Author name

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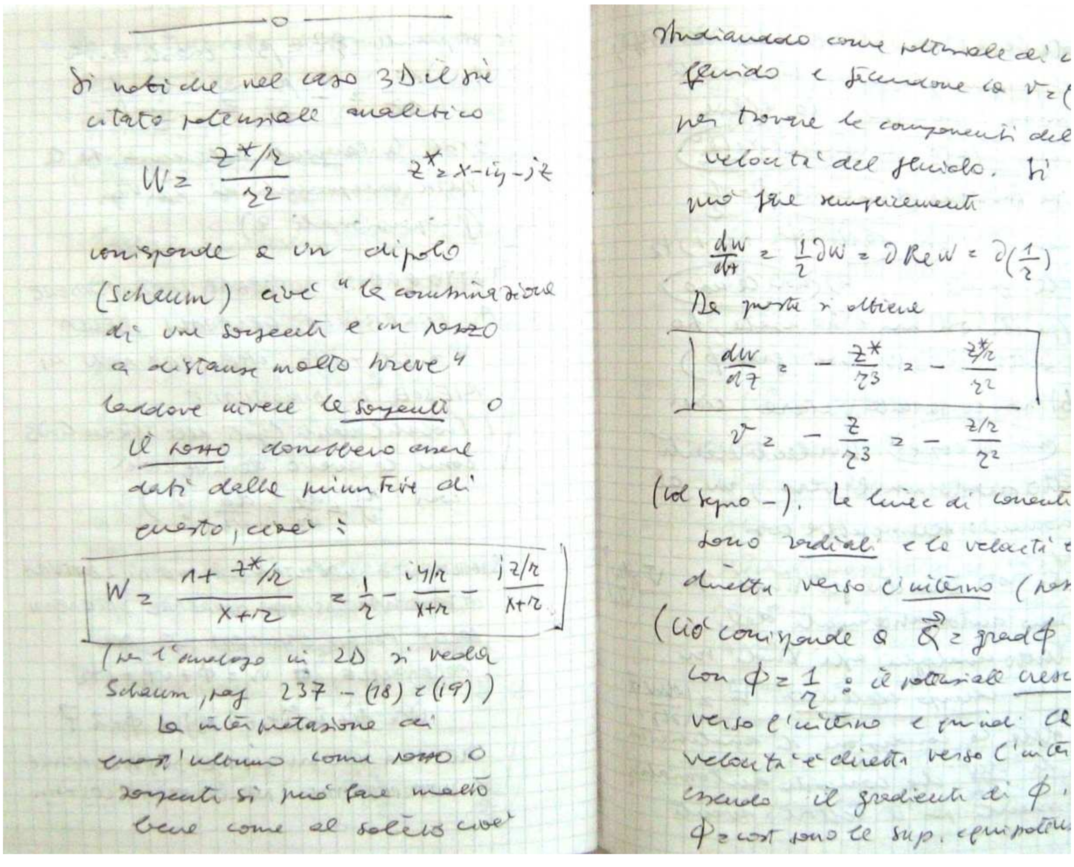
Title

Electrostatics and fluid flow with 3D analytic functions.

Abstract

I present examples of applications of 3-dimensional analytic functions to electrostatics and potential flows, mainly devoted to engineers and physicists.
Of course, the paper only suggests areas of future development, despite that a persistent idea, from Sommerfeld on, seems to be “The powerful tool of the theory of complex functions cannot be used in three-dimensional potential theory” (Sommerfeld, “Mechanics of Deformable Bodies”, Academic Press, 1950)
I summarize here unpublished manuscripts dated 1994.



A survey of basic properties

(Note: in all this paper I'll name the complex quantities $(x+iy)$ or $(x+iy+jz)$ as Z in order to avoid ambiguity with the usual Cartesian coordinate z).

Summary of basic formulas.
2 dimensions:

$$\partial^* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$
$$\partial = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$
$$\frac{\partial}{\partial Z} = \frac{1}{2}\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$$

$$\frac{\partial}{\partial Z^*} = \frac{1}{2}\partial^* = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

3 dimensions:

$$\partial^* = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z}$$

$$\partial = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - j\frac{\partial}{\partial z}$$

The complex notation I've adopted elsewhere for Clifford algebra reduces in this case to "real 1 and imaginary i , j, squared -1".
In fact the only property you must remember here and in the following is (so as for the imaginary i) $j^* = -j$ and $j^2 = -1$.
More, i and j anticommute, $ij = -ji$. Full stop.
I'll only consider 3 - dimensional analytic functions in the form $U = (U_1 + iU_2 + jU_3)$.
Analiticity condition (Cauchy – Riemann) $\partial^* U = 0$ gives:

$$U = (U_1 + iU_2 + jU_3)$$
$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} + j\frac{\partial}{\partial z}\right)(U_1 + iU_2 + jU_3) = 0$$

and developing:

1 $$\frac{\partial U_1}{\partial x} - \frac{\partial U_2}{\partial y} - \frac{\partial U_3}{\partial z} = 0$$

i $$\frac{\partial U_2}{\partial x} + \frac{\partial U_1}{\partial y} = 0$$

j $$\frac{\partial U_3}{\partial x} + \frac{\partial U_1}{\partial z} = 0$$

ij $$\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z} = 0$$

For the vector $\vec{U}$ having components the conjugate of U, ie $\vec{U} = (U_1, -U_2, -U_3)$ this means $rot, div = 0$.

