Physics as Generalized Number Theory: Classical Number Fields

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Abstract

Physics as a generalized number theory program involves three threads: various p-adic physics and their fusion together with real number based physics to a larger structure, the attempt to understand basic physics in terms of classical number fields discussed in this article, and infinite primes whose construction is formally analogous to a repeated second quantization of an arithmetic quantum field theory.

In this article the connection between standard model symmetries and classical number fields is discussed. The basis vision is that the geometry of the infinite-dimensional WCW ("world of classical worlds") is unique from its mere existence. This leads to its identification as union of symmetric spaces whose Kähler geometries are fixed by generalized conformal symmetries. This fixes space-time dimension and the decomposition $M^4 \times S$ and the idea is that the symmetries of the Kähler manifold S make it somehow unique. The motivating observations are that the dimensions of classical number fields are the dimensions of partonic 2-surfaces, space-time surfaces, and imbedding space and M^8 can be identified as hyper-octonions- a sub-space of complexified octonions obtained by adding a commuting imaginary unit. This stimulates some questions.

Could one understand $S = CP_2$ number theoretically in the sense that M^8 and $H = M^4 \times CP_2$ be in some deep sense equivalent ("number theoretical compactification" or $M^8 - H$ duality)? Could associativity define the fundamental dynamical principle so that space-time surfaces could be regarded as associative or co-associative (defined properly) sub-manifolds of M^8 or equivalently of H.

One can indeed define the associative (co-associative) 4-surfaces using octonionic representation of gamma matrices of 8-D spaces as surfaces for which the modified gamma matrices span an associate (co-associative) sub-space at each point of space-time surface. Also $M^8 - H$ duality holds true if one assumes that this associative sub-space at each point contains preferred plane of M^8 identifiable as a preferred commutative or co-commutative plane (this condition generalizes to an integral distribution of commutative planes in M^8). These planes are parametrized by CP_2 and this leads to $M^8 - H$ duality.

WCW itself can be identified as the space of 4-D local sub-algebras of the local Clifford algebra of M^8 or H which are associative or co-associative. An open conjecture is that this characterization of the space-time surfaces is equivalent with the preferred extremal property of Kähler action with preferred extremal identified as a critical extremal allowing infinite-dimensional algebra of vanishing second variations.

Keywords: Classical number fields, quaternions, octonions, complexified octonions, associativity, local Clifford algebra, quaternionic sub-manifolds.

Contents

1 Introduction

This article is second one in the series devoted to the vision about TGD as a generalized number theory. The basic theme is the role of classical number fields [25, 26, 27] in quantum TGD. A central notion is $M^8 - H$ duality which might be also called number theoretic compactification. This duality allows to identify imbedding space equivalently either as M^8 or $M^4 \times CP_2$ and explains the symmetries of standard model number theoretically. These number theoretical symmetries induce also the symmetries dictating the geometry of the "world of classical worlds" (WCW) as a union of symmetric spaces [17]. This infinite-dimensional Kähler geometry is expected to be highly unique from the mere requirement of its existence requiring infinite-dimensional symmetries provided by the generalized conformal symmetries of the light-cone boundary $\delta M^4_+ \times S$ and of light-like 3-surfaces and the answer to the question what makes 8-D imbedding space and $S = CP_2$ so unique would be the reduction of these symmetries to number theory.

Zero energy ontology has become the corner stone of also number theoretical vision. In zero energy ontology either light-like or space-like 3-surfaces can be identified as the fundamental dynamical objects, and the extension of general coordinate invariance leads to effective 2-dimensionality (strong form of holography) in the sense that the data associated with partonic 2-surfaces and the distribution of 4-D tangent spaces at them located at the light-like boundaries of causal diamonds (CDs) defined as intersections of future and past directed light-cones code for quantum physics and the geometry of WCW. Also the hierarchy of Planck constants [11] plays a role but not so important one.

The basic number theoretical structures are complex numbers, quaternions [26] and octonions [27], and their complexifications obtained by introducing additional commuting imaginary unit $\sqrt{-1}$. Hyper-octonionic (-quaternionic,-complex) sub-spaces for which octonionic imaginary units are multiplied by commuting $\sqrt{-1}$ have naturally Minkowskian signature of metric. The question is whether and how the hyper-structures could allow to understand quantum TGD in terms of classical number fields. The answer which looks the most convincing one relies on the existence of octonionic representation of 8-D gamma matrix algebra.

1. The first guess is that associativity condition for the sub-algebras of the local Clifford algebra defined in this manner could select 4-D surfaces as associative (hyper-quaternionic) sub-spaces of this algebra and define WCW purely number theoretically. The associative sub-spaces in question would be spanned by the modified gamma matrices defined by the modified Dirac action fixed by the variational principle (Kähler action) selecting space-time surfaces as preferred extremals [7].

2. This condition is quite not enough: one must strengthen it with the condition that a preferred commutative and thus hyper-complex sub-algebra is contained in the tangent space of the space-time surface. This condition actually generalizes somewhat since one can introduce a family of so called Hamilton-Jacobi coordinates for M^4 allowing an integrable distribution of decompositions of tangent space to the space of non-physical and physical polarizations [12]. The physical interpretation is as a number theoretic realization of gauge invariance selecting a preferred local commutative plane of non-physical polarizations.

3. Even this is not yet the whole story: one can define also the notions of co-associativity and co-commutativity applying in the regions of space-time surface with Euclidian signature of the induced metric. The basic unproven conjecture is that the decomposition of space-time surfaces to associative and co-associative regions containing preferred commutative *resp.* co-commutative 2-plane in the 4-D tangent plane is equivalent with the preferred extremal property of Kähler action and the hypothesis that space-time surface allows a slicing by string world sheets and by partonic 2-surfaces [7].

1.1 Hyper-octonions and hyper-quaternions

The discussions for years ago with Tony Smith [28] stimulated very general ideas about space-time surface as an associative, quaternionic sub-manifold of octonionic 8-space: in what sense remained however an open question. Also the observation that quaternionic and octonionic primes have norm squared equal to prime in complete accordance with p-adic length scale hypothesis, led to suspect that the notion of primeness for quaternions, and perhaps even for octonions, might be fundamental for the formulation of quantum TGD. The original idea was that space-time surfaces could be regarded as four-surfaces in 8-D imbedding space with the property that the tangent spaces of these spaces can be locally regarded as 4- *resp.* 8-dimensional quaternions and octonions.

It took some years to realize that the difficulties related to the realization of Lorentz invariance might be overcome by replacing quaternions and octonions with hyper-quaternions and hyper-octonions. Hyper-quaternions *resp.* -octonions is obtained from the algebra of ordinary quaternions and octonions by multiplying the imaginary part with $\sqrt{-1}$ and can be regarded as a sub-space of complexified quaternions *resp.* octonions. The transition is the number theoretical counterpart of the transition from Riemannian to pseudo-Riemannian geometry performed already in Special Relativity. The loss of number field and even sub-algebra property is not fatal and has a clear physical meaning. The notion of primeness is inherited from that for complexified quaternions *resp.* octonions.

Complexified number fields make also sense p-adically unlike the notions of number fields themselves unless restricted to be algebraic extensions of rational variants of number fields. What deserves separate emphasis is that the basic structure of the standard model would reduce to number theory.

1.2 Number theoretical compactification and $M^8 - H$ duality

The notion of hyper-quaternionic and octonionic manifold makes sense but it not plausible that $H = M^4 \times CP_2$ could be endowed with a hyper-octonionic manifold structure. Situation changes if H is replaced with hyper-octonionic M^8. Suppose that $X^4 \subset M^8$ consists of hyper-quaternionic and co-hyper-quaternionic regions. The basic observation is that the hyper-quaternionic sub-spaces of M^8 with a fixed hyper-complex structure (containing in their tangent space a fixed hyper-complex subspace M^2 or at least one of the light-like lines of M^2) are labeled by points of CP_2. Hence each hyper-quaternionic and co-hyper-quaternionic four-surface of M^8 defines a 4-surface of $M^4 \times CP_2$.

One can loosely say that the number-theoretic analog of spontaneous compactification occurs: this of course has nothing to do with dynamics as in super string model.

This picture was still too naive and it became clear that not all known extremals of Kähler action contain fixed $M^2 \subset M^4$ or light-like line of M^2 in their tangent space.

1. The first option represents the minimal form of number theoretical compactification. M^8 is interpreted as the tangent space of H. Only the 4-D tangent spaces of light-like 3-surfaces X_l^3 (wormhole throats or boundaries) are assumed to be hyper-quaternionic or co-hyper-quaternionic and contain fixed M^2 or its light-like line in their tangent space. Hyper-quaternionic regions would naturally correspond to space-time regions with Minkowskian signature of the induced metric and their co-counterparts to the regions for which the signature is Euclidian. What is of special importance is that this assumption solves the problem of identifying the boundary conditions fixing the preferred extremals of Kähler action since in the generic case the intersection of M^2 with the 3-D tangent space of X_l^3 is 1-dimensional. The surfaces $X^4(X_l^3) \subset M^8$ would be hyper-quaternionic or co-hyper-quaternionic but would not allow a local mapping between the 4-surfaces of M^8 and H.

