

Source-free electromagnetism's canonical fields reveal the free-photon Schrödinger equation

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Abstract

Classical equations of motion that are first-order in time and conserve energy can only be quantized after their variables have been transformed to canonical ones, i.e., variables in which the energy is the system's Hamiltonian. The source-free version of Maxwell's equations is purely dynamical, first-order in time and has a well-defined nonnegative conserved field energy, but is decidedly noncanonical. That should long ago have made source-free Maxwell equation canonical Hamiltonization a research priority, and afterward, standard textbook fare, but textbooks seem unaware of the issue. The opposite parities of the electric and magnetic fields and consequent curl operations that typify Maxwell's equations are especially at odds with their being canonical fields. Transformation of the magnetic field into the transverse part of the vector potential helps but is not sufficient; further simple nonnegative symmetric integral transforms, which commute with all differential operators, are needed for both fields; such transforms also supplant the curls in the equations of motion. The canonical replacements of the source-free electromagnetic fields remain transverse-vector fields, but are more diffuse than their predecessors, albeit less diffuse than the transverse vector potential. Combined as the real and imaginary parts of a complex field, the canonical fields prove to be the transverse-vector wave function of a time-dependent Schrödinger equation whose Hamiltonian operator is the quantization of the free photon's square-root relativistic energy. Thus proper quantization of the source-free Maxwell equations is identical to second quantization of free photons that have normal square-root energy. There is no physical reason why first and second quantization of any relativistic free particle ought not to proceed in precise parallel, utilizing the square-root Hamiltonian operator. This natural procedure leaves no role for the completely artificial Klein-Gordon and Dirac equations, as accords with their grossly unphysical properties.

Introduction

Notwithstanding the approximately century and a half which has passed since both the development of Hamiltonian classical dynamics and also the final codification of the laws which govern electromagnetic fields in configuration space, canonical Hamiltonian formulation of the purely dynamical source-free instance of electromagnetic field theory in configuration space unaccountably still lingers as essentially terra incognita. This is all the stranger insofar as proper canonical Hamiltonization of a classical dynamical system is a fundamental prerequisite to its rigorous unambiguous quantization, whether by the Hamiltonian phase-space path integral [1], or by the self-consistent (slight) extension of Dirac's canonical commutation rule [2].

A crucial aspect of correct canonical Hamiltonization of the source-free instance of Maxwell's field equations in configuration space is that it is subject *not only* to those equations *themselves*, but *also* to the fact that the conserved energy content of the source-free (transverse) dynamical electromagnetic fields is a very specific *known functional* of those fields. This specific conserved field-energy functional *necessarily* becomes the field system's *Hamiltonian functional* when its fields have been transformed to *canonical* ones, i.e., we shall have transformed the fields to canonical ones when the field equations expressed in terms of

the transformed fields *exactly match* the Hamiltonian equations of motion for the transformed fields which result from taking the Hamiltonian to be the conserved field-energy functional as expressed in terms of the transformed fields. Therefore the fact that the conserved field-energy functional is *known a priori* places a *strong constraint* on just what the canonical fields can be.

In the source-free case, Maxwell’s field equations in configuration space are strictly linear and homogeneous, so it is not surprising that we can restrict ourselves to purely linear transformations of the dynamical magnetic and transverse electric fields in the search for properly canonical fields. (The longitudinal electric field is *never* dynamical in character, and it *vanishes identically* in the source-free case.) It as well turns out to be unnecessary to mix the magnetic field with the transverse electric field in the course of this search. But the axial/polar dichotomy between the magnetic field and the transverse electric field turns out to be incompatible with fields that are properly canonical; this failing is readily rectified by a one-to-one orthogonal linear mapping of the axial-vector magnetic field onto a transverse polar-vector counterpart. The resulting transformed field equations, however, still fail to match the Hamiltonian equations of motion which follow from the transformed conserved field-energy functional, but a further nonnegative symmetric linear transformation of all fields, which commutes with all partial derivative operators and doesn’t affect the field equations, repairs this disparity. This last transformation is decidedly *nonlocal* however: the still transverse *canonical* fields turn out to be *diffused* relative to the transverse electric and magnetic fields—albeit to a *lesser extent* than the transverse *vector potential* is *likewise diffused* relative to the magnetic field (the Aharonov-Bohm effect is a quite dramatic manifestation of that particular relative diffusion).

The transverse canonical vector fields are not only diffused relative to the transverse dynamical electric and magnetic fields; they also have a different dimensionality, namely that of the square root of action density rather than that of the square root of energy density, which is the dimensionality of electric and magnetic fields. If the two transverse canonical vector fields are divided by $(2\hbar)^{\frac{1}{2}}$ and then combined as the real and imaginary parts of a complex transverse-vector field, it is seen that this complex-valued field satisfies the natural time-dependent Schrödinger equation in configuration representation whose Hamiltonian operator is $\hbar c(-\nabla^2)^{\frac{1}{2}}$, namely $|c\hat{\mathbf{p}}|$, the Hamiltonian operator for a first-quantized free solitary ultrarelativistic zero-mass particle that is the zero-mass limit of the natural correspondence-principle-mandated square-root relativistic free-particle Hamiltonian $(m^2c^4 + |c\hat{\mathbf{p}}|^2)^{\frac{1}{2}}$. Moreover, the dimensionality of the complex-valued transverse-vector field in this Schrödinger equation is that of the square root of probability density, which is appropriate for a Schrödinger equation wave function. In other words, the two transverse *canonical* vector fields of source-free electromagnetic field theory comprise the *natural basis* for the complex-valued transverse-vector *wave-function description* of the *first-quantized free solitary photon*. Source-free electromagnetic field theory *is* in this way precisely the time-dependent Schrödinger-equation description of the first-quantized free solitary photon with its natural square-root Hamiltonian operator, but that stark fact simply *does not become technically manifest* until this field theory has been *properly canonically Hamiltonized*!

It is delightfully amazing that James Clerk Maxwell, building atop the foundation laid by Michael Faraday’s experimental results, effectively discovered the Schrödinger equation, indeed for a tricky transversely-polarized spin 1 ultrarelativistic particle, very long before Erwin Schrödinger’s own nonrelativistic quantum insights. Remarkably, Maxwell could accomplish this feat with no knowledge whatsoever of the quantum of action—whose discovery by Max Planck still lay well in the future—because of the intriguing happenstance that in configuration representation the photon’s zero-mass property “releases” a factor of \hbar from its Hamiltonian operator that neatly *cancels out* the factor of \hbar which is a *fixture* of the left-hand side of the time-dependent Schrödinger equation. Thus does ultrarelativistic quantum mechanics chameleon-like metamorphose into “classical field theory”! The *technical details* of this connection *can not*, of course, be fully laid bare until Maxwell’s effective formulation of the theory in terms of the axial magnetic and transverse-polar electric fields has been *properly canonically Hamiltonized*—the two corresponding transverse canonical fields have the same (not the opposite) parity, are relatively somewhat more diffused than their antecedent magnetic and electric fields, albeit less so than is the case for the transverse vector potential relative to its antecedent magnetic field, and are joined together in a single complex-valued transverse-vector wave function that describes the first-quantized free solitary photon.

