Path-Integral Gauge Invariant Mapping: From Abelian Gauge Anomalies to the Generalized Stueckelberg Mechanism

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Abstract

Reviewing a path-integral procedure of recovering gauge invariance from anomalous effective actions developed by Harada and Tsutsui in the 80's, it is shown that there is another way to achieve gauge symmetry, besides the one presented by the authors, which may be anomaly-free, preserving current conservation. It is also shown that the generalization of Harada-Tsutsui technique to other models which are not anomalous but do not exhibit gauge invariance allows the identification of the gauge invariant formulation of the Proca model with the Stueckelberg model, leading to the interpretation of the gauge invariant mapping as a generalization of the Stueckelberg mechanism.

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I. INTRODUCTION

In the eighties, an amount of discussion about anomalous gauge models in quantum field theory was presented. The central role of the discussion was played by consistence of such theories. Although some theorists considered such models as inconsistent, some authors produced works to support the idea that they are not so.

In this sense, we must cite the work of Jackiw and Rajaraman [1], in which it was shown that a gauge anomalous two-dimensional theory could be well defined and able to provide a mass generation mechanism from chiral anomalies. This work was soon followed by the one of Faddeev and Shatashvili [2], who noticed that quantum gauge invariance could be restored by the introduction of new quantum degrees of freedom, that transform second class constraints into first class ones. In adding these extra fields, the effective anomalous action is mapped into a gauge invariant one. Then, the works of Babelon, Schaposnik and Viallet [3] and Harada and Tsutsui [4] showed independently that these new degrees of freedom could emerge quite naturally by the application of Fadeev-Popov's method through the non-factorization of the integration over the gauge group. Soon after, Harada and Tsutsui recognized that the same procedure could be applied to the Proca model [5], leading to a possible generalization of their technique. Recently, it was inferred that the gauge anomaly has null expectation value if the entire quantum theory is considered[1] [6], and by imposing gauge invariance of the bosonic measure.

We can recognize the main strategy to give consistence to these models with the introduction of the new degrees of freedom, which recovers gauge invariance. In this sense, it seems useful to analyze such mapping and explore its potential. Although restoring gauge symmetry at the final effective action, one may ask whether such technique is able to provide current conservation or it just preserves the quantum anomaly.

This work is intended to elucidate this question for the particular case of abelian gauge models. In this sense, in section II, the origin of abelian gauge anomaly is briefly reviewed in path integral approach. In section III, the gauge invariant formalism developed by Harada and Tsutsui is rederived by redefining the vacuum functional multiplying it by the gauge volume, instead of proceeding with Fadeev-Popov's method, and it is shown that the anom-

[1] That obtained by considering the gauge fields as quantum fields.

aly is preserved in the original form proposed by the authors. Section IV is intended to show that their procedure can be used to obtain an alternative abelian gauge invariant formulation which may provide an anomaly-free model. In section V, the Harada-Tsutsui gauge invariant formulation of the Proca model is rederived. Finally, in section VI, a correspondence between the Proca's gauge invariant mapping and the Stueckelberg mechanism is pointed out, leading to the interpretation of the enhanced mapping as a generalization of the Stueckelberg mechanism [7]. The conclusion is, then, presented in section VII.

II. THE ORIGIN OF ABELIAN GAUGE ANOMALY IN PATH INTEGRAL APROACH

Consider an abelian gauge theory described by the action

$$I[\psi,\overline{\psi},A_\mu] \equiv I_M[\psi,\overline{\psi},A_\mu] + I_G[A_\mu], \tag{1}$$

where $I_M[\psi,\overline{\psi},A_\mu]$ is the matter action minimally coupled to the gauge abelian field A_μ, and $I_G[A_\mu]$ is the free bosonic action. If the action is said to be invariant under local gauge transformations

$$\psi \to \psi^\theta = \exp\left(i\theta(x)\right)\psi \tag{2}$$

$$\overline{\psi} \to \overline{\psi}^\theta = \exp\left(-i\theta(x)\right)\overline{\psi} \tag{3}$$

$$A_\mu \to A_\mu^\theta = A_\mu + \frac{1}{e}\partial_\mu\theta(x), \tag{4}$$

one can say that, classically, the theory exhibits a conserved current given by

$$J^\mu = -\frac{1}{e}\frac{\delta I_M}{\delta A_\mu}. \tag{5}$$

Now, if we proceed the quantization of the fermionic fields, then, after integrating them out, we will arrive at an effective action given by

$$\exp\left(iW\left[A\right]\right) = \int d\psi d\overline{\psi} \exp\left(iI\left[\psi,\overline{\psi},A\right]\right). \tag{6}$$