2. One can also consider a more local map of $X^4(X_l^3) \subset H$ to $X^4(X_l^3) \subset M^8$. The idea is to allow $M^2 \subset M^4 \subset M^8$ to vary from point to point so that $S^2 = SO(3)/SO(2)$ characterizes the local choice of M^2 in the interior of X^4. This leads to a quite nice view about strong geometric form of $M^8 - H$ duality in which M^8 is interpreted as tangent space of H and $X^4(X_l^3) \subset M^8$ has interpretation as tangent for a curve defined by light-like 3-surfaces at X_l^3 and represented by $X^4(X_l^3) \subset H$. Space-time surfaces $X^4(X_l^3) \subset M^8$ consisting of hyper-quaternionic and co-hyper-quaternionic regions would naturally represent a preferred extremal of E^4 Kähler action. The value of the action would be same as CP_2 Kähler action. $M^8 - H$ duality would apply also at the induced spinor field and at the level of configuration space.

3. Strong form of $M^8 - H$ duality satisfies all the needed constraints if it represents Kähler isometry between $X^4(X_l^3) \subset M^8$ and $X^4(X_l^3) \subset H$. This implies that light-like 3-surface is mapped to light-like 3-surface and induced metrics and Kähler forms are identical so that also Kähler action and field equations are identical. The only differences appear at the level of induced spinor fields at the light-like boundaries since due to the fact that gauge potentials are not identical.

4. The map of $X_l^3 \subset H \rightarrow X_l^3 \subset M^8$ would be crucial for the realization of the number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as those preferred coordinates in which the points of imbedding space are rational/algebraic. Thus the point of $X^4 \subset H$ is algebraic if it is mapped to algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not have practical meaning. Number theoretical compactication could thus be motivated by the number theoretical universality.

5. The possibility to use either M^8 or H picture might be extremely useful for calculational purposes. In particular, M^8 picture based on $SO(4)$ gluons rather than $SU(3)$ gluons could perturbative description of low energy hadron physics. The strong $SO(4)$ symmetry of low energy hadron physics can be indeed seen direct experimental support for the $M^8 - H$ duality.

1.3 Notations

Some notational conventions are in order before continuing. The fields of quaternions *resp.* octonions having dimension 4 *resp.* 8 and will be denoted by Q and O. Their complexified variants will be denoted by Q_C and O_C. The sub-spaces of hyper-quaternions HQ and hyper-octonions HO are obtained by multiplying the quaternionic and octonionic imaginary units by $\sqrt{-1}$. These sub-spaces are very intimately related with the corresponding algebras, and can be seen as Euclidian and Minkowkian variants of the same basic structure. Also the Abelianized versions of the hyper-quaternionic and -octonionic sub-spaces can be considered: these algebras have a representation in the space of spinors of imbedding space $H = M^4 \times CP_2$.

2 Quaternion and octonion structures and their hyper counterparts

In this introductory section the notions of quaternion and octonion structures and their hyper counterparts are introduced with strong emphasis on the physical interpretation. Literature contains several variants of these structures (Hyper-Kähler structure [19] and quaternion Kähler structure possessed also by CP_2 [21]). The notion introduced here is inspired by the physical motivations coming from TGD. As usual the first proposal based on the notions of (hyper-)quaternion and (hyper-)octonion analyticity was not the correct one. Much later a local variant of the notion based on tangent space emerged.

2.1 Octonions and quaternions

In the following only the basic definitions relating to octonions and quaternions are given. There is an excellent article by John Baez [27] describing octonions and their relations to the rest of mathematics and physics.

Octonions can be expressed as real linear combinations $\sum_k x^k I_k$ of the octonionic real unit $I_0 = 1$ (counterpart of the unit matrix) and imaginary units I_a, $a = 1, ..., 7$ satisfying

$$\begin{aligned} I_0^2 &= I_0 \equiv 1 \ , \\ I_a^2 &= -I_0 = -1 \ , \\ I_0 I_a &= I_a \ . \end{aligned} \qquad (2.1)$$

Octonions are closed with respect to the ordinary sum of the 8-dimensional vector space and with respect to the octonionic multiplication, which is neither commutative ($ab \neq ba$ in general) nor associative ($a(bc) \neq (ab)c$ in general).

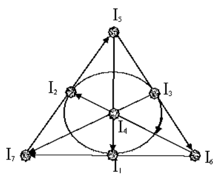

Figure 1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

A concise manner to summarize octonionic multiplication is by using octonionic triangle. Each line (6 altogether) containing 3 octonionic imaginary units forms an associative triple which together with $I_0 = 1$ generate a division algebra of quaternions. Also the circle spanned by the 3 imaginary units at the middle of the sides of the triangle is associative triple. The multiplication rules for each associative triple are simple:

$$I_a I_b = \epsilon_{abc} I_c \ , \tag{2.2}$$

where ϵ_{abc} is 3-dimensional permutation symbol. $\epsilon_{abc} = 1$ for the clockwise sequence of vertices (the direction of the arrow along the circumference of the triangle and circle). As a special case this rule gives the multiplication table of quaternions. A crucial observation for what follows is that any pair of imaginary units belongs to one associative triple.

The non-vanishing structure constants ${d_{ab}}^c$ of the octonionic algebra can be read directly from the octonionic triangle. For a given pair I_a, I_b one has

$$\begin{aligned} I_a I_b &= {d_{ab}}^c I_c \ , \\ d_{ab} \quad c &= {\epsilon_{ab}}^c \ , \\ I_a^2 &= {d_{aa}}^0 I_0 = -I_0 \ , \\ I_0^2 &= {d_{00}}^0 I_0 \ , \\ I_0 I_a &= {d_{0a}}^a I_a = I_a \ . \end{aligned} \tag{2.3}$$

For ϵ_{abc} c belongs to the same associative triple as ab.

Non-associativity means that is not possible to represent octonions as matrices since matrix product is associative. Quaternions can be represented and the structure constants provide the defining representation as $I_a \rightarrow d_{abc}$, where b and c are regarded as matrix indices of 4×4 matrix. The algebra automorphisms of octonions form 14-dimensional group G_2, one of the so called exceptional Lie-groups. The isotropy group of imaginary octonion unit is the group $SU(3)$. The Euclidian inner product of the two octonions is defined as the real part of the product $\overline{x}y$

$$\begin{aligned} (x,y) &= Re(\overline{x}y) = \sum_{k=0,..7} x_k y_k \ , \\ \overline{x} &= x^0 I_0 - \sum_{i=1,..,7} x^k I_k \ , \end{aligned} \tag{2.4}$$

and is just the Euclidian norm of the 8-dimensional space.

2.2 Hyper-octonions and hyper-quaternions

The Euclidicity of the quaternion norm suggests that octonions are not a sensible concept in TGD context. One can imagine two manners to circumvent this conclusion.

1. Minkowskian metric for octonions and quaternions is obtained by identifying Minkowski inner product xy as the real counterpart of the product

$$x \cdot y \equiv Re(xy) = x^0 y^0 - \sum_k x^k y^k \ . \tag{2.5}$$

   $SO(1,7)$ ($SO(1,3)$ in quaternionic case) Lorentz invariance appears completely naturally as the symmetry of the real part of the octonion (quaternion) product and hence of octonions/quaternions and there is no need to perform the complexification of the octonion algebra. Furthermore, only the signature $(1,7)$ ($(1,3)$ in the quaternionic case) is possible and this would raise $M_+^4 \times CP_2$ in a preferred position.

   This norm does not give rise to a number theoretic norm defining a homomorphism to real numbers. Indeed, the number theoretic norm defined by the determinant of the linear mapping defined by the multiplication with quaternion or octonion, is inherently Euclidian. This is in conflict with the idea that quaternionic and octonionic primes and their infinite variants should have key role in TGD [14, A7].

2. Hyper-octonions and hyper-quaternions provide a possible solution to these problems. These are obtained by multiplying imaginary units by commutative and associative $\sqrt{-1}$. These numbers form a sub-space of complexified octonions/quaternions and the cross product of imaginary parts leads out from this sub-space. In this case number theoretic norm induced from Q_C/O_C gives the fourth/eighth power of Minkowski length and Lorentz group acts as its symmetries. Light-like hyper-quaternions and -octonions causing the failure of the number field property have also a clear physical interpretation.

   A criticism against the notion of hyper-quaternionic and octonionic primeness is that the tangent space as an algebra property is lost and the notion of primeness is inherited from Q_C/O_C. Also non-commutativity and non-associativity could cause difficulties.

Zero energy ontology leads to a possible physical interpretation of complexified octonions. The moduli space for causal diamonds corresponds to a Cartesian product of $M^4 \times CP_2$ whose points label the position of either tip of $CD \times CP_2$ and space I whose points label the relative positive of the second tip with respect to the first one. p-Adic length scale hypothesis results if one assumes that the proper time distance between the tips comes in powers of two so that one has union of hyperboloids $H_n \times CP_2$, $H_n = \{m \in M^4_+ | a = 2^n a_0)\}$. A further quantization of hyperboloids H_n is obtained by replacing it with a lattice like structure is highly suggestive and would correspond to an orbit of a point of H_n under a subgroup of $SL(2, Q_C)$ or $SL(2, Z_C)$ acting as Lorentz transformations in standard manner. Also algebraic extensions of Q_C and Z_C can be considered. Also in the case of CP_2 discretization is highly suggestive so that one would have an orbit of a point of CP_2 under a discrete subgroup of $SU(3, Q)$.

The outcome could be interpreted by saying that the moduli space in question is $H \times I$ such that H corresponds to hyper-octonions and I to a discretized version of $\sqrt{-1}H$ and thus a subspace of complexified octonions. An open question whether the quantization has some deeper mathematical meaning.

2.3 Basic constraints

Before going to details it is useful to make clear the constraints on the concept of the hyper-octonionic structure implied by TGD view about physics.

$M^4 \times CP_2$ cannot certainly be regarded as having any global octonionic structure (for instance in the sense that it could be regarded as a coset space associated with some exceptional group). There are however clear indications for the importance of the hyper-quaternionic and -octonionic structures.