In the world of quantum theory, canonical Hamiltonization becomes merely the required *prelude* to quantization; this of course also applies to the quantization of source-free electromagnetic field theory. After completion of canonical Hamiltonization, canonical quantization parlays the Poisson bracket relations of the fundamental canonical dynamical variables into commutation relations. In canonically Hamiltonized source-free electromagnetic field theory, the fundamental canonical dynamical variables are the vector components of the two transverse-vector canonical fields themselves. Thus it is the components of these transverse-vector canonical fields themselves that quantization promotes into noncommuting Hermitian operators; these

operators inherit from their simple mutual Poisson bracket relations equally simple mutual commutation relations. The vector components of the free solitary-photon wave functions, being straightforward complex linear combinations of the corresponding vector components of the two real transverse-vector canonical fields, become non-Hermitian noncommuting operators such that there are as well simple commutation relations between the vector components of the quantized wave function and those of its *Hermitian conjugate* (the *transverse character* of these canonical vector fields and complex vector wave functions subtracts an annoying longitudinal projection-operator term from their formal configuration-representation commutators, which creates an ugly and potentially confusing distraction that is merely technical in nature). The Hamiltonian functional *also* becomes quantized via its bilinear dependence on the real canonical fields (or, as well, via its linear dependence on both the complex photon wave function and its complex conjugate), and is a Hermitian operator. As is the normal case in quantum theory, the Heisenberg picture with respect to this Hamiltonian operator manifests *the same dynamical equations for the quantized field operators as they obeyed prior to their quantization*. Thus the *quantized* wave function operator *still satisfies the very same Schrödinger equation* that it satisfied *prior* to its quantization, *which can thus properly be termed second quantization*. This Schrödinger equation in its *second-quantized* form obviously *still features the very same* $m \rightarrow 0$ limit of the relativistic *square-root* Hamiltonian operator $(m^2c^4 + |\widehat{\mathbf{c}}\mathbf{p}|^2)^{\frac{1}{2}}$ (which comes to $\hbar c(-\nabla^2)^{\frac{1}{2}}$ in configuration representation) that it featured *prior* to that quantization! Finally, the simple commutation relation between the quantized non-Hermitian wave function operator and its Hermitian conjugate is such that the quantized wave function operator is interpretable as the annihilation operator for free photon states in the underlying Hilbert space (commonly called Fock space), while its Hermitian conjugate is likewise interpretable as the creation operator for such free photon states.

This *second-quantized* theory of *arbitrarily many* free photons, or, *equivalently*, quantized theory of source-free electromagnetism, is not, of course, the end of the physics story. We have so far *postponed* consideration of the coupling of *charged matter* to the dynamical transverse part of the electromagnetic fields, i.e., to photons. In the rather global technical language of the Maxwell equations, this coupling (in fact an inhomogeneous driving term) can be mathematically abstracted in terms of the transverse part of the current-density vector field. From a microscopic charged-particle perspective, however, relativistic coupling to electromagnetism occurs via the four-vector potential A^μ . The part of A^μ which is related to the dynamical transverse electromagnetic fields is, naturally enough, the transverse part of the vector potential \mathbf{A} , which we denote as \mathbf{A}_T . Indeed the precise linear relationship of \mathbf{A}_T to the complex-valued transverse-vector photon wave function and its complex conjugate can be readily traced. The *remaining* A^0 and *longitudinal* part of \mathbf{A} are related 1) to the nondynamical longitudinal part of the electric field, which is a mere homogeneous functional of the global charge density, and 2) to a nonphysical “gauge degree of freedom”. Thus they bear no direct relation to the photon wave function. Because the photon wave function and its complex conjugate become operators upon second quantization, \mathbf{A}_T becomes a linearly related operator as well, in fact a Hermitian one that can either create or annihilate a free photon state in the underlying Fock space. This detailed tying of \mathbf{A}_T to the free-photon annihilation and creation operators is the technical way in which the direct interaction between photons and charged particles finds effect in interacting second-quantized theory. Such a theory must of course as well include the above-described second-quantized Hamiltonian functional operator of source-free electromagnetism, i.e., the Hamiltonian operator for the for an arbitrary number of noninteracting free photons.

What about A^0 and the longitudinal part of \mathbf{A} , which we write as \mathbf{A}_L ? These are also coupled to charged particles, but have no direct relation to *dynamical* transverse electromagnetism or photons. They encompass the nondynamical physics of \mathbf{E}_L , the longitudinal part of the electric field, which can be deduced from Maxwell’s equations to be *entirely the creature of the global charge density*, and therefore is properly termed *coulombic*, and the “nonphysics” of the choice of gauge function. Given the global Hamiltonian functional for the charged particles, specifically *including in particular* their formal interaction with electromagnetism via arbitrary A^μ , we readily obtain the global charge density ρ as the functional derivative of that Hamiltonian functional with respect to A^0 . The longitudinal part of the electric field, namely \mathbf{E}_L , is then obtained as a closed-form linear homogeneous functional of ρ , but this is not sufficient to pin down *both* A^0 and \mathbf{A}_L , which, additionally, calls for a choice of gauge. The simplest choice is the Coulomb gauge, which rather brusquely puts \mathbf{A}_L to zero, oblivious to even the slightest pretense of special-relativistic finesse. Thereupon A^0 also becomes a closed-form linear homogeneous functional of ρ via the static Coulomb kernel. Aside from the blatantly nonrelativistic character of a potential A^0 that arises from the static Coulomb kernel, two other subtleties present themselves: 1) since A^0 couples to the very same particles that give rise to the global ρ , one must compensate for double counting by the usual expedient of halving A^0 relative to its naive value, and 2) if it should turn out that ρ itself still retains a formal dependence on A^0 , one will have only obtained

an *implicit equation* for A^0 , one that requires solution (probably only an approximate one via successive iteration will be feasible).

A relativistically more plausible gauge [3] is rooted in the Lorentz condition ($c\nabla \cdot \mathbf{A}_L + \dot{A}^0 = 0$) conjoined to the requirement that A^0 be linear and homogeneous in ρ , but *retarded* by $(|\mathbf{r} - \mathbf{r}'|/c)$. This can be achieved because the Lorentz condition imposes on A^0 the requirement that it be related to ρ by the c -speed compatible second-order partial differential equation ($\ddot{A}^0/c^2 - \nabla^2 A^0 = \rho$). The upshot is that A^0 is a closed-form integral that is linear and homogeneous in ρ via the static Coulomb kernel, but for ρ evaluated at the *retarded time* ($t - |\mathbf{r} - \mathbf{r}'|/c$). With A^0 in hand, \mathbf{A}_L can be determined from \dot{A}^0 via the Lorentz condition. In the special case that $\dot{\rho} = 0$, this “retarded Lorentz gauge” is identical to the Coulomb gauge. The above caveats concerning halving A^0 relative to its naive value and the possibility of obtaining only an implicit equation for A^0 still apply.

