The gauge theory's expansion in the Electro-Magnetic field

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ABSTRACT

In the special relativity theory, study the gauge theory in the Electro-magnetic field theory.Using that the Electro-magnetic potential is 4-vector, treat invariant potential. And the Electro-Magnetic field theory's the gauge theory expand.

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I.Introduction

In the special relativity theory, Electro-magnetic potential $(\phi, \vec{A})$ is 4-vector likely time-space.

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{c\partial t} \ , \ \vec{B} = \vec{\nabla} \times \vec{A} \quad (1)$$

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)A^\alpha = \frac{4\pi}{c} j^\alpha = \frac{4\pi}{c}\rho_0 \frac{dx^\alpha}{d\tau} \quad (2)$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \gamma(\frac{1}{c}\frac{\partial}{\partial t'} - \frac{\vec{v}}{c}\cdot\vec{\nabla}'), \vec{\nabla} = \vec{\nabla}' - \gamma\frac{\vec{v}}{c}\frac{1}{c}\frac{\partial}{\partial t'} - (1-\gamma)\frac{\vec{v}}{v^2}\cdot\vec{\nabla}'\vec{v} \quad (3)$$

$$ct = \gamma(ct' + \frac{\vec{v}}{c}\cdot\vec{x}'), \ \vec{x} = \vec{x}' + \gamma\frac{\vec{v}}{c}ct' - (1-\gamma)\frac{\vec{v}\cdot\vec{x}'}{v^2}\vec{v} \quad (4)$$

$$\phi = \gamma(\phi' + \frac{\vec{v}}{c}\cdot\vec{A}') \quad (5)$$

$$\vec{A} = \vec{A}' + \gamma\frac{\vec{v}}{c}\phi' - (1-\gamma)\frac{\vec{v}\cdot\vec{A}'}{v^2}\vec{v} \ (6), \gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$$

II.Additional chapter

An inertial coordinate system S's Electro-magnetic potential $(\phi, \vec{A})$ and the other inertial coordinate system S''s Electro-magnetic potential $(\phi', \vec{A}')$'s the gauge transformation is

$$\phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}, \ \vec{A} \to \vec{A} + \vec{\nabla}\Lambda \quad (7)$$

Λ is a function of S

$$\phi' \to \phi' - \frac{1}{c}\frac{\partial\Lambda'}{\partial t'}, \ \vec{A}' \to \vec{A}' + \vec{\nabla}'\Lambda' \quad (8)$$

Λ' is a function of S'

Therefore, the formula (5), the formula (6) be used by the formula (7) and the formula (8)

$$\begin{aligned}\phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} &= \gamma\{\phi' - \frac{1}{c}\frac{\partial\Lambda'}{\partial t'} + \frac{\vec{v}}{c}\cdot(\vec{A}' + \vec{\nabla}'\Lambda')\} \\ &= \gamma\{\phi' + \frac{\vec{v}}{c}\cdot\vec{A}'\} + \gamma\{-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'} + \frac{\vec{v}}{c}\cdot\vec{\nabla}'\Lambda'\} \end{aligned} \quad (9)$$

$$\begin{aligned}\vec{A} + \vec{\nabla}\Lambda &= \vec{A}' + \vec{\nabla}'\Lambda' + \gamma\frac{\vec{v}}{c}(\phi' - \frac{1}{c}\frac{\partial\Lambda'}{\partial t'}) - (1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{A}' + \vec{\nabla}'\Lambda')\vec{v} \\ &= \{\vec{A}' + \gamma\frac{\vec{v}}{c}\phi' - (1-\gamma)\frac{\vec{v}}{v^2}\cdot\vec{A}'\vec{v}\} + \{\vec{\nabla}'\Lambda' + \gamma\frac{\vec{v}}{c}(-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}) - (1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{\nabla}'\Lambda')\vec{v} \ (10)\end{aligned}$$

In this time, the formula (9) and the formula (10) be used by the formula (5) and the formula (6) and the gauge function Λ and Λ''s relation is

$$-\frac{1}{c}\frac{\partial\Lambda}{\partial t}=\gamma(-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}+\frac{\vec{v}}{c}\cdot\vec{\nabla}'\Lambda')=\gamma(-\frac{1}{c}\frac{\partial}{\partial t'}+\frac{\vec{v}}{c}\cdot\vec{\nabla}')\Lambda' \quad (11)$$

$$\vec{\nabla}\Lambda=\vec{\nabla}'\Lambda'+\gamma\frac{\vec{v}}{c}(-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})-(1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{\nabla}'\Lambda')\vec{v}$$

$$=[\vec{\nabla}'+\gamma\frac{\vec{v}}{c}(-\frac{1}{c}\frac{\partial}{\partial t'})-(1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{\nabla}')\vec{v}]\Lambda' \quad (12)$$

In this time, it has to that the formula (3) coincide the formula (11) and formula (12) therefore it has to that a function Λ of S, coincide Λ' of S', Hence $\Lambda=\Lambda'$.