To find the quantum version of the current conservation law, first we make a change of variables in the fermion fields

$$\begin{aligned} \exp\left(iW\left[A\right]\right) &= \int d\psi d\overline{\psi} \exp\left(iI\left[\psi,\overline{\psi},A\right]\right) \\ &= \int d\psi^\theta d\overline{\psi}^\theta \exp\left(iI\left[\psi^\theta,\overline{\psi}^\theta,A\right]\right), \end{aligned} \tag{7}$$

and then, just as in the classical case, we make use of the invariance of the action by noticing that $I\left[\psi^{\theta},\overline{\psi}^{\theta},A\right]=I\left[\psi,\overline{\psi},A^{-\theta}\right]$

$$\exp\left(iW\left[A\right]\right)=\int d\psi^{\theta}d\overline{\psi}^{\theta}\exp\left(iI\left[\psi,\overline{\psi},A^{-\theta}\right]\right) \tag{8}$$

Now, a subtle difference between the classical and the quantum gauge theory arises: if the quantum measure is *locally* gauge invariant, *i. e.*, if

$$d\psi^{\theta}d\overline{\psi}^{\theta}=d\psi d\overline{\psi}, \tag{9}$$

then, by considering $\theta(x)$ as an infinitesimal parameter, we will have

$$\begin{aligned}\exp\left(iW\left[A\right]\right)&=\int d\psi^{\theta}d\overline{\psi}^{\theta}\exp\left(iI\left[\psi,\overline{\psi},A^{-\theta}\right]\right)\\&=\int d\psi d\overline{\psi}\exp\left(iI\left[\psi,\overline{\psi},A_{\mu}-\frac{1}{e}\partial_{\mu}\theta(x)\right]\right)\\&=\exp\left(iW\left[A\right]\right)-\int dxi\theta(x)\int d\psi d\overline{\psi}\partial_{\mu}\left(-\frac{1}{e}\frac{\delta I}{\delta A_{\mu}}\right)\exp\left(iI\left[\psi,\overline{\psi},A_{\mu}\right]\right)\end{aligned} \tag{10}$$

$$\Rightarrow\int d\psi d\overline{\psi}\partial_{\mu}\left(-\frac{1}{e}\frac{\delta I}{\delta A_{\mu}}\right)\exp\left(iI\left[\psi,\overline{\psi},A_{\mu}\right]\right)=0. \tag{11}$$

But gauge invariance of the free bosonic action implies that $\partial_{\mu}\left(\frac{\delta I_G}{\delta A_{\mu}}\right)\equiv 0$, therefore,

$$\int d\psi d\overline{\psi}\partial_{\mu}\left(-\frac{1}{e}\frac{\delta I_M}{\delta A_{\mu}}\right)\exp\left(iI\left[\psi,\overline{\psi},A_{\mu}\right]\right)=0. \tag{12}$$

Equation (12) is the quantum version of current conservation. However, it was necessary to impose invariance of the fermionic measure (9) to get the above result. If, instead of (9), we had

$$d\psi^{\theta}d\overline{\psi}^{\theta}=\exp\left(i\alpha_1\left[A,\theta\right]\right)d\psi d\overline{\psi} \tag{13}$$

then, instead of (12), we would arrive at

$$\begin{aligned}\exp\left(iW\left[A\right]\right)&=\int d\psi^{\theta}d\overline{\psi}^{\theta}\exp\left(iI\left[\psi,\overline{\psi},A^{-\theta}\right]\right)\\&=\int d\psi d\overline{\psi}\exp\left(iI\left[\psi,\overline{\psi},A^{-\theta}\right]+i\alpha_1\left[A,\theta\right]\right)\\&=\int d\psi d\overline{\psi}\exp\left\{iI\left[\psi,\overline{\psi},A_{\mu}\right]+i\int dx\partial_{\mu}\theta(x)\left(-\frac{1}{e}\frac{\delta I}{\delta A_{\mu}}\right)\right.\\&\left.+i\alpha_1\left[A,0\right]+i\int dx\left.\frac{\delta\alpha_1}{\delta\theta}\right|_{\theta=0}\theta(x)\right\},\end{aligned}$$

but $\partial_\mu\left(-\frac{1}{e}\frac{\delta I}{\delta A_\mu}\right) = \partial_\mu\left(-\frac{1}{e}\frac{\delta I_M}{\delta A_\mu}\right)$ and $\alpha_1[A,0]=0$, therefore