The following sections provide the technical/mathematical details that explicitly underpin and demonstrate what is described and discussed in the foregoing paragraphs.

Longitudinal nondynamical and transverse dynamical electromagnetic fields

The four Maxwell equations for the electromagnetic field (\mathbf{E}, \mathbf{B}) with four-current source $(\rho, \mathbf{j}/c)$ are comprised of Coulomb’s law,

$$\nabla \cdot \mathbf{E} = \rho, \quad (1a)$$

Faraday’s law,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}}/c, \quad (1b)$$

Gauss’ law,

$$\nabla \cdot \mathbf{B} = 0, \quad (1c)$$

and Maxwell’s law,

$$\nabla \times \mathbf{B} = (\mathbf{j} + \dot{\mathbf{E}})/c. \quad (1d)$$

The Coulomb and Gauss laws do not involve time derivatives of the electromagnetic fields, i.e., they are nondynamical in character. This raises the possibility that some part of the electromagnetic field (\mathbf{E}, \mathbf{B}) may itself be of nondynamical character, i.e. determinable without reference to any initial conditions. This in fact turns out to be the case for the longitudinal part of the electric field, which it is therefore very useful to detach from the rest of the electromagnetic field. The ability to separate a vector field into its longitudinal and transverse parts in a linear and unique fashion is in fact extremely useful *throughout* electromagnetic theory, so we turn first to a discussion of how that is carried out.

We shall now indicate that any vector field $\mathbf{F}(\mathbf{r})$ which is continuously differentiable, and for which $|\nabla \cdot \mathbf{F}(\mathbf{r})|/|\mathbf{r}|$ is integrable over all space, has a unique decomposition *achieved by linear operation* into the *sum* of its longitudinal part $\mathbf{F}_L(\mathbf{r})$ with its transverse part $\mathbf{F}_T(\mathbf{r})$, where $\nabla \cdot \mathbf{F}_L = \nabla \cdot \mathbf{F}$, $\nabla \times \mathbf{F}_L = \mathbf{0}$ and $\nabla \cdot \mathbf{F}_T = 0$.

Since we require that $\nabla \times \mathbf{F}_L = \mathbf{0}$, it will need to be the case that $\mathbf{F}_L(\mathbf{r}) = -\nabla S(\mathbf{r})$, where $S(\mathbf{r})$ is a scalar field. Since we also require that $\nabla \cdot \mathbf{F}_L = \nabla \cdot \mathbf{F}$, we must have that $-\nabla^2 S = \nabla \cdot \mathbf{F}$, whose general solution is,

$$S(\mathbf{r}) = c_0 + \mathbf{k}_0 \cdot \mathbf{r} + [(-\nabla^2)^{-1}(\nabla \cdot \mathbf{F})](\mathbf{r}),$$

where c_0 and \mathbf{k}_0 are arbitrary constants, and the natural notation $(-\nabla^2)^{-1}$ denotes the integral operator whose configuration-representation kernel $\langle \mathbf{r} | (-\nabla^2)^{-1} | \mathbf{r}' \rangle$ is,

$$\langle \mathbf{r} | (-\nabla^2)^{-1} | \mathbf{r}' \rangle = (4\pi |\mathbf{r} - \mathbf{r}'|)^{-1},$$

because, as is well-known,

$$(-\nabla_{\mathbf{r}}^2)(4\pi |\mathbf{r} - \mathbf{r}'|)^{-1} = \delta^{(3)}(\mathbf{r} - \mathbf{r}') = \langle \mathbf{r} | \mathbf{r}' \rangle.$$

Since $\mathbf{F}_L = -\nabla S$, we obtain $\mathbf{F}_L = \mathbf{k}_0 - \nabla [(-\nabla^2)^{-1}(\nabla \cdot \mathbf{F})]$, where \mathbf{k}_0 is an arbitrary constant. Because we require the decomposition of \mathbf{F} into the sum of \mathbf{F}_L with \mathbf{F}_T to be achieved by *linear operation*, \mathbf{k}_0 must vanish identically, and therefore,

$$\mathbf{F}_L = -\nabla [(-\nabla^2)^{-1}(\nabla \cdot \mathbf{F})],$$

or

$$\mathbf{F}_L(\mathbf{r}) = -\nabla_{\mathbf{r}} \int (4\pi |\mathbf{r} - \mathbf{r}'|)^{-1} (\nabla_{\mathbf{r}'} \cdot \mathbf{F}(\mathbf{r}')) d^3 \mathbf{r}'.$$

We now note from the above two equations the *key fact* that \mathbf{F}_L is *entirely determined* by $\nabla \cdot \mathbf{F}$. Since, of course, $\mathbf{F}_T = \mathbf{F} - \mathbf{F}_L$, the fact that $\nabla \cdot \mathbf{F}_L = \nabla \cdot \mathbf{F}$ implies that $\nabla \cdot \mathbf{F}_T = 0$, as is required for \mathbf{F}_T .

We shall now systematically apply this *linear decomposition* into its longitudinal and transverse parts to *each vector field* in each of the four Maxwell equations of Eqs. (1). We consequently obtain,

$$\nabla \cdot \mathbf{E}_L = \rho, \quad (2a)$$

$$\nabla \times \mathbf{E}_T = -\dot{\mathbf{B}}_T/c, \quad (2b)$$

$$\dot{\mathbf{B}}_L = \mathbf{0}, \quad (2c)$$

$$\mathbf{B}_L = \mathbf{0}, \quad (2d)$$

$$\nabla \times \mathbf{B}_T = (\mathbf{j}_T + \dot{\mathbf{E}}_T)/c, \quad (2e)$$

and,

$$\mathbf{j}_L + \dot{\mathbf{E}}_L = \mathbf{0}. \quad (2f)$$

From Eq. (2a) and the representation for the longitudinal part of any vector field obtained above, we obtain \mathbf{E}_L in closed form,

$$\mathbf{E}_L(\mathbf{r}, t) = -\nabla_{\mathbf{r}} [(-\nabla^2)^{-1} \rho](\mathbf{r}) = -\nabla_{\mathbf{r}} \int (4\pi |\mathbf{r} - \mathbf{r}'|)^{-1} \rho(\mathbf{r}', t) d^3 \mathbf{r}'. \quad (3a)$$

