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Ordered Weighted Average Optimization in Multiobjective Spanning Tree Problems

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Abstract

Multiobjective Spanning Tree Problems are studied in this paper. The ordered median objective function is used as an averaging operator to aggregate the vector of objective values of feasible solutions. This leads to the Ordered Weighted Average Spanning Tree Problem, a nonlinear combinatorial optimization problem. Different mixed integer linear programs are proposed, based on the most relevant minimum cost spanning tree models in the literature. These formulations are analyzed and several enhancements presented. Their empirical performance is tested over a set of randomly generated benchmark instances. The results of the computational experiments show that the choice of an appropriate formulation allows to solve larger instances with more objectives than those previously solved in the literature.

Keywords: Combinatorial Optimization, Multiobjective optimization, Ordered median, Ordered weighted average, Spanning trees.

1 Introduction

Optimization problems related to spanning trees, or simply spanning tree problems are among the core problems in combinatorial optimization. On the one hand, the combinatorial object that represents spanning trees has important structural properties. On the other hand, from a practitioner point of view, spanning trees are found in a wide range of applications in many fields (e.g. computer networks design, telecommunications networks, transportation, etc). Furthermore, they often appear as subproblems of other more complex optimization problems.

The most relevant property of trees is their matroid structure. This implies that the basic problem of finding a minimum cost spanning tree, can be solved efficiently (Prim, 1957; Kruskal, 1956). This also implies that formulations with the integrality property can be obtained, which allow to solve the minimum cost Spanning Tree Problem (STP) with linear programming tools. However, these good

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features can be lost for several reasons. For instance, when the objective function does not preserve Gale optimality, i.e., it is not monotonic on the edges costs (Lawler, 1966; Fernández et al., 2014), as it happens in the Optimum Communication Spanning Tree Problem (Hu, 1974). The reader may refer to Landete and Marín (2014) for a description of alternative objective functions for STP. An optimization STP also becomes a hard problem when several objectives are considered simultaneously (Ehrgott, 2005). In such cases, no efficient combinatorial algorithm is known so the choice of an appropriate mathematical programming representation of the combinatorial object may become crucial. In this sense, formulations for STP with good properties can be outperformed by other formulations in the new environment.

From a different point of view, in the multiobjective case, it is widely accepted that the use of order and aggregation functions may yield compromise solutions for the different criteria. The literature includes many works on this area related to combinatorial optimization. Some examples, among many others, include minimax problems (Hansen, 1980; Schrijver, 1983), combining minisum and minimax (Averbakh and Berman, 1995; Hansen and Labbé, 1988; Hansen et al., 1991; Minoux, 1989; Punnen et al., 1995; Tamir et al., 2002), k -centrum optimization (Garfinkel et al., 2006; Kalcsics et al., 2002; Punnen, 1992; Slater, 1978a,b; Tamir, 2000), lexicographic optimization (Calvete and Mateo, 1998; Croce et al., 1999), k -th best solutions (Lawler, 1972; Martello et al., 1984; Pascoal et al., 2003; Yen, 1971), most uniform solutions (Galil and Schieber, 1998; López de los Mozos et al., 2008), minimum-envy solutions (Espejo et al., 2009), solutions with minimum deviation (Gupta et al., 1990), regret solutions (Averbakh, 2001; Conde, 2004; Puerto and Rodríguez-Chía, 2003), equity measures (Gupta and Punnen, 1988; López de los Mozos et al., 2008; Mesa et al., 2003; Punnen and Aneja, 1997), discrete ordered median location problems (Boland et al., 2006; Marín et al., 2009; Puerto, 2008; Puerto and Tamir, 2005; Puerto and Rodríguez-Chía, 2015), ordered weighted average objectives (Fernández et al., 2013, 2014; Galand and Spanjaard, 2012), and covering objectives (Balas and Padberg, 1972; Breuer, 1970; Christofides and Korman, 1974; Kelly, 1944; Lawler, 1966). This paper elaborates on a specific form of aggregation criterion in multicriteria optimization, called Ordered Weighted Average operator (OWA). It is well known that this family parameterizes the aggregation function used when the decision-maker seeks the simultaneous satisfaction of all the criteria, to the case when the individual satisfaction of any of the criteria is sought. This fact is particularly relevant, due to its generality, as it includes as particular cases most of the above mentioned operators. This observation connects our results with the multicriteria literature and has been made already explicit in Fernández et al. (2014).

Multiobjective STPs have already been studied by some authors, mostly for the biobjective case (see Hamacher and Ruhe, 1994; Andersen et al., 1996; Ramos et al., 1998; Sourd and Spanjaard, 2008; Steiner and Radzik, 2008). In this paper we address the Multiobjective STP under the perspective of the OWA operator for a general number of objectives. This problem will be referred to as OWA Spanning Tree Problem (OWASTP). In OWASTP the optimality of traditional combinatorial algorithms is no longer guaranteed. Furthermore, formulations adapted from good STP formulations lose the integrality property. Thus alternative formulations that originally do not exhibit such good properties, may now outperform them. In Galand et al. (2010) OWASTP was addressed using Choquet optimization and Galand and Spanjaard (2012) presented a first ordered median Mixed Integer Linear Programming (MILP) formulation. Our goal in this paper is to exploit properties of alternative formulations for OWASTP. As we will see, an appropriate formulation allows us to solve larger instances and with more objectives than those previously solved in the literature (Galand and Spanjaard, 2012), with up to 100 nodes and 10 objective functions. The contributions of this paper are (1) to provide new formulations for OWASTP combining appropriate formulations for STP and for OWA problem; (2) to prove a new complexity result showing that OWASTP is NP-complete even for cactus graphs and two objectives; (3) to establish a theoretical and empirical comparison between the new formulations and previous existing ones; and, (4) to provide reinforcements that together with

the new OWASTP formulations are able to outperform previous results in the literature.

The structure of the paper is the following. In Section 2 we formally define OWASTP and prove our new complexity result. Section 3 presents the catalogue of STP formulations that we study for OWASTP. One such formulation has already been used in Galand and Spanjaard (2012). We will use it as a reference for the alternative formulations that we present. The empirical performance of the resulting OWASTP formulations is analyzed in Section 4, where we present extensive numerical results and a comparison with existing ones. Finally, some conclusions are summarized in Section 5.

2 Problem definition

The Ordered Weighted Average operator is defined over a feasible set $Q \subseteq \mathbb{R}^m$. Let $C \in \mathbb{R}^{p \times m}$ be a given matrix, whose rows, denoted by C^i , are associated with the cost vectors of p objective functions. The index set for the rows of C is denoted by $P = \{1, \dots, p\}$. Let also $\omega \in \mathbb{R}^{p+}$ denote a vector of non-negative weights. For $x \in Q$, the vector $y = Cx \in \mathbb{R}^p$ is referred to as the outcome vector relative to C . In the following we assume $y = Cx$, with $x \in Q$. For a given y , let σ be a permutation of the indices of $i \in P$ such that $y_{\sigma_1} \geq \dots \geq y_{\sigma_p}$. Feasible solutions $x \in Q$ are evaluated with an operator defined as $OWA_{(C,\omega)}(x) = \omega^t y_\sigma$. The OWA optimization Problem (OWAP) is to find $x \in Q$ of minimum value with respect to the above operator, that is

$$\text{OWAP: } \min_{x \in Q} OWA_{(C,\omega)}(x)$$

The OWA is a very general operator, which has as particular cases well-known objective functions namely the Ordered Median Objective and the Vector Assignment Ordered Median (see Fernández et al., 2014). In addition, the OWA operator allows to model various aggregation functions according to the vector of weights w (see, e.g., Ogryczak and Olender, 2012). Some examples are the minimum, maximum, median, center or k -centrum functions. Therefore, the selection of non/monotonic or non/symmetric w -weights is directly connected with the problem structure and thus with its complexity (Kasperski and Zielinski, 2013).

As defined, the OWA operator is indeed not linear. Moreover, in general, it is not convex either. For the case of monotonic weights, its convexity is known (Kalcics, Nickel, Puerto, 2003; Puerto and Rodríguez-Chía, 2015) and some elegant linearization of OWA functions have been proposed in the literature (see, e.g., Ogryczak and Sliwinski, 2003; Ogryczak and Tamir, 2003). Depending on the type of monotonicity, the problems are simpler (with decreasing weights in the case of minimization) or harder (with increasing weights in the case of minimization). In this paper we focus on OWASTP with arbitrary weights. Two well-known particular cases of the OWA operator with arbitrary weights are the Hurwicz criterion (Hurwicz, 1951) defined as $\alpha \max_{i \in P} y_i + (1 - \alpha) \min_{i \in P} y_i$ and the k -trimmed mean defined as $\sum_{i=k+1}^{p-k} (p - 2k)^{-1} y_i$. These criteria are of special interest for being non-monotonic and non-convex (Grzybowski et al., 2011, Puerto and Tamir, 2005) and have already been considered when analyzing the behavior of OWA operators in multiobjective optimization (see, e.g., Galand and Spanjaard, 2012).

OWASTP is defined as follows. Let $G = (V, E)$ be an undirected connected graph with set of nodes V , $|V| = n$, and set of edges E , $|E| = m$. In the following we assume that G contains at least one cycle, that is $m > n - 1$, as otherwise the problem becomes trivial. A spanning tree of G is a subgraph $T = (V, E')$ where $E' \subset E$ is a minimal set of edges connecting the set of nodes V . Let \mathcal{T} denote the set of spanning trees defined on G . Then, OWASTP can be defined as

$$\text{OWASTP: } \min_{x \in \mathcal{T}} \text{OWA}_{(C, \omega)}(x).$$

Example 1. Consider the graph $G = (N, E)$ depicted in Figure 1-(a) and the 3-cost vectors on E , whose values are represented next to each edge. The optimal solution to OWASTP with $\omega^t = (0.4, 0, 0.6)$ is depicted in Figure 1-(b) and has a value of 8.8. When the weights are $\omega^t = (0.8, 0, 0.2)$ the optimal OWASTP value is 10.4, corresponding to the tree depicted in Figure 1-(c).

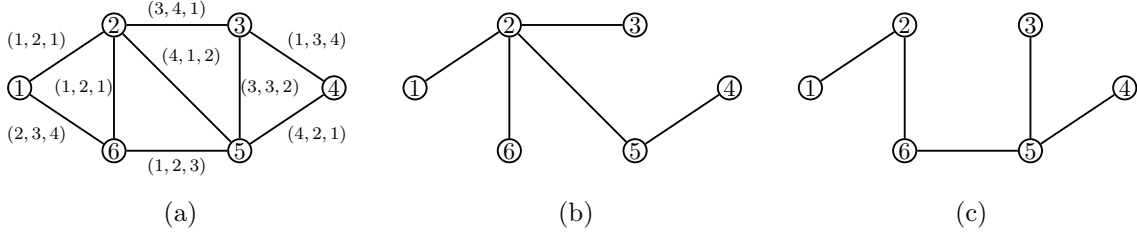


Figure 1: Graph with edge costs (a) and OWASTP solutions for $\omega^t = (0.4, 0, 0.6)$ (b) and $\omega^t = (0.8, 0, 0.2)$ (c).

□

OWASTP is known to be NP-hard on general graphs since, for a weight vector $w_1 = 1$ and $w_i = 0$, $i = 2, \dots, p$, the problem reduces to the bicriteria Min-Max Spanning tree which is NP-hard on general graphs (Hamacher and Ruhe, 1994). This complexity result can be strengthened following Yu (1998), where the author proved that the bicriteria Min-Max Spanning tree problem is NP-hard on grid graphs and $p = 2$. Consequently, OWASTP is NP-hard in this special class of graphs, as well. However, we can further refine the latter complexity by showing that OWASTP remains NP-hard on the very special class of *cactus* graphs, for $p = 2$, and assuming a strictly decreasing weights vector.

Given a connected graph $G = (V, E)$, a node $v \in V$ is a cut node if after removing v and all edges incident to it the remaining graph is no longer connected. A graph without cut vertices is called *nonseparable*. A *block* is a maximal nonseparable graph. A cactus graph is a graph in which every block is an edge or a cycle of three or more nodes (see, e.g., Brandstadt et al., 2000). In Brandstadt et al. (2000) cactus graphs are considered as *almost trees* and classified in the lowest level difficulty class of graphs, just after the acyclic ones. The decision version of the convex OWASTP on cactus graphs can be stated as follows:

Input: A cactus graph $G = (V, E)$ with positive integer weights (c_e^1, c_e^2) assigned to each edge $e \in E$. A vector $w \geq 0$ of rational components (w_1, w_2) , with $w_1 > w_2$, and a rational $K \geq 0$. Let $f_1(T)$ and $f_2(T)$ be the weights of any spanning tree T of G computed with respect to c_e^1 and c_e^2 , respectively. Let σ be an ordering of $f_1(T)$ and $f_2(T)$ such that $f_{\sigma_1}(T) \geq f_{\sigma_2}(T)$.

Output: Is there a spanning tree T of G such that, for the ordering σ , $w_1 f_{\sigma_1}(T) + w_2 f_{\sigma_2}(T) \leq K$?

Claim Problem OWASTP is NP-complete on cactus graphs and $p = 2$.

The reduction is from Partition with Disjoint Pairs (PDP) which is the following problem: Given n pairs of integers (a_i, b_i) , $i = 1, \dots, n$, is there a subset $S \subset [1, \dots, n]$ (a bi-partition) of the set of indexes such that: $\sum_{i \in S} a_i + \sum_{i \notin S} b_i = \frac{Q}{2}$, where $\sum_{i=1}^n (a_i + b_i) = Q$?