$$\begin{aligned}
\exp\left(iW\left[A\right]\right) &= \int d\psi d\overline{\psi}\exp\left\{iI\left[\psi,\overline{\psi},A_\mu\right] - i\int dx\theta(x)\left[\partial_\mu\left(-\frac{1}{e}\frac{\delta I_M}{\delta A_\mu}\right) - \left.\frac{\delta\alpha_1}{\delta\theta}\right|_{\theta=0}\right]\right\} \\
&= \int d\psi d\overline{\psi}\exp\left(iI\left[\psi,\overline{\psi},A_\mu\right]\right)\left\{1 - i\int dx\theta(x)\left[\partial_\mu\left(-\frac{1}{e}\frac{\delta I_M}{\delta A_\mu}\right) - \left.\frac{\delta\alpha_1}{\delta\theta}\right|_{\theta=0}\right]\right\} \\
&= \exp\left(iW\left[A\right]\right) - i\int dx\theta(x)\int d\psi d\overline{\psi}\exp\left(iI\left[\psi,\overline{\psi},A_\mu\right]\right)\left[\partial_\mu\left(-\frac{1}{e}\frac{\delta I_M}{\delta A_\mu}\right) - \left.\frac{\delta\alpha_1}{\delta\theta}\right|_{\theta=0}\right]
\end{aligned}$$

$$\Rightarrow \int d\psi d\overline{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I_M}{\delta A_\mu}\right)\exp\left(iI\left[\psi,\overline{\psi},A_\mu\right]\right) = \mathcal{A}\exp\left(iW\left[A\right]\right), \tag{14}$$

and we see that, instead of (12), we would have a nonzero right-hand side in equation (14), where

$$\mathcal{A} \equiv \left.\frac{\delta\alpha_1}{\delta\theta}\right|_{\theta=0} \tag{15}$$

is called the anomaly and the theory is said to be anomalous.

It is convenient, to our purposes, to rewritte the anomaly (15) by noticing that

$$\begin{aligned}
\left.\frac{\delta\alpha_1}{\delta\theta}\right|_{\theta=0} &= \left.\frac{\delta W\left[A^\theta\right]}{\delta\theta}\right|_{\theta=0} \\
&= \int d^n x \left.\frac{\delta W\left[A^\theta\right]}{\delta A_\mu^\theta(y)}\right|_{\theta=0}\frac{\delta A_\mu^\theta(y)}{\delta\theta(x)} \\
&= \int d^n x\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(y)}\partial_\mu\left[\delta\left(x-y\right)\right] \\
&= \partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right),
\end{aligned}$$

and, therefore

$$\mathcal{A} \equiv \left.\frac{\delta\alpha_1}{\delta\theta}\right|_{\theta=0} = \partial_\mu\left(-\frac{1}{e}\frac{\delta W\left[A\right]}{\delta A_\mu(x)}\right). \tag{16}$$

III. GAUGE INVARIANT FORMULATION OF ANOMALOUS MODELS

The anomaly arises from de non-invariance of the effective action. To see this, we notice that

$$\begin{aligned}\exp\left(iW\left[A^{\theta}\right]\right) &= \int d\psi d\overline{\psi}\exp\left(iI\left[\psi,\overline{\psi},A^{\theta}\right]\right)\\ &= \int d\psi^{\theta}d\overline{\psi}^{\theta}\exp\left(iI\left[\psi^{\theta},\overline{\psi}^{\theta},A^{\theta}\right]\right)\\ &= \int d\psi d\overline{\psi}\exp\left(iI\left[\psi,\overline{\psi},A\right]+i\alpha_{1}\left[A,\theta\right]\right)\\ &= \exp\left(iW\left[A\right]+i\alpha_{1}\left[A,\theta\right]\right), \end{aligned} \tag{17}$$

that is,

$$\Rightarrow \alpha_{1}\left[A,\theta\right] = W\left[A^{\theta}\right]-W\left[A\right]. \tag{18}$$

Therefore, from (14) it seems that current conservation at quantum level may be obtained only for theories with gauge invariant effective actions.

A gauge invariant formulation of anomalous theories was developed by Harada and Tsutsui in [4]. We will derive the same results in a different way that is more convenient to our purposes. It is considered the full quantum theory, described by the vacuum functional

$$\begin{aligned} Z &= \int d\psi d\overline{\psi}dA_{\mu}\exp\left(iI\left[\psi,\overline{\psi},A\right]\right)\\ &= \int dA_{\mu}\exp\left(iW\left[A\right]\right). \end{aligned} \tag{19}$$

The functional can be redefined by multiplying it by the gauge volume and, then, a change of variables in the gauge field is performed

$$\begin{aligned} Z &= \int d\theta dA_{\mu}\exp\left(iW\left[A\right]\right)\\ &= \int d\theta dA_{\mu}^{\theta}\exp\left(iW\left[A^{\theta}\right]\right). \end{aligned} \tag{20}$$