Eq. (2d) makes Eq. (2c) redundant. In consequence of Eq. (2d), $\mathbf{B} = \mathbf{B}_T$, which we shall simply *always bear in mind*. That enables us to drop all references to \mathbf{B}_T . Therefore Eq. (2b) can be written,

$$\nabla \times \mathbf{E}_T = -\dot{\mathbf{B}}/c, \quad (3b)$$

and Eq. (2e) can likewise be written,

$$\nabla \times \mathbf{B} = (\mathbf{j}_T + \dot{\mathbf{E}}_T)/c. \quad (3c)$$

With regard to Eq. (2f), we recall that the longitudinal part of a vector field is *completely determined* by its divergence. Therefore it is sufficient to simply work with the divergence of Eq. (2f), from which we deduce that $\nabla \cdot \mathbf{j} + \nabla \cdot \dot{\mathbf{E}}_L = 0$. Now there is *nothing more* to be learned about \mathbf{E}_L , as it is given *in closed form in terms of* ρ in Eq. (3a). Therefore we use Eq. (2a) to eliminate its presence, and thus obtain the celebrated *constraint* due to charge/current conservation:

$$\nabla \cdot \mathbf{j} + \dot{\rho} = 0. \quad (3d)$$

Eqs. (3), plus the fact that \mathbf{B} is purely transverse, completely replace the Maxwell equation system of Eqs. (1), and are clearly far more informative than that system. In particular, it is crystal-clear from Eq. (3a) that \mathbf{E}_L is a completely nondynamical variable which is *entirely independent of the choice of initial conditions*. It is *only the transverse fields* \mathbf{E}_T and \mathbf{B} that are actual dynamical variables, and *only these* can legitimately be incorporated into a standard dynamical framework!

Source-free electromagnetic theory

We shall now do away with the charged matter sources, namely put ρ and \mathbf{j} to zero, and thereby deal with purely self-sustaining radiation. Eq. (3a) thereupon becomes $\mathbf{E}_L = \mathbf{0}$, and thus can be dropped entirely along with Eq. (3d), which reduces to the trivial identity $0 = 0$. We are left with the following two dynamical equations, which involve only the two purely transverse fields \mathbf{E}_T and \mathbf{B} ,

$$\dot{\mathbf{B}} = -c \nabla \times \mathbf{E}_T, \quad (4a)$$

and,

$$\dot{\mathbf{E}}_T = c \nabla \times \mathbf{B}. \quad (4b)$$

In addition to these equations of motion, source-free electromagnetism has a very well-known conserved nonnegative field-energy functional [4,5], which is given by,

$$E[\mathbf{E}_T, \mathbf{B}] = \frac{1}{2} \int [|\mathbf{E}_T(\mathbf{r}, t)|^2 + |\mathbf{B}(\mathbf{r}, t)|^2] d^3 \mathbf{r}. \quad (4c)$$

We can readily calculate the time rate of change of $E[\mathbf{E}_T, \mathbf{B}]$ by applying the two field equations of motion, namely Eqs. (4a) and (4b), which yields,

$$dE/dt = c \int [(\nabla \times \mathbf{B}) \cdot \mathbf{E}_T - \mathbf{B} \cdot (\nabla \times \mathbf{E}_T)] d^3\mathbf{r} \quad (4d)$$

Now it is an *identity* that,

$$\nabla \cdot (\mathbf{B} \times \mathbf{E}_T) = (\nabla \times \mathbf{B}) \cdot \mathbf{E}_T - \mathbf{B} \cdot (\nabla \times \mathbf{E}_T),$$

which implies that,

$$dE/dt = c \int \nabla \cdot (\mathbf{B} \times \mathbf{E}_T) d^3\mathbf{r}, \quad (4e)$$

and the integral over all space of such a divergence will, of course, vanish under normal circumstances. Thus the nonnegative field-energy functional $E[\mathbf{E}_T, \mathbf{B}]$ is indeed conserved,

$$dE/dt = 0. \quad (4f)$$

A useful corollary of this demonstration is that under normal circumstances,

$$\int (\nabla \times \mathbf{F}) \cdot \mathbf{G} d^3\mathbf{r} = \int \mathbf{F} \cdot (\nabla \times \mathbf{G}) d^3\mathbf{r}. \quad (5)$$

Now the correct conserved energy of a dynamical system becomes the system's Hamiltonian whenever that system is described by correct canonical variables. Here it is immediately clear that $(\mathbf{E}_T, \mathbf{B})$ are *not* correct canonical fields for this dynamical electromagnetic system, because treating the system's correct energy functional of Eq. (4c) as the system's Hamiltonian functional results in the putative Hamiltonian equations of motion $\dot{\mathbf{E}}_L = \mathbf{B}$ and $\dot{\mathbf{B}} = -\mathbf{E}_L$, which blatantly disagree with the actual equations of motion that are given by Eqs. (4b) and (4a)—there is *no trace* of the very prominent curl operations of the equations of motion to be found in the much more austere straightforward Hamiltonian equations that flow from the nonnegative field-energy functional.

Like parities and additional diffuseness of the canonical fields

We are now clearly obliged to search for correct canonical fields for this source-free electromagnetic system by trying out transformations of $(\mathbf{E}_T, \mathbf{B})$. Since we are dealing with linear equations and a bilinear energy functional, it is clear that we can restrict ourselves to linear transformations. Both the form of the bilinear energy functional and that of the present equations of motion of Eqs. (4b) and (4a) strongly suggest that we not look at any transformations that mix the electric with the magnetic field. In light of the form of the bilinear energy functional, however, it seems urgent to find a transformation that eliminates the curl operations from the equations of motion. The presence of the curl operations is entwined with the fact that \mathbf{B} is an axial transverse vector while \mathbf{E}_L is a polar transverse vector; thus it seems important to find a way map \mathbf{B} onto a polar transverse vector without harming any physically important information carried by \mathbf{B} .

The vector potential \mathbf{A} is a polar vector whose transverse part appears to carry all the physical information that is contained in the transverse axial vector \mathbf{B} . This is so because $\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \mathbf{A}_T$. However, \mathbf{A}_T itself has a different dimensionality from \mathbf{B} , and is also considerably more diffuse, as the Aharonov-Bohm effect pointedly illustrates. It would be good to be able to apply a transformation to \mathbf{A}_T that leaves its transverse polar nature intact, yet compensates for its differences from \mathbf{B} in dimensionality and diffuseness. In this regard, it is interesting to note that $\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}_T) = -\nabla^2 \mathbf{A}_T$, which is *less* diffuse than \mathbf{B} , and errs in dimensionality relative to \mathbf{B} in the *opposite* direction from the dimensionality error made by \mathbf{A}_T . These considerations strongly suggest that the object we would *truly like* to have is $(-\nabla^2)^{\frac{1}{2}} \mathbf{A}_T$. Since, as we have noted just above, $\nabla \times \mathbf{B} = -\nabla^2 \mathbf{A}_T$, we have that, $(-\nabla^2)^{\frac{1}{2}} \mathbf{A}_T = (-\nabla^2)^{-\frac{1}{2}} \nabla \times \mathbf{B}$, i.e., we can conveniently express the entity we want *entirely* in terms of \mathbf{B} *itself*, without *any need* to make reference to \mathbf{A}_T . It is convenient to invent a shorthand notation for this desired transformation of \mathbf{B} from an axial transverse vector to a polar transverse vector that otherwise mimics \mathbf{B} *itself* just as *closely as possible*,