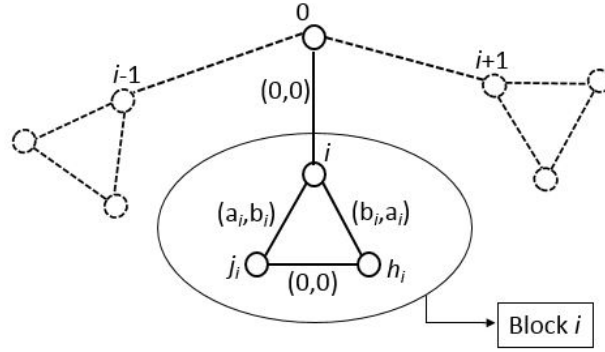


Figure 2: The cactus graph used in the proof of the NP-completeness claim.

Problem PDP is NP-hard (see, e.g., Karp, 1972; Richey, 1990; Richey and Punnen, 1992). We observe that PDP is also known as *Alternating Partition* which was proved to be NP-complete in Garey and Johnson (1979).

Proof.

Given an instance of PDP, construct the (very simple) cactus graph in Figure 2. Set $w_1 = 1$, $w_2 = 1 - \delta$, with $0 < \delta < 1$ and rational. Let $K = Q(1 - \frac{\delta}{2})$.

For each block i assign to edge (i, j_i) weights (a_i, b_i) ; while to edge (i, h_i) assign weights (b_i, a_i) . To all other edges of G assign weights $(0, 0)$. Given a solution of PDP, we can construct a solution of OWASTP as follows: if $i \in S$ the corresponding edge in the spanning tree T in block i is (i, j_i) , otherwise add to T (i, h_i) . Then, since T must be a spanning tree of G , we must add edges (j_i, h_i) and all the edges $(i, 0)$, $i = 1, \dots, n$. The weight of T w.r.t. the first component of the edge weights is $f_1(T) = \sum_{i \in S} a_i + \sum_{i \notin S} b_i = \frac{Q}{2}$. Similarly, $f_2(T) = \sum_{i \notin S} a_i + \sum_{i \in S} b_i = \frac{Q}{2}$. Hence, for the ordering σ we have:

$$f_{\sigma_1}(T) + (1 - \delta)f_{\sigma_2}(T) = Q(1 - \frac{\delta}{2}) = K.$$

Conversely, if we have a solution of OWASTP, T , there must exist a subset S of the n blocks for which edges (a_i, b_i) belong to T , $i \in S$; while for the other blocks (i.e., $i \notin S$) edges (b_i, a_i) belong to T .

Actually, since $\sum_{i=1}^n (a_i + b_i) = Q$, for any spanning tree T such that $f_1(T) \neq f_2(T)$, given the ordering σ we have $f_{\sigma_1}(T) > Q/2$ and $f_{\sigma_2}(T) < Q/2$. Suppose that $f_{\sigma_1}(T) = \frac{Q}{2} + \Gamma$ and $f_{\sigma_2}(T) = \frac{Q}{2} - \Gamma$, $\Gamma > 0$. Then, computing the objective function of OWASTP we have:

$$f_{\sigma_1}(T) + (1 - \delta)f_{\sigma_2}(T) = \frac{Q}{2} + \Gamma + (1 - \delta)(\frac{Q}{2} - \Gamma) = Q(1 - \frac{\delta}{2}) + \delta\Gamma > K.$$

Since OWASTP belongs to NP, then OWASTP is NP-complete. □

We conclude this section by recalling that, according to Yu (1998), if p is unbounded the Min-Max Spanning tree problem is strongly NP-hard even for grid graphs (reduction from 3-partition problem which is strongly NP-hard). Following this result, OWASTP is also strongly NP-hard with unbounded number of criteria for grid graphs and this would prevent to find a FPTAS.

3 OWASTP formulations

In this section we present several formulations for OWASTP. All of them are MILP formulations, which integrate a mixed integer linear programming STP formulation within a generic mathematical programming formulation for an OWA combinatorial problem (Fernández et al., 2014). We start with the catalogue of STP formulations and then we give the mathematical programming formulations for OWA combinatorial optimization problems that we have used.

3.1 Mixed Integer Linear Programming formulations for STP

Many alternative MILP formulations have been proposed for STP. For an overview of the possible alternatives and the properties in each case, the interested reader is addressed to the excellent book chapter by Magnanti and Wolsey (1995) where many of them are presented and compared.

It is well-known that STP formulations exist with the integrality property. Unfortunately, when they are embedded within an OWAP framework the integrality property is lost, so explicit integrality conditions are needed. Alternative STP formulations without such property may now be superior. This explains why some of the formulations we have used lack the integrality property. The criterion that has guided the selection of the formulations is either their good theoretical properties or some characteristic that seemed useful as, for instance, a small number of variables or constraints.

We start with two well-known models, the first one derived from the matroid polyhedron (Edmonds, 1970, 1971) and the second one proposed by Martin (1991), both of which having the integrality property. Then we present three existing formulations without the integrality property, based, respectively, on cutset inequalities, flow balance equations and Miller-Tucker-Zemlin inequalities (Miller et al., 1960). We present another STP formulation based on a relaxation of the formulation proposed by Martin (1991), which uses considerably fewer variables.

All formulations use design variables x to represent the edges of the spanning trees. Let $x_e, e \in E$ be a binary variable equal to 1 if edge $e = (u, v)$ is in the spanning tree, and zero otherwise. Some formulations use additional variables related to the arcs of the directed network, $D = (V, A)$ with the same node set as the original undirected G and set of arcs A , containing two arcs associated with each edge of E , i.e., $A = \{(u, v), (v, u) \mid (u, v) \in E\}$.

Throughout we will use the following standard notation. Given a subset of nodes $S \subset V$, $E(S)$ and $A(S)$ respectively denote the subsets of edges of E and arcs of A with both end-nodes in S , i.e., $E(S) = \{e = (u, v) \in E : u, v \in S\}$ and $A(S) = \{(u, v) \in A : u, v \in S\}$. The *cut-set associated with* $S \subset V$, $\delta(S) = \{e = (u, v) \in E \mid (u \in S, v \in V \setminus S) \text{ or } (v \in S, u \in V \setminus S)\}$, contains all edges with one node in S and the other node outside S . When working on the directed network D , for $S \subset V$, we let $\delta^+(S) = \{(u, v) \in A \mid u \in S, v \in V \setminus S\}$ denote the cutset directed out of S and $\delta^-(S) = \{(u, v) \in A \mid u \in V \setminus S, v \in S\}$ the cutset directed into S . Directed cuts will also be referred to as *dicuts*.

Next we focus on the domains that characterize feasible solutions in each case.

The domain in the subtour elimination formulation is:

$$\mathcal{T}^{sub} : \quad \sum_{e \in E} x_e = n - 1 \quad (1a)$$

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad \emptyset \neq S \subset V \quad (1b)$$

$$x_e \geq 0 \quad e \in E \quad (1c)$$

The cardinality constraint (1a) imposes that exactly $n - 1$ edges are chosen. Constraints (1b) ensure that the solution contains no cycle. The number of such constraints is exponential on the number of nodes. However, they can be separated in polynomial time by solving a series of minimum (s, t) -cut problems. An effective algorithm can be implemented using a Gomory-Hu cut tree (Hu, 1974).

It is well-known that all the extreme points in the above domain are integer and that formulation \mathcal{T}^{sub} is stronger than the formulation where inequalities (1b) are replaced by the cut-set constraints $\sum_{e \in \delta(S)} x_e \geq 1$, that we denote \mathcal{T}^{cut} , which may have fractional extreme points (Magnanti and Wolsey, 1995).

The extended formulation of Martin (1991) models an arborescence rooted at each node $k \in V$, in which arcs follow the direction from the leaves to the root. The arcs of such arborescences are then related to the design x variables. For $k \in V, (u, v) \in E$, let q_{kuv} and q_{kvu} be decision variables that, respectively, indicate whether or not arcs (u, v) and $(v, u) \in A$ belong to the arborescence rooted at k . The domain of the K. Martin (KM) formulation is the following:

$$\mathcal{T}^{km} : \quad \sum_{e \in E} x_e = n - 1 \quad (2a)$$

$$q_{kuv} + q_{kvu} = x_{uv} \quad k \in V, (u, v) \in E \quad (2b)$$

$$\sum_{(u,v) \in \delta^+(u)} q_{kuv} \leq 1 \quad k \in V, u \in V : u \neq k \quad (2c)$$

$$\sum_{(k,v) \in \delta^+(k)} q_{kkv} \leq 0 \quad k \in V \quad (2d)$$

$$x_e \geq 0 \quad e \in E \quad (2e)$$

$$q_{kuv} \geq 0 \quad k \in V, (u, v) \in A \quad (2f)$$

Constraint (2a) ensures that the tree has $n - 1$ edges. On the other hand, constraints (2b) indicate that the arcs that are used in the arborescences are precisely the ones associated with the $n - 1$ selected undirected edges. In other words, the underlying undirected graph supporting all the arborescences is exactly the same, so all the arborescences use exactly $n - 1$ arcs, and the only differences among arborescences are the directions of the arcs, but not the edges on the undirected graph that are used. For each arborescence, (2c) impose that no more than one arc leaves any node different from the root k , while (2d) forbids any arc leaving the root node k . Hence, these constraints imply that for each arborescence, the component containing the root node does not contain any cycle. Since each node is the root of one arborescence, (2b) guarantee that the selected undirected edges contain no cycle and, by (2a), the solutions define spanning trees.

While formulation \mathcal{T}^{km} has the integrality property, it has an $O(n^3)$ number of both q variables and constraints (2b). As the size of the graph increases, this number can be prohibitive. When the

integrality property is lost because of the addition of new constraints, the computational burden for solving a formulation with such a large number of variables and constraints may become too high.

The Miller-Tucker-Zemlin (MTZ) inequalities are an alternative to the exponential size family of constraints in (1b), to guarantee the connectivity of the solutions and thus prevent cycles. These constraints were initially proposed by Miller et al. (1960) in the context of the Traveling Salesman Problem. They have been adapted to other problems and reinforced by different authors (see, e.g. Laporte, 1992, Landete and Marín, 2014). In particular, they have been used by Gouveia (1995) for the Hop-Constrained Spanning Tree Problem, which is a generalization of STP in which the paths starting at a specified root node r are restricted to have no more than p edges. The MTZ formulation for STP builds an arborescence rooted at a specified node $r \in V$, in which arcs follow the direction from the root to the leaves. It uses binary variables to represent the arcs of the arborescence. Each edge $(u, v) \in E$, is associated with a pair of binary variables, y_{uv} and y_{vu} , which take the value 1 if and only if arcs (u, v) and $(v, u) \in A$ belong to the arborescence, respectively. In addition, it uses continuous variables l_u , denoting the position that node u occupies in the arborescence with respect to r . Since, in principle, there is no pre-specified root node, below r denotes any arbitrarily selected node. The domain of this formulation is given by the following set of constraints:

$$\mathcal{T}^{mtz} : \quad \sum_{e \in E} x_e = n - 1 \quad (3a)$$

$$\sum_{(v,u) \in \delta^-(u)} y_{vu} = 1 \quad u \in V \setminus \{r\} \quad (3b)$$

$$y_{uv} + y_{vu} = x_{uv} \quad (u, v) \in E \quad (3c)$$

$$l_v \geq l_u + 1 - n(1 - y_{uv}) \quad (u, v) \in A \quad (3d)$$

$$l_u = 1 \quad u = r \quad (3e)$$

$$2 \leq l_u \leq n \quad u \in V \setminus \{r\} \quad (3f)$$

$$y_{uv} \in \{0, 1\} \quad (u, v) \in A \quad (3g)$$

$$x_e \in \{0, 1\} \quad e \in E \quad (3h)$$

Constraint (3a) ensures that the tree has $n - 1$ edges. Equations (3b) impose that each node apart from the root is reached by one single arc, while (3c) guarantee that an edge is selected if any of its two arcs is selected. Constraints (3d) state that if an arc (u, v) is selected the position in the tree of v is higher than the position of u . Finally, (3e) and (3f) assign appropriate bounds to variables l_u , to ensure that the relative position of the root node in the tree is 1 and that the position of any other node is greater than or equal to 2 and does not exceed the number of nodes.

The flow-based STP formulation we present below (see Magnanti and Wolsey, 1995) is based on the formulation of Gavish (1983) for the capacitated minimal directed tree problem, and was used by Galand and Spanjaard (2012) for OWASTP. In addition to the binary design variables x , the formulation uses continuous flow variables φ defined on the arcs of the directed network $D = (V, A)$. There is a single source node, which is an arbitrarily selected node $r \in V$, with inflow $n - 1$. All other nodes have a demand of one unit. For each $(u, v) \in A$ the decision variable φ_{uv} represents the amount

of flow through arc (u, v) . Then the domain of the flow formulation for STP is:

$$\mathcal{T}^{flow} : \sum_{e \in E} x_e = n - 1 \quad (4a)$$

$$\sum_{(r,v) \in \delta^+(r)} \varphi_{rv} - \sum_{(u,r) \in \delta^-(r)} \varphi_{ur} = n - 1 \quad (4b)$$

$$\sum_{(u,v) \in \delta^+(u)} \varphi_{uv} - \sum_{(v,u) \in \delta^-(u)} \varphi_{vu} = -1 \quad u \in V \setminus \{r\} \quad (4c)$$

$$\varphi_{uv} \leq (n - 1)x_{uv} \quad (u, v) \in E \quad (4d)$$

$$\varphi_{vu} \leq (n - 1)x_{uv} \quad (u, v) \in E \quad (4e)$$

$$\varphi_{uv} \geq 0 \quad (u, v) \in A \quad (4f)$$

$$x_e \in \{0, 1\} \quad e \in E \quad (4g)$$

Again, constraint (4a) ensures that exactly $n - 1$ edges are selected. The block of constraints (4b)–(4c) guarantees that $n - 1$ units of flow leave the source node r and that at least one unit of flow arrives to every other node. The main role of these constraints is to guarantee that the graph induced by the arcs through which the flow circulates is connected and all nodes are “covered”. Constraints (4d)–(4e) extend these two properties to the graph induced by the x variables, by imposing that all the edges used for sending flow in some direction are activated.