Now we use the fact that the boson measure *is* gauge invariant, that is $dA_{\mu}=dA_{\mu}^{\theta}$, and we arrive at a theory containing a scalar field θ, besides the gauge field A_{μ}

$$\begin{aligned} Z &= \int d\theta dA_{\mu}\exp\left(iW'\left[A,\theta\right]\right)\\ &= \int dA_{\mu}\exp\left(iW_{eff}\left[A\right]\right), \end{aligned} \tag{21}$$

where

$$W'\left[A,\theta\right]\equiv W\left[A^{\theta}\right] \text{ and } \exp\left(iW_{eff}\left[A\right]\right)\equiv\int d\theta\exp\left(iW'\left[A,\theta\right]\right) \tag{22}$$

It is easy to see that the new effective action $W_{eff}[A]$ is gauge invariant. To do this, we notice that

$$\begin{aligned}
\exp\left(iW_{eff}\left[A^{\lambda}\right]\right) &= \int d\theta \exp\left(iW'\left[A^{\lambda},\theta\right]\right) \\
&= \int d\theta \exp\left(iW'\left[A,\theta+\lambda\right]\right) \\
&= \int d\left(\theta+\lambda\right) \exp\left(iW'\left[A,\theta+\lambda\right]\right) \\
&= \exp\left(iW_{eff}\left[A\right]\right)
\end{aligned} \tag{23}$$

One could ask if, after this procedure, the anomaly would survive, and we can say that it depends on the starting action. Indeed, one may choose an initial action by noticing that

$$\begin{aligned}
Z &= \int d\theta dA_{\mu} \exp\left(iW'\left[A,\theta\right]\right) \\
&= \int d\theta dA_{\mu} \exp\left(iW\left[A^{\theta}\right]\right) \\
&= \int d\theta dA_{\mu} \exp\left(iW\left[A\right] + i\alpha_1\left[A,\theta\right]\right) \\
&= \int d\theta d\psi d\overline{\psi} dA_{\mu} \exp\left(iI\left[\psi,\overline{\psi},A\right] + i\alpha_1\left[A,\theta\right]\right).
\end{aligned} \tag{24}$$

The action in eq. (24), with the addition of the Wess-Zumino term $\alpha_1[A,\theta]$ [9], is known as the standard action [4]

$$I_{st}\left[\psi,\overline{\psi},A,\theta\right] = I\left[\psi,\overline{\psi},A\right] + \alpha_1\left[A,\theta\right]. \tag{25}$$

As one could notice, although the final effective action $W_{eff}[A]$ is gauge invariant, the standard one $I_{st}\left[\psi,\overline{\psi},A,\theta\right]$ *is not*, since $\alpha_1[A,\theta]$ breaks gauge invariance. To understand what it means, we see that, if we search for a kind of conserved current from this theory, we need to start from the gauge invariance of the effective action, which leads to

$$\partial_{\mu}\left(-\frac{1}{e}\frac{\delta W_{eff}\left[A\right]}{\delta A_{\mu}(x)}\right) = 0. \tag{26}$$

Then we have

$$
\begin{aligned}
&\partial_\mu \left(-\frac{1}{e} \frac{\delta W_{eff}[A]}{\delta A_\mu(x)} \right) \exp\left(iW_{eff}[A]\right) \\
&= \frac{i}{e}\partial_\mu \left\{ \frac{\delta}{\delta A_\mu(x)} \exp\left(iW_{eff}[A]\right) \right\} \\
&= \frac{i}{e}\partial_\mu \left\{ \frac{\delta}{\delta A_\mu(x)} \int d\theta d\psi d\overline{\psi} \exp\left(iI_{st}\left[\psi,\overline{\psi},A,\theta\right]\right) \right\} \\
&= \int d\theta d\psi d\overline{\psi} \partial_\mu \left(-\frac{1}{e} \frac{\delta I_{st}}{\delta A_\mu(x)} \right) \exp\left(iI_{st}\left[\psi,\overline{\psi},A,\theta\right]\right) \\
&= \int d\theta d\psi d\overline{\psi} \partial_\mu \left(-\frac{1}{e} \frac{\delta I_M\left[\psi,\overline{\psi},A\right]}{\delta A_\mu(x)} - \frac{1}{e} \frac{\delta \alpha_1[A,\theta]}{\delta A_\mu(x)} \right) \exp\left(iI_{st}\left[\psi,\overline{\psi},A,\theta\right]\right) = 0,
\end{aligned}
\tag{27}
$$

and, since $\alpha_1[A,\theta]$ is not gauge invariant, one can not say that $\partial_\mu \left(-\frac{1}{e}\frac{\delta\alpha_1[A,\theta]}{\delta A_\mu(x)} \right) = 0$, which would lead to the current conservation law. Instead, we have

$$
\begin{aligned}
&\int d\theta d\psi d\overline{\psi} \partial_\mu J^\mu \exp\left(iI_{st}\left[\psi,\overline{\psi},A,\theta\right]\right) \\
&= \int d\theta d\psi d\overline{\psi} \partial_\mu \left(\frac{1}{e} \frac{\delta \alpha_1[A,\theta]}{\delta A_\mu(x)} \right) \exp\left(iI_{st}\left[\psi,\overline{\psi},A,\theta\right]\right) \neq 0.
\end{aligned}
\tag{28}
$$

