Puzzle corner

Norman Do*

Welcome to the Australian Mathematical Society *Gazette*'s Puzzle Corner. Each issue will include a handful of entertaining puzzles for adventurous readers to try. The puzzles cover a range of difficulties, come from a variety of topics, and require a minimum of mathematical prerequisites to be solved. And should you happen to be ingenious enough to solve one of them, then the first thing you should do is send your solution to us.

In each Puzzle Corner, the reader with the best submission will receive a book voucher to the value of $50, not to mention fame, glory and unlimited bragging rights! Entries are judged on the following criteria, in decreasing order of importance: accuracy, elegance, difficulty, and the number of correct solutions submitted. Please note that the judge's decision — that is, my decision — is absolutely final. Please e-mail solutions to ndo@math.mcgill.ca or send paper entries to: Gazette of the AustMS, Birgit Loch, Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, Qld 4350, Australia.

The deadline for submission of solutions for Puzzle Corner 13 is 1 September 2009. The solutions to Puzzle Corner 13 will appear in Puzzle Corner 15 in the November 2009 issue of the *Gazette*.

Digital deduction

The numbers 2^{2009} and 5^{2009} are written on a piece of paper in decimal notation. How many digits are on this piece of paper?

Square, triangle and circle

Let $ABCD$ be a unit square and ABX an equilateral triangle with X outside the square. What is the radius of the circle passing through C, D and X?

Piles of stones

There are 25 stones sitting in a pile next to a blackboard. You are allowed to take a pile and divide it into two smaller piles of size a and b, but then you must write the number $a \times b$ on the blackboard. You continue to do this until you are left with 25 piles, each with one stone. What is the maximum possible sum of the numbers written on the blackboard?

Photo: Craig Jewell

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Do you know my number now?

Two geniuses are each assigned a positive integer and are told that the two numbers differ by 1. They then take turns to ask each other, 'Do you know my number now?'. If the geniuses always respond to questions truthfully, prove that one of them will eventually answer affirmatively.

Fun with floors

Prove the following interesting identity for every integer n greater than 1, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

$$\lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor + \cdots + \lfloor \sqrt[n]{n} \rfloor = \lfloor \log_2 n \rfloor + \lfloor \log_3 n \rfloor + \cdots + \lfloor \log_n n \rfloor$$

Noodling around

In front of you is a bowl containing 100 noodles. You randomly pick two ends of noodles and join them together until there are no more ends left and only noodle loops remaining.

(1) What is the probability that there is only one loop in the bowl?
(2) What is the probability that there are k loops in the bowl?
(3) What is the expected number of loops in the bowl?

Photo: James Rubio

Solutions to Puzzle Corner 11

The \$50 book voucher for the best submission to Puzzle Corner 11 is awarded to Ivan Guo.

Bags and eggs

Solution by Ben Carr: The bag with the most eggs must contain at least nineteen of them. To see that the task is possible with nineteen eggs, place bag 1 and one egg into bag 2, then place bag 2 and one egg into bag 3, then place bag 3 and one egg into bag 4, and so on.

Area identity

Solution by Ivan Guo: Take the identity that we wish to prove and add $\text{Area}(AMP) + \text{Area}(CDR)$ to both sides to obtain the equivalent identity

$$\text{Area}(ABN) + \text{Area}(CDN) = \text{Area}(CMD).$$

This is in turn equivalent to the equation

$$\text{Area}(ABCD) - \text{Area}(AND) = \text{Area}(CMD),$$

which is true since $\text{Area}(AND) = \text{Area}(CMD) = \frac{1}{2}\text{Area}(ABCD)$. Note that the points M and N could have been anywhere on the sides AB and BC and the result would still hold.

Factorial fun

Solution by Laura McCormick and Rick Mabry: The square-free part of a positive integer x is the number $\frac{x}{n^2}$, where n^2 is the largest perfect square dividing x. We will write $x \equiv y$ whenever x and y have the same square-free parts. It is easy to see that this is an equivalence relation and we have the following.

$$\begin{aligned}\prod_{k=1}^{4n} k! &= 4n \times (4n-1)^2 \times (4n-2)^3 \times \cdots \times 3^{4n-2} \times 2^{4n-1} \times 1^{4n} \\ &\equiv 4n \times (4n-2) \times \cdots \times 4 \times 2 = 2^{2n}(2n)! \equiv (2n)!\end{aligned}$$

This means that, for every positive integer n, the number

$$\frac{1}{(2n)!}\prod_{k=1}^{4n} k!$$

is a perfect square. For our particular problem $n = 25$ and hence, we must erase 50! so that the product of the remaining 99 numbers is a perfect square.

In fact, Laura and Rick managed to solve the far more general problem of determining for which positive integers $m \leq n$ the following number is a perfect square.

$$\frac{1}{m!}\prod_{k=1}^{n} k!$$

Highway construction

Solution by Stephen Muirhead: For the sake of contradiction, assume that the highway never reaches 100 kilometres in length. Let X_k denote the length of highway, in kilometres, built in the first k months of construction. Then $X_k < 100$, so we have $\frac{1}{X_k^{100}} > \frac{1}{100^{100}}$ for all k. In other words, more than $\frac{1}{100^{100}}$ kilometres of highway are built every month. Thus, after 100^{101} months, at least 100 kilometres of highway have been built in total, a contradiction.

Busy bee

Solution by Ivan Guo: Suppose that the bee starts at the point A and, after flying for two metres, is at the point B. Let the midpoint of AB be O and consider the sphere of radius one metre with centre O. If the bee visited the point P outside the sphere during its flight from A to B, then let P' be the point such that O is the midpoint of PP'. Note that $APBP'$ is a parallelogram with O as its centre. Since the shortest path between two points is a line segment, the distance the bee flew between A and B is at least $AP + PB = P'B + PB \geq PP' > 2$. But this contradicts the fact that the bee flew two metres from A to B. The same argument shows that the bee could not have escaped the sphere on its return flight from B to A.

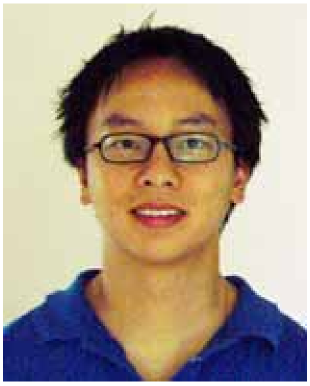
Coin-flipping games

Solution by Jamie Simpson:

(1) Let the two players each toss the coin once, with a player winning if they obtain heads while their opponent obtains tails. In the event that the outcomes are the same, simply repeat the procedure until the outcomes differ. Clearly, no player has an advantage in this game so each player has probaility $\frac{1}{2}$ of winning.

(2) Let the two players take turns tossing the coin, with a player winning if they obtain the first head. Then the first player's probability of winning is $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots = \frac{2}{3}$ and the second player's probaility of winning is $\frac{1}{3}$.

(3) Let the binary expansion of $\frac{1}{\pi}$ be $0.b_1b_2b_3\ldots$. Let the two players take turns tossing the coin until a head appears and suppose this happens on toss k. If $b_k = 1$, we say that the first player wins and if $b_k = 0$, then we say that the second player wins. Then the first player's probability of winning is $\frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots = \frac{1}{\pi}$.



Norman Do is currently a CRM-ISM Postdoctoral Fellow at McGill University in Montreal. He is an avid solver, collector and distributor of mathematical puzzles. When not playing with puzzles, Norman performs research in geometry and topology, with a particular focus on moduli spaces of curves.