Concerning domain \mathcal{T}^{flow} note that, because of the flow constraints (4b)–(4c), the removal of constraint (4a) would not change the set of optimal solutions (as opposed to the case of the maximum STP). However, constraint (4a) reinforces considerably the linear relaxation of formulation \mathcal{T}^{flow} , so it is kept in the formulation. Another improvement consists of replacing (4d) and (4e) by the tighter set of constraints:

$$\varphi_{uv} + \varphi_{vu} \leq (n - 1)x_{uv} \quad (u, v) \in E : u = r \vee v = r \quad (4d')$$

$$\varphi_{uv} + \varphi_{vu} \leq (n - 2)x_{uv} \quad (u, v) \in E : u \neq r \wedge v \neq r \quad (4e')$$

3.1.1 An alternative formulation for STP

Below we present an alternative formulation for STPs, which inherits some of the ideas behind the \mathcal{T}^{km} formulation without requiring $O(n^3)$ variables. In particular, instead of building one arborescence for each node, we arbitrarily set one single root node $r \in V$ and build one single arborescence rooted at r . The arcs of such an arborescence are determined by the subset of variables q_{ruv} , $(u, v) \in A$. Since r is fixed, in the following we remove the first index and simply denote these variables by q_{uv} , $(u, v) \in A$. Indeed, equality (2a), plus the subset of constraints (2b), (2c) and (2d) associated with $k = r$ defines a relaxation to formulation \mathcal{T}^{km} , which uses $O(n^2)$ variables. Unfortunately, such relaxation is not valid for STPs, as it may produce solutions which are not associated with connected sets of arcs. Luckily, this weakness can be easily overcome by including the following dicut inequalities:

$$\sum_{(u,v) \in \delta^+(S)} q_{uv} \geq 1, \quad S \subseteq V \setminus \{r\},$$

which guarantee the connectivity of the obtained solutions (at least one arc will *exit* from any subset of nodes S not containing the root node) and thus, the validity of the formulation. The formulation

is then as follows:

$$\mathcal{T}^{km2} : \quad \sum_{(u,v) \in E} x_{uv} = n - 1 \quad (6a)$$

$$q_{uv} + q_{vu} = x_{uv} \quad (u, v) \in E \quad (6b)$$

$$\sum_{(u,v) \in \delta^+(u)} q_{uv} \leq 1 \quad u \in V \setminus \{r\} \quad (6c)$$

$$\sum_{(r,v) \in \delta^+(r)} q_{rv} \leq 0 \quad (6d)$$

$$\sum_{(u,v) \in \delta^+(S)} q_{uv} \geq 1 \quad \emptyset \neq S \subset V \setminus \{r\} \quad (6e)$$

$$x_{uv} \geq 0 \quad (u, v) \in E \quad (6f)$$

$$q_{uv} \geq 0 \quad u, v \in V \quad (6g)$$

Remark 1.

- (a) The only difference between formulations \mathcal{T}^{mtz} and \mathcal{T}^{km2} is the way in which subtours are prevented. The former uses the Miller-Tucker-Zemlin inequalities, which are known to be weaker than cut-type constraints used in the latter. This indicates that formulation \mathcal{T}^{mtz} is weaker than \mathcal{T}^{km2} . Below we provide a stronger evidence of the superiority of \mathcal{T}^{km2} over \mathcal{T}^{mtz} , as we will see that \mathcal{T}^{km2} has the integrality property, even if some redundancies are eliminated.
- (b) For any $u \in V \setminus \{r\}$ the constraint (6e) corresponding to the set $S = \{u\}$ reduces to $\sum_{(u,v) \in \delta^+(u)} q_{uv} \geq 1$. Together with constraints (6c) this implies that $\sum_{(u,v) \in \delta^+(u)} q_{uv} = 1$ for all $u \in V \setminus \{r\}$. Observe, however, that the new set of constraints (6e) together with (6a) and (6b) already imply that $\sum_{(u,v) \in \delta^+(u)} q_{uv} = 1$ for all $u \in V \setminus \{r\}$. To see this, note first that if we add all the constraints (6e) associated with singletons $S = \{u\}$ with $u \in V \setminus \{r\}$ we get

$$\sum_{u \in V \setminus \{r\}} \sum_{(u,v) \in \delta^+(u)} q_{uv} = \sum_{(u,r) \in \delta^-(r)} q_{ur} + \sum_{(u,v) \in A(V \setminus \{r\})} q_{uv} \geq n - 1.$$

Thus, we have,

$$\begin{aligned} n - 1 &\leq \sum_{(u,r) \in \delta^-(r)} q_{ur} + \sum_{(u,v) \in A(V \setminus \{r\})} q_{uv} \leq \\ &\sum_{(r,v) \in \delta^+(r)} q_{rv} + \sum_{(u,r) \in \delta^-(r)} q_{ur} + \sum_{(u,v) \in A(V \setminus \{r\})} q_{uv} = \sum_{(u,v) \in A} q_{uv} = \sum_{(u,v) \in E} x_{uv} = n - 1, \end{aligned}$$

where the last two equalities follow from constraints (6b) and (6a), respectively.

Hence, we can conclude that $\sum_{(u,v) \in \delta^+(r)} q_{uv} = 0$ and $\sum_{(u,v) \in \delta^+(u)} q_{uv} = 1$ for all $u \in V \setminus \{r\}$, since otherwise we would reach a contradiction.

The above remark indicates that the dicut constraints (6e) make the sets of constraints (6c) and (6d)

unnecessary. Hence, STP formulation which emanates from the above discussion is:

$$\mathcal{T}^{dc} : \quad \sum_{(u,v) \in E} x_{uv} = n - 1 \quad (7a)$$

$$q_{uv} + q_{vu} = x_{uv} \quad (u, v) \in E \quad (7b)$$

$$\sum_{(u,v) \in \delta^+(S)} q_{uv} \geq 1 \quad \emptyset \neq S \subset V \setminus \{r\} \quad (7c)$$

$$x_{uv} \geq 0 \quad (u, v) \in E \quad (7d)$$

$$q_{uv} \geq 0 \quad u, v \in V \quad (7e)$$

The reader may observe that formulation \mathcal{T}^{dc} can be readily transformed into the *directed cut* formulation of Magnanti and Wolsey (1995) by just changing the directions of the arcs of the arborescence and, thus, directing the arcs from the root r to the leaves, instead of from the leaves to the root. Since the directed cut formulation of Magnanti and Wolsey (1995) has the integrality property, so does formulation \mathcal{T}^{dc} . In its turn, this implies the integrality of the domain of \mathcal{T}^{km2} .

The number of dicut constraints (7c) is exponential on $|V|$. Nevertheless, they can be incorporated into the formulation only if needed via an efficient separation oracle, as they can be separated in polynomial time by finding the Gomory-Hu cut tree (Hu, 1974).

3.1.2 Comparison of formulations

Let $P(\mathcal{T}^{(\cdot)})$ denote the polyhedron associated with the linear programming relaxation of formulation $\mathcal{T}^{(\cdot)}$. Except for formulation \mathcal{T}^{sub} , all other formulations above are extended formulations, in the sense that, besides the design x variables, additional sets of variables are used. For comparing all the formulations in the same space we project the polyhedra associated with the extended formulations onto the space of the x variables, and denote by $P_x(\mathcal{T}^{(\cdot)})$ the projected polyhedron associated with formulation $\mathcal{T}^{(\cdot)}$.

Several of the formulations described above have the integrality property, namely formulations \mathcal{T}^{sub} , \mathcal{T}^{km} and \mathcal{T}^{km2} . This means that $P_x(\mathcal{T}^{sub}) = P_x(\mathcal{T}^{km}) = P_x(\mathcal{T}^{km2})$. In its turn, each of these formulations is tighter than any of the formulations without integrality property. That is, $P_x(\mathcal{T}^{km2}) \subset P_x(\mathcal{T}^{mtz})$ and $P_x(\mathcal{T}^{km2}) \subset P_x(\mathcal{T}^{flow})$. Below we compare $P_x(\mathcal{T}^{mtz})$ and $P_x(\mathcal{T}^{flow})$, as we have not seen such comparison in the literature.

The example of Figure 3 illustrates that $P_x(\mathcal{T}^{flow}) \not\subseteq P_x(\mathcal{T}^{mtz})$. The components of a x vector such that $\sum_{e \in E} x_e = n - 1$ are given next to each edge. Taking $r = 5$ as the root node, the flow $\varphi_{53} = \varphi_{54} = 2$, $\varphi_{31} = \varphi_{42} = 1$, together with x , define a feasible solution to formulation $P_x(\mathcal{T}^{flow})$. However, there is no feasible y vector that together with the depicted x vector satisfies constraints (3b) and (3c).

On the other hand, the example depicted in Figure 4 shows that $P_x(\mathcal{T}^{mtz}) \not\subseteq P_x(\mathcal{T}^{flow})$, i.e. the two formulations are not related in that there exist feasible solutions to $P_x(\mathcal{T}^{flow})$ that do not give rise to feasible solutions to $P_x(\mathcal{T}^{mtz})$ and the other way around.

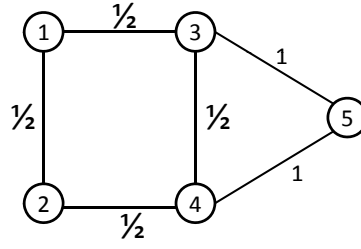


Figure 3: Fractional x solution with $\sum_{e \in E} x_e = n - 1$.

Consider a complete graph with $n = 5$ nodes and cost matrix:

$$C = \begin{pmatrix} 0 & 31 & 19 & 33 & 67 \\ 31 & 0 & 57 & 40 & 38 \\ 19 & 57 & 0 & 2 & 18 \\ 33 & 40 & 2 & 0 & 13 \\ 67 & 38 & 18 & 13 & 0 \end{pmatrix}$$

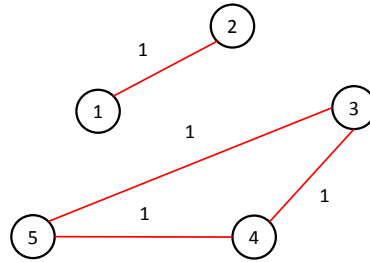


Figure 4: Solution x of \mathcal{T}^{mtz} formulation in the complete graph ($n = 5$) of the above example.

The optimal solution to the linear relaxation of \mathcal{T}^{mtz} is given by: $x_{12} = 1, x_{34} = 1, x_{35} = 1, x_{45} = 1;$ $y_{12} = 1, y_{34} = 0.5, y_{35} = 0.5, y_{43} = 0.5, y_{45} = 0.5, y_{53} = 0.5, y_{54} = 0.5;$ and $\ell_1 = 1, \ell_2 = 2, \ell_3 = 2, \ell_4 = 2, \ell_5 = 3.5$.

It is clear that the above solution to \mathcal{T}^{mtz} does not induce a feasible solution to \mathcal{T}^{flow} since the vector x does not produce a connected solution in the graph. Thus we have the following result:

Corollary 1.

$$P_x(\mathcal{T}^{sub}) = P_x(\mathcal{T}^{km}) = P_x(\mathcal{T}^{km2}) \subseteq \begin{cases} P_x(\mathcal{T}^{mtz}) \\ \neq \\ P_x(\mathcal{T}^{flow}) \end{cases}$$

3.2 Mixed Integer Linear Programming formulations for OWAP

This section presents the OWA formulation that we use for OWASTP. The choice is based on our preliminary experiments for STP and on previous results of Fernández et al. (2014), that show that this formulation outperforms other alternatives when the embedded combinatorial object is the shortest path or the perfect matching problem. In the formulations below we assume that we use the same polyhedron to represent the combinatorial object \mathcal{T} , namely the set of spanning trees. This is expressed as $x \in \mathcal{T}$.

Consider the following binary variables that define the specific positions in the ordering of the sorted

cost function values:

$$z_{ij} = \begin{cases} 1 & \text{if cost function } i \text{ occupies position } j, \\ 0 & \text{otherwise} \end{cases}$$

For each $j \in P$, let also θ_j be a variable representing the value of the objective function sorted at position j . Then, OWAP can be formulated as:

$$F^\theta : \quad V = \min \sum_{j \in P} \omega_j \theta_j \quad (8a)$$

$$s.t. \quad \sum_{i \in P} z_{ij} = 1 \quad j \in P \quad (8b)$$

$$\sum_{j \in P} z_{ij} = 1 \quad i \in P \quad (8c)$$

$$C^i x \leq \theta_j + M(1 - \sum_{k \geq j} z_{ik}) \quad i, j \in P \quad (8d)$$

$$\theta_j \geq \theta_{j+1} \quad j \in P : j < p \quad (8e)$$

$$x \in \mathcal{T} \quad (8f)$$

$$\theta_j \geq 0 \quad j \in P \quad (8g)$$

$$z \in \{0, 1\}^{p \times p} \quad (8h)$$

The objective function (8a) minimizes the weighted average of sorted objective function values, provided that θ_j , $j \in P$, are enforced to take on the appropriate values. Constraints (8b)–(8c) define a permutation of the cost functions, by placing one single cost function at each position and each cost function at one single position of the sequence. Constraints (8d) relate cost function values with the values placed in the sorted sequence. Constraints (8e) are optimality cuts which help the resolution of F^θ , as explained in Fernández et al. (2014).