Now, we can perform integration over the $\theta - field$ in the right-hand side of (28), using (18), (6) and the gauge invariance of $W_{eff}[A]$. It is straightforward to find

$$
\int d\theta d\psi d\overline{\psi} \partial_\mu J^\mu \exp\left(iI_{st}\left[\psi,\overline{\psi},A,\theta\right]\right) = \mathcal{A} \exp\left(iW_{eff}[A]\right), \tag{29}
$$

and we see that the standard formulation still preserves the anomaly. This may be explained by the switching of gauge symmetry breakdown from the effective action to the starting one, namely, the standard action. On the other hand, it remains to be verified whether the anomaly still survives after imposed the equations of motion of this modified theory.

IV. RECOVERING CURRENT CONSERVATION

The standard action is not the only one that can provide the gauge invariant effective theory given by (22). Indeed, from (21) we have

$$
\begin{aligned}
Z &= \int d\theta dA_\mu \exp\left(iW'[A,\theta]\right) \\
&= \int d\theta dA_\mu \exp\left(iW\left[A^\theta\right]\right) \\
&= \int d\theta d\psi d\overline{\psi} dA_\mu \exp\left(iI\left[\psi,\overline{\psi},A^\theta\right]\right).
\end{aligned}
\tag{30}
$$

Thus, we can see that the same procedure that leads to (21) and (22) can be performed by means of the *enhanced* action, defined by

$$I_{en}\left[\psi,\overline{\psi},A,\theta\right]\equiv I\left[\psi,\overline{\psi},A^{\theta}\right]. \tag{31}$$

The advantage of the enhanced action is that it is really gauge invariant. Moreover, if we start from the gauge invariance of $W_{eff}[A]$ and proceed the same calculations which lead to (28), we will arrive at

$$\begin{aligned}
&\partial_\mu\left(-\frac{1}{e}\frac{\delta W_{eff}[A]}{\delta A_\mu(x)}\right)\exp\left(iW_{eff}[A]\right)\\
&=\int d\theta d\psi d\overline{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I_{en}}{\delta A_\mu(x)}\right)\exp\left(iI_{en}\left[\psi,\overline{\psi},A,\theta\right]\right)\\
&=\int d\theta d\psi d\overline{\psi}\partial_\mu\left(-\frac{1}{e}\frac{\delta I\left[\psi,\overline{\psi},A^{\theta}\right]}{\delta A_\mu(x)}\right)\exp\left(iI_{en}\left[\psi,\overline{\psi},A,\theta\right]\right)=0
\end{aligned} \tag{32}$$

In fermionic theories, generally the gauge fields are coupled linearly to the fermions. So, expanding the matter action to the first order, we will obtain

$$\begin{aligned}
I\left[\psi,\bar{\psi},A\right]&=I_M\left[\psi,\bar{\psi},A\right]+I_G[A]\\
&=I_F\left[\psi,\bar{\psi}\right]+\int d^nx\frac{\delta I_M[\psi,\bar{\psi},A]}{\delta A_\mu(x)}A_\mu(x)+I_G[A],
\end{aligned} \tag{33}$$

where $I_F\left[\psi,\bar{\psi}\right]\equiv I_M\left[\psi,\bar{\psi},0\right]$ corresponds to the free fermionic action. However, $\frac{\delta I_M[\psi,\bar{\psi},A]}{\delta A_\mu(x)}=-eJ^\mu(x)$, therefore

$$I\left[\psi,\bar{\psi},A\right]=I_F\left[\psi,\bar{\psi}\right]+I_G[A]-e\int d^nxJ^\mu(x)A_\mu(x) \tag{34}$$

$$I_M\left[\psi,\bar{\psi},A\right]=I_F\left[\psi,\bar{\psi}\right]-e\int d^nxJ^\mu(x)A_\mu(x). \tag{35}$$

Thus, evidently

$$-\frac{1}{e}\frac{\delta I_M\left[\psi,\bar{\psi},A^{\theta}\right]}{\delta A_\mu(x)}=-\frac{1}{e}\frac{\delta I_M\left[\psi,\bar{\psi},A^{\theta}\right]}{\delta A^{\theta}_\mu(x)}=-\frac{1}{e}\frac{\delta I_M\left[\psi,\bar{\psi},A\right]}{\delta A_\mu(x)}=J^\mu. \tag{36}$$

Since $I_G[A]$ is gauge invariant, which means that $\partial_\mu\left(-\frac{1}{e}\frac{\delta I_G[A]}{\delta A_\mu(x)}\right)=0$, eq. (32) leads to

$$\partial_\mu\left(-\frac{1}{e}\frac{\delta W_{eff}[A]}{\delta A_\mu(x)}\right)=0\Leftrightarrow\int d\theta d\psi d\overline{\psi}\partial_\mu J^\mu(x)\exp\left(iI_{en}\left[\psi,\overline{\psi},A,\theta\right]\right)=0 \tag{37}$$

Eq. (37) means that the current is conserved in this alternative approach with no quantum breakdown. Therefore, we see that the gauge invariant formulation constructed by Harada and Tsutsui, but alternatively built with the enhanced action, instead of the standard one, provides a theory with no current conservation breakdown and, thus, anomaly-free.