$$\mathbf{B}^\ddagger \stackrel{\text{def}}{=} (-\nabla^2)^{-\frac{1}{2}} \nabla \times \mathbf{B}, \quad (6a)$$

which has the marvelous property that,

$$(\mathbf{B}^\ddagger)^\ddagger = \mathbf{B}, \quad (6b)$$

i.e., this linear operation is actually a *conjugation* when it is *restricted to transverse vector fields*. We can dub it “polar-axial conjugation”. To quell any lingering doubts as to the calculational “meat” on its somewhat abstract “bone”, we explicitly exhibit the integral-operator kernel of $(-\nabla^2)^{-\frac{1}{2}}$ in configuration representation,

$$\langle \mathbf{r} | (-\nabla^2)^{-\frac{1}{2}} | \mathbf{r}' \rangle = (2\pi)^{-3} \int |\mathbf{k}|^{-1} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} d^3\mathbf{k} = \frac{1}{2\pi^2 |\mathbf{r} - \mathbf{r}'|^2}. \quad (6c)$$

It is, moreover, very satisfying to note that the physically key transverse part of the vector potential is neatly related to this conjugate of \mathbf{B} ,

$$\mathbf{A}_T = (-\nabla^2)^{-\frac{1}{2}} \mathbf{B}^\ddagger. \quad (6d)$$

Finally, just as a conjugation operation *ought* to be, polar-axial conjugation is *orthogonal* in the natural Hilbert space of *transverse vector fields*. Thus the conserved nonnegative field-energy functional $E[\mathbf{E}_T, \mathbf{B}]$ of Eq. (4c) is *invariant* under this conjugation. In particular, from the corollary given by Eq. (5), the *symmetric nature* of the linear operator $(-\nabla^2)^{-\frac{1}{2}}$, the fact that *it commutes with differential operators* and the fact that its *square equals* $(-\nabla^2)^{-1}$, it follows that,

$$\int \mathbf{B}^\ddagger \cdot \mathbf{B}^\ddagger d^3\mathbf{r} = \int \mathbf{B} \cdot \nabla \times [\nabla \times (-\nabla^2)^{-1} \mathbf{B}] d^3\mathbf{r} = \int \mathbf{B} \cdot \mathbf{B} d^3\mathbf{r}. \quad (6e)$$

Although the polar-axial conjugation of \mathbf{B} leaves the conserved nonnegative field-energy functional $E[\mathbf{E}_T, \mathbf{B}]$ of Eq. (4c) *form-invariant*, it *does* abolish the curl operations from the *equations of motion*. Thus Eq. (4b) can immediately be rewritten in terms of \mathbf{B}^\ddagger as,

$$\dot{\mathbf{E}}_T = c(-\nabla^2)^{\frac{1}{2}} \mathbf{B}^\ddagger. \quad (7a)$$

Writing Eq. (4a) in terms of \mathbf{B}^\ddagger involves slightly more work, in that the curl operator must first be applied to both sides before the translation in terms of \mathbf{B}^\ddagger can be made,

$$\dot{\mathbf{B}}^\ddagger = -c(-\nabla^2)^{\frac{1}{2}} \mathbf{E}_T. \quad (7b)$$

The operator $(-\nabla^2)^{\frac{1}{2}}$ is most easily handled in Fourier transform, as it is distribution-valued (has locally singular behavior) in configuration representation. Its kernel there is given by,

$$\langle \mathbf{r} | (-\nabla^2)^{\frac{1}{2}} | \mathbf{r}' \rangle = (2\pi)^{-3} \int |\mathbf{k}| e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} d^3\mathbf{k} = \lim_{\epsilon \rightarrow 0} \frac{(3\epsilon^2 - |\mathbf{r} - \mathbf{r}'|^2)}{\pi^2(\epsilon^2 + |\mathbf{r} - \mathbf{r}'|^2)^3}. \quad (7c)$$

Having knocked the curl operations out of the equations of motion, Eqs. (7a) and (7b), we are much closer to our goal of achieving canonical fields, but are not quite there yet. However, what still needs to be done is now relatively easy to light on. The operator factor with dimensionality of frequency that now uniformly emerges in both equations of motion, namely $c(-\nabla^2)^{\frac{1}{2}}$ must *also* be persuaded to show its face in the transformed field-energy functional: our polar-axial conjugation transformation changed the equations of motion and left the field-energy functional form-invariant, but our *next* transformation must do precisely the *opposite*. Multiplying *both* of \mathbf{E}_T and \mathbf{B} by a *power* of the operator $[c^2(-\nabla^2)]$ doesn't change the *form* of Eqs. (7a) and (7b) at all, but it causes that operator to the *negative of twice that power* to appear in the field-energy functional. To have canonical consistency between the equations of motion and the field-energy functional, the power of the operator $[c^2(-\nabla^2)]$ that must appear explicitly in the field-energy functional is one half. That requires the canonical fields to be equal to the present \mathbf{E}_T and \mathbf{B}^\ddagger fields times this operator to the power of minus one quarter.

Our canonical fields are now “in the bag”. They are explicitly,

$$\Phi = [c^2(-\nabla^2)]^{-\frac{1}{4}} \mathbf{E}_T, \quad \Pi = [c^2(-\nabla^2)]^{-\frac{1}{4}} \mathbf{B}^\ddagger. \quad (8a)$$

The equations of motion don't change their form from that of Eqs. (7a) and (7b), notwithstanding our having changed to the canonical fields Φ and Π , because the common operator factor of $[c^2(-\nabla^2)]^{\frac{1}{4}}$ simply factors through and out of those equations. Thus the equations for Φ and Π are,

$$\dot{\Phi} = c(-\nabla^2)^{\frac{1}{2}} \Pi, \quad \dot{\Pi} = -c(-\nabla^2)^{\frac{1}{2}} \Phi. \quad (8b)$$

When the field-energy functional is written in terms of the canonical fields Φ and Π , it *does* change its form; it now becomes a Hamiltonian that is *consistent* with the equations of motion of Eq. (8b). This Hamiltonian is,

$$H[\Phi, \Pi] = \frac{1}{2} \int \left[\Phi \cdot \left(c(-\nabla^2)^{\frac{1}{2}} \Phi \right) + \Pi \cdot \left(c(-\nabla^2)^{\frac{1}{2}} \Pi \right) \right] d^3\mathbf{r}. \quad (8c)$$

We noted in Eq. (6d) that $\mathbf{A}_T = (-\nabla^2)^{-\frac{1}{2}} \mathbf{B}^\dagger$. Now from Eq. (8a) we deduce that $\mathbf{B}^\dagger = c^{\frac{1}{2}}(-\nabla^2)^{\frac{1}{4}} \Pi$. Therefore we obtain that,

$$\mathbf{A}_T = c^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}} \Pi \quad (8d)$$

Eq. (8a) shows that the canonical fields are somewhat more diffuse than the corresponding electric and magnetic fields, and while those fields have dimensionality of the square root of energy density, the *canonical* fields have dimensionality of the square root of *action density*.