1. $SU(3)$ is the only simple 8-dimensional Lie-group and acts as the group of isometries of CP_2: if $SU(3)$ had some kind of octonionic structure, CP_2 would become unique candidate for the space S. The decomposition $SU(3) = h + t$ to $U(2)$ sub-algebra and its complement corresponds rather closely to the decomposition of (hyper-)octonions to (hyper-)quaternionic sub-space and its complement. The electro-weak $U(2)$ algebra has a natural 1+3 decomposition and generators allow natural hyper-quaternionic structure. The components of the Weyl tensor of CP_2 behave with respect to multiplication like quaternionic imaginary units but only one of them is covariantly constant so that hyper Kähler structure [19] with three covariantly constant quaternionic imaginary units represented by Kähler forms is not possible. These tensors and metric tensor however define quaternionic structure [21].

2. M^4_+ has a natural 1+3 decomposition and a unique cosmic time coordinate defined as the light cone proper time. Hyper-quaternionic structure is consistent with the Minkowskian signature of the inner product and hyper quaternion units have a natural representation in terms of covariantly constant self-dual symplectic forms [22] and their contractions with sigma matrices. It is not however clear whether this representation is physically interesting.

2.4 How to define hyper-quaternionic and hyper-octonionic structures?

I have considered several proposals for how to define quaternionic and octonionic structures and their hyper-counterparts.

1. (Hyper-)octonionic manifolds would obtained by gluing together coordinate patches using (hyper-)octonion analytic functions with real Laurent coefficients (this guarantees associativity and commutativity). This definition does not yet involve metric or any other structures (such as Kähler structure). This approach does not seem to be physically realistic.

2. Second option is based on the idea of representing quaternionic and octonionic imaginary units as antisymmetric tensors. This option makes sense for quaternionic manifolds [20] and CP_2 indeed represents an example represents of this kind of manifold. The problem with the octonionic structure is that antisymmetric tensors cannot define non-associative product.

3. If the manifold is endowed with metric, octonionic structure should be defined as a local tangent space structure analogous to eight-bein structure and local gauge algebra structures. This can be achieved by contracting octo-bein vectors with the standard octonionic basis to get octonion form I_k. Each vector field a^k defines naturally octonion field $A = a^k I_k$. The product of two vector fields can be defined by the octonionic multiplication and this leads to the introduction of a tensor field d_{klm} of these structure constants obtained as the contraction of the octobein vectors with the octonionic structure constants d_{abc}. Hyper-octonion structure can defined in a completely analogous manner.

   It is possible to induce octonionic structure to any 4-dimensional space-time surface by forming the projection of I_k to the space-time surface and redefining the products of I_k:s by dropping away that part of the product, which is orthogonal to the space-time surface. This means that the structure constants of the new 4-dimensional algebra are the projections of d_{klm} to the space-time surface. One can also define similar induced algebra in the 4-dimensional normal space of the space-time surface. The hypothesis would be that the induced tangential is associative or hyper-quaternionic algebra. Also co-associativity defined as associativity of the normal space algebra is possible. This property would give for the 4-dimensionality of the space-time surface quite special algebraic meaning. The problem is now that there is no direct connection with quantum TGD proper- in particular the connection with the classical dynamics defined by Kähler action is lacking.

4. 8-dimensional gamma matrices allow a representation in terms of tensor products of octonions and 2×2 matrices. Genuine matrices are of course not in question since the product of the gamma matrices fails to be associative. An associative representation is obtained by restricting the matrices to a quaternionic plane of complex octonions. If the space-time surface is hyper-quaternionic in the sense that induced gamma matrices define a quaternionic plane of complexified octonions at each point of space-time surface the resulting local Clifford algebra is associative and structure constants define a matrix representation for the induced gamma matrices.

   A more general definition allows gamma matrices to be modified gamma matrices defined by Kähler action appearing in the modified Dirac action and forced both by internal consistency and super-conformal symmetry [6, 7]. The modified gamma matrices associated with Kähler action do not in general define tangent space of the space-time surface as the induced gamma matrices do. Also co-associativity can be considered if one can identify a preferred imaginary unit such that the multiplication of the modified gamma matrices with this unit gives a quaternionic basis. This condition makes sense only if the preferred extremals of the action are hyper-quaternionic surfaces in the sense defined by the action. That this is true for Kähler action at least is an is an unproven conjecture.

In the sequel only the fourth option will be considered.

2.5 How to end up to quantum TGD from number theory?

An interesting possibility is that quantum TGD could emerge from a condition that a local version of hyper-finite factor of type II_1 represented as a local version of infinite-dimensional Clifford algebra exists. The conditions are that "center or mass" degrees of freedom characterizing the position of CD separate uniquely from the "vibrational" degrees of freedom being represented in terms of octonions and that for physical states associativity holds true. The resulting local Clifford algebra would be identifiable as the local Clifford algebra of WCW (being an analog of local gauge groups and conformal fields [30]).

The uniqueness of M^8 and $M^4 \times CP_2$ as well as the role of hyper-quaternionic space-time surfaces as fundamental dynamical objects indeed follow from rather weak conditions if one restricts the consideration to gamma matrices and spinors instead of assuming that M^8 coordinates are hyper-octonionic as was done in the first attempts.

1. The unique feature of M^8 and any 8-dimensional space with Minkowski signature of metric is that it is possible to have an octonionic representation of the complexified gamma matrices [6, 8] and of spinors. This does not require octonionic coordinates for M^8. The restriction to a quaternionic plane for both gamma matrices and spinors guarantees the associativity.

2. One can also consider a local variant of the octonionic Clifford algebra in M^8. This algebra contains associative sub-algebras for which one can assign to each point of M^8 a hyper-quaternionic plane. It is natural to assume that this plane is either a tangent plane of 4-D manifold defined naturally by the induced gamma matrices defining a basis of tangent space or more generally, by modified gamma matrices defined by a variational principle (these gamma matrices do not define tangent space in general). Kähler action defines a unique candidate for the variational principle in question. Associativity condition would automatically select sub-algebras associated with 4-D hyper-quaternionic space-time surfaces.

3. This vision bears a very concrete connection to quantum TGD. In [8] the octonionic formulation of the modified Dirac equation is studied and shown to lead to a highly unique general solution ansatz for the equation working also for the matrix representation of the Clifford algebra. An open question is whether the resulting solution as such defined also solutions of the modified Dirac equation for the matrix representation of gammas. Also a possible identification for 8-dimensional counterparts of twistors as octo-twistors follows: associativity implies that these twistors are very closely related to the ordinary twistors. In TGD framework octo-twistors provide an attractive manner to get rid of the difficulties posed by massive particles for the ordinary twistor formalism.

4. Associativity implies hyperquaternionic space-time surfaces (in a more general sense as usual) and this leads naturally to the notion of WCW and local Clifford algebra in this space. Number theoretic arguments imply $M^8 - H$ duality. The resulting infinite-dimensional Clifford algebra would differ from von Neumann algebras in that the Clifford algebra and spinors assignable to the center of mass degrees of freedom of causal diamond CD would be expressed in terms of octonionic units although they are associative at space-time surfaces. One can therefore say that quantum TGD follows by assuming that the tangent space of the imbedding space corresponds to a classical number field with maximal dimension.

5. The slicing of the Minkowskian space-time surface inside CD by stringy world sheets and by partonic 2-surfaces inspires the question whether the modified gamma matrices associated with the stringy world sheets *resp.* partonic 2-surfaces could be could commutative *resp.* co-commutative. Commutativity would also be seen as the justification for why the fundamental objects are effectively 2-dimensional.

This formulation is undeniably the most convincing one found hitherto since the notion of hyper-quaternionic structure is local and has elegant formulation in terms of modified gamma matrices.

3 Number theoretical compactification and $M^8 - H$ duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to number theory. In strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as surfaces of H or as surfaces of M^8 composed of associative and co-associative regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric.

3.1 Basic idea behind $M^8 - M^4 \times CP_2$ duality

The observation that $M^4 \times CP_2$ does not allow octonionic structure in the sense that transition functions would be octonion analytic functions with real coefficients forced to ask whether four-surfaces $X^4 \subset M^8$ could under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly so that the spontaneous compactification of super string models would correspond in TGD to two different manners to interpret the space-time surface. The following arguments suggest that this is indeed the case. One could end up to the duality also from the attempt to understand $M^4 \times CP_2$ decomposition number theoretically.

The hard mathematical fact behind number theoretical compactification is that the quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space) are parameterized by CP_2 just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group.

1. The space of complex structures of the octonion space is parameterized by S^6. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Hyper-quaternions can be identified as $U(2)$ Lie-algebra but it is obvious that hyper-octonions do not allow an identification as $SU(3)$ Lie algebra. Rather, octonions decompose as $1 \oplus 1 \oplus 3 \oplus \overline{3}$ to the irreducible representations of $SU(3)$.

2. Geometrically the choice of a preferred complex (quaternionic) structure means fixing of complex (quaternionic) sub-space of octonions. The fixing of a hyper-quaternionic structure of hyper-octonionic M^8 means a selection of a fixed hyper-quaternionic sub-space $M^4 \subset M^8$ implying the decomposition $M^8 = M^4 \times E^4$. If M^8 is identified as the tangent space of $H = M^4 \times CP_2$, this decomposition results naturally. It is also possible to select a fixed hyper-complex structure, which means a further decomposition $M^4 = M^2 \times E^2$.