For comparison purposes in our computational experiments, below we present the formulation used by Galand and Spanjaard (2012) for OWASTP. This formulation uses the above z binary variables plus an additional set of continuous variables $y = (y_{ij})_{i,j \in P} \in \mathbb{R}^{p \times p}$, where y_{ij} denotes the value of

cost function i if it occupies the j -th position in the ordering. The formulation is as follows:

$$F^{GS} : \quad V = \min \sum_{j \in P} \omega_j \sum_{i \in P} y_{ij} \quad (9a)$$

$$s.t. \quad \sum_{i \in P} z_{ij} = 1 \quad j \in P \quad (9b)$$

$$\sum_{j \in P} z_{ij} = 1 \quad i \in P \quad (9c)$$

$$\sum_{i \in P} y_{ij} \geq \sum_{i \in P} y_{ij+1} \quad j \in P : j < p \quad (9d)$$

$$y_{ij} \leq M z_{ij} \quad i, j \in P \quad (9e)$$

$$\sum_{j \in P} y_{ij} = C^i x \quad i \in P \quad (9f)$$

$$x \in \mathcal{T} \quad (9g)$$

$$y_{ij} \geq 0 \quad i, j \in P \quad (9h)$$

$$z \in \{0, 1\}^{p \times p} \quad (9i)$$

The objective function (9a) minimizes the weighted average of sorted objective function values. Constraints (9b)–(9c) are a copy of constraints (8b)–(8c) respectively, and thus define a cost function permutation. Constraints (9d) impose that the sorted values are ordered non-increasingly. Constraints (9e) relate cost function values with the values placed in the sorted sequence. Constraints (9f) ensure that one of the y_{ij} variables gives precisely the value of the objective function i .

Note that the relationship between θ in formulation F^θ and the y variables in F^{GS} is:

$$\theta_j = \sum_{i \in P} y_{ij} \quad j \in P : j > 1. \quad (10)$$

The reader should observe that F^θ and F^{GS} differ in the way we represent the OWA operator, although as mentioned at the beginning of this section the combinatorial object \mathcal{T} is represented by means of the same polyhedron.

Next, we prove two results concerning formulations F^θ and F^{GS} . Let us denote by Ω^θ and Ω^{GS} the domains defined by their respective sets of constraints. We first prove that F^θ and F^{GS} have the same set of optimal solutions although $\Omega^{GS} \subset \Omega^\theta$. This property no longer holds for the respective relaxations, where everything remains unchanged except for the z variables, which are allowed to take continuous values, i.e. $z_{ij} \geq 0$, $i, j \in P$. In particular, we will see that $\Omega_{LR}^{GS} \subset \Omega_{LR}^\theta$, where Ω_{LR}^θ and Ω_{LR}^{GS} denote the respective continuous relaxed domains. Moreover, in general, the sets of optimal solutions of the linear relaxations for the objective functions (8a) and (9a) do not coincide.

Property 2. *Every optimal solution to F^{GS} is also optimal to F^θ and conversely.*

Proof.

Given the relationship (10) between θ and y variables, in Ω^θ we can substitute Constraints (8d) by $C^i x \leq \sum_{i \in P} y_{ij} + M(1 - \sum_{k \geq j} z_{ik})$, $i, j \in P$.

- We prove first that $\Omega^{GS} \subseteq \Omega^\theta$, that is, every solution $(x, z, y) \in \Omega^{GS}$ (not necessarily optimal) is such that $(x, z, y) \in \Omega^\theta$. Observe that it suffices to prove that every $(x, z, y) \in \Omega^{GS}$, with y and θ related by (10), satisfies

$$C^i x \leq \sum_{i \in P} y_{ij} + M(1 - \sum_{k \geq j} z_{ik}) \quad i, j \in P. \quad (9d')$$

Let \hat{x} be a feasible solution in \mathcal{T} and \hat{z} a permutation that sorts the cost function values of x . Then, for fixed \hat{x} and \hat{z} values there is a unique \hat{y} since, according to (9e)–(9f) there is at most one $j \in P$ such that $y_{ij} \neq 0$ for each $i \in P$. It follows that such $(\hat{x}, \hat{z}, \hat{y})$ verifies (9d').

- Next we prove that every optimal solution of F^θ , satisfies that $(x, z, y) \in \Omega^{GS}$, after performing the change of variable given by (10). For this, it is sufficient to prove that every optimal solution $(x, z, y) \in \Omega^\theta$ verifies (9e) and (9f). Let \hat{x} be a feasible solution in \mathcal{T} . Then, there exists a permutation \hat{z} that sorts the cost function values of \hat{x} in non increasing order. Now, we give values to the $\hat{\theta}$ variables according to this ordered sequence, and we determine the \hat{y} values by means of $\hat{y}_{ij} = \hat{\theta}_j \hat{z}_{ij}$ $i, j \in P$. From here, it follows that $(\hat{x}, \hat{z}, \hat{y})$ verifies (9d)–(9f). In addition, we note that, in general, for fixed \hat{x} and \hat{z} , the polyhedron given by (8d)–(8e) is unbounded and thus $\Omega^\theta \not\subseteq \Omega^{GS}$.

□

Property 3. $\Omega_{LR}^{GS} \subset \Omega_{LR}^\theta$.

Proof.

First, we observe that $\Omega_{LR}^\theta \not\subseteq \Omega_{LR}^{GS}$ since, otherwise, the optimal solution of F_{LR}^θ for the graph in Example 1 (with value 8.6 when $\omega = (0.8, 0, 0.2)$) could not have a smaller value than the optimal solution of F_{LR}^{GS} (with value 9.4).

Next, we prove that every feasible solution $(x, z, y) \in \Omega_{LR}^{GS}$ is such that $(x, z, y) \in \Omega_{LR}^\theta$, once the change of variable given by (10) is done.

Indeed, we have to prove that any $(x, z, y) \in \Omega_{LR}^{GS}$ verifies

$$C^i x \leq \sum_{i \in P} y_{ij} + M(1 - \sum_{k \geq j} z_{ik}) \quad i, j \in P. \quad (9d')$$

Let \hat{x} be a feasible solution in \mathcal{T} and \hat{z} a fractional vector. Since $C^i x = \sum_{i \in P} y_{ij}$ and $(1 - \sum_{k \geq j} z_{ik})$ is greater than or equal to zero, it is clear that constraint (9d') is verified.

□

The above result proves that the linear relaxation of Ω^{GS} is stronger than that of Ω^θ , although the two formulations share the same optimal integer values. Nevertheless, as we shall show in the computational experiments, formulation F^θ provides much better results in terms of running times and number of optimal solutions found. The reason may be the smaller number of variables used in the second formulation.

To conclude this section we state the relationships between the different formulations that derived from the combination of some OWA representation and any of STP polytopes described above. To this end, let us denote by $P_{xz}(\Omega^{(\cdot)})$ the projection onto the space of the x, z variables of the linear relaxation of an OWA polytope built on the corresponding $\mathcal{T}^{(\cdot)}$ polytope for STP. The following property states the relationships among them.

Property 4.

$$P_{xz}(\Omega^{sub}) = P_{xz}(\Omega^{km}) = P_{xz}(\Omega^{km2}) \subseteq \begin{cases} P_{xz}(\Omega^{flow}) \\ \neq \\ P_{xz}(\Omega^{mtz}) \end{cases} \quad (11)$$

3.2.1 Enhancements and valid inequalities for OWAP

Formulation F^θ admits other enhancements like removing some redundant variables, adding valid inequalities, etc. First, we observe that since system (8b)–(8c) contains exactly $2p - 1$ linearly independent equations, the above permutation can also be represented without variables z_{i1} , for all $i \in P$, which can be replaced by $1 - \sum_{j \in P: j > 1} z_{ij}$. In this way, system (8b)–(8c) can also be rewritten as

$$\sum_{i \in P} z_{ij} = 1 \quad j \in P : j > 1, \quad (12)$$

$$\sum_{j \in P} z_{ij} \leq 1 \quad i \in P. \quad (13)$$

Second, constraints (8c) and (8e) can be removed from F^θ without changing the set of optimal solutions. We denote by F_{R1}^θ formulation $F^\theta \setminus \{(8e)\}$ and by F_{R2}^θ formulation $F^\theta \setminus \{(8c), (8e)\}$. Note that formulations F_{R1}^θ and F_{R2}^θ admit some solutions that are infeasible in F^θ (e.g. a solution where $\theta_j \leq \theta_{j+1}$ for some j). However, these two new formulations have fewer constraints and could be more efficient in a branch-and-bound algorithm.

Finally, we present some valid inequalities that can be added to the above OWAP in order to improve the bound of the linear relaxation and/or to reduce the search space in the branch-and-bound tree.

- *Constraints related to bounds of cost function values.* Let $l_i(u_i)$ denote the minimum (maximum) objective value relative to cost function $i \in P$, respectively. It is clear that $l_i(u_i)$ are valid lower (upper) bounds on the value of objective i , independently of the position of cost function i in the ordering. Therefore we obtain the following two sets of constraints which are valid for OWAP:

$$l_i \leq C^i x \leq u_i \quad i \in P \quad (14)$$

- *Constraints related to bounds of values in specific positions.* Let $l_j^\pi(u_j^\pi)$ denote the j -th lowest (largest) value of $l_i(u_i)$. Then, $l_j^\pi(u_j^\pi)$ is a valid lower (upper) bound of the objective function sorted in position j , that is

$$l_j^\pi \leq \theta_j \leq u_j^\pi \quad j \in P \quad (15)$$

- There are also different bounds on the value of the cost function i and the value of the cost function sorted in position j :

$$\sum_{j \in P} \max\{l_i, l_j^\pi\} z_{ij} \leq C^i x \leq \sum_{j \in P} \min\{u_i, u_j^\pi\} z_{ij} \quad i \in P \quad (16)$$

4 Computational experience

Next, we report on the results of some computational experiments we have run, in order to compare empirically the proposed formulations and reinforcements. We have studied OWASTP combining the different formulations proposed for STP and OWAP. First of all we have chosen the best formulation, according to Fernández et al. (2014), among those proposed for OWAP, namely F_{R2}^θ . We recall that the goal of this paper is the analysis of some STP formulations within the OWAP framework.

In our computational experience we study three particular cases of the OWA operator already considered in multiobjective optimization (see, e.g., Galand and Spanjaard, 2012). We study first the k -centrum criterion (Tamir, 2000) that evaluates the sum of the k greatest objective functions. This criterion is monotonic and convex since the sorting weights ($w_1 = \dots = w_k = 1/k$, $w_{k+1} = \dots = w_p = 0$) are in decreasing order. For that reason this operator can be modelled using the linearization of the OWA function with monotonic weights given by Ogryczak and Sliwinski (2003) and Ogryczak and Tamir (2003), which avoids the use of binary variables. Then, we study two non-monotonic and non-convex criteria namely the k -trimmed mean (Galand and Spanjaard, 2012) defined as $\sum_{i=k+1}^{p-k} (p-2k)^{-1} y_i$ and the Hurwicz criterion (Hurwicz, 1951), defined as $\alpha \max_{i \in P} y_i + (1-\alpha) \min_{i \in P} y_i$. These criteria are of special interest for being non-convex since the sorting weights ($w_1 = \alpha$, $w_2 = \dots = w_{p-1} = 0$, $w_p = 1-\alpha$ and $w_1 = \dots = w_k = 0$, $w_{k+1} = \dots = w_{p-k-1} = 1/(p-2k)$, $w_{p-k} = \dots = w_p = 0$, respectively) are not in non-increasing order (Grzybowski et al., 2011, Puerto and Tamir, 2005).

For our computational experiments we have followed the design of Galand and Spanjaard, 2012. Thus, the number of objectives ranges in $|P| \in \{5, 8, 10\}$ and the considered values of k, α are

- k -centrum: $|P| = 5$ and $k \in \{1, 3, 4\}$, $|P| = 8$ and $k \in \{2, 4, 7\}$, $|P| = 10$ and $k \in \{3, 5, 8\}$,
- Hurwicz: $|P| \in \{5, 8, 10\}$ and $\alpha \in \{0.4, 0.6, 0.8\}$,
- k -trimmed: $|P| = 5$ and $k \in \{1, 2\}$, $|P| = 8$ and $k \in \{2, 3\}$.

Graphs are complete with $|V| \in \{20, 30, 40, 50, 60, 70, 80, 100\}$ and the components of the cost vectors randomly drawn from a uniform distribution on $[1, 100]$. In addition, for each selection of the parameters $(|V|, p)$, 10 instances were randomly generated. All instances were solved with the MIP Xpress 7.7 optimizer, under a Windows 7 environment in an Intel(R) Core(TM)i7 CPU 2.93 GHz processor and 16 GB RAM. Default values were initially used for all parameters of Xpress solver and a time limit of 3600 seconds was set. We have also tested different combinations of parameters for the solver cut strategy and intensity of heuristics but, unless it is specified, the best results were obtained with the parameters of the solver set to the default values.

Throughout the section F^{GS} denotes the formulation of Galand and Spanjaard (2012) for OWASTP. Otherwise, we denote by $F^{(\cdot)}$ the combination of the OWA F_{R2}^θ formulation together with a STP $\mathcal{T}^{(\cdot)}$. We report results of formulations F^{GS} , F^{km} , F^{cut} , F^{mtz} , F^{flow} , and F^{km2} .

The separation of the cutset inequalities in formulation F^{cut} was implemented using a max-flow based algorithm (Gusfield, 1990). Heuristics in Xpress solver were configured with intensity 2 (out of 3) and an initial solution was given to the problem. The initial solution was the minimum cost spanning tree obtained using as edge costs the average costs among all objectives.

We have summarized the results in five sets of three tables each. In each set, the first table refers to OWA 1 (k -centrum), the second one to OWA 2 (Hurwicz) and the third one to OWA 3 (trimmed mean), respectively.

Tables 1.c, $c \in \{1, 2, 3\}$ show results with OWA c for formulations F^{GS} , F^{km} , F^{cut} , F^{mtz} , F^{flow} , and F^{km2} . Tables 2.c, $c \in \{1, 2, 3\}$ give results of the best strengthening for F^{km2} , which consist of inequalities (14), (15) and (16). Analogously, Tables 3.c, $c \in \{1, 2, 3\}$ report the same information but referred to F^{mtz} . Tables 4.c, $c \in \{1, 2, 3\}$, display a comparison of our two best formulations with the results reported in Galand and Spanjaard (2012). The results of a last series of experiments with larger graphs with up to $|V| = 100$ nodes and with up to $|P| = 10$ objectives are presented in Tables 5.c, $c \in \{1, 2\}$. We do not give results of OWA 3 (k -trimmed) since Table 4.3 indicates that the instance sizes limit for this criterion is already reached for $|P| = 8$ objectives.

