,u,z;k)=e^{uz}$. Hence $\widetilde{H}(x,y,u,z;k)=H(x,y,u,z;k)$. □

Let $w\in B_n$ with exactly one fixed point. Suppose that $\mathrm{cyc}\,(w)=k$ and $w(\ell)=\ell$, i.e., ℓ is the fixed point of w. Then the standard form of w can be written as $w=C_1C_2\cdots C_k$, where $C_i=(c_{i1},\cdots,c_{ij})$, $1\leqslant i\leqslant k$ and $1\leqslant j\leqslant n$. The *reduction* of w is defined by

$$\mathrm{red}\,(w)=\mathrm{red}\,(C_1)\mathrm{red}\,(C_2)\cdots\mathrm{red}\,(C_k).$$

If $C_i=(\ell)$, then $\mathrm{red}\,(C_i)=\emptyset$, i.e., we delete the fixed point of w. If $\#C_i\geqslant 2$, then let $\mathrm{red}\,(C_i)=(\widetilde{c}_{i1},\cdots,\widetilde{c}_{ij})$. For $1\leqslant s\leqslant j$, the elements $\widetilde{c}_{is}$ are defined as follows:

- If $|c_{is}|<\ell$, then $\widetilde{c}_{is}=c_{is}$;
- If $c_{is}>\ell$, then $\widetilde{c}_{is}=c_{is}-1$;
- If $c_{is}<0$ and $|c_{is}|>\ell$, then $\widetilde{c}_{is}=c_{is}+1$.

It should be noted that $\mathrm{red}\,(w)\in B_{n-1}$ with no fixed points and the reduction map of w does not change the numbers of excedances and anti-excedances of w.

Proof of Theorem 9. Comparing (6) with (29), we immediately get (14). In the following, we shall prove (15). We first consider a change of the grammar given in Lemma 15. Note that $D_{G_4}(I) = Iy + Iu$ and

$$D_{G_4}(Iy) = I(y^2 + yu + 2kxy) = Iy(x + y) + Iyu + (2k - 1)Ixy.$$

Setting $a = xy,\ b = x + y$ and $c = Iy$, we get $D_{G_4}(a) = 2kab, D_{G_4}(b) = 4ka$,

$$D_{G_4}(I) = c + uI, D_{G_4}(c) = (b + u)c + (2k - 1)aI, D_{G_4}(u) = 2ka.$$

Consider the grammar

$$G_5 = \{I \to c + uI, c \to (b + u)c + (2k - 1)aI, u \to 2ka, a \to 2kab, b \to 4ka\}.$$

Note that $D_{G_5}(I) = c + Iu$ and $D^2_{G_5}(I) = (b + 2u)c + (u^2 + (4k - 1)a)I$. By induction, it is routine to verify that there exist nonnegative integers $p(n, i, j; k)$ and $q(n, i, j; k)$ such that

$$D^n_{G_5}(I) = \sum_{i=0}^{n} u^i \left(\sum_{j=0}^{\lfloor \frac{n-1-i}{2} \rfloor} p(n, i, j; k)a^j b^{n-1-i-2j} c + \sum_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} q(n, i, j; k)a^j b^{n-i-2j} I \right). \quad (31)$$

Combining (31) and Lemma 15, we immediately obtain

$$\begin{aligned}
&\sum_{w \in B_n} x^{\mathrm{exc}\,(w)} y^{\mathrm{aexc}\,(w)} u^{\mathrm{fix}\,(w)} k^{n-\mathrm{cyc}\,(w)} \\
&= \sum_{i=0}^{n} u^i \sum_{j=0}^{\lfloor \frac{n-1-i}{2} \rfloor} p(n, i, j; k)(xy)^j (x + y)^{n-1-i-2j} y + \\
&\sum_{i=0}^{n} u^i \sum_{j=0}^{\lfloor \frac{n-i}{2} \rfloor} q(n, i, j; k)(xy)^j (x + y)^{n-i-2j}.
\end{aligned}$$

Since $D_{G_5}(I) = c + Iu$, we have $p(1, 0, 0; k) = q(1, 1, 0; k) = 1$, $p(1, i, j; k) = 0$ if $(i, j) \neq (0, 0)$ and $q(1, i, j, ; k) = 0$ if $(i, j) \neq (1, 0)$. By induction, it is routine to verify that $p(n, i, j; k) = q(n, i, j; k) = 0$ if i and j are outside the bounds given in (31).