To finish this section, we shall analyze the *classical* equations of motion obtained from original abelian anomalous models

$$\frac{\delta I[\psi,\overline{\psi},A_\mu]}{\delta\psi} = \frac{\delta I_M[\psi,\overline{\psi},A_\mu]}{\delta\psi} = 0 \tag{38}$$

$$\frac{\delta I[\psi,\overline{\psi},A_\mu]}{\delta\overline{\psi}} = \frac{\delta I_M[\psi,\overline{\psi},A_\mu]}{\delta\overline{\psi}} = 0 \tag{39}$$

$$\frac{\delta I}{\delta A_\mu} = \frac{\delta I_M}{\delta A_\mu} + \frac{\delta I_G}{\delta A_\mu} = 0 \tag{40}$$

and compare them with those from the enhanced theory $I_{en}\left[\psi,\overline{\psi},A,\theta\right] \equiv I\left[\psi,\overline{\psi},A^\theta\right]$

$$\frac{\delta I\left[\psi,\bar{\psi},A^\theta_\mu\right]}{\delta\psi(x)} = \frac{\delta I_M\left[\psi,\bar{\psi},A^\theta_\mu\right]}{\delta\psi(x)} = 0 \tag{41}$$

$$\frac{\delta I\left[\psi,\bar{\psi},A^\theta_\mu\right]}{\delta\overline{\psi}} = \frac{\delta I_M\left[\psi,\bar{\psi},A^\theta_\mu\right]}{\delta\overline{\psi}} = 0 \tag{42}$$

$$\frac{\delta I\left[\psi,\bar{\psi},A^\theta_\mu\right]}{\delta A_\mu} = \frac{\delta I_M\left[\psi,\bar{\psi},A^\theta_\mu\right]}{\delta A_\mu(x)} + \frac{\delta I_G\left[A^\theta_\mu\right]}{\delta A_\mu(x)} = \frac{\delta I_M\left[\psi,\bar{\psi},A_\mu\right]}{\delta A_\mu(x)} + \frac{\delta I_G\left[A_\mu\right]}{\delta A_\mu(x)} = 0 \tag{43}$$

$$\frac{\delta I}{\delta\theta} = \partial_\mu\left(-\frac{1}{e}\frac{\delta I_M[\psi,\bar{\psi},A]}{\delta A_\mu(x)}\right) = \partial_\mu J^\mu = 0 \tag{44}$$

As one could see, the equation (44) for θ is redundant, since it is just the current conservation law imposed by global gauge invariance. The equation of motion for the gauge field is the same in both theories, since it is gauge invariant. Finally, the equations for the fermionic fields are reducible one to the other by a simple redefinition of the gauge field which is nothing but a generic gauge transformation $A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$ that does not change the other equations. Thus, classically both formulations are completely equivalent one to the other, and the scalar is not even noted. On the other hand, at quantum level, the simple original theory is anomalous, while the enhanced one, with the addition of the $\theta - field$, is not.

V. GAUGE INVARIANT FORMULATION APPLIED TO NON-ANOMALOUS THEORIES - THE PROCA MODEL

As shown by the authors in the work of ref. [5] to the case of the massive vector field, the Harada-Tsutsui procedure of mapping a theory that does not exhibit quantum gauge symmetry into a gauge invariant one does not need to be tied to the particular class of classically symmetric models. Indeed, to proceed this mapping, it was only necessary to consider the exponential of the effective action $\exp(iW[A])$, gauge transforming it into $\exp\left(iW\left[A^{\theta}\right]\right)$, and then to perform an integration over the $\theta - field$ to obtain, finally, the exponential of the gauge invariant effective action $W_{eff}[A]$. But *any* action that does not exhibit gauge invariance could, in principle, be attached to this procedure. Lets us reconsider, for instance, the massive vector field interacting with fermions, whose action is

$$I\left[\psi,\bar{\psi},A_{\mu}\right] = I_M\left[\psi,\bar{\psi},A_{\mu}\right] - \frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}\int d^4x A^{\mu}A_{\mu}. \tag{45}$$