In order to facilitate the comparison among all tables, best results in each table are marked in bold. In all tables each row summarizes the results corresponding to a group of instances with the same parameters ($|P|, |V|, \alpha$). Columns are grouped in blocks. The first block contains three columns with the values of the instances parameters. In Tables 1.c, 2.c and 3.c, $c \in \{1, 2, 3\}$ rows correspond to groups of 10 instances. The first block of columns is followed by a block of four columns for each tested formulation. The columns of each such block are the following. Columns $gapR$ give the percentage relative gap at the root node, computed as $100(z^* - z_R)/z_R$, where z^* denotes the value of the best solution found and z_R the optimal value of the linear relaxation at the root node. Columns $t/gap(\#)$ report average computing times in seconds over the 10 instances of the row (denoted by t). When t is smaller than 10 seconds we report one additional precision digit. If at least one instance reaches the time limit, the number of instances in the group solved to optimality within the time limit is given in brackets ($\#$). In such a case, t is computed using the time limit for all unsolved instances. If no instance was solved to optimality, instead of t we report the average optimality gap relative to the lower bound at termination over all the instances of the group, (denoted by gap). Columns t^*/gap^* show the maximum computing time in seconds over the 10 instances of the row (t^*). If at least one instance reached the time limit, instead of t^* we report the maximum optimality gap, over all the instances of the group (denoted by gap^*). Finally, columns nod indicate the average number of nodes explored in the branch-and-bound tree in compact format denoting $a * 10^n$ as aen . The caption just below each block gives the formulation the block refers to.

Tables 4.c, $c \in \{1, 2, 3\}$, display the results obtained and reported in Galand and Spanjaard (2012) for OWASTP and our two best formulations. Results provided by Galand and Spanjaard (2012) correspond to minimum, average and maximum running times for groups of 30 instances with the same parameters ($|P|, |V|$). An entry with the symbol “-” indicates that the average execution time was beyond 15min (900s). Block F^{GS} shows the results reported by Galand and Spanjaard (2012) in IBM ILOG CPLEX 12 without any preprocessing whereas, blocks F_{P1}^{GS} and F_{P2}^{GS} give the running times after applying two different preprocessings (shavings) described in that paper. In that case, all the algorithms were implemented in C++ and were run on either an Intel Xeon 2.5GHz personal computer with a memory of 4GB for five objectives, or an Intel Core 3.0 GHz personal computer with a memory of 8GB for eight objectives. For the sake of comparison and according with <https://www.spec.org/cgi-bin/osgresults>, the performance indices of the Intel Core I7 3GHz, used in Galand and Spanjaard (2012) and the Intel(R) Core(TM)i7 CPU 2.93 GHz, used in our computational results, are rather similar; whereas the performance of the Intel Xeon 2.5GHz is slightly inferior.

The meaning of the columns of Tables 5.c, $c \in \{1, 2\}$ is like in the first three sets of tables.

All data instances and disaggregated results of all experiments are available via e-mail upon request to the authors.

4.1 Tests with the k -centrum criterion

In Table 1.1, the results of block F^{GS} exhibit small values of gapR for all instance sizes. However, many of these instances remain unsolved after 1 hour of CPU time leaving integrality gaps around 1%. The results of block F^{km} show that, according to the low average number of explored nodes in the B&B tree, solving the LP relaxation of the problem becomes quite hard so the corresponding gaps at termination remain quite large in comparison with F^{GS} . We recall that F^{km} uses $O(n^3)$ variables and constraints, which can be too high in large graphs. The results of block F^{cut} indicate that, in general, this formulation is outperformed by all formulations. Block F^{flow} behaves similarly to F^{GS} which suggests that flow MST formulations embedded within an OWA framework for the k -centrum criterion does not exhibit a good performance. On the contrary, block F^{mtz} shows the best performance in terms of number of instances solved to optimality, running times and optimality gaps. Block F^{km2} shows a good performance but not as good as F^{mtz} .

Next we report on the three most promising strengthening found, which consist on adding valid inequalities (14), (15) to formulations F^{km2} and F^{mtz} . Note that reinforcement (16) cannot be applied to the k -centrum criteria since we are using here the formulation of Ogryczak and Tamir (2003) that does not require binary variables z_{ij} . Table 2.1 shows that the performance of F^{km2} is slightly improved when constraints (15) are added. On the contrary, Table 3.1 shows that the performance of F^{mtz} is not improved in general adding any of these valid inequalities.

Figure 5 summarizes the main results of Tables 1.1, 2.1 and 3.1 as follows. To better illustrate the results we have shown in a single figure the information relative to computing time and optimality gap at termination. For this, we have defined the *Time/Gap (TG) performance index* of an instance as $0.5(\frac{time}{3600} + \frac{gap}{100})$. This index is represented in a plot where the horizontal axis depicts the size of the instances, namely $(|P|, |V|)$, and the vertical axis corresponds to the scaled values of the index. The reader should observe that whenever for a given instance the gap at termination is null this index reflects its computing time and is always represented below the horizontal line $(3600, 0\%)$ because the corresponding instance must have been solved within the time limit. Analogously, a point above the horizontal line $(3600, 0\%)$, indicates an instance with a strictly positive gap at the time limit. The average TG performance index of a set of instances is the average of their TG performance values and the worst TG index is the maximum TG value among the instances in the set. For each instance size $(|P|, |V|)$, Figures 5(a) and (b) plot the average and worst TG performance index, respectively. Note that this index has been scaled on the y-axis in order to illustrate a measure with the time and gap obtained. We focus on the comparison with the formulation that already exists in the literature, F^{GS} , and those with a better performance in our study. From Figure 5 (a) we conclude that the best performance for OWASTP is obtained with formulations F^{mtz} and $F^{km2} + (15)$, being $F^{km2} + (15)$ slightly outperformed by F^{mtz} . In terms of TG index, Figure 5 (b) shows that F^{mtz} is more stable with 5 objectives but the maximum TG value is stabilized for instances of 8 objectives and $|V| = \{50, 60\}$.

Table 4.1 shows that, in nearly all cases, F^{mtz} and $F^{km2} + (16)$ have a better performance than F^{GS} , $F_{P_1}^{GS}$ and $F_{P_2}^{GS}$ (Galand and Spanjaard, 2012).

Table 5.1 shows our results with formulations F^{mtz} and $F^{km2} + (16)$ for larger graphs of sizes up to 100 nodes and with up to 10 objectives. These instances are larger than the largest ones reported so far in the literature with the k -centrum criterion, which, to the best of our knowledge have up to 60 nodes and up to 8 objectives (Galand and Spanjaard, 2012). We can observe that, when $|V| \geq 80$, after 1h of computing time there are already some unsolved instances. Nevertheless, the performance of F^{mtz} and $F^{km2} + (16)$ is remarkable, as the biggest gaps at termination are always around 1%.

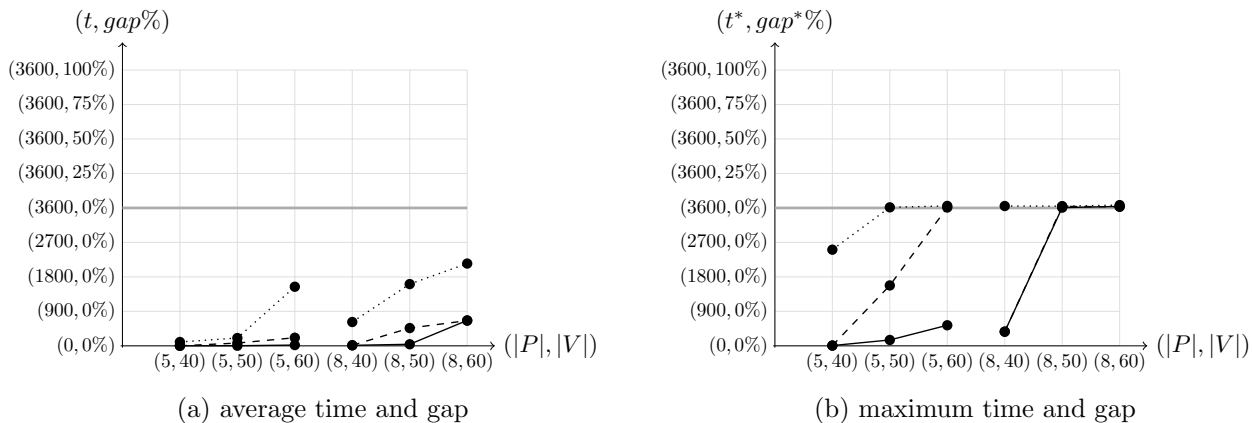


Figure 5: TG performance profiles for each instance size $(|P|, |V|)$, k -centrum criterion and some selected formulations: F^{GS} (dotted line), F^{mtz} (solid line), $F^{km2} + (15)$ (dashed line).

4.2 Tests with the Hurwicz criterion

In Table 1.2, the results of block F^{GS} indicate that the OWASTP formulation of Galand and Spanjaard (2012) produces the smallest gaps at the root node ($gapR$), although only few instances could be solved to optimality, and the gaps remaining at termination (gap^*) are outperformed by most of the other formulations. The results of block F^{km} show that the number of instances solved to optimality is higher than that of F^{GS} , although the gaps in the unsolved instances are higher. We recall that F^{km} uses $O(n^3)$ variables and constraints, which can be too high in large graphs. This also explains the low average number of explored nodes in the B&B tree. The results of block F^{cut} indicate that, in general, this formulation outperforms both F^{GS} and F^{km} . As can be seen in block F^{flow} this formulation improves the average running times and gaps of F^{cut} but the maximum optimality gaps at termination are still competitive in F^{cut} against F^{flow} . The blocks F^{mtz} and F^{km2} show the best performance in terms of number of instances solved to optimality, running times and optimality gaps.

Table 2.2 shows that the performance of F^{km2} is improved when constraints (16) are added. On the contrary, Table 3.2 shows that the performance of F^{mtz} is almost not improved adding any of these valid inequalities.

Figure 6 summarizes, in terms of the TG index, the main results of Tables 1.2, 2.2 and 3.2. From Figure 6 (a) we conclude that the best performance for OWASTP is obtained with formulations F^{mtz} and $F^{km2} + (16)$, being F^{mtz} slightly outperformed by $F^{km2} + (16)$. In terms of maximum TG index, Figure 6 (b) shows the best performance for F^{mtz} and $F^{km2} + (16)$.

In Table 4.2 our results in columns F^{mtz} and $F^{km2} + (16)$ show a better performance as compared with the results reported in Galand and Spanjaard (2012) in almost all cases. Table 4.2 shows that, in nearly all cases, F^{mtz} and $F^{km2} + (16)$ have a better performance than F^{GS} , F_{P1}^{GS} and F_{P2}^{GS} .

Table 5.2 shows our results with formulations F^{mtz} and $F^{km2} + (16)$ with respect to the Hurwicz criterion with larger graphs of sizes up to 100 nodes and with up to 10 objectives. To the best of our knowledge the largest OWASTP instances reported so far in the literature with this criterion have up to 60 nodes and up to 5 objectives. We can observe that, when $|V| \geq 80$, after 1h of computing time there are still some unsolved instances. Nevertheless, the performance of F^{mtz} and $F^{km2} + (16)$ is remarkable, as the biggest gaps at termination are always below 1%.

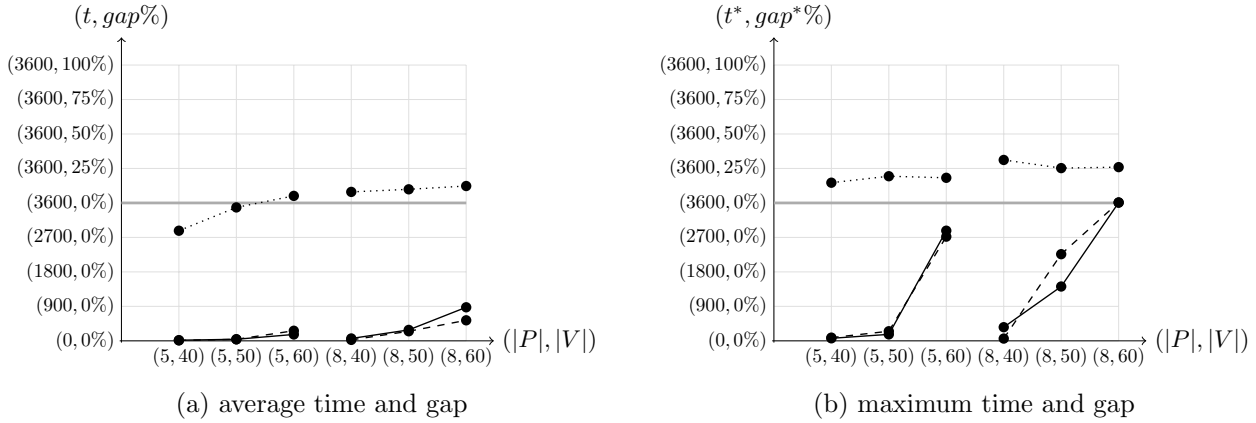


Figure 6: TG performance profiles for each instance size $(|P|, |V|)$, Hurwicz criterion and some selected formulations: F^{GS} (dotted line), F^{mtz} (solid line), $F^{km2} + (16)$ (dashed line).

4.3 Tests with the k -trimmed criterion

The results of the k -trimmed criterion indicate that this is the hardest objective function as compared to the k -centrum and Hurwicz. This difficulty lies on the fact that the LP bound at the root node of the B&B tree is rather poor (zero in nearly all cases) which produces percentage optimality gaps at the root node ($gapR$) of almost 100% for all formulations without reinforcements.