Extracting the coefficients of $a^j b^{n-2j} c$ and $a^j b^{n+1-2j} I$ on both sides of the expression

$$D^{n+1}_{G_5}(I) = D_{G_5} \left(\sum_{i,j} p(n, i, j; k)u^i a^j b^{n-1-i-2j} c + \sum_{i,j} q(n, i, j; k)u^i a^j b^{n-i-2j} I \right),$$

we obtain the following recurrence system:

$$\begin{aligned}
p(n + 1, 0, j; k) = &(1 + 2kj)p(n, 0, j; k) + 4k(n - 2j + 1)p(n, 0, j - 1; k) + \\
&2kp(n, 1, j - 1; k) + q(n, 0, j; k), \\
q(n + 1, 0, j; k) = &2kjq(n, 0, j; k) + 4k(n - 2j + 2)q(n, 0, j - 1; k) + 2kq(n, 1, j - 1; k) + \\
&(2k - 1)p(n, 0, j - 1; k).
\end{aligned}$$

Let $w \in B_n$ and $\operatorname{fix}(w) = 1$. Then $\operatorname{exc}(w) + \operatorname{aexc}(w) = n - \operatorname{fix}(w) = n - 1$. Recall that we always write w in standard cycle decomposition. Since w has only one fixed point, there are n choices for the fixed point of w. For the numbers $p(n,i,j;k)$ and $q(n,i,j;k)$, by comparing (28) with (31), we see that the index i only marks the number of fixed points, and the index j only depends on the numbers of excedances and anti-excedances. Let w' be the reduction of w. Then $w' \in B_{n-1}$, $\operatorname{cyc}(w') = \operatorname{cyc}(w) - 1$ and $\operatorname{fix}(w') = 0$. Moreover, we have $\operatorname{exc}(w) = \operatorname{exc}(w')$ and $\operatorname{aexc}(w) = \operatorname{aexc}(w')$. By using the properties of the reduction map, we get

$$\sum_{\substack{w \in B_n \\ \operatorname{fix}(w)=1}} x^{\operatorname{exc}(w)} y^{\operatorname{aexc}(w)} k^{n-\operatorname{cyc}(w)} = n \sum_{\substack{w' \in B_{n-1} \\ \operatorname{fix}(w')=0}} x^{\operatorname{exc}(w')} y^{\operatorname{aexc}(w')} k^{n-1-\operatorname{cyc}(w')}. \tag{32}$$

By using (31), we see that

$$\begin{aligned} &\sum_{\substack{w \in B_n \\ \operatorname{fix}(w)=1}} x^{\operatorname{exc}(w)} y^{\operatorname{aexc}(w)} k^{n-\operatorname{cyc}(w)} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} p(n,1,j;k)(xy)^j (x+y)^{n-2-2j} y + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} q(n,1,j;k)(xy)^j (x+y)^{n-1-2j}, \end{aligned}$$

$$\begin{aligned} &\sum_{\substack{w' \in B_{n-1} \\ \operatorname{fix}(w')=0}} x^{\operatorname{exc}(w')} y^{\operatorname{aexc}(w')} k^{n-1-\operatorname{cyc}(w')} \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - 1} p(n-1,0,j;k)(xy)^j (x+y)^{n-2-2j} y + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} q(n-1,0,j;k)(xy)^j (x+y)^{n-1-2j}. \end{aligned}$$

Replacing j with $j-1$ while adjusting the bounds of summation, and then equating appropriate coefficients yields the following relations:

$$p(n,1,j-1;k) = np(n-1,0,j-1;k), \;\; q(n,1,j-1;k) = nq(n-1,0,j-1;k).$$

Therefore, by setting $p(n,0,j;k) = p(n,j;k)$ and $q(n,0,j;k) = q(n,j;k)$, we obtain the recurrence system of the numbers $p(n,j;k)$ and $q(n,j;k)$.

Setting $u = 0$ in (31) and then taking $a = x$, $b = 1+x$ and $c = I$, we get the symmetric decomposition of the polynomials $d_n^B(x;k)$. Clearly, when $k \geqslant 1/2$, the numbers $p(n,j;k)$ and $q(n,j;k)$ are nonnegative, and so the polynomials $d_n^B(x;k)$ are bi-γ-positive. This completes the proof. □

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References

[1] C.A. Athanasiadis. Gamma-positivity in combinatorics and geometry. *Sém. Lothar. Combin.*, 77:Article B77i, 2018.

[2] M. Beck, A. Stapledon. On the log-concavity of Hilbert series of Veronese subrings and Ehrhart series. *Math. Z.*, 264:195–207, 2010.