3. The basic result behind number theoretic compactification and $M^8 - H$ duality is that hyper-quaternionic sub-spaces $M^4 \subset M^8$ containing a fixed hyper-complex sub-space $M^2 \subset M^4$ or its light-like line $M_{\pm}$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

4. Space-time surface $X^4 \subset M^8$ is by the standard definition hyper-quaternionic if the tangent spaces of X^4 are hyper-quaternionic planes. Co-hyper-quaternionicity means the same for normal spaces. The presence of fixed hyper-complex structure means at space-time level that the tangent space of X^4 contains fixed M^2 at each point. Under this assumption one can map the points $(m, e) \in M^8$ to points $(m, s) \in H$ by assigning to the point (m, e) of X^4 the point (m, s), where $s \in CP_2$ characterize $T(X^4)$ as hyper-quaternionic plane. This definition is not the only one and even the appropriate one in TGD context the replacement of the tangent plane with the 4-D plane spanned by modified gamma matrices defined by Kähler action is a more natural choice. This plane is not parallel to tangent plane in general. In the sequel $T(X^4)$ denotes the preferred 4-plane which co-incides with tangent plane of X^4 only if the action defining modified gamma matrices is 4-volume.

5. The choice of M^2 can be made also local in the sense that one has $T(X^4) \supset M^2(x) \subset M^4 \subset H$. It turns out that strong form of number theoretic compactification requires this kind of generalization. In this case one must be able to fix the convention how the point of CP_2 is assigned to a hyper-quaternionic plane so that it applies to all possible choices of $M^2 \subset M^4$. Since $SO(3)$ hyper-quaternionic rotation relates the hyper-quaternionic planes to each other, the natural assumption is hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of CP_2. Under this assumption it is possible to map hyper-quaternionic surfaces of M^8 for which $M^2 \subset M^4$ depends on point of X^4 to H.

3.2 Hyper-octonionic Pauli "matrices" and modified definition of hyper-quaternionicity

Hyper-octonionic Pauli matrices suggest an interesting possibility to define precisely what hyper-quaternionicity means at space-time level (for background see [9]).

1. According to the standard definition space-time surface X^4 is hyper-quaternionic if the tangent space at each point of X^4 in $X^4 \subset M^8$ picture is hyper-quaternionic. What raises worries is that this definition involves in no manner the action principle so that it is far from obvious that this identification is consistent with the vacuum degeneracy of Kähler action. It also unclear how one should formulate hyper-quaternionicity condition in $X^4 \subset M^4 \times CP_2$ picture.

2. The idea is to map the modified gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k \gamma_A$, to hyper-octonionic Pauli matrices σ^α by replacing γ_A with hyper-octonion unit. Hyper-quaternionicity would state that the hyper-octonionic Pauli matrices σ^α obtained in this manner span complexified quaternion sub-algebra at each point of space-time. These conditions would provide a number theoretic manner to select preferred extremals of Kähler action. Remarkably, this definition applies both in case of M^8 and $M^4 \times CP_2$.

3. Modified Pauli matrices span the tangent space of X^4 if the action is four-volume because one has $\frac{\partial L_K}{\partial h^k_\alpha} = \sqrt{g} g^{\alpha\beta} \partial h^l_\beta h_{kl}$. Modified gamma matrices reduce to ordinary induced gamma matrices in this case: 4-volume indeed defines a super-conformally symmetric action for ordinary gamma matrices since the mass term of the Dirac action given by the trace of the second fundamental form vanishes for minimal surfaces.

4. For Kähler action the hyper-quaternionic sub-space does not coincide with the tangent space since $\frac{\partial L_K}{\partial h^k_\alpha}$ contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. Modified gamma matrices are required by super conformal symmetry for the extremals of Kähler action and they also guarantee that vacuum extremals defined by surfaces in $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2, are trivially hyper-quaternionic surfaces. The modified definition of hyper-quaternionicity does not affect in any manner $M^8 \leftrightarrow M^4 \times CP_2$ duality allowing purely number theoretic interpretation of standard model symmetries.

A side comment not strictly related to hyper-quaternionicity is in order. The anti-commutators of the modified gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

3.3 Minimal form of $M^8 - H$ duality

The basic problem in the construction of quantum TGD has been the identification of the preferred extremals of Kähler action playing a key role in the definition of the theory. The most elegant manner to do this is by fixing the 4-D tangent space $T(X^4(X^3_l))$ of $X^4(X^3_l)$ at each point of X^3_l so that the boundary value problem is well defined. What I called number theoretical compactification allows to achieve just this although I did not fully realize this in the original vision. The minimal picture is following.

1. The basic observations are following. Let M^8 be endowed with hyper-octonionic structure. For hyper-quaternionic space-time surfaces in M^8 tangent spaces are by definition hyper-quaternionic. If they contain a preferred plane $M^2 \subset M^4 \subset M^8$ in their tangent space, they can be mapped to 4-surfaces in $M^4 \times CP_2$. The reason is that the hyper-quaternionic planes containing preferred the hyper-complex plane M^2 of $M_\pm \subset M^2$ are parameterized by points of CP_2. The map is simply $(m, e) \to (m, s(m, e))$, where m is point of M^4, e is point of E^4, and $s(m, 2)$ is point of

CP_2 representing the hyperquaternionic plane. The inverse map assigns to each point (m, s) in $M^4 \times CP_2$ point m of M^4, undetermined point e of E^4 and 4-D plane. The requirement that the distribution of planes containing the preferred M^2 or $M_\pm$ corresponds to a distribution of planes for 4-D surface is expected to fix the points e. The physical interpretation of M^2 is in terms of plane of non-physical polarizations so that gauge conditions have purely number theoretical interpretation.

2. In principle, the condition that $T(X^4)$ contains M^2 can be replaced with a weaker condition that either of the two light-like vectors of M^2 is contained in it since already this condition assigns to $T(X^4)$ M^2 and the map $H \to M^8$ becomes possible. Only this weaker form applies in the case of massless extremals [12] as will be found.

3. The original idea was that hyper-quaternionic 4-surfaces in M^8 containing $M^2 \subset M^4$ in their tangent space could correspond to preferred extremals of Kähler action. This condition does not seem to be consistent with what is known about the extremals of Kähler action. The weaker form of the hypothesis is that hyper-quaternionicity holds only for 4-D tangent spaces of $X^3_l \subset H = M^4 \times CP_2$ identified as wormhole throats or boundary components lifted to 3-surfaces in 8-D tangent space M^8 of H. The minimal hypothesis would be that only $T(X^4(X^3_l))$ at X^3_l is associative that is hyper-quaternionic for fixed M^2. $X^3_l \subset M^8$ and $T(X^4(X^3_l))$ at X^3_l can be mapped to $X^3_l \subset H$ if tangent space contains also $M_\pm \subset M^2$ or $M^2 \subset M^4 \subset M^8$ itself having interpretation as preferred hyper-complex plane. This condition is not satisfied by all surfaces X^3_l as is clear from the fact that the inverse map involves local E^4 translation. The requirements that the distribution of hyper-quaternionic planes containing M^2 corresponds to a distribution of 4-D tangent planes should fix the E^4 translation to a high degree.

4. A natural requirement is that the image of $X^3_l \subset H$ in M^8 is light-like. The condition that the determinant of induced metric vanishes gives an additional condition reducing the number of free parameters by one. This condition cannot be formulated as a condition on CP_2 coordinate characterizing the hyper-quaternionic plane. Since M^4 projections are same for the two representations, this condition is satisfied if the contributions from CP_2 and E^4 and projections to the induced metric are identical: $s_{kl}\partial_\alpha s^k \partial_\beta s^l = e_{kl}\partial_\alpha e^k \partial_\beta e^l$. This condition means that only a subset of light-like surfaces of M^8 are realized physically. One might argue that this is as it must be since the volume of E^4 is infinite and that of CP_2 finite: only an infinitesimal portion of all possible light-like 3-surfaces in M^8 can can have H counterparts. The conclusion would be that number theoretical compactification is 4-D isometry between $X^4 \subset H$ and $X^4 \subset M^8$ at X^3_l. This unproven conjecture is unavoidable.

5. $M^2 \subset T(X^4(X^3_l))$ condition fixes $T(X^4(X^3_l))$ in the generic case by extending the tangent space of X^3_l, and the construction of configuration space spinor structure fixes boundary conditions completely by additional conditions necessary when X^3_l corresponds to a light-like 3 surfaces defining wormhole throat at which the signature of induced metric changes. What is especially beautiful that only the data in $T(X^4(X^3_l))$ at X^3_l is needed to calculate the vacuum functional of the theory as Dirac determinant: the only remaining conjecture (strictly speaking un-necessary but realistic looking) is that this determinant gives exponent of Kähler action for the preferred extremal and there are excellent hopes for this by the structure of the basic construction.

The basic criticism relates to the condition that light-like 3-surfaces are mapped to light-like 3-surfaces guaranteed by the condition that $M^8 - H$ duality is isometry at X^3_l.

3.4 Strong form of $M^8 - H$ duality

The proposed picture is the minimal one. One can of course ask whether the original much stronger conjecture that the preferred extrema of Kähler action correspond to hyper-quaternionic surfaces could make sense in some form. One can also wonder whether one could allow the choice of the plane M^2 of non-physical polarization to be local so that one would have $M^2(x) \subset M^4 \subset M^4 \times E^4$, where M^4 is fixed hyper-quaternionic sub-space of M^8 and identifiable as M^4 factor of H.

1. If M^2 is same for all points of X^3_l, the inverse map $X^3_l \subset H \to X^3_l \subset M^8$ is fixed apart from possible non-uniquencess related to the local translation in E^4 from the condition that hyper-quaternionic planes represent light-like tangent 4-planes of light-like 3-surfaces. The question is

whether not only X_l^3 but entire four-surface $X^4(X_l^3)$ could be mapped to the tangent space of M^8. By selecting suitably the local E^4 translation one might hope of achieving the achieving this. The conjecture would be that the preferred extrema of Kähler action are those for which the distribution integrates to a distribution of tangent planes.