The complex-valued photon wave function and Schrödinger equation

Now let us make the transition to the standard *complex* transverse vector field which has the dimensionality of the square root of *probability density*. This field is,

$$\Psi = (\Pi - i\Phi)/(2\hbar)^{\frac{1}{2}}. \quad (9a)$$

We note from Eq. (8b) that,

$$(\dot{\Pi} - i\dot{\Phi}) = c(-\nabla^2)^{\frac{1}{2}} (-\Phi - i\Pi) = -ic(-\nabla^2)^{\frac{1}{2}} (\Pi - i\Phi).$$

Multiplying both sides of the above result through by $i\hbar$, we deduce from Eq. (9a) that,

$$i\hbar\dot{\Psi} = \hbar c(-\nabla^2)^{\frac{1}{2}} \Psi. \quad (9b)$$

Eq. (9b) is a Schrödinger equation with square-root Hamiltonian operator $\hbar c(-\nabla^2)^{\frac{1}{2}} = |c\hat{\mathbf{p}}|$, which is that of a massless free particle. This Schrödinger equation's wave function Ψ has the dimensionality of the square root of probability density, and it is a complex transverse vector field. In other words, after its proper canonical Hamiltonization and field complexification, source-free electromagnetism perfectly describes the solitary, first-quantized free photon, i.e., it is revealed to be first-quantized photodynamics. Let us now tie down the last detail of this identification by rewriting the canonical Hamiltonian functional $H[\Phi, \Pi]$ of Eq. (8c) in terms of Ψ and its complex conjugate Ψ^* . The result is,

$$H[\Psi^*, \Psi] = \int \Psi^* \cdot \left(\hbar c(-\nabla^2)^{\frac{1}{2}} \Psi \right) d^3\mathbf{r}. \quad (9c)$$

This is indeed the Hamiltonian functional that corresponds to the configuration-space Schrödinger equation of Eq. (9b), whose complex-valued transverse-vector wave function Ψ is now completely ready for second quantization. Before carrying this out, let us express \mathbf{A}_T , the transverse part of the vector potential, in terms of the complex-valued photon wave function Ψ . To do this we combine the result of Eq. (8d) with $\Pi = (\hbar/2)^{\frac{1}{2}} (\Psi + \Psi^*)$, which follows from Eq. (9a), to obtain,

$$\mathbf{A}_T = (\hbar c/2)^{\frac{1}{2}}(-\nabla^2)^{-\frac{1}{4}} (\Psi + \Psi^*). \quad (9d)$$

This reveals the transverse part of the vector potential to be somewhat more diffuse than the photon wave function, which itself is revealed by Eqs. (9a) and (8a) to be equally more diffuse than the electromagnetic fields.

Second-quantized photodynamics

If the photon wave function $\Psi(\mathbf{r})$ were a complex-valued *full*-vector field, it would be quantized by being changed into a non-Hermitian operator $\hat{\Psi}(\mathbf{r})$ whose commutation relation with its Hermitian conjugate $\hat{\Psi}^\dagger(\mathbf{r})$ would be given by,

$$[(\hat{\Psi}(\mathbf{r}))_i, (\hat{\Psi}^\dagger(\mathbf{r}'))_j] = \delta_{ij} \delta^{(3)}(\mathbf{r} - \mathbf{r}') = \langle \mathbf{r} | \delta_{ij} | \mathbf{r}' \rangle, \quad i, j = 1, 2, 3, \quad (10a)$$

which is the straightforward consequence of Hermitian quantization of the real *canonical* vector fields $\Phi(\mathbf{r})$ and $\Pi(\mathbf{r})$ that accords with the Dirac relation of commutators to classical Poisson brackets for the quantized components of the classical phase-space vector. Here, however, we must contend with the technical/mathematical annoyance imposed by the *transverse*-vector character of the real canonical fields $\Phi(\mathbf{r})$ and $\Pi(\mathbf{r})$, which effectively *removes* one third of the classical phase-space degrees of freedom that would have been present had $\Phi(\mathbf{r})$ and $\Pi(\mathbf{r})$ been real canonical *full*-vector fields. What is *missing* from the classical *full*-vector field phase space is, of course, its *longitudinal part*, i.e., all those vector fields which can be written as the gradient of a scalar field. The *ij* components of the *projection operator* into this *longitudinal part* of the space of vector fields are given by the integro-differential operators $(-\partial_i(-\nabla^2)^{-1}\partial_j)$. This operator is readily verified to map any vector field into the gradient of a scalar, to be a symmetric linear operator on the natural Hilbert space of real vector fields and to be equal to the square of itself. Therefore the *ij* components of the projection operator into the *transverse part* of the space of vector fields are given by the operators $(\delta_{ij} + \partial_i(-\nabla^2)^{-1}\partial_j)$. Thus to be mathematically consistent with the *transverse*-vector nature of the classical canonical and quantum operator fields, we need to replace Eq. (10a) by the commutation relation,

$$[(\widehat{\Psi}(\mathbf{r}))_i, (\widehat{\Psi}^\dagger(\mathbf{r}'))_j] = \langle \mathbf{r} | (\delta_{ij} + \partial_i(-\nabla^2)^{-1}\partial_j) | \mathbf{r}' \rangle = \delta_{ij}\delta^{(3)}(\mathbf{r} - \mathbf{r}') - (2\pi)^{-3} \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} (\mathbf{k})_i |\mathbf{k}|^{-2} (\mathbf{k})_j d^3\mathbf{k}, \quad i, j = 1, 2, 3. \quad (10b)$$

This commutation relation has the well-known interpretation that $\widehat{\Psi}^\dagger(\mathbf{r})$ creates a free-photon state localized at \mathbf{r} , while $\widehat{\Psi}(\mathbf{r})$ annihilates such a free-photon state, so that the underlying Hilbert space, called Fock space, accommodates arbitrarily large numbers of free photons [6]. The real-valued bilinear Hamiltonian functional $H[\Psi^*, \Psi]$ of Eq. (9c) must be correspondingly quantized to become the Hermitian Hamiltonian *operator* $\widehat{H}[\widehat{\Psi}^\dagger, \widehat{\Psi}]$ that governs the time evolution of this second-quantized photodynamical system of arbitrarily-large numbers of free-photons. An very important (and normally expected) consequence of this is that in the Heisenberg picture the non-Hermitian field $\widehat{\Psi}$ *obeys the selfsame Schrödinger equation* of Eq. (9b) *that was its equation of motion before it was second-quantized*. In other words, the occurrence *at the quantized-field level* of the free-photon Schrödinger equation of Eq. (9b) is *an unavoidable consequence of correct quantization of the source-free Maxwell equations*.