Clearly, the massive term breaks gauge invariance. If we consider the quantum version of this model and proceed the gauge invariant mapping, we will get

$$\begin{aligned}\int d\theta \exp\left(iW\left[A^{\theta}\right]\right) &= \int d\theta d\psi d\overline{\psi} \exp\left(iI\left[\psi,\overline{\psi},A^{\theta}\right]\right) \\ &= \int d\theta d\psi d\overline{\psi} \exp\left(I_M\left[\psi,\bar{\psi},A^{\theta}_{\mu}\right] - \frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}\int d^4x A^{\theta\mu}A^{\theta}_{\mu}\right) \\ &= \int d\theta d\psi^{\theta} d\overline{\psi}^{\theta} \exp\left(I_M\left[\psi^{\theta},\bar{\psi}^{\theta},A^{\theta}_{\mu}\right] - \frac{1}{4}\int d^4x F^{\mu\nu}F_{\mu\nu} + \frac{m^2}{2}\int d^4x A^{\theta\mu}A^{\theta}_{\mu}\right),\end{aligned} \tag{46}$$

and if the theory is not anomalous, that is, if $d\psi^{\theta}d\overline{\psi}^{\theta} = d\psi d\overline{\psi}$, we will arrive at an enhanced model given by

$$\exp\left(iW_{eff}[A]\right) = \int d\theta d\psi d\overline{\psi} \exp\left(iI_{en}\left[\psi,\overline{\psi},A,\theta\right]\right), \tag{47}$$

where

$$I_{en}\left[\psi,\bar{\psi},A_{\mu},\theta\right] = I_M\left[\psi,\bar{\psi},A_{\mu}\right] + \int d^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\frac{m^2}{e^2}\partial^{\mu}\theta\partial_{\mu}\theta + \frac{1}{2}m^2A^{\mu}A_{\mu} + \frac{m^2}{e}A^{\mu}\partial_{\mu}\theta\right). \tag{48}$$

If we proceed integration over the gauge parameter, we will find

$$\int d\theta \exp\left\{\frac{i}{2}m^2\int dx\left(\frac{2}{e}A_\mu\partial^\mu\theta+\frac{1}{e^2}\partial_\mu\theta\partial^\mu\theta\right)\right\}$$
$$=\exp\left(-\frac{i}{2}m^2\int dxA_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right)\int d\theta\exp\left\{-i\frac{m^2}{2e}\int dx\left[\left(\frac{e}{\Box}\partial^\mu A_\mu+\theta\right)\Box\left(\frac{e}{\Box}\partial^\nu A_\nu+\theta\right)\right]\right\}. \tag{49}$$

Performing the change of variables $\theta\rightarrow\theta'=\theta+\frac{1}{\Box}\partial^\mu A_\mu$; $d\theta'=d\theta$, we will arrive at

$$\int d\theta\exp\left\{\frac{i}{2}m^2\int dx\left(2A_\mu\partial^\mu\theta+\partial_\mu\theta\partial^\mu\theta\right)\right\}\sim\exp\left(-\frac{i}{2}m^2\int dxA_\mu\frac{\partial^\mu\partial^\nu}{\Box}A_\nu\right). \tag{50}$$

Using this result into (47), we finally obtain

$$\int d\theta d\psi d\overline{\psi}\exp\left(iI_{en}\left[\psi,\bar{\psi},A_\mu,\theta\right]\right)=\int d\psi d\overline{\psi}\exp\left(iI'\left[\psi,\bar{\psi},A_\mu\right]\right), \tag{51}$$

with

$$I'\left[\psi,\bar{\psi},A_\mu\right]\equiv I_M\left[\psi,\bar{\psi},A_\mu\right]+\int d^nx\left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}m^2A_\mu\left(\eta^{\mu\nu}-\frac{\partial^\mu\partial^\nu}{\Box}\right)A_\nu\right\}. \tag{52}$$

It is easy to see that, classically, the gauge invariant formulation of Proca model (52) may be thought as equivalent to its correlate (45), since one is reducible to the other, with no loss of physical meaning, by the Lorentz gauge choice $\partial_\mu A^\mu=0$ in (52). Therefore, this example clearly shows that the Harada-Tsutsui technique may be used as a procedure to map a theory with no gauge symmetry into a gauge invariant one even in some cases where we are dealing with classical models.

VI. THE ENHANCED FORMALISM AND THE STUECKELBERG MECHANISM

In the enhanced anomalous model's formalism, we start with a gauge invariant action $I_{en}\left[\psi,\bar{\psi},A,\theta\right]$, and reach an effective one $W_{eff}\left[A\right]$ which is also gauge invariant. However, there is an intermediate action $W'\left[A,\theta\right]=W\left[A^\theta\right]$ with no gauge symmetry. Nevertheless, it is obviously invariant under generalized gauge transformations

$$A_\mu\rightarrow A_\mu+\frac{1}{e}\partial_\mu\Lambda$$
$$\theta\rightarrow\theta-\Lambda \tag{53}$$

It means that we can set $\theta(x) = k$ by a simple gauge choice and get back to the original formalism. In other words, classically, θ is not noted, but must exist and be quantized in order to provide an anomaly-free model. In section 4, we saw that the classical equations of motion of the enhanced version of anomalous models are reducible to those of the original one by a simple redefinition of the gauge boson. By the generalized gauge symmetry point of view (53) above, thus, it simply means a gauge choice where the scalar is set constant.