2. There is however a problem. What is known about extremals of Kähler action is not consistent with the assumption that fixed M^2 of $M_\pm \subset M^2$is contained in the tangent space of X^4. This suggests that one should relax the condition that $M^2 \subset M^4 \subset M^8$ is a fixed hyper-complex plane associated with the tangent space or normal space X^4 and allow M^2 to vary from point to point so that one would have $M^2 = M^2(x)$. In $M^8 \to H$ direction the justification comes from the observation (to be discussed below) that it is possible to uniquely fix the convention assigning CP_2 point to a hyper-quaternionic plane containing varying hyper-complex plane $M^2(x) \subset M^4$.

   Number theoretic compactification fixes naturally $M^4 \subset M^8$ so that it applies to any $M^2(x) \subset M^4$. Under this condition the selection is parameterized by an element of $SO(3)/SO(2) = S^2$. Note that M^4 projection of X^4 would be at least 2-dimensional in hyper-quaternionic case. In co-hyper-quaternionic case E^4 projection would be at least 2-D. $SO(2)$ would act as a number theoretic gauge symmetry and the $SO(3)$ valued chiral field would approach to constant at X_l^3 invariant under global $SO(2)$ in the case that one keeps the assumption that M^2 is fixed ad X_l^3.

3. This picture requires a generalization of the map assigning to hyper-quaternionic plane a point of CP_2 so that this map is defined for all possible choices of $M^2 \subset M^4$. Since the $SO(3)$ rotation of the hyper-quaternionic unit defining M^2 rotates different choices parameterized by S^2 to each other, a natural assumption is that the hyper-quaternionic planes related by $SO(3)$ rotation correspond to the same point of CP_2. Denoting by M^2 the standard representative of M^2, this means that for the map $M^8 \to H$ one must perform $SO(3)$ rotation of hyper-quaternionic plane taking $M^2(x)$ to M^2 and map the rotated plane to CP_2 point. In $M^8 \to H$ case one must first map the point of CP_2 to hyper-quaternionic plane and rotate this plane by a rotation taking $M^2(x)$ to M^2.

4. In this framework local M^2 can vary also at the surfaces X_l^3, which considerably relaxes the boundary conditions at wormhole throats and light-like boundaries and allows much more general variety of light-like 3-surfaces since the basic requirement is that M^4 projection is at least 1-dimensional. The physical interpretation would be that a local choice of the plane of non-physical polarizations is possible everywhere in $X^4(X_l^3)$. This does not seem to be in any obvious conflict with physical intuition.

These observation provide support for the conjecture that (classical) $S^2 = SO(3)/SO(2)$ conformal field theory might be relevant for (classical) TGD.

1. General coordinate invariance suggests that the theory should allow a formulation using any light-like 3-surface X^3 inside $X^4(X_l^3)$ besides X_l^3 identified as union of wormhole throats and boundary components. For these surfaces the element $g(x) \in SO(3)$ would vary also at partonic 2-surfaces X^2 defined as intersections of $\delta CD \times CP_2$ and X^3 (here CD denotes causal diamond defined as intersection of future and past directed light-cones). Hence one could have $S^2 = SO(3)/SO(2)$ conformal field theory at X^2 (regarded as quantum fluctuating so that also $g(x)$ varies) generalizing to WZW model for light-like surfaces X^3.

2. The presence of E^4 factor would extend this theory to a classical $E^4 \times S^2$ WZW model bringing in mind string model with 6-D Euclidian target space extended to a model of light-like 3-surfaces. A further extension to X^4 would be needed to integrate the WZW models associated with 3-surfaces to a full 4-D description. General Coordinate Invariance however suggests that X_l^3 description is enough for practical purposes.

3. The choices of $M^2(x)$ in the interior of X_l^3 is dictated by dynamics and the first optimistic conjecture is that a classical solution of $SO(3)/SO(2)$ Wess-Zumino-Witten model obtained by coupling $SO(3)$ valued field to a covariantly constant $SO(2)$ gauge potential characterizes the choice of $M^2(x)$ in the interior of $M^8 \supset X^4(X_l^3) \subset H$ and thus also partially the structure of the preferred extremal. Second optimistic conjecture is that the Kähler action involving also E^4 degrees of freedom allows to assign light-like 3-surface to light-like 3-surface.

4. The best that one can hope is that M^8-H duality could allow to transform the extremely non-linear classical dynamics of TGD to a generalization of WZW-type model. The basic problem is to understand how to characterize the dynamics of CP_2 projection at each point.

In H picture there are two basic types of vacuum extremals: CP_2 type extremals representing elementary particles and vacuum extremals having CP_2 projection which is at most 2-dimensional Lagrange manifold and representing say hadron. Vacuum extremals can appear only as limiting cases of preferred extremals which are non-vacuum extremals. Since vacuum extremals have so decisive role in TGD, it is natural to requires that this notion makes sense also in M^8 picture. In particular, the notion of vacuum extremal makes sense in M^8.

This requires that Kähler form exist in M^8. E^4 indeed allows full S^2 of covariantly constant Kähler forms representing quaternionic imaginary units so that one can identify Kähler form and construct Kähler action. The obvious conjecture is that hyper-quaternionic space-time surface is extremal of this Kähler action and that the values of Kähler actions in M^8 and H are identical. The elegant manner to achieve this, as well as the mapping of vacuum extremals to vacuum extremals and the mapping of light-like 3-surfaces to light-like 3-surfaces is to assume that M^8-H duality is Kähler isometry so that induced Kähler forms are identical.

This picture contains many speculative elements and some words of warning are in order.

1. Light-likeness conjecture would boil down to the hypothesis that M^8-H correspondence is Kähler isometry so that the metric and Kähler form of X^4 induced from M^8 and H would be identical. This would guarantee also that Kähler actions for the preferred extremal are identical. This conjecture is beautiful but strong.

2. The slicing of $X^4(X^3_l)$ by light-like 3-surfaces is very strong condition on the classical dynamics of Kähler action and does not make sense for pieces of CP_2 type vacuum extremals.

3.4.1 Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative

The 8-dimensionality of M^8 allows to consider both associativity (hyper-quaternionicity) of the tangent space and associativity of the normal space- let us call this co-assosiativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p\simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p\propto\sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k\propto\sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p\rightarrow k$ duality.

3.4.2 Are the known extremals of Kähler action consistent with the strong form of M^8-H duality

It is interesting to check whether the known extremals of Kähler action [12] are consistent with strong form of M^8-H duality assuming that M^2 or its light-like ray is contained in $T(X^4)$ or normal space.

1. CP_2 type vacuum extremals correspond cannot be hyper-quaternionic surfaces but co-hyper-quaternionicity is natural for them. In the same manner canonically imbedded M^4 can be only hyper-quaternionic.

2. String like objects are associative since tangent space obviously contains $M^2(x)$. Objects of form $M^1\times X^3\subset M^4\times CP_2$ do not have M^2 either in their tangent space or normal space in H. So that the map from $H\rightarrow M^8$ is not well defined. There are no known extremals of Kähler action of this type. The replacement of M^1 random light-like curve however gives vacuum extremal

with vanishing volume, which need not mean physical triviality since fundamental objects of the theory are light-like 3-surfaces.

3. For canonically imbedded CP_2 the assignment of $M^2(x)$ to normal space is possible but the choice of $M^2(x) \subset N(CP_2)$ is completely arbitrary. For a generic CP_2 type vacuum extremals M^4 projection is a random light-like curve in $M^4 = M^1 \times E^3$ and $M^2(x)$ can be defined uniquely by the normal vector $n \in E^3$ for the local plane defined by the tangent vector dx^μ/dt and acceleration vector d^2x^μ/dt^2 assignable to the orbit.

4. Consider next massless extremals. Let us fix the coordinates of X^4 as $(t, z, x, y) = (m^0, m^2, m^1, m^2)$. For simplest massless extremals CP_2 coordinates are arbitrary functions of variables $u = k \cdot m = t - z$ and $v = \epsilon \cdot m = x$, where $k = (1, 1, 0, 0)$ is light-like vector of M^4 and $\epsilon = (0, 0, 1, 0)$ a polarization vector orthogonal to it. Obviously, the extremals defines a decomposition $M^4 = M^2 \times E^2$. Tangent space is spanned by the four H-vectors $\nabla_\alpha h^k$ with M^4 part given by $\nabla_\alpha m^k = \delta^k_\alpha$ and CP_2 part by $\nabla_\alpha s^k = \partial_u s^k k_\alpha + \partial_v s^k \epsilon_\alpha$.

   The normal space cannot contain M^4 vectors since the M^4 projection of the extremal is M^4. To realize hyper-quaternionic representation one should be able to from these vector two vectors of M^2, which means linear combinations of tangent vectors for which CP_2 part vanishes. The vector $\partial_t h^k - \partial_z h^k$ has vanishing CP_2 part and corresponds to M^4 vector $(1, -1, 0, 0)$ fix assigns to each point the plane M^2. To obtain M^2 one would need $(1, 1, 0, 0)$ too but this is not possible. The vector $\partial_y h^k$ is M^4 vector orthogonal to ϵ but M^2 would require also $(1, 0, 0, 0)$. The proposed generalization of massless extremals allows the light-like line $M_\pm$ to depend on point of M^4 [12], and leads to the introduction of Hamilton-Jacobi coordinates involving a local decomposition of M^4 to $M^2(x)$ and its orthogonal complement with light-like coordinate lines having interpretation as curved light rays. $M^2(x) \subset T(X^4)$ assumption fails fails also for vacuum extremals of form $X^1 \times X^3 \subset M^4 \times CP_2$, where X^1 is light-like random curve. In the latter case, vacuum property follows from the vanishing of the determinant of the induced metric.