At this second-quantized level, \mathbf{A}_T , the transverse part of the electromagnetic potential, *also* becomes an operator, in fact a Hermitian one. From Eq. (9d) we see that its operator form will be given by,

$$\widehat{\mathbf{A}}_T = (\hbar c/2)^{\frac{1}{2}} (-\nabla^2)^{-\frac{1}{4}} (\widehat{\Psi} + \widehat{\Psi}^\dagger). \quad (10c)$$

It is apparent that $\widehat{\mathbf{A}}_T$ is the object which couples photons to charged particles. From its form as given by Eq. (10c) it is clear that charged particles can both absorb and emit photons via $\widehat{\mathbf{A}}_T$. We now cast a backward glance at what has been accomplished in the foregoing sections, and then take to its completion the discussion we have just begun on how to theoretically set up the electromagnetic interactions that affect charged particles.

Photodynamical and coulombic interactions of charged particles

The source-free case of electromagnetism has, after its meticulous canonical Hamiltonization, which is an absolute necessity that has been essentially universally honored only in the breach for the past century and a half, effortlessly yielded up the completely natural first- and second-quantized theories of free transverse-vector photons, with the later governed by the bilinear-field Hamiltonian operator $\widehat{H}[\widehat{\Psi}^\dagger, \widehat{\Psi}]$, which is the straightforward quantized version of the c-number Hamiltonian functional $H[\Psi^*, \Psi]$ of Eq. (9c). The quantized-field Hamiltonian $\widehat{H}[\widehat{\Psi}^\dagger, \widehat{\Psi}]$ yields in the Heisenberg picture an equation of motion for the quantized field $\widehat{\Psi}$ that is *form-identical* to Eq. (9b), the Schrödinger equation for the first-quantized photon wave function Ψ . This *second-quantized* form of the Schrödinger equation for $\widehat{\Psi}$ obviously *still features the very same* $m \rightarrow 0$ limit of the relativistic *square-root* Hamiltonian operator $(m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}$ (which comes to $\hbar c(-\nabla^2)^{\frac{1}{2}}$ in configuration representation) that it featured as it applied to Ψ . Thus the emergence of this free-photon Schrödinger equation with the just-mentioned *natural square-root relativistic Hamiltonian operator* is the *unavoidable consequence* of quantization of source-free electromagnetism!

Maxwell's equations indicate that charged matter interacts with the dynamical transverse part of electromagnetism via a global transverse current density, as is seen from Eq. (3c). Microscopic theories of charged particles, however, indicate that their interaction with transverse electromagnetism occurs via \mathbf{A}_T ,

the transverse part of the electromagnetic potential, whose quantized form $\widehat{\mathbf{A}}_T$ is given directly in terms of the Hermitian part of the non-Hermitian quantized photon wave-function field $\widehat{\Psi}$ by Eq. (10c).

To describe charged particles and electromagnetism participating in mutual interaction, we require a Hamilton which consists of the *sum* of $\widehat{H}[\widehat{\Psi}^\dagger, \widehat{\Psi}]$ with a Hamiltonian that describes the charged particles, *specifically including* the effects on them which an *arbitrary external c-number four-vector potential* $A^\mu = (A^0, \mathbf{A})$ produces. After writing $\mathbf{A} = \mathbf{A}_L + \mathbf{A}_T$ by applying the now familiar unique linear longitudinal/traverse decomposition of any vector field, we *concretely replace* \mathbf{A}_T by the Hermitian photon field operator $\widehat{\mathbf{A}}_T$ that is given by Eq. (10c). The *remaining* A^0 and \mathbf{A}_L *have no direct relation to photons*: in light of the basic definition $\mathbf{E}_L = (-\nabla A^0 - \dot{\mathbf{A}}_L/c)$ and Eq. (3a), they are determined 1) by the global charge density ρ of the system, *which exerts a coulombic effect on itself*, and 2) by one's choice of gauge. The system's global charge density ρ is given by the functional derivative of the charged-particle Hamiltonian with respect to A^0 . If it should happen that this ρ *itself* depends on A^0 , then, after making the choice of gauge, one will *still not have in hand the actual result for* A^0 , but only an *implicit equation* for that result, whose solution can probably only be successively approximated by iteration. So far as choice of gauge is concerned, by far the simplest is the Coulomb gauge, which, via its requirement that $\nabla \cdot \mathbf{A} = 0$, neatly implies that \mathbf{A}_L vanishes, albeit this is *prima facie* disrespectful of special relativistic precepts. That notwithstanding, there does not seem to be any physical reason to eschew the Coulomb gauge, since Maxwell's equations yield the result of the coulombic effect to be the Eq. (3a) form of \mathbf{E}_L in terms of ρ , which is *very closely related* to the Coulomb gauge result below for A^0 , when A^0 is *additionally* required to be linear and *homogeneous* in ρ , namely,

$$A^0(\mathbf{r}, t) = \frac{1}{2} \int (4\pi|\mathbf{r} - \mathbf{r}'|)^{-1} \rho(\mathbf{r}', t) d^3\mathbf{r}'. \quad (11)$$

An *unusual factor of one half* has been inserted into Eq. (11) to *compensate the double counting* that would otherwise occur because this particular A^0 inherently interacts coulombically *with the very same charge density* ρ that gives rise to it.

If, in spite of the very close relationship of the Coulomb gauge result of Eq. (11) to the Maxwell equation result of Eq. (3a) for \mathbf{E}_L , a relativistically more plausible gauge is nonetheless thought to be desirable, the *retarded Lorentz gauge* of Ref. [3] would seem to be an excellent choice. Scrapping the Coulomb gauge requirement that $\mathbf{A}_L = \mathbf{0}$, but imposing the relativistically impeccable Lorentz condition $(c\nabla \cdot \mathbf{A}_L + \dot{A}^0) = 0$ implies that A^0 satisfies the second-order in time partial differential equation $(\ddot{A}^0/c^2 - \nabla^2 A^0) = \rho$. Imposing the *further* requirements that A^0 be linear and *homogeneous* in ρ , and that it respond to *changes in* ρ *only after the c-speed retardation time* $|\mathbf{r} - \mathbf{r}'|/c$, turns out to yield an A^0 which is *different* from that of Eq. 11 *only in that* $\rho(\mathbf{r}', t)$ on its right-hand side *is replaced by* $\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)$. With that retarded-Lorentz-gauge result for A^0 in terms of ρ *in hand*, one then determines the retarded-Lorentz-gauge \mathbf{A}_L from the retarded-Lorentz-gauge \dot{A}^0 and the Lorentz condition, which explicitly yields $\mathbf{A}_L = \nabla[(-\nabla^2)^{-1}(\dot{A}^0/c)]$. For the case that ρ is *time-independent*, this full retarded-Lorentz-gauge result *reduces to that of the Coulomb gauge*.