Likely this upper case

$$\phi^2-\vec{A}\cdot\vec{A}=\phi'^2-\vec{A}'\cdot\vec{A}' \quad (13)$$

That the formula (13) be used by the formula (7) ,the formula (8) of the gauge transformation, be used by the formula(3),the formula (5),the formula (6) and the formula(11),the formula(12)

$$(\phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t})^2-(\vec{A}+\vec{\nabla}\Lambda)\cdot(\vec{A}+\vec{\nabla}\Lambda)$$

$$=(\phi^2-\vec{A}\cdot\vec{A})-2(\frac{1}{c}\frac{\partial\Lambda}{\partial t}\phi+\vec{A}\cdot\vec{\nabla}\Lambda)+[\frac{1}{c^2}(\frac{\partial\Lambda}{\partial t})^2-\vec{\nabla}\Lambda\cdot\vec{\nabla}\cdot\Lambda]$$

$$=(\phi'^2-\vec{A}'\cdot\vec{A}')-2(\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}\phi'+\vec{A}'\cdot\vec{\nabla}'\Lambda')+[\frac{1}{c^2}(\frac{\partial\Lambda'}{\partial t'})^2-\vec{\nabla}'\Lambda'\cdot\vec{\nabla}'\cdot\Lambda']$$

$$=(\phi'-\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})^2-(\vec{A}'+\vec{\nabla}'\Lambda')\cdot(\vec{A}'+\vec{\nabla}'\Lambda') \quad (14)$$

$$(\phi^2-\vec{A}\cdot\vec{A})=(\phi'^2-\vec{A}'\cdot\vec{A}'), \quad (15)$$

$$(\frac{1}{c}\frac{\partial\Lambda}{\partial t}\phi+\vec{A}\cdot\vec{\nabla}\Lambda)=(\frac{1}{c}\frac{\partial\Lambda'}{\partial t'}\phi'+\vec{A}'\cdot\vec{\nabla}'\Lambda') \quad (16)$$

$$(\frac{1}{c}\frac{\partial\Lambda}{\partial t})^2-\vec{\nabla}\Lambda\cdot\vec{\nabla}\Lambda=(\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})^2-\vec{\nabla}'\Lambda'\cdot\vec{\nabla}'\Lambda' \quad (17),\quad \Lambda=\Lambda'$$

The formula (13) is invariant by the formula (14) about the gauge transformation, the formula (7),the formula (8).The formula (13) is invariant about the special relativistic transformation.Therefore, it thinks an invariant Electro-Magnetic potential $\overline{\phi}$ likely the coordinate system's the invariant time.

$$\overline{\phi}^2=\phi^2-\vec{A}\cdot\vec{A}=\phi'^2-\vec{A}'\cdot\vec{A}' \quad (18)$$

In the formula (2), Electro-Magnetic potential A^α transform likely the differential coordinate dx^α

$$d\tau^2=dt^2-\frac{1}{c^2}(d\vec{x}\cdot d\vec{x}) \quad (19)$$

$$d\tau^2 = dt^2(1-\frac{u^2}{c^2}) \ , \ \frac{d\vec{x}}{dt} = \vec{u} \quad (20)$$

$$\overline{\phi}^2 = \phi^2(1-\frac{u^2}{c^2}), \ \overline{\phi} = \phi\sqrt{1-\frac{u^2}{c^2}} \quad (21)$$

$$\frac{\vec{A}}{\phi} = \frac{\vec{u}}{c}, \ \vec{A} = \frac{\vec{u}}{c}\phi \quad (22)$$

An example of the formula (22)'s potential is Lienard-Wiechert potential that made by moving point charge.[3,4]

In the inertial coordinate system and the other inertial coordinate system,

$$\overline{\phi}^2 = \phi^2(1-\frac{u^2}{c^2}) = \phi'^2(1-\frac{u'^2}{c^2}), \ \frac{d\vec{x}}{dt} = \vec{u}, \ \frac{d\vec{x}'}{dt'} = \vec{u}'$$

$$\frac{\vec{A}}{\phi} = \frac{\vec{u}}{c}, \ \vec{A} = \frac{\vec{u}}{c}\phi, \quad \frac{\vec{A}'}{\phi'} = \frac{\vec{u}'}{c}, \ \vec{A}' = \frac{\vec{u}'}{c}\phi'$$

$$d\vec{x} = d\vec{x}' + \gamma\frac{\vec{v}}{c}cdt' - (1-\gamma)\frac{\vec{v}\cdot d\vec{x}'}{v^2}\vec{v}, \ dt = \gamma(dt' + \frac{\vec{v}}{c^2}\cdot d\vec{x}')$$