5. The deformations of string like objects to magnetic flux quanta are basic conjectural extremals of Kähler action and the proposed picture supports this conjecture. In hyper-quaternionic case the assumption that local 4-D plane of X^3 defined by modified gamma matrices contains $M^2(x)$ but that $T(X^3)$ does not contain it, is very strong. It states that $T(X^4)$ at each point can be regarded as a product $M^2(x) \times T^2$, $T^2 \subset T(CP_2)$, so that hyper-quaternionic X^4 would be a collection of Cartesian products of infinitesimal 2-D planes $M^2(x) \subset M^4$ and $T^2(x) \subset CP_2$. The extremals in question could be seen as local variants of string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$, where X^2 is minimal surface and Y^2 holomorphic surface of CP_2. One can say that X^2 is replaced by a collection of infinitesimal pieces of $M^2(x)$ and Y^2 with similar pieces of homologically non-trivial geodesic sphere $S^2(x)$ of CP_2, and the Cartesian products of these pieces are glued together to form a continuous surface defining an extremal of Kähler action. Field equations would pose conditions on how $M^2(x)$ and $S^2(x)$ can depend on x. This description applies to magnetic flux quanta, which are the most important must-be extremals of Kähler action.

3.4.3 Geometric interpretation of strong $M^8 - H$ duality

In the proposed framework $M^8 - H$ duality would have a purely geometric meaning and there would nothing magical in it.

1. $X^4(X^3_l) \subset H$ could be seen a curve representing the orbit of a light-like 3-surface defining a 4-D surface. The question is how to determine the notion of tangent vector for the orbit of X^3_l. Intuitively tangent vector is a one-dimensional arrow tangential to the curve at point X^3_l. The identification of the hyper-quaternionic surface $X^4(X^3_l) \subset M^8$ as tangent vector conforms with this intuition.

2. One could argue that M^8 representation of space-time surface is kind of chart of the real space-time surface obtained by replacing real curve by its tangent line. If so, one cannot avoid the question under which conditions this kind of chart is faithful. An alternative interpretation is that a representation making possible to realize number theoretical universality is in question.

3. An interesting question is whether $X^4(X^3_l)$ as orbit of light-like 3-surface is analogous to a geodesic line -possibly light-like- so that its tangent vector would be parallel translated in the sense that $X^4(X^3)$ for any light-like surface at the orbit is same as $X^4(X^3_l)$. This would give justification for the possibility to interpret space-time surfaces as a geodesic of configuration space: this is one of the first -and practically forgotten- speculations inspired by the construction of configuration space geometry. The light-likeness of the geodesic could correspond at the level of X^4 the possibility to decompose the tangent space to a direct sum of two light-like spaces and 2-D transversal space producing the foliation of X^4 to light-like 3-surfaces X^3_l along light-like curves.

4. M^8-H duality would assign to X^3_l classical orbit and its tangent vector at X^3_l as a generalization of Bohr orbit. This picture differs from the wave particle duality of wave mechanics stating that once the position of particle is known its momentum is completely unknown. The outcome is however the same: for X^3_l corresponding to wormhole throats and light-like boundaries of X^4, canonical momentum densities in the normal direction vanish identically by conservation laws and one can say that the the analog of (q,p) phase space as the space carrying wave functions is replaced with the analog of subspace consisting of points $(q,0)$. The dual description in M^8 would not be analogous to wave functions in momentum space space but to those in the space of unique tangents of curves at their initial points.

3.4.4 The Kähler and spinor structures of M^8

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action. As found, this leads to the conclusion that the$M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The free Kähler forms could thus allow to produce M^8 counterparts of these gauge potentials possessing same couplings as their H counterparts. This picture would produce parity breaking in M^8 picture correctly.

4. Only the charged parts of classical electro-weak gauge fields would be absent. This would conform with the standard thinking that charged classical fields are not important. The predicted classical W fields is one of the basic distinctions between TGD and standard model and in this framework. A further prediction is that this distinction becomes visible only in situations, where H picture is necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

5. Also super-symmetries of quantum TGD crucial for the construction of configuration space geometry force this picture. In the absence of coupling to Kähler gauge potential all constant spinor fields and their conjugates would generate super-symmetries so that M^8 would allow $N = 8$ super-symmetry. The introduction of the coupling to Kähler gauge potential in turn means that all covariantly constant spinor fields are lost. Only the representation of all three neutral parts of electro-weak gauge potentials in terms of three independent Kähler gauge potentials allows right-handed neutrino as the only super-symmetry generator as in the case of H.

6. The $SO(3)$ element characterizing $M^2(x)$ is fixed apart from a local $SO(2)$ transformation, which suggests an additional $U(1)$ gauge field associated with $SO(2)$ gauge invariance and representable as Kähler form corresponding to a quaternionic unit of E^4. A possible identification of this gauge field would be as a part of electro-weak gauge field.

3.4.5 M^8 dual of configuration space geometry and spinor structure?

If one introduces M^8 spinor structure and preferred extremals of M^8 Kähler action, one cannot avoid the question whether it is possible or useful to formulate the notion of configuration space geometry and spinor structure for light-like 3-surfaces in M^8 using the exponent of Kähler action as vacuum functional.

1. The isometries of the configuration space in M^8 and H formulations would correspond to symplectic transformation of $\delta M^4_\pm \times E^4$ and $\delta M^4_\pm \times CP_2$ and the Hamiltonians involved would belong to the representations of $SO(4)$ and $SU(3)$ with 2-dimensional Cartan sub-algebras. In H picture color group would be the familiar $SU(3)$ but in M^8 picture it would be $SO(4)$. Color confinement in both $SU(3)$ and $SO(4)$ sense could allow these two pictures without any inconsistency.

2. For $M^4 \times CP_2$ the two spin states of covariantly constant right handed neutrino and antineutrino spinors generate super-symmetries. This super-symmetry plays an important role in the proposed construction of configuration space geometry. As found, this symmetry would be present also in M^8 formulation so that the construction of M^8 geometry should reduce more or less to the replacement of CP_2 Hamiltonians in representations of $SU(3)$ with E^4 Hamiltonians in representations of $SO(4)$. These Hamiltonians can be taken to be proportional to functions of E^4 radius which is $SO(4)$ invariant and these functions bring in additional degree of freedom.

3. The construction of Dirac determinant identified as a vacuum functional can be done also in M^8 picture and the conjecture is that the result is same as in the case of H. In this framework the construction is much simpler due to the flatness of E^4. In particular, the generalized eigen modes of the Dirac operator $D_K(Y^3_l)$ restricted to the X^3_l correspond to a situation in which one has fermion in induced Maxwell field mimicking the neutral part of electro-weak gauge field in H as far as couplings are considered. Induced Kähler field would be same as in H. Eigen modes are localized to regions inside which the Kähler magnetic field is non-vanishing and apart from the fact that the metric is the effective metric defined in terms of canonical momentum densities via the formula $\hat{\Gamma}^\alpha = \partial L_K/\partial h^k_\alpha \Gamma_k$ for effective gamma matrices. This in fact, forces the localization of modes implying that their number is finite so that Dirac determinant is a product over finite number eigenvalues. It is clear that M^8 picture could dramatically simplify the construction of configuration space geometry.

4. The eigenvalue spectra of the transversal parts of D_K operators in M^8 and H should identical. This motivates the question whether it is possible to achieve a complete correspondence between H and M^8 pictures also at the level of spinor fields at X^3 by performing a gauge transformation eliminating the classical W gauge boson field altogether at X^3_l and whether this allows to transform the modified Dirac equation in H to that in M^8 when restricted to X^3_l. That something like this might be achieved is supported by the fact that in Coulomb gauge the component of gauge potential in the light-like direction vanishes so that the situation is effectively 2-dimensional and holonomy group is Abelian.

3.4.6 Why $M^8 - H$ duality is useful?

Skeptic could of course argue that $M^8 - H$ duality produces only an inflation of unproven conjectures. There are however strong reasons for $M^8 - H$ duality: both theoretical and physical.

1. The map of $X^3_l \subset H \to X^3_l \subset M^8$ and corresponding map of space-time surfaces would allow to realize number theoretical universality. $M^8 = M^4 \times E^4$ allows linear coordinates as natural coordinates in which one can say what it means that the point of imbedding space is rational/algebraic. The point of $X^4 \subset H$ is algebraic if it is mapped to an algebraic point of M^8 in number theoretic compactification. This of course restricts the symmetry groups to their rational/algebraic variants but this does not have practical meaning. Number theoretical compactification could in fact be motivated by the number theoretical universality.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action since the Kähler form in E^4 has constant components. If the spinor connection in E^4 is combination of the three Kähler forms mimicking neutral part of electro-weak gauge potential, the eigenvalue spectrum for the modified Dirac operator would correspond to that for a fermion in $U(1)$ magnetic field defined by an Abelian magnetic field whereas in $M^4 \times CP_2$ picture $U(2)_{ew}$ magnetic fields would be present.

3. $M^8 - H$ duality provides insights to low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

3.5 $M^8 - H$ duality and low energy hadron physics

The description of $M^8 - H$ at the configuration space level can be applied to gain a view about color confinement and its dual for electro-weak interactions at short distance limit. The basic idea is that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quark color using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since configuration space degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [15].