Conclusion

It has recently been strenuously argued on the basis of the correspondence principle that the *only physically sensible* Hamiltonian operator for a solitary, relativistic first-quantized free particle of positive mass m is the square root operator $(m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}$ [7]. The extension of this idea to massless particles of course assigns them the Hamiltonian operator $|c\widehat{\mathbf{p}}|$, which is $\hbar c(-\nabla^2)^{\frac{1}{2}}$ in configuration representation. One naturally turns to a long and firmly established theory, namely source-free electromagnetism, which is supposed to include a massless free particle, namely the free photon, in its ambit, to see how this Hamiltonian-operator cum Schrödinger equation idea fares in the context of its quantization. The foregoing work shows that it fares absolutely brilliantly, with the expected Schrödinger equation and its associated Hamiltonian operator being perfectly maintained right to the level of the quantized field (or, more precisely, quantized photon wave function) *if one does not neglect* to properly canonically Hamiltonize the source-free Maxwell equations (whose electric and magnetic fields are *very far* from being properly canonical!) *before* embarking on quantization. In fact, this Schrödinger equation's appearance at the *first-quantized* level turns out to be *no more* than a simple, direct *by-product* of merely the *proper canonical Hamiltonization* of Maxwell's source-free equations!

These results for source-free electromagnetism lend impressive support to the almost ridiculously straightforward idea that the *correspondence-principle-mandated* relativistic free-particle square-root Hamiltonian operator $(m^2c^4 + |c\widehat{\mathbf{p}}|^2)^{\frac{1}{2}}$ and associated time-dependent Schrödinger equation is *without exception* the correct *starting point* for relativistic quantum theory (a glance at the section just above, and perhaps even more so at the latter sections of Ref. [7], reveals a taste of the mind-boggling complexity and richness which this *mere*

starting point rapidly gives way to). Who would be prepared to *for an instant* contest the completely parallel assertion that the correspondence-principle-mandated nonrelativistic free-particle kinetic-energy Hamiltonian operator $|\hat{\mathbf{p}}|^2/(2m)$ and associated time-dependent Schrödinger equation is *without exception* the correct *starting point* for nonrelativistic quantum theory? It may not, in this very regard, have escaped the reader's attention that the correspondence-principle-mandated *square root* relativistic Hamiltonian has the *delicately subtle* property that as $c \rightarrow \infty$,

$$[(m^2c^4 + |\hat{\mathbf{c}}\hat{\mathbf{p}}|^2)^{\frac{1}{2}} - mc^2] \rightarrow |\hat{\mathbf{p}}|^2/(2m).$$

For the linearized Dirac Hamiltonian $\vec{\alpha} \cdot \hat{\mathbf{p}}c + \beta mc^2$, there is simply *no remotely similar* property. The *very best* in this regard which can be done with the Dirac Hamiltonian is to subtract away its value at $\hat{\mathbf{p}} = \mathbf{0}$, leaving $\vec{\alpha} \cdot \hat{\mathbf{p}}c$, which *diverges* as $c \rightarrow \infty$! Klein-Gordon theory fails altogether to present us with a Hamiltonian (which is the proximate cause of its quantum-theoretic downfall [7]), but it does give us the *square* of the correspondence-principle-mandated relativistic one, namely $(m^2c^4 + |\hat{\mathbf{c}}\hat{\mathbf{p}}|^2)$. Subtracting away its value at $\hat{\mathbf{p}} = \mathbf{0}$ leaves $|\hat{\mathbf{c}}\hat{\mathbf{p}}|^2$, which *also diverges* as $c \rightarrow \infty$! The relativistic free-particle *square root Hamiltonian* $(m^2c^4 + |\hat{\mathbf{c}}\hat{\mathbf{p}}|^2)^{\frac{1}{2}}$ is *very subtly tuned* indeed! The square root *itself* is in fact theoretically *completely entwined* with the square roots that are *archetypal* of the Lorentz transformation [7].

In contrast to this square root, which is *uniquely fathered* for the free particle by the Lorentz transformation *itself* [7], both the Klein-Gordon and Dirac equations were *artificially concocted* for the *express purpose* of evading nonlocal integral operators in the configuration representation of relativistic quantum mechanics [8, 7]. In light of this fact, it is a truly monumental irony that while the source-free Maxwell equations written in terms of the electric and magnetic fields achieve *exactly the configuration-representation locality* so willfully *insisted upon* by Klein, Gordon and Dirac, those equations are thereby *inconsistent with quantization*, being *far from properly canonically Hamiltonized*, and when this issue is duly attended to, they *utterly lose* that “precious” configuration-representation locality to *precisely* one of the square-root operators which Klein, Gordon and Dirac so abjured! An extremely powerful lesson emerges from this: there quite simply can be no physically legitimate union of special relativity with quantum theory *without* those square-root operators. The efforts of Klein, Gordon and Dirac to rid relativistic quantum theory of these square-root operators and achieve configuration-representation locality merely engenders a dismal list of theoretically inappropriate or grotesquely unphysical consequences [7]. The Dirac free particle theory, for example, presents a number of operators that are *without question* physical observables, such as the three components of velocity and the energy (i.e., the Dirac Hamiltonian itself), which nonetheless *fail* to mutually commute when the limit $\hbar \rightarrow 0$ is taken! That is grotesquely unphysical, and quite enough to permanently consign Dirac theory to the dustbin. But just for good measure, these commutators *diverge* in the nonrelativistic limit $c \rightarrow \infty$! Bereft of a Hamiltonian, Klein-Gordon theory has no time evolution operator and no Heisenberg picture. For the same underlying reason, it manifests negative probabilities. In short, it is so hopelessly crippled that it cannot be regarded as quantum theory at all. These are items from the dismal list of Dirac and Klein-Gordon equation shortcomings, but there is not so much as a *single item* in that pejorative list which pertains to the *square root Hamiltonian*! With this *utterly lopsided* accounting of the theoretical pros and cons, it is far past time for the theoretical physics community to finally awaken to just what in its repertoire needs to be revised [7] to give the operator $(m^2c^4 + |\hat{\mathbf{c}}\hat{\mathbf{p}}|^2)^{\frac{1}{2}}$ *exactly the same standing* in relativistic quantum physics that the operator $|\hat{\mathbf{p}}|^2/(2m)$ properly has in nonrelativistic quantum physics.

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