Table 1.3 shows that for $|P| = 5$ the best results in terms of average and maximum gaps at termination are obtained with F^{GS} . In contrast the results of block F^{km} indicate that solving the LP relaxation of the instances in less than 1 hour of computing time, becomes nearly impossible (observe that, in general, the number of explored nodes in the B&B tree is very small). Once again, we recall that F^{km} uses $O(n^3)$ variables and constraints, which can be too demanding for large graphs. The results of block F^{cut} indicate that, in general, this formulation outperforms both F^{GS} and F^{km} for $|P| = 8$. As can be seen, formulations F^{flow} and F^{mtz} exhibit a similar performance, which is not competitive against F^{GS} and F^{km2} . Formulation F^{km2} shows the best performance for $|P| = 8$ in terms of number of instances solved to optimality, running times and optimality gaps, and the second best for $|P| = 5$, just after F^{GS} .

Table 2.3 shows that the performance of F^{km2} for $|P| = 8$ improves when constraints (14) are added to strengthen the formulation. On the contrary, the best performance for F^{km2} and $|P| = 5$ is obtained for $F^{km2} + (16)$. We can observe that both F^{GS} and $F^{km2} + (16)$ produce the best results for the k -trimmed criterion and $|P| = 5$. Table 3.3 shows that the performance of F^{mtz} only improves when constraints (14) are added, but this improvement is not enough to outperform F^{GS} or $F^{km2} + (16)$.

Figure 7 summarizes in terms of the TG index, the main results of Tables 1.3, 2.3 and 3.3 as follows. From Figure 7 (a) we conclude that the best performance for OWASTP and $|P| = 5$ is obtained with formulations F^{GS} and $F^{km2} + (16)$, being $F^{km2} + (16)$ slightly outperformed by F^{GS} . When the number of objectives increases to $|P| = 8$, the best performance is attained by $F^{km2} + (16)$. Similarly, with respect to worse cases (Figure 7(b)), F^{GS} produces the best the TG index for $|P| = 5$ and $F^{km2} + (16)$ for $|P| = 8$.

Finally, we observe in Table 4.3 that, in contrast with the results obtained for the k -centrum and Hurwicz criteria, formulations $F^{mtz} + (14)$ and $F^{km2} + (16)$ do not outperform the results reported by Galand and Spanjaard (2012) for the k -trimmed criterion. The reader may note that already for

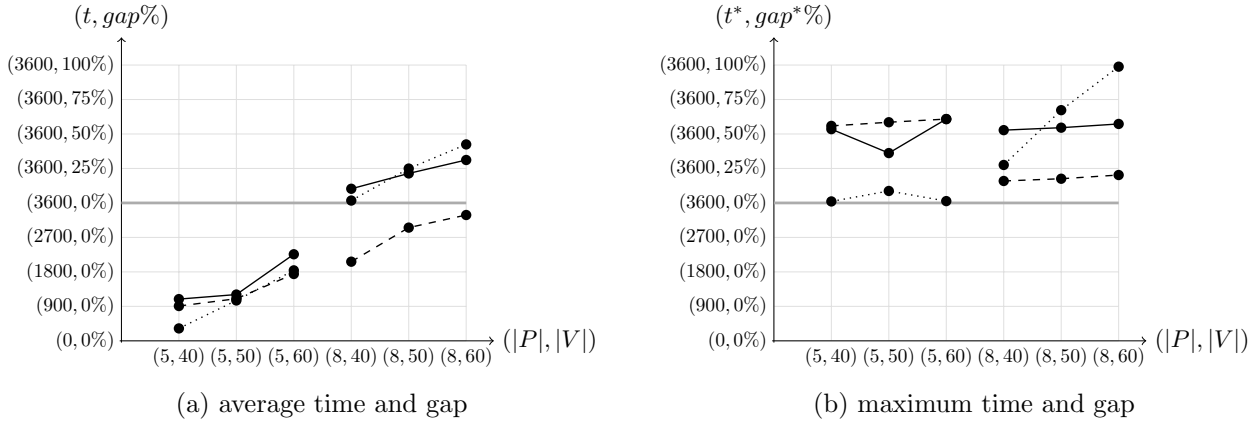


Figure 7: TG performance profiles for each instance size $(|P|, |V|)$, k -trimmed criterion and some selected formulations: F^{GS} (line dotted), F^{mtz} (line solid), F^{km2} (line dashed).

$|P| = 8$ there are several k -trimmed instances which could not be solved in one hour (see Table 4.3). This shows that $|P| = 8$ is the size limit of the instances that can be solved in one hour. This has been the reason for not including a third table with larger instances for the k -trimmed criterion.

4.4 Performance summary

In summary, we observe that the performance of the OWAST formulations largely depend on the choice for the objective function (that is, the weights vector). In particular we conclude, from our computational experience, that for the k -centrum and $|P| \in \{5, 8\}$, it is convenient to apply formulations F^{mtz} and $F^{km2} + (15)$, being $F^{km2} + (15)$ slightly outperformed by F^{mtz} . When $|P| = 10$ the performance of both F^{mtz} and $F^{km2} + (16)$ is remarkable, as the biggest gaps at termination are always around 1%. The conclusion for the Hurwicz criterion is different, since the best performance for OWASTP is obtained with formulations F^{mtz} and $F^{km2} + (16)$, being F^{mtz} slightly outperformed by $F^{km2} + (16)$. When $|P| = 10$ the performance of both F^{mtz} and $F^{km2} + (16)$ is remarkable, as the biggest gaps at termination are always around 1%. However, in this case, F^{mtz} is outperformed by $F^{km2} + (16)$. Finally, we conclude that the best performance for the k -trimmed criterion and $|P| = 5$ is obtained with formulations F^{GS} and $F^{km2} + (16)$, being $F^{km2} + (16)$ slightly outperformed by F^{GS} . When the number of objectives increases to $|P| = 8$, the best performance is attained by $F^{km2} + (16)$. Therefore, we cannot conclude that a specific formulation is superior to all the others regardless of the objective function considered. This reinforces the interest of the catalog of formulations and valid inequalities, and the extensive numerical results from computational experiments presented in this paper.

5 Conclusions

In this paper we have presented reinforced mathematical programming formulations for OWASTP as well as alternative new formulations which reduces the number of decision variables. These new formulations reinforced with appropriate constraints have shown to be very effective for efficiently solving medium size OWASTP instances. However, from the obtained results it is also clear that for solving larger OWASTP instances with more objective functions further improvements are needed.

Our current research focuses on the design of more sophisticated elimination tests as well as on alternative formulations leading to tighter LP bounds.

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Table 1.1: Results for the k -centrum criterion

$ P \setminus k $	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$						
5 40 1	3.0	429	2510	3e5	3.6	184	404	3e3	18.2	3449(1)	33.4%	5e6	3.0	479(9)	0.1%	3e5	7.2	3e3	0.7	2.1	6.2	5e3
5 40 3	2.7	192	1034	1e5	0.3	148	456	4e2	10.0	3345(1)	26.7%	3e6	2.7	137	769	1e5	2.0	3e3	0.4	1.4	6.4	2e3
5 40 4	2.3	8.6	56	8e3	0.1	42	384	15	2.8	3241(1)	6.1%	3e6	2.3	8.8	65	7e3	2.8	2.0	0.2	0.5	1.7	2e2
5 50 1	2.8	1200(8)	0.4%	3e5	3.2	1224	2171	5e3	29.2	28.9%(0)	32.8%	3e6	2.8	429	1624	3e5	0.5	22	0.6	417(9)	0.5%	6e5
5 50 3	2.6	262	1658	6e4	6.0	819(9)	27.6%	6e2	15.5	15.2%(0)	31.6%	3e6	2.6	7.2	268	5e4	0.4	4.4	0.5	173	1636	3e5
5 50 4	2.4	146	1345	6e4	0.1	165	1591	16	3.8	3291(1)	6.8%	2e6	2.4	20	137	2e4	0.2	1.5	0.2	3.2	19	2e3
5 60 1	3.0	3386(1)	1.5%	5e5	23.3	2939(2)	32.0%	2e3	29.3	29.2%(0)	33.2%	2e6	2.9	2502(4)	1.1%	1e6	0.5	75	0.6	694(9)	0.2%	7e5
5 60 3	2.7	2331(5)	0.8%	3e5	19.9	2532(3)	30.5%	2e2	24.9	24.7%(0)	33.0%	2e6	2.7	1857(6)	0.7%	6e5	0.4	39	0.4	295	2536	3e5
5 60 4	2.4	744(9)	0.5%	2e5	3.2	369(9)	31.7%	4	3.9	3.5%(0)	6.3%	2e6	2.4	430(9)	0.2%	3e5	0.2	1.3	0.3	8.3	57	5e3
8 40 2	2.4	3085(3)	1.4%	1e6	5.2	2297(6)	0.5%	3e4	16.0	15.5%(0)	24.8%	4e6	2.3	3256(3)	0.9%	2e6	0.9	146	0.9	182	639	2e5
8 40 4	2.1	1230(8)	0.5%	5e5	0.6	956	3125	2e4	21.0	10.5%(0)	20.9%	5e6	2.1	1051(8)	0.7%	6e5	0.7	25	0.7	47	256	6e4
8 40 7	1.4	12	77	7e3	0.0	6.0	13	8	11.0	3437(1)	4.2%	2e6	1.4	7.4	36	6e3	0.1	1.1	0.1	1.1	4.6	5e2
8 50 2	2.1	3424(1)	1.3%	5e5	5.4	3043(2)	1.4%	6e3	17.5	17.2%(0)	27.7%	9e6	2.0	3293(1)	0.7%	9e5	0.7	849(9)	0.3%	6e5	0.9%	1e6
8 50 4	1.8	2754(4)	0.9%	5e5	0.2	2085(9)	0.1%	2e4	13.8	13.4%(0)	25.7%	9e6	1.8	2185(5)	0.6%	7e5	0.5	79	503	0.8%	7e5	7e5
8 50 7	1.4	460(9)	0.0%	1e5	0.1	207	1949	62	2.4	2.0%(0)	4.7%	8e6	1.4	102	718	7e4	0.1	3.3	0.2	53	387	6e4
8 60 2	2.4	0.8%(0)	2.1%	3e5	25.3	3490(1)	29.8%	0	24.9	24.7%(0)	29.2%	5e6	2.1	0.5%(0)	1.1%	9e5	0.6	1832(6)	1.1%	8e5	0.7%	1e6
8 60 4	2.1	0.5%(0)	1.2%	3e5	11.7	11.6%(0)	26.0%	8e3	21.2	21.1%(0)	28.3%	5e6	2.0	3442(1)	0.9%	8e4	0.5	1159(7)	0.7%	5e5	1.7%	7e5
8 60 7	1.6	330	2388	7e4	3.0	375(9)	29.7%	17	2.8	2.5%(0)	5.4%	5e6	1.6	114	842	4e4	0.1	5.4	0.1	192	1744	1e5

Table 1.2: Results for the Hurwicz criterion

$ P \setminus k $	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$								
5 40 0.4	41.5	3303(1)	14.7%	5e6	37.5	3102(2)	48.9%	1e4	48.5	350	1629	3e3	48.5	108	707	1e5	47.3	16	66	1e3	21	114	3e3	
5 40 0.6	26.9	2432(4)	13.5%	4e6	46.0	2366(4)	36.1%	1e4	37.4	140	491	1e3	37.3	19	46	3e4	35.9	13	40	1e3	40	1e3	3e3	
5 40 0.8	8.2	2301(4)	3.6%	2e6	2.6	3043(2)	19.7%	5e3	22.2	685	2807	6e3	22.2	83	371	8e4	20.4	16	20	2e3	22	47	3e3	
5 50 0.4	40.4	3207(2)	19.4%	3e6	48.0	3113(2)	49.6%	5e3	48.8	429	945	2e3	48.8	239	816	2e5	47.6	33	70	2e3	49	163	3e3	
5 50 0.6	27.2	3276(1)	13.3%	4e6	36.5	3337(1)	37.8%	5e3	37.5	820(9)	0.1%	3e3	37.5	772(9)	0.1%	6e5	36.0	34	73	6e3	36.1	48	191	3e3
5 50 0.8	7.4	3310(2)	1.6%	2e6	20.3	2542(5)	20.0%	4e3	22.2	1661(8)	0.4%	6e3	22.1	512	2560	3e5	20.5	55	169	2e4	97	354	9e3	
5 60 0.4	41.6	13.3%(0)	18.3%	2e6	48.3	3086(4)	50.3%	3e3	49.2	1728(7)	0.5%	3e3	49.2	1470(7)	1.0%	7e5	47.9	81	249	2e4	48.0	163	703	4e3
5 60 0.6	27.4	3312(1)	12.2%	2e6	36.4	3341(2)	37.3%	2e3	37.8	1881(6)	0.3%	3e3	37.8	1334(7)	0.5%	6e5	36.3	58	82	4e3	36.3	170	617	6e3
5 60 0.8	8.6	1.6%(0)	4.0%	2e6	18.9	3572(1)	20.1%	2e5	22.3	3383(2)	0.9%	5e3	22.2	2822(4)	0.9%	1e6	20.3	361	2875	2e5	20.4	787(9)	0.3%	4e4
8 40 0.4	35.4	6.4%(0)	13.8%	4e6	41.7	41.5%(0)	43.1%	1e4	39.4	1590	3188	1e4	39.4	679(9)	1.2%	5e5	38.5	51	202	3e4	38.5	42	165	1e4
8 40 0.6	26.5	10.1%(0)	31.2%	3e6	32.2	31.9%(0)	34.1%	2e4	30.3	1727(9)	0.3%	2e4	30.2	718(9)	0.4%	5e5	29.2	70	356	5e4	29.2	52	1e4	
8 40 0.8	14.7	7.5%(0)	10.7%	3e6	19.8	19.5%(0)	20.1%	2e4	19.8	1367(8)	0.9%	8e5	19.8	1367(8)	0.9%	8e5	18.6	64	212	5e4	18.6	57	146	2e4
8 50 0.4	36.5	9.4%(0)	12.9%	2e6	42.1	42.0%(0)	46.2%	5e3	40.4	3021(3)	1.2%	1e4	40.5	3021(2)	1.9%	1e6	39.6	179	1079	1e5	39.7	179	461	5e4
8 50 0.6	27.2	11.5%(0)	25.3%	2e6	32.1	32.1%(0)	33.9%	5e3	31.3	0.5%(0)	1.1%	1e4	31.4	2897(4)	2.4%	2e6	30.3	277	1226	1e5	30.4	288	1233	1e5
8 50 0.8	14.9	8.8%(0)	12.6%	2e6	20.1	20.0%(0)	20.4%	6e3	20.1	0.4%(0)	0.9%	1e4	20.1	2353(5)	0.5%	9e5	19.0	363	1418	2e5	19.1	496	2625	2e5
8 60 0.4	36.5	12.5%(0)	14.8%	1e6	41.3	41.2%(0)	49.0%	2e3	39.6	0.6%(0)	1.0%	6e3	39.8	3310(1)	3.5%	1e6	38.5	296	517	9e4	38.5	358	2050	9e4
8 60 0.6	27.2	13.4%(0)	26.0%	1e6	32.1	32.0%(0)	34.8%	2e3	30.8	0.5%(0)	0.9%	6e3	31.0	0.9%(0)	2.3%	1e6	29.6	766(9)	0.1%	2e5	29.6	527	1470	2e5
8 60 0.8	15.7	10.4%(0)	13.7%	1e6	19.8	19.8%(0)	20.7%	3e3	20.1	0.5%(0)	0.7%	7e3	20.1	0.4%(0)	0.7%	1e6	18.8	1575(9)	0.2%	6e5	18.8	1293(9)	0.3%	3e5