$$\vec{u} = \frac{d\vec{x}}{dt} = \frac{1}{\gamma}\frac{\vec{u}' + \gamma\vec{v} - (1-\gamma)\frac{\vec{v}\cdot\vec{u}'}{v^2}\vec{v}}{(1+\frac{\vec{v}}{c^2}\cdot\vec{u}')}, \ \vec{u}' = \frac{d\vec{x}'}{dt'}, \phi = \gamma(\phi' + \frac{\vec{v}}{c}\cdot\vec{A}')$$

$$\vec{A} = \frac{\vec{u}}{c}\phi = \frac{1}{\gamma}[\frac{\vec{u}' + \gamma\vec{v} - (1-\gamma)\frac{\vec{v}\cdot\vec{u}'}{v^2}\vec{v}}{(1+\frac{\vec{v}}{c^2}\cdot\vec{u}')}]\cdot\gamma\frac{1}{c}[\phi' + \frac{\vec{v}}{c}\cdot\vec{A}') \ , \ \vec{A}' = \frac{\vec{u}'}{c}\phi'$$

$$= \frac{1}{\gamma}[\frac{\vec{u}' + \gamma\vec{v} - (1-\gamma)\frac{\vec{v}\cdot\vec{u}'}{v^2}\vec{v}}{(1+\frac{\vec{v}}{c^2}\cdot\vec{u}')}]\cdot\gamma\frac{1}{c}\phi'[1+\frac{\vec{v}}{c}\cdot\frac{\vec{u}'}{c}]$$

$$= \frac{\vec{u}'}{c}\phi' + \gamma\vec{v}\frac{\phi'}{c} - (1-\gamma)\frac{\vec{v}}{v^2}\cdot(\vec{u}'\frac{\phi'}{c})\vec{v}, \vec{A}' = \frac{\vec{u}'}{c}\phi'$$

$$= \vec{A}' + \gamma\frac{\vec{v}}{c}\phi' - (1-\gamma)\frac{\vec{v}\cdot\vec{A}'}{v^2}\vec{v} \quad (23), \gamma = 1/\sqrt{1-\frac{v^2}{c^2}}$$

According to the formula (22), Electro-Magnetic field is

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{c\partial t} = -\vec{\nabla}\phi - \frac{\partial}{c\partial t}(\frac{\vec{u}}{c}\phi) = -\vec{\nabla}\phi - \frac{\phi}{c}\frac{\partial\vec{u}}{c\partial t} - \frac{\vec{u}}{c}\frac{\partial\phi}{c\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\frac{\vec{u}}{c}\phi) = \phi(\vec{\nabla}\times\frac{\vec{u}}{c}) - \frac{\vec{u}}{c}\times\vec{\nabla}\phi \quad (24)$$

The formula (2)'s Electro-Magnetic potential equation is

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2)\phi=\frac{4\pi}{c}j^0=4\pi\rho=\frac{4\pi}{c}\rho_0\frac{dx^0}{d\tau}=\frac{4\pi}{c}\rho_0\frac{cdt}{d\tau}=4\pi\bar{\gamma}\rho_0$$

$$j^0=c\rho \ , \ \rho=\bar{\gamma}\rho_0 \ , \ \bar{\gamma}=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (25)$$

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2)\vec{A}=(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2)\phi\frac{\vec{u}}{c}=\frac{4\pi}{c}\vec{j}=\frac{4\pi}{c}\rho\vec{u}$$

$$\vec{A}=\frac{\vec{u}}{c}\phi \ , \ \vec{j}=\rho\vec{u} \ , \rho=\bar{\gamma}\rho_0 \ , \ \bar{\gamma}=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (26)$$

By the formula (21), Electro-Magnetic invariant potential $\overline{\phi}$ is

$$\bar{\gamma}\overline{\phi}=\phi, \ \bar{\gamma}=\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad (27)$$

Therefore by the formula (25) and the formula (27), Electro-Magnetic invariant potential $\overline{\phi}$'s equation is

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2)\bar{\gamma}\overline{\phi}=4\pi\bar{\gamma}\rho_0$$

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2)\overline{\phi}=4\pi\rho_0 \qquad (28)$$

III.Conclusion

About the gauge functions Λ and Λ', by the formula (17)

$$(\frac{1}{c}\frac{\partial\Lambda}{\partial t})^2-\vec{\nabla}\Lambda\cdot\vec{\nabla}\Lambda=(\frac{1}{c}\frac{\partial\Lambda'}{\partial t'})^2-\vec{\nabla}'\Lambda'\cdot\vec{\nabla}'\Lambda' \ (29), \ \Lambda=\Lambda'$$

and Lorentz gauge is

$$\frac{1}{c}\frac{\partial\phi}{\partial t}+\vec{\nabla}\cdot\vec{A}=0 \ , \ \frac{1}{c}\frac{\partial\phi'}{\partial t'}+\vec{\nabla}'\cdot\vec{A}'=0 \ (30)$$

Therefore the gauge functions Λ and Λ' is satisfied

$$(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2)\Lambda=(\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}-\nabla'^2)\Lambda'=0 \ (31), \ \Lambda=\Lambda'$$

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