On the other hand, the pure enhanced Proca model, which is also invariant under (53), is described by

$$I_P\left[A,\theta\right] = \int d^n x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\frac{m^2}{e^2}\partial^\mu\theta\partial_\mu\theta + \frac{1}{2}m^2 A^\mu A_\mu + \frac{m^2}{e}A^\mu\partial_\mu\theta\right). \qquad (54)$$

If we simply redefine the $\theta - field$ by a multiplicative constant

$$B(x) \equiv \frac{m}{e}\theta(x), \qquad (55)$$

then we will just find the Stueckelberg action [7]

$$I_{Stueck}\left[A,B\right] = \int d^n x \left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(mA^\mu + \partial^\mu B\right)\left(mA_\mu + \partial_\mu B\right)\right\}, \qquad (56)$$

and (53) becomes Pauli's gauge transformations [8]

$$A_\mu \to A_\mu + \partial_\mu\Lambda(x) \qquad (57)$$

$$B \to B - m\Lambda(x). \qquad (58)$$

Therefore, we see that the Harada-Tsutsui formalism, using the enhanced form in the case of abelian anomalous models, may be in closed conection with the Stueckelberg mechanism, and may be viewed as its generalization, which might be stated as follows: *every gauge boson has to be accompanied by a scalar in such a way that its gradient must be added up to the gauge boson itself.*

The biggest advantage of the Stueckelberg massive abelian model, which coincides exactly with the enhanced formulation of the Proca model, is that it was rigorously proved to be renormalizable and unitary [10].

We started by the integration over what we called the gauge parameter, but now we can reinterpret it by saying that it is not the gauge parameter which is actually integrated, but the Stueckelberg scalar, a compensating quantum field which is hidden in conventional gauge symmetry, but becomes necessary in order to recover it when the symmetry is broken, and able to provide an abelian anomaly-free theory as well as a renormalizable massive vector model.

VII. CONCLUSION

Revisiting a procedure to transform effective actions of anomalous generic models into gauge invariant ones, built in the 80's, it was found that it might be much more proficuous than it might have seem to be at a first sight. Indeed, the gauge invariant formulation is not only able to map an anomalous model into a gauge invariant one, but it may also be able to remove abelian gauge anomalies, which simply disappear when the $\theta - field$ is introduced into the theory by gauge transforming the gauge field. Moreover, it provides a bridge between the gauge invariant formulation of anomalous models and a generalization of the Stueckelberg mechanism, where the $\theta - field$, identified as the Stueckelberg scalar, may be present together with the gauge field in *any* abelian theory, instead of being present only in the particular case of the massive vector model.

On the other hand, such discussion may raise a paradox: If one formalism is mapped into another one by simple manipulations over the functional integral, which would suggest that both formalisms are equivalent, how, in the anomalous case, one might present current conservation breakdown while the other has it conserved? As we have seen, the original formalism is anomalous, which would mean that it is closer to the standard formalism, that preserves its anomaly, than to the enhanced one. In this sense, one might ask which of the two gauge invariant formalisms may be equivalent to the original one. Work is in progress in order to clarify this question.

The relevance of the Stueckelberg mechanism is that it is able to deal with gauge symmetry breaking and, since it is renormalizable, it provides a mechanism alternative to the Higgs [11]. Moreover, it can be recovered in a rather singular limit of the Higgs mechanism [12]. In our generalization of the Stueckelberg mechanism, we saw that it is also able to provide a gauge anomaly-free model. On the other hand, It is well known, for the simplest case of the anomalous Jackiw-Rajaraman model, that there is an alternative mass-generation mechanism to the gauge boson from quantum corrections of anomalous $2 - D$ chiral fermions [1]. Perhaps it is not mere coincidence that a breaking in the gauge symmetry in both cases is related to vector boson's mass generation, and that it may be recovered by an introdution of a scalar. The generalization of the Jackiw-Rajaraman mass generation mechanism from chiral anomalies to higher dimensions is under consideration.

Finally, we can point out that, besides the correspondence between gauge invariant mapping and the Stueckelberg mechanism, this kind of procedure might be generalized to other symmetries than gauge one, and it may be a road to a technique that leads to restore other kind of symmetry breakdowns, like chiral or gravitational anomalies, for example.

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