Table 1.3: Results for the k -trimmed criterion

$ P \setminus k $	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$	$gapR_t / gap(\#) t^* / gap^* nod$								
5 40 1	87.6	538(9)	1.1%	6e5	95.7	1368(8)	48.9%	2e3	96.7	1803(5)	31.2%	1e6	96.6	125	756	1e5	95.7	750(8)	30.3%	3e5	95.7	836(8)	27.9%	9e5
5 40 2	99.9	107	244	1e5	99.9	99.9%(0)	99.9%	2e2	99.9	1093(7)	39.2%	1e6	99.9	2358(5)	99.6%	1e6	99.9	2548(3)	95.8%	6e5	99.9	2173(4)	51.6%	1e6
5 50 1	96.6	716(9)	4.9%	5e5	98.8	2880(4)	89.7%	2e2	99.9	1449(8)	27.2%	9e5	99.9	1168(7)	39.3%	3e5	98.7	1813(5)	32.2%	7e5	98.7	1813(5)	32.2%	7e5
5 50 2	99.9	1322(8)	8.7%	7e5	99.9	99.9%(0)	99.9%	1	99.9	786(8)	43.6%	6e5	99.9	3259(1)	99.9%	6e5	99.9	2172(4)	60.3%	4e5	99.9	1685(6)	62.1%	8e5
5 60 1	99.9	2131(5)	0.8%	8e5	100.0	97.1%(0)	99.9%	0	100.0	1482(7)	32.6%	5e5	100.0	1482(7)	32.6%	5e5	99.9	817(8)	39.5%	1e5	99.9	1540(6)	34.6%	5e5
5 60 2	99.9	1530(8)	1.3%	6e5	99.9	99.9%(0)	99.9%	0	99.9	1490(6)	28.5%	9e5	99.9	3246(1)	99.9%	9e5	99.9	1499(6)	89.9%	8e4	99.9	2225(4)	70.7%	4e5
8 40 2	80.5	3312(1)	3.4%	2e6	97.0	67.5%(0)	99.9%	9e3	96.3	2016(5)	15.8%	2e6	96.3	3451(1)	41.4%	1e6	95.9	3320(1)	35.8%	2e6	99.9	1846(6)	6.9%	2e6
8 40 3	99.9	10.0%(0)	27.5%	1e6	99.9	99.9%(0)	99.9%	7e2	99.9	2618(3)	36.1%	4e6	99.9	93.5%(0)	99.8%	1e6	99.9	91.8%(0)	99.8%	1e6	99.9	1901(7)	21.7%	1e6
8 50 2	86.3	2.8% (0)	5.5%	2e6	97.6	85.4%(0)	99.9%	5e2	97.4	3274(2)	17.6%	3e6	97.4	31.6%(0)	48.5%	1e6	97.0	38.4%(0)	50.0%	1e6	97.0	2492(4)	11.8%	1e6
8 60 2	99.9	47.1%(0)	67.3%	4e5	99.9	99.9%(0)	99.9%	1																

Table 3.1: Results for the k -centrum criterion

$ P \setminus V \setminus k$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$						
5 40 1	0.7	3.0	7.2	3e3	0.7	4.9	13	6e3	0.7	3.1	7.3	3e3
5 40 3	0.4	1.7	2.8	5e2	0.4	1.4	2.1	4e2	0.4	1.8	3.2	5e2
5 40 4	0.2	0.8	2.0	8	0.2	0.9	2.1	11	0.2	0.8	2.1	8
5 50 1	0.5	22	153	2e4	0.5	19	100	2e4	0.5	23	157	2e4
5 50 3	0.4	4.4	16	6e3	0.4	3.4	10	3e3	0.4	4.5	16	6e3
5 50 4	0.2	1.5	5.6	7e2	0.2	1.5	4.4	6e2	0.2	1.5	5.8	7e2
5 60 1	0.5	75	537	8e4	0.5	270	2002	1e5	0.5	77	551	8e4
5 60 3	0.4	39	179	3e4	0.4	26	121	5e4	0.4	40	185	3e4
5 60 4	0.2	1.3	3.0	3e2	0.2	1.7	4.0	3e2	0.2	1.4	3.2	3e2
8 40 2	0.9	146	372	2e5	0.9	495	1465	3e5	0.9	148	376	2e5
8 40 4	0.7	25	126	4e4	0.7	63	152	5e4	0.7	25	128	4e4
8 40 7	0.1	1.2	3.1	3e2	0.1	1.2	3.6	2e2	0.1	1.2	3.3	3e2
8 50 2	0.7	849(9)	0.3%	6e5	0.7	1685(6)	0.4%	8e5	0.7	734(9)	0.2%	7e5
8 50 4	0.5	79	503	8e4	0.5	435	2467	2e5	0.5	80	511	8e4
8 50 7	0.1	3.3	18	2e3	0.1	2.6	5.9	6e2	0.1	3.4	19	2e3
8 60 2	0.6	1832(6)	1.1%	8e5	0.6	2462(4)	1.1%	9e5	0.6	1826(6)	1.1%	8e5
8 60 4	0.5	1289(7)	0.7%	5e5	0.5	1622(7)	0.5%	4e5	0.5	1232(7)	0.7%	6e5
8 60 7	0.1	5.4	24	4e3	0.1	9.4	55	5e3	0.1	4.2	19	4e3

$$F_{mtz} + (14)$$

$$F_{mtz} + (15)$$

$$F_{mtz} + (16)$$

Table 3.2: Results for the Hurwicz criterion

$ P \setminus V \setminus \alpha$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$					
5 40 0.4	47.3	16	66	1e3	37.5	17	66	8e2	47.3	16	66	1e3
5 40 0.6	35.9	17	65	1e3	35.9	19	65	1e3	35.9	17	65	1e3
5 40 0.8	20.4	16	65	2e3	20.4	17	64	3e3	20.4	16	65	2e3
5 50 0.4	47.6	33	70	2e3	47.6	35	69	4e3	47.6	36	74	3e3
5 50 0.6	36.0	34	73	6e3	36.0	56	176	2e4	36.0	42	116	9e3
5 50 0.8	20.5	55	169	2e4	20.3	64	279	2e4	17.8	53	133	1e4
5 60 0.4	47.9	81	249	2e4	47.9	105	318	2e4	38.9	89	160	1e4
5 60 0.6	36.3	58	82	4e3	36.3	78	163	7e3	31.3	61	81	4e3
5 60 0.8	20.3	361	2875	2e5	20.3	482(9)	0.1%	1e5	18.0	345	2759	2e5
8 40 0.4	38.5	51	202	3e4	38.5	42	121	2e4	29.4	84	300	4e4
8 40 0.6	29.2	70	356	5e4	29.2	93	424	7e4	24.5	160	969	9e4
8 40 0.8	18.6	64	212	5e4	18.6	127	723	7e4	16.6	93	297	4e4
8 50 0.4	39.6	217	1079	1e5	39.6	455	1708	1e5	32.2	445	1841	1e5
8 50 0.6	30.3	277	1226	1e5	30.3	494	2859	2e5	26.1	483	2260	1e5
8 50 0.8	19.0	363	1418	2e5	19.0	633	2669	2e5	17.2	700	2746	2e5
8 60 0.4	38.5	296	517	9e4	38.5	673(9)	0.1%	1e5	30.7	1111(9)	0.1%	2e5
8 60 0.6	29.6	766(9)	0.1%	2e5	29.6	701	2928	3e5	25.6	1243(8)	0.2%	2e5
8 60 0.8	18.8	1575(9)	0.2%	6e5	18.8	2070(7)	0.3%	7e5	17.7	2225(6)	1.4%	5e5

$$F_{mtz} + (14)$$

$$F_{mtz} + (15)$$

$$F_{mtz} + (16)$$

Table 3.3: Results for the k -trimmed criterion

$ P \setminus V \setminus k$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$	$gapR \ t / gap(\#) \ t / gap^* \ nod$					
5 40 1	95.7	750(8)	30.3%	3e5	95.7	1084(7)	34.9%	7e5	79.8	2912(2)	55.5%	1e6
5 40 2	99.9	2548(3)	95.8%	6e5	99.9	568(9)	53.5%	1e5	68.9	1096(7)	64.7%	2e5
5 50 1	98.7	1168(7)	39.3%	3e5	98.7	1467(6)	36.2%	7e5	81.3	3332(1)	44.8%	6e5
5 50 2	99.9	2172(4)	60.3%	4e5	99.9	397(9)	19.2%	5e4	71.0	1113(7)	61.5%	1e5
5 60 1	99.9	817(8)	39.5%	1e5	99.9	1815(5)	44.7%	4e5	83.6	2819(3)	59.6%	3e5
5 60 2	99.9	1499(6)	89.9%	8e4	99.9	1523(6)	60.8%	2e5	72.5	600(9)	54.4%	4e4
8 40 2	95.9	3320(1)	35.8%	2e6	95.9	21.1%(0)	36.1%	3e6	85.0	13.6%(0)	24.1%	2e6
8 40 3	99.9	91.8%(0)	99.8%	1e6	99.9	2887(2)	52.8%	2e6	83.3	21.9%(0)	34.5%	1e6
8 50 2	97.0	38.4%(0)	59.0%	1e6	97.0	3258(1)	40.0%	1e6	86.9	20.3%(0)	28.5%	6e5
8 50 3	99.9	95.7%(0)	100.0%	4e5	99.9	31.7%(0)	54.6%	1e6	85.0	42.0%(0)	56.0%	4e5
8 60 2	98.1	54.8%(0)	65.4%	6e5	98.1	25.5%(0)	41.0%	8e5	88.8	43.4%(0)	66.0%	2e5
8 60 3	99.9	96.2%(0)	99.9%	2e5	99.9	34.2%(0)	57.3%	6e5	86.9	68.9%(0)	74.9%	2e5