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

4 Quaternions, octonions, and modified Dirac equation

Classical number fields define one vision about quantum TGD. This vision about quantum TGD has evolved gradually and involves several speculative ideas.

1. The hard core of the vision is that space-time surfaces as preferred extremals of Kähler action can be identified as what I have called hyper-quaternionic surfaces of M^8 or $M^4 \times CP_2$. This requires only the mapping of the modified gamma matrices to octonions or to a basis of subspace of complexified octonions. This means also the mapping of spinors to octonionic spinors. There is no need to assume that imbedding space-coordinates are octonionic.

2. I have considered also the idea that quantum TGD might emerge from the mere associativity.

   (a) Consider Clifford algebra of WCW. Treat "vibrational" degrees of freedom in terms second quantized spinor fields and add center of mass degrees of freedom by replacing 8-D gamma matrices with their octonionic counterparts - which can be constructed as tensor products of octonions providing alternative representation for the basis of 7-D Euclidian gamma matrix algebra - and of 2-D sigma matrices. Spinor components correspond to tensor products of octonions with 2-spinors: different spin states for these spinors correspond to leptons and baryons.

   (b) Construct a local Clifford algebra by considering Clifford algebra elements depending on point of M^8 or H. The octonionic 8-D Clifford algebra and its local variant are non-accociative. Associative sub-algebra of 8-D Clifford algebra is obtained by restricting the elements so any quaternionic 4-plane. Doing the same for the local algebra means restriction of the Clifford algebra valued functions to any 4-D hyper-quaternionic sub-manifold of M^8 or H which means that the gamma matrices span complexified quaternionic algebra at each point of space-time surface. Also spinors must be quaternionic.

   (c) The assignment of the 4-D gamma matrix sub-algebra at each point of space-time surface can be done in many manners. If the gamma matrices correspond to the tangent space of space-time surface, one obtains just induced gamma matrices and the standard definition of quaternionic sub-manifold. In this case induced 4-volume is taken as the action principle. If Kähler action defines the space-time dynamics, the modified gamma matrices do not span the tangent space in general.

   (d) An important additional element is involved. If the M^4 projection of the space-time surface contains a preferred subspace M^2 at each point, the quaternionic planes are labeled by points of CP_2 and one can equivalently regard the surfaces of M^8 as surfaces of $M^4 \times CP_2$ (number-theoretical "compactification"). This generalizes: M^2 can be replaced with a distribution of planes of M^4 which integrates to a 2-D surface of M^4 (for instance, for string like objects this is necessarily true). The presence of the preferred local plane M^2

corresponds to the fact that octonionic spin matrices Σ_{AB} span 14-D Lie-algebra of $G_2 \subset SO(7)$ rather than that 28-D Lie-algebra of $SO(7,1)$ whereas octonionic imaginary units provide 7-D fundamental representation of G_2. Also spinors must be quaternionic and this is achieved if they are created by the Clifford algebra defined by induced gamma matrices from two preferred spinors defined by real and preferred imaginary octonionic unit. Therefore the preferred plane $M^3 \subset M^4$ and its local variant has direct counterpart at the level of induced gamma matrices and spinors.

(e) This framework implies the basic structures of TGD and therefore leads to the notion of world of classical worlds (WCW) and from this one ends up with the notion WCW spinor field and WCW Clifford algebra and also hyper-finite factors of type II_1 and III_1. Note that M^8 is exceptional: in other dimensions there is no reason for the restriction of the local Clifford algebra to lower-dimensional sub-manifold to obtain associative algebra.

The above line of ideas leads naturally to (hyper-)quaternionic sub-manifolds and to basic quantum TGD (note that the "hyper" is un-necessary if one accepts just the notion of quaternionic sub-manifold formulated in terms of modified gamma matrices). One can pose some further questions.

1. Quantum TGD reduces basically to the second quantization of the induced spinor fields. Could it be that the theory is integrable only for 4-D hyper-quaternionic space-time surfaces in M^8 (equivalently in $M^4 \times CP_2$) in the sense than one can solve the modified Dirac equation exactly only in these cases?

2. The construction of quantum TGD -including the construction of vacuum functional as exponent of Kähler function reducing to Kähler action for a preferred extremal - should reduce to the modified Dirac equation defined by Kähler action. Could it be that the modified Dirac equation can be solved exactly only for Kähler action.

3. Is it possible to solve the modified Dirac equation for the octonionic gamma matrices and octonionic spinors and map the solution as such to the real context by replacing gamma matrices and sigma matrices with their standard counterparts? Could the associativity conditions for octospinors and modified Dirac equation allow to pin down the form of solutions to such a high degree that the solution can be constructed explicitly?

4. Octonionic gamma matrices provide also a non-associative representation for 8-D version of Pauli sigma matrices and encourage the identification of 8-D twistors as pairs of octonionic spinors conjectured to be highly relevant also for quantum TGD. Does the quaternionicity condition imply that octo-twistors reduce to something closely related to ordinary twistors as the fact that 2-D sigma matrices provide a matrix representation of quaternions suggests?

In the following I will try to answer these questions by developing a detailed view about the octonionic counterpart of the modified Dirac equation and proposing explicit solution ansätze for the modes of the modified Dirac equation.

4.1 The replacement of $SO(7,1)$ with G_2

The basic implication of octonionization is the replacement of $SO(7,1)$ as the structure group of spinor connection with G_2. This has some rather unexpected consequences.

4.1.1 Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1 \times \sigma_1 \ \ , \ \ \gamma^i = \gamma^i \otimes \sigma_2 \ \ , i = 1,..,7 \ . \tag{4.1}$$

7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i , i = 1, ..., 6 \ , \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6} \gamma^{6)}_i \ . \qquad (4.2)$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1 \times \sigma_1 \ , \ \gamma_i = e_i \otimes \sigma_2 \ . \qquad (4.3)$$

where e_i are the octonionic units. $e_i^2 = -1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod \gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = e_i \times \sigma_3 \ , \ \Sigma_{ij} = f_{ij}{}^k e_k \otimes 1 \ . \qquad (4.4)$$

These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension of the G_2 algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [27] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I^3_L = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [A8]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

4.1.2 Some physical implications of $SO(7,1) \rightarrow G_2$ reduction

This has interesting physical implications if one believes that the octonionic description is equivalent with the standard one.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics. This might provide a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

   Also the realization of $M^8 - H$ duality led to the conclusion M^8 spinor connection should have only neutral components. The isospin matrix associated with the electromagnetic charge is $e_1 \times 1$ and represents the preferred imaginary octonionic unit so that that the image of the electro-weak gauge algebra respects associativity condition. An open question is whether octonionization is part of M^8-H duality or defines a completely independent duality. The objection is that information is lost in the mapping so that it becomes questionable whether the same solutions to the modified Dirac equation can work as a solution for ordinary Clifford algebra.

2. If $SU(2)_R$ were mapped to zero only left handed parts of the gauge fields would remain. All classical gauge fields would remain in the spectrum so that information would not be lost. The identification of the electro-weak gauge fields as three covariantly constant quaternionic units would be possible in the case of M^8 allowing Hyper-Kähler structure [19], which has been speculated to be a hidden symmetry of quantum TGD at the level of WCW. This option would lead to difficulties with associativity since the action of the charged gauge potentials would lead out from the local quaternionic subspace defined by the octonionic spinor.

3. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom! Since the resulting interactions are of gravitational character, one might say that electro-weak interactions are mapped to manifestly gravitational interactions. Since $SU(2)$ corresponds to rotational group one cannot say that spinor connection would give rise only to left or right handed couplings, which would be obviously a disaster.

4.1.3 Octo-spinors and their relation to ordinary imbedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\begin{aligned} \Psi_{L,i} &= e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \\ \Psi_{q,i} &= e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \end{aligned} \tag{4.5}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1 + 1 + 3 + \overline{3}$ as representations of $SU(3) \subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1 \pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin } 1/2 \ , \\ \{e_2 \pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -}1/2 \ , \\ \{e_4 \pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ , \\ \{e_6 \pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin } 1/2 \ . \end{array} \tag{4.6}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon = \pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon = 1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon = -1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions. The only difference between quarks and leptons is that the induced Kähler gauge potentials couple to them differently.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic modified Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2 \subset M^4$.

4.2 Octonionic counterpart of the modified Dirac equation

The solution ansatz for the octonionic counterpart of the modified Dirac equation discussed below makes sense also for ordinary modified Dirac equation which raises the hope that the same ansatz, and even same solution could provide a solution in both cases.

4.2.1 The general structure of the modified Dirac equation

In accordance with quantum holography and the notion of generalized Feynman diagram, the modified Dirac equation involves two equations which must be consistent with each other.

1. There is 3-dimensional generalized eigenvalue equation for which the modified gamma matrices are defined by Chern-Simons action [31] defined by the CP_2 Kähler form J.

$$\begin{aligned} D_3\Psi &= [D_{C-S} + Q_{C-S}]\,\Psi = \lambda^k \gamma_k \Psi \ , \\ Q_{C-S} &= Q_\alpha \hat{\Gamma}^\alpha_{C-S} \ , \ Q_\alpha = Q_A g^{AB} j_{B\alpha} \ . \end{aligned} \tag{4.7}$$

   The gamma matrices γ_k are M^4 gamma matrices in standard Minkowski coordinates and thus constant. Given eigenvalue λ_k defines pseudo momentum which is some function of the genuine momenta p_k and other quantum numbers via the boundary conditions associated with the generalized eigenvalue equation.