[3] P. Brändén, L. Solus. Symmetric decompositions and real-rootedness. *Int Math. Res Notices*, rnz059, 2019.

[4] F. Brenti. Unimodal polynomials arising from symmetric functions. *Proc. Amer. Math. Soc.*, 108:1133–1141, 1990.

[5] F. Brenti. q-Eulerian polynomials arising from Coxeter groups. *European J. Combin.*, 15:417–441, 1994.

[6] F. Brenti. A class of q-symmetric functions arising from plethysm. *J. Combin. Theory Ser. A*, 91:137–170, 2000.

[7] T.-W. Chao, J. Ma, S.-M. Ma, Y.-N. Yeh. $1/k$-Eulerian polynomials and k-inversion sequences. *Electron. J. Combin.*, 26(3):#P3.35, 2019.

[8] W.Y.C. Chen. Context-free grammars, differential operators and formal power series. *Theoret. Comput. Sci.*, 117:113–129, 1993.

[9] W.Y.C. Chen, R.L. Tang, A.F.Y. Zhao. Derangement polynomials and excedances of type B. *Electron. J. Combin.*, 16(2): #R15, 2009.

[10] W.Y.C. Chen, A.M. Fu. Context-free grammars for permutations and increasing trees. *Adv. in Appl. Math.*, 82:58–82, 2017.

[11] C.-O. Chow. On derangement polynomials of type B, II. *J. Combin. Theory Ser. A*, 116:816–830, 2009.

[12] D. Dumont. Grammaires de William Chen et dérivations dans les arbres et arborescences. *Sém. Lothar. Combin.*, 37: Art. B37a 1–21, 1996.

[13] D. Foata, M. P. Schützenberger. Théorie géometrique des polynômes eulériens. *Lecture Notes in Math.*, vol. 138, Springer, Berlin, 1970.

[14] D. Foata, V. Strehl. Euler numbers and variations of permutations in *Colloquio Internazionale sulle Teorie Combinatorie* (Roma, 1973), Tomo I, Atti dei Convegni Lincei, No. **17**, Accad. Naz. Lincei, Rome, pp. 119–131, 1976.

[15] S.R. Gal. Real root conjecture fails for five and higher-dimensional spheres. *Discrete Comput. Geom.*, 34:269–284, 2005.

[16] Z. Lin, J. Zeng. The γ-positivity of basic Eulerian polynomials via group actions. *J. Combin. Theory Ser. A*, 135:112–129, 2015.

[17] Lily L. Liu, Yi Wang. A unified approach to polynomial sequences with only real zeros. *Adv. in Appl. Math.*, 38:542–560, 2007.

[18] S.-M. Ma, T. Mansour. The $1/k$-Eulerian polynomials and k-Stirling permutations. *Discrete Math.*, 338:1468–1472, 2015.

[19] S.-M. Ma, J. Ma, Y.-N. Yeh, B.-X. Zhu. Context-free grammars for several polynomials associated with Eulerian polynomials. *Electron. J. Combin.*, 25(1):#P1.31, 2018.

[20] S.-M. Ma, J. Ma, Y.-N. Yeh. γ-positivity and partial γ-positivity of descent-type polynomials. *J. Combin. Theory Ser. A*, 167:257–293, 2019.

[21] T.K. Petersen. Enriched P-partitions and peak algebras. *Adv. Math.*, 209(2):561–610, 2007.

[22] D.P. Roselle. Permutations by number of rises and successions. *Proc. Amer. Math. Soc.*, 19:8–16, 1968.

[23] C.D. Savage, G. Viswanathan. The $1/k$-Eulerian polynomials. *Electron J. Combin.*, 19(1):#P9, 2012.

[24] H. Shin, J. Zeng. The symmetric and unimodal expansion of Eulerian polynomials via continued fractions. *European J. Combin.*, 33:111–127, 2012.

[25] H. Shin, J. Zeng. Symmetric unimodal expansions of excedances in colored permutations. *European J. Combin.*, 52: 174–196, 2016.

[26] N.J.A. Sloane. The On-Line Encyclopedia of Integer Sequences, published electronically at http://oeis.org, 2010.

[27] R.P. Stanley. A survey of alternating permutations. *Contemp. Math.*, 531 (2010), 165–196.

[28] J. Stembridge. Enriched P-partitions. *Trans. Amer. Math. Soc.*, 349(2):763–788, 1997.

[29] Y. Zhuang. Eulerian polynomials and descent statistics. *Adv. in Appl. Math.*, 90:86–144, 2017.