$$F_{mtz} + (14)$$

$$F_{mtz} + (15)$$

$$F_{mtz} + (16)$$

Table 3: OWASTP results for F_{mtz} including valid inequalities

Table 4.1: Results for the k -centrum criterion

$P V k$	t_*	t	t^*	t	t^*	t	t^*	t	t^*	t	t^*	t	t^*		
5 20 1	0.3	1.1	1.11	1.3	1.1	93	0.6	1.1	1.18	0.6	0.8	1.5	0.1	0.2	0.6
5 20 3	0.2	0.8	2.0	1.4	2.3	3.5	0.2	0.8	2.1	0.0	0.5	0.9	0.0	0.1	0.2
5 20 4	0.1	0.4	2.1	0.7	1.0	1.3	0.0	0.3	2.2	0.0	0.1	0.6	0.0	0.0	0.1
5 30 1	0.9	9.1	339	14	66	229	0.8	86	334	0.7	3.1	15	0.1	1.3	4.7
5 30 3	1.2	3.1	593	1.0	24	225	1.0	37	833	0.1	0.8	2.0	0.1	0.4	1.1
5 30 4	0.3	1.2	184	4.3	5.6	9.9	0.2	4.1	64	0.0	0.3	1.1	0.0	0.1	0.3
5 40 1	-	-	-	63	628	2638	-	-	-	1.3	3.0	7.2	0.4	1.7	7.5
5 40 3	1.2	78	610	41	87	443	1.2	65	617	0.8	1.7	2.8	0.3	0.6	1.6
5 40 4	0.4	13	225	15	19	30	0.2	14	273	0.2	0.8	2.0	0.1	0.3	1.0
5 50 1	-	-	-	-	-	-	-	-	-	1.7	22	153	1.1	406	3600
5 50 3	8.5	2039	-	119	420	7360	6.1	1897	-	1.0	4.4	16	0.6	88	766
5 50 4	0.9	243	4108	46	60	103	0.5	58	366	0.2	1.5	5.6	0.1	1.0	3.9
5 60 1	-	-	-	-	-	-	-	-	-	1.8	75	537	1.5	509	3600
5 60 3	-	-	-	62	660	5826	-	-	-	1.5	39	179	0.8	323	2885
5 60 4	8.9	1116	11016	19	394	3673	3.7	843	9777	0.3	1.3	3.0	0.2	5.0	29
5 70 1	-	-	-	-	-	-	-	-	-	2.5	30	92	2.2	412	3600
5 70 3	-	-	-	-	-	-	-	-	-	1.2	6.3	28	0.6	45	365
5 70 4	-	-	-	-	-	-	-	-	-	0.4	3.3	15	0.1	32	311
8 20 2	5.2	57	436	3.9	50	393	4.7	59	381	1.1	3.3	14	0.3	1.8	12
8 20 4	1.7	23	135	1.8	17	81	2.3	21	112	0.6	1.5	4.5	0.1	0.3	1.5
8 20 7	0.3	1.4	7.9	0.5	0.9	2.5	0.1	0.8	3.1	0.0	0.1	0.6	0.0	0.0	0.1
8 30 2	37	1279	6648	25	1024	6000	22	1269	5584	6.4	34	140	1.5	27	7.1
8 30 4	3.1	295	3414	15	189	1992	14	275	3109	1.2	7.1	38	0.6	5.3	20
8 30 7	0.5	3.0	26	1.5	2.6	8.4	0.1	2.3	23	0.1	0.6	1.4	0.0	0.2	0.8
8 40 2	-	-	-	-	-	-	-	-	-	6.5	146	372	7.3	81	270
8 40 4	95	1776	11820	67	1040	5029	76	1697	10283	1.0	25	126	0.5	21	106
8 40 7	0.9	122	3134	4.4	18	321	0.5	78	1959	0.2	1.2	3.1	0.1	0.4	1.3
8 50 2	-	-	-	-	-	-	-	-	-	23	849	3600	25	1100	3600
8 50 4	-	-	-	-	-	-	-	-	-	4.7	79	503	2.4	652	3600
8 50 7	3.5	157	1532	15	35	159	1.5	100	876	0.5	3.3	18	0.2	26	177
8 60 2	-	-	-	-	-	-	-	-	-	11	1832	3600	12	1549	3600
8 60 4	-	-	-	-	-	-	-	-	-	3.7	1289	3600	2.5	1138	3600
8 60 7	12	480	7599	23	74	526	9.8	278	3609	0.3	5.4	24	0.2	55	472
8 70 2	-	-	-	-	-	-	-	-	-	41	1919	3600	147	1886	3600
8 70 4	-	-	-	-	-	-	-	-	-	20	1268	3600	46	928	3600
8 70 7	-	-	-	-	-	-	-	-	-	0.4	121	1043	0.1	371	3600

Table 4.2: Results for the Hurwicz criterion

$P V k$	t_*	t	t^*	t	t^*	t	t^*	t	t^*	t	t^*	t	t^*		
5 20 0.4	0.3	1.2	2.3	0.5	1.5	2.6	0.3	1.1	2.3	0.9	2.7	9.4	0.5	0.5	0.7
5 20 0.6	0.9	1.8	3.4	1.1	2.1	3.4	0.6	1.7	2.8	0.9	2.4	5.4	0.5	0.6	0.8
5 20 0.8	0.6	1.7	4.3	0.5	1.5	3.4	0.5	1.5	3.1	1.0	3.2	7.7	0.3	0.7	1.3
5 30 0.4	2.4	4.6	10	3.6	6.0	12	2.2	4.5	10	2.9	6.0	12	1.4	2.3	3.9
5 30 0.6	1.9	8.9	42	3.1	10	44	1.6	9.1	44	2.3	4.9	8.1	0.9	5.2	23
5 30 0.8	1.5	28	105	1.2	19	61	0.5	18	91	2.3	9.4	34	1.4	6.1	19
5 40 0.4	6.5	18	51	12	24	57	6.6	18	45	7.8	16	66	4.2	15	84
5 40 0.6	7.9	46	156	13	52	164	7.7	46	156	8.3	17	65	4.0	7.2	15
5 40 0.8	6.5	71	212	7.1	53	185	4.1	52	226	7.7	16	65	9.9	14	28
5 50 0.4	22	124	324	39	143	353	22	125	336	13	33	70	10	29	70
5 50 0.6	26	367	2374	42	385	2404	26	368	2394	15	34	73	8.2	25	99
5 50 0.8	9.7	298	3665	23	218	1972	15	225	1979	16	55	169	9.7	66	251
5 60 0.4	4.1	46.1	4131	87	512	4175	41	461	4093	28	81	249	20	148	733
5 60 0.6	-	-	-	-	-	-	-	-	-	18	58	82	21	115	748
5 60 0.8	6.4	1725	16779	25	867	9427	2.6	1586	29575	37	361	2875	70	526	2720

Table 4.3: Results for the k -trimmed criterion

$P V k$	t_*	t	t^*	t	t^*	t	t^*	t	t^*	t	t^*	t	t^*
5 20 1	0.5	1.4	2.4	0.5	1.5	2.4	0.8	1.1	1.9	0.5	0.6	0.9	0.9
5 20 2	1.2	2.5	5.2	1.2	2.5	5.4	0.7	1.0	1.9	1.3	1.7	3.0	3.0
5 30 1	6.8	101	2187	6.8	102	2209	1.2	1441	3600	0.6	1081	3600	3600
5 30 2	8.6	468	9778	8.7	434	8720	1.3	42	392	3.2	138	1344	1344
5 40 1	9.5	1160	12874	9.6	1120	13655	1.9	1084	3600	2.4	1088	3600	3600
5 40 2	20	168	1029	20	169	1039	2.1	568	3600	4.7	33	196	196
5 50 1	29	733	15269	29	732	15239	5.8	1467	3600	1.5	1447	3600	3600
5 50 2	49	191	1215	49	191	1214	4.0	397	3600	9.1	813	3600	3600
5 60 1	45	379	1505	46	379	1503	6.2	1815	3600	5.5	1828	3600	3600
5 60 2	83	829	6407	83	828	6383	19	1523	3600	27	526	3600	3600

Table 4: Comparison with the results obtained by Galand and Spanjaard, 2012

Table 5.1: Results for the k -centrum criterion

$P V $	α	$gapR$	t/gap	t^*/gap^*	nod	$gapR$	t/gap	t^*/gap^*	nod	
5	40	1	0.7	3.0	3e3	0.7	1.7	7.5	5e3	
5	40	3	0.4	1.7	2.8	0.4	1.6	1e3	1e3	
5	40	4	0.2	0.8	2.0	0.2	0.3	1.0	3e2	
5	50	1	0.5	22	153	0.6	406(9)	0.3%	9e5	
5	50	3	0.4	4.4	16	6e3	0.5	88	766	
5	50	4	0.2	1.5	5.6	7e2	0.2	1.0	3.9	
5	60	1	0.5	75	537	0.6	509(9)	0.3%	8e5	
5	60	3	0.4	39	179	3e4	0.4	323	2885	
5	60	4	0.2	1.3	3.0	3e2	0.3	5.0	29	
5	80	1	0.3	230	1033	8e4	0.4	1453(6)	0.3%	1e6
5	80	3	0.2	18	148	7e3	0.3	846(8)	0.2%	7e5
5	80	4	0.1	2.0	6.4	1e2	0.1	19	95	
5	100	1	0.2	750	0.2%	1e5	0.3	1114(9)	1.0%	6e5
5	100	3	0.2	188	955	3e4	0.2	568	9)	
5	100	4	0.1	6.6	45	7e2	0.1	130	1186	
8	40	2	0.9	146	372	2e5	0.9	81	270	
8	40	4	0.7	25	126	4e4	0.7	21	106	
8	40	7	0.1	1.2	3.1	3e2	0.1	0.4	1.3	
8	50	2	0.7	849	0.3%	6e5	0.8	1100(8)	0.9%	2e6
8	50	4	0.5	79	503	8e4	0.6	652(9)	0.6%	1e6
8	50	7	0.1	3.3	18	2e3	0.2	26	177	
8	60	2	0.6	1832(6)	1.1%	8e5	0.6	1549	0.6%	
8	60	4	0.5	1289(7)	0.7%	5e5	0.5	1138	0.6%	
8	60	7	0.1	5.4	24	4e3	0.1	55	472	
8	80	2	0.3	2520	0.5%	2e6	0.4	3281(1)	0.8%	2e6
8	80	4	0.3	293	1449	2e5	0.3	1292(8)	0.6%	1e6
8	80	7	0.1	3.5	13	9e2	0.1	379(9)	0.3%	2e5
8	100	2	0.4	3591	1%	9e5	0.4	3%	0)	
8	100	4	0.3	2606	4%	7e5	0.3	2944(3)	0.8%	1e6
8	100	7	0.1	17	82	1e3	0.1	368(9)	0.2%	9e4
10	40	2	1.0	2229(5)	0.4%	2e6	1.0	1062(9)	0.4%	2e6
10	40	4	0.8	470	949	6e5	0.8	267	1872	
10	40	7	0.4	7.9	27	9e3	0.4	14	109	
10	50	2	0.8	3336(1)	0.5%	2e6	0.9	2910(4)	0.7%	4e6
10	50	4	0.7	2651(5)	0.3%	1e6	0.7	1323	0.6%	
10	50	7	0.4	142	481	7e4	0.5	746(9)	0.3%	1e6
10	60	2	0.7	0.4%	0.7%	1e6	0.7	0.4%	0.9%	
10	60	4	0.6	3331(2)	0.5%	2e6	0.6	2273	0.5%	
10	60	7	0.3	873	9%	4e5	0.4	1488(7)	0.3%	2e6
10	80	2	0.7	0.5%(0)	1.1%	4e5	0.6	0.4%	0.9%	
10	80	4	0.5	0.3%(0)	0.7%	9e5	0.4	3289	0.2%	
10	80	7	0.3	2403(5)	0.5%	4e5	0.3	2014	0.6%	
10	100	2	0.5	0.4%	0.7%	5e5	0.5	3519	1%	
10	100	4	0.4	3265(1)	0.6%	5e5	0.4	3259	1.2%	
10	100	7	0.2	2119	0.3%	2e5	0.3	2358(4)	0.6%	7e5

$$F_{km2} + (15)$$

Table 5: OWASTP results for large instances

Table 5.2: Results for the Hurwicz criterion

$P V $	α	$gapR$	t/gap	t^*/gap^*	nod	$gapR$	t/gap	t^*/gap^*	nod
5	40	0.4	47.3	16	66	47.3	15	84	2e3
5	40	0.6	35.9	17	65	1e3	35.9	7.2	15
5	40	0.8	20.4	16	65	2e3	20.4	14	28
5	50	0.4	47.6	33	70	2e3	47.6	29	70
5	50	0.6	36.0	34	73	6e3	36.1	25	99
5	50	0.8	20.5	55	169	2e4	20.4	66	251
5	60	0.4	47.9	81	249	2e4	48.0	148	733
5	60	0.6	36.3	58	82	4e3	36.3	115	748
5	60	0.8	20.3	361	2875	2e5	20.4	526	2720
5	80	0.4	47.4	295	1201	5e4	47.7	711(9)	0.4%
5	80	0.6	32.5	148	307	1e4	32.5	480	2446
5	80	0.8	20.2	259	1164	3e4	20.2	1066(8)	0.2%
5	100	0.4	47.8	825	9%	1e5	47.8	945(9)	0.2%
5	100	0.6	36.2	196	532	1e4	36.2	1076(8)	0.1%
5	100	0.8	20.1	954	0.0%	1e5	20.1	1713(7)	0.3%
8	40	0.4	38.5	51	202	3e4	38.5	21	59
8	40	0.6	29.2	70	356	5e4	29.2	23	50
8	40	0.8	18.6	64	212	5e4	18.6	32	53
8	50	0.4	39.6	217	1079	1e5	39.7	122	345
8	50	0.6	30.3	277	1226	1e5	30.4	249	1493
8	50	0.8	19.0	363	1418	2e5	19.1	379	2262
8	60	0.4	38.5	296	766(9)	0.1%	2e5	38.5	225
8	60	0.6	29.6	296	6e5	0.2%	2e5	29.6	371
8	60	0.8	18.8	1575	9%	6e5	18.8	874	9%
8	80	0.4	38.5	1823(7)	0.4%	3e5	38.6	1375(8)	0.7%
8	80	0.6	29.6	1761(7)	0.2%	3e5	29.6	1514(8)	0.1%
8	80	0.8	18.8	1611	3032	3e5	18.8	1119	2357
8	100	0.4	38.7	2758(4)	0.5%	2e5	38.1	2313	6%
8	100	0.6	29.7	2733	4%	0.5%	2e5	29.7	2653
8	100	0.8	18.9	3449(1)	0.3%	3e5	19.0	3005(4)	0.7%
10	40	0.4	35.8	154	786	1e5	35.8	82	352
10	40	0.6	27.2	320	1028	2e5	27.2	150	443
10	40	0.8	17.5	545	1936	5e5	17.5	435	1158
10	50	0.4	36.0	1012(8)	0.2%	5e5	36.0	454	2961
10	50	0.6	27.5	1541(7)	0.2%	7e5	28.9	705	2695
10	50	0.8	18.0	2618(5)	0.4%	1e6	18.0	2191(6)	0.3%
10	60	0.4	35.7	1847(7)	0.2%	6e5	35.7	1361(8)	0.2%
10	60	0.6	27.1	2622(5)	0.5%	8e5	27.2	2247(9)	0.4%
10	60	0.8	17.7	3544(1)	0.5%	1e6	17.7	3439(2)	0.3%
10	80	0.4	35.0	3388(2)	0.9%	4e5	35.0	2743(4)	0.4%
10	80	0.6	27.0	3448(1)	0.3%	4e5	27.0	3123(3)	0.4%
10	80	0.8	17.7	0.4%(0)	0.8%	4e5	17.7	0.2%(0)	0.6%
10	100	0.4	35.0	0.3%(0)	0.7%	2e5	34.9	0.2%(0)	0.4%
10	100	0.6	26.9	0.3%(0)	0.8%	2e5	26.8	0.2%(0)	0.4%
10	100	0.8	17.6	0.3%(0)	0.5%	2e5	17.6	0.2%(0)	0.4%

$$F_{km2} + (16)$$