   The charges Q_A correspond to real four-momentum and charges in color Cartan algebra. The term Q can be rather general since it provides a representation for the measurement interaction by mapping observables to Cartan algebra of isometry group and to the infinite hierarchy of conserved currents implied by quantum criticality. The operator O characterizes the quantum critical conserved current. The surface Y^3_l can be chosen to be any light-like 3-surface "parallel" to the wormhole throat in the slicing of X^4: this means an additional symmetry. Formally the measurement interaction term can be regarded as an addition of a gauge term to the Kähler gauge potential associated with the Kähler form J or CP_2.

   The square of the equation gives the spinor analog of d'Alembert equation and generalized eigenvalue as the analog of mass squared. The propagator associated with the wormhole throats is formally massless Dirac propagator so that standard twistor formalism applies also without the octonionic representation of the gamma matrices although the physical particles propagating along the opposite wormhole throats are massive on mass shell particles with both signs of energy [7].

2. Second equation is the 4-D modified Dirac equation defined by Kähler action.

$$D_K\Psi = 0 \ . \tag{4.8}$$

   The dimensional reduction of this operator to a sum corresponding to $D_{K,3}$ acting on light-like 3-surfaces and 1-D operator $D_{K,1}$ acting on the coordinate labeling the 3-D light-like 3-surfaces in the slicing would allow to assign eigenvalues to $D_{K,3}$ as analogs of energy eigenvalues for ordinary Schrödinger equation. One proposal has been that Dirac determinant could be identified as the product of these eigen values. Another and more plausible identification is as the product of pseudo masses assignable to D_3 defined by Chern-Simons action. It must be however made clear that the identification of the exponent of the Kähler function to Chern-Simons term makes the identification as Dirac determinant un-necessary.

3. There are two options depending on whether one requires that the eigenvalue equation applies only on the wormhole throats and at the ends of the space-time surface or for all 3-surfaces in the slicing of the space-time surface by light-like 3-surfaces. In the latter case the condition that the pseudo four-momentum is same for all the light-like 3-surfaces in the slicing gives a consistency condition stating that the commutator of the two Dirac operators vanishes for the solutions in the case of preferred extremals, which depend on the momentum and color quantum numbers also:

$$[D_K, D_3]\,\Psi = 0 \ . \tag{4.9}$$

This condition is quite strong and there is no deep reason for it since λ_k does not correspond to the physical conserved momentum so that its spectrum could depend on the light-like 3-surface in the slicing. On the other hand, if the eigenvalues of D_3 belong to the preferred hyper-complex plane M^2, D_3 effectively reduces to a 2-dimensional algebraic Dirac operator $\lambda^k\gamma_k$ commuting with D_K: the values of λ^k cannot depend on slice since this would mean that D_K does not commute with D_3.

4.2.2 About the hyper-octonionic variant of the modified Dirac equation

What gives excellent hopes that the octonionic variant of modified Dirac equation could lead to a provide precise information about the solution spectrum of modified Dirac equation is the condition that everything in the equation should be associative. Hence the terms which are by there nature non-associative should vanish automatically.

1. The first implication is that the besides octonionic gamma matrices also octonionic spinors should belong to the local quaternionic plane at each point of the space-time surface. Spinors are also generated by quaternionic Clifford algebra from two preferred spinors defining a preferred plane in the space of spinors. Hence spinorial dynamics seems to mimic very closely the space-time dynamics and one might even hope that the solutions of the modified Dirac action could be seen as maps of the space-time surface to surfaces of the spinor space. The reduction to quaternionic sub-algebra suggest that some variant of ordinary twistors emerges in this manner in matrix representation.

2. The octonionic sigma matrices span G_2 where as ordinary sigma matrices define $SO(7,1)$. On the other hand, the holonomies are identical in the two cases if right-handed charge matrices are mapped to zero so that there are indeed hopes that the solutions of the octonionic Dirac equation cannot be mapped to those of ordinary Dirac equation. If left-handed charge matrices are mapped to zero, the resulting theory is essentially the analog of electrodynamics coupled to gravitation at classical level but it is not clear whether this physically acceptable. It is not clear whether associativity condition leaves only this option under consideration.

3. The solution ansatz to the modified Dirac equation is expected to be of the form $\Psi = D_K(\Psi_0 u_0 + \Psi_1 u_1)$, where u_0 and u_1 are constant spinors representing real unit and the preferred unit e_1. Hence constant spinors associated with right handed electron and neutrino and right-handed d and u quark would appear in Ψ and Ψ_i could correspond to scalar coefficients of spinors with different charge. This ansatz would reduce the modified Dirac equation to $D_K^2\Psi_i = 0$ since there are no charged couplings present. The reduction of a d'Alembert type equation for single scalar function coupling to $U(1)$ gauge potential and $U(1)$ "gravitation" would obviously mean a dramatic simplification raising hopes about integrable theory.

4. The condition $D_K^2\Psi = 0$ involves products of three octonions and involves derivatives of the modified gamma matrices which might belong to the complement of the quaternionic sub-space. The restriction of Ψ to the preferred hyper-complex plane M^2 simplifies the situation dramatically but $(D_K^2)D_K\Psi = D_K(D_K^2)\Psi = 0$ could still fail. The problem is that the action of D_K is not algebraic so that one cannot treat reduce the associativity condition to $(AA)A = A(AA)$.

4.3 Could the notion of octo-twistor make sense?

Twistors have led to dramatic successes in the understanding of Feynman diagrammatics of gauge theories, $N = 4$ SUSYs, and $N = 8$ supergravity [35, 36, 37]. This motivated the question whether they might be applied in TGD framework too [9] - at least in the description of the QFT limit. The basic problem of the twistor program is how to overcome the difficulties caused by particle massivation and TGD framework suggests possible clues in this respect.

1. In TGD it is natural to regard particles as massless particles in 8-D sense and to introduce 8-D counterpart of twistors by relying on the geometric picture in which twistors correspond to a pair of spinors characterizing light-like momentum ray and a point of M^8 through which the ray traverses. Twistors would consist of a pair of spinors and quark and lepton spinors define the natural candidate for the spinors in question. This approach would allow to handle massive on-mass-shell states but cannot cope with virtual momenta massive in 8-D sense.

2. The emergence of pseudo momentum λ_k from the generalized eigenvalue equation for D_{C-S} suggest a dramatically simpler solution to the problem. Since propagators are effectively massless propagators for pseudo momenta, which are functions of physical on shell momenta (with both signs of energy in zero energy ontology) and of other quantum numbers, twistor formalism can be applied in its standard form. An attractive assumption is that also λ^k are conserved in the vertices but a good argument justifying this is lacking. One can ask whether also $N = 4$ SUSY, $N = 8$ super-gravity, and even QCD could have similar interpretation.

This picture should apply also in the case of octotwistors with minor modifications and one might hope that octotwistors could provide new insights about what happens in the real case.

1. In the case of ordinary Clifford algebra unit matrix and six-dimensional gamma matrices γ_i, $i = 1, ..., 6$ and $\gamma_7 = \prod_i \gamma_i$ would define the variant of Pauli sigma matrices as $\sigma_0 = 1$, $\sigma_k = \gamma_k$, $k = 1, .., 7$ The problem is that masslessness condition does not correspond to the vanishing of the determinant for the matrix $p_k \sigma^k$.

2. In the case of octo-twistors Pauli sigma matrices σ^k would correspond to hyper-octonion units $\{\sigma_0, \sigma_k\} = \{1, ie^k\}$ and one could assign to $p_k \sigma^k$ a matrix by the linear map defined by the multiplication with $P = p_k \sigma^k$. The matrix is of form $P_{mn} = p^k f_{kmn}$, where f_{kmn} are the structure constants characterizing multiplication by hyper-octonion. The norm squared for octonion is the fourth root for the determinant of this matrix. Since $p_k \sigma^k$ maps its octonionic conjugate to zero so that the determinant must vanish (as is easy to see directly by reducing the situation to that for hyper-complex numbers by considering the hyper-complex plane defined by P).

3. Associativity condition for the octotwistors requires that the gamma matrix basis appearing in the generalized eigenvalue equation for Chern-Simons Dirac operator must differs by a local G_2 rotation from the standard hyper-quaternionic gamma matrix for M^4 so that it is always in the local hyper-quaternionic plane. This suggests that octo-twistor can be mapped to an ordinary twistor by mapping the basis of hyper-quaternions to Pauli sigma matrices. A stronger condition guaranteeing the commutativity of D_3 with $\lambda^k \gamma_k$ is that λ_k belongs to a preferred hyper-complex plane M^2 assignable to a given CD. Also the two spinors should belong to this plane for the proposed solution ansatz for the modified Dirac equation. Quaternionization would also allow to assign momentum to the spinors in standard manner.

   The spectrum of pseudo-momenta would be 2-dimensional (continuum at worst) and this should certainly improve dramatically the convergence properties for the sum over the non-conserved pseudo-momenta in propagators which in the worst possible of worlds might destroy the manifest finiteness of the theory based on the generalized Feynman diagrams with the throats of wormholes carrying always on mass shell momenta. This effective 2-dimensionality should apply also in the real case and would have no catastrophic consequences since pseudo momenta are in question.

   As a matter fact, the assumption the decomposition of quark momenta to longitudinal and transversal parts in perturbative QCD might have interpretation in terms of pseudo-momenta if they are conserved.

4. $M^8 - H$ duality suggests a possible interpretation of the pseudo-momenta as M^8 momenta which by purely number theoretical reasons must be commutative and thus belong to M^2 hyper-complex plane. One ends up with the similar outcome as one constructs a representation for the quantum states defined by WCW spinor fields as superpositions of real units constructed as ratios of infinite hyper-octonionic integers with precisely defined number theoretic anatomy and transformation properties under standard model symmetries having number theoretic interpretation [14, A7].

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