Developing ∂U^* instead you have:

$$\partial U^* = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - j\frac{\partial}{\partial z}\right)(U_1 - iU_2 - jU_3) = 0$$

1 $$\frac{\partial U_1}{\partial x} - \frac{\partial U_2}{\partial y} - \frac{\partial U_3}{\partial z} = 0$$

i $$-\frac{\partial U_2}{\partial x} - \frac{\partial U_1}{\partial y} = 0$$

j $$-\frac{\partial U_3}{\partial x} - \frac{\partial U_1}{\partial z} = 0$$

ij $$\frac{\partial U_3}{\partial y} - \frac{\partial U_2}{\partial z} = 0$$

As you can see, $\partial U^*$is not $\partial^* U$, but if $\partial^* U = 0$ you have also $\partial U^* = 0$.
This allows to apply, in 3D, the same property as in 2D

$$\partial U = \partial U + \partial U^* = \partial(U + U^*) = 2\partial Re(U)$$

Ie for the ∂ derivative of an analytic function $U = (U_1 + iU_2 + jU_3)$ you may make the derivative only of the real part:

$$\frac{1}{2}\partial U = \partial Re(U)$$

Of course, due to the analyticity of $U, \partial^* U = 0$ and then $\partial U^* = 0$, you can also write:

$$\partial U = \partial U - \partial U^* = \partial(U - U^*) = 2\partial Im(U)$$

and so:

$$\frac{1}{2}\partial U = \partial Im(U)$$

Summarizing:

$$\begin{cases} \frac{1}{2}\partial U = \partial U_1 \\ \frac{1}{2}\partial U = \partial(iU_2 + jU_3) \end{cases}$$

Another interesting property is (as in 2 – dimensions):

$$\frac{1}{2}\partial U = \frac{\partial U}{\partial x}$$

Thus the ∂ derivative of an analytic function (ie $\frac{1}{2}\partial$) is equal to the derivative in space in the direction of the real axis x.
This obviously follows from $\partial U = \partial U + \partial^* U = (\partial + \partial^*)U = 2\frac{\partial}{\partial x}U$.

Summary:
everything goes as in 2 D (almost everything, unfortunately an analytic function is no longer a conformal transformation).
In fact the main difference is that we can't write for the derivative the expression $\frac{d}{dZ}$ (however its role is played in part by $\frac{1}{2}\partial$). You may note for example that $Z = x + iy + jz$ identifies a point in space, but is not analytic, neither is $\frac{1}{2}\partial Z = 1$. If you want an analytic function whose derivative $\frac{1}{2}\partial$ is one, you have $x + i\frac{y}{2} + j\frac{z}{2}$ (check).
In the same time you cannot build up analytic function neither with the rule $f(Z) = f(x + iy + jz)$, nor $f(\,x + i\frac{y}{2} + j\frac{z}{2})$.
It is easy to build up analytic functions with the following property, however general:

If A is harmonic, $U = \partial A$ is analytic.

In fact, A harmonic means:

$$\partial\partial^* A = 0$$

However $\partial\partial^* = \partial^*\partial$, so

$$\partial^*\partial A = 0$$

ie, if we take $U = \partial A$, it follows $\partial^* U = 0$, so U is analytical QED.
This extends the concept of "potential" of a field, where the field is the ∂ derivative. But we can also, again, see this as a empirical way to build up analytic functions.
Note: aside from mathematics, in a physical problem you must be aware of the sign, example, if Φ is the electric potential, $\vec{E} = -grad\Phi$, but in these considerations and in the following I don't care of this sign. I will simply take for granted that an exchange $\pm q$ in charge (or an exchange source/sink) means an exchange in the field sign.

Exercises

Exercise 1 - Flow around a sphere.

In 2 D for the flow around a "sphere" it considers the analytic potential

$$A = Z + \frac{a^2}{Z}$$

Developing:

$$A = u + iv = r(cos\varphi + isin\varphi) + \frac{a^2}{r}(cos\varphi - isin\varphi)$$

$$\begin{cases} u = (r + \frac{a^2}{r})cos\varphi \\ v = (r - \frac{a^2}{r})sin\varphi \end{cases}$$

Call $A = u + iv = \Phi + i\Psi$ where

$$\begin{cases} \Phi \quad velocity\ potential \\ \Psi \quad \ stream\ function \end{cases}$$

The imaginary part Ψ is zero on the "sphere" contour $r = a$. Let $v = \frac{dA}{dZ}$, which is analytic.
The analyticity for v means $rot\vec{v}, div\vec{v} = 0$ for $\vec{v}$ having components the conjugate of, v ie $\vec{v} = (v_1, -v_2)$, see also [1]. So we can assume as velocity field (analytic) the derivative $v = \frac{dA}{dZ}$.

THIS SITUATION IMMEDIATELY GENERALIZES TO 3D WHERE THE ANALYTICITY CONDITION $\partial^* U = 0$ MEANS $rot\vec{U}, div\vec{U} = 0$ FOR THE VECTOR $\vec{U}$ HAVING FOR COMPONENTS THE CONJUGATE OF U, IE $\vec{U} = (U_1, -U_2, -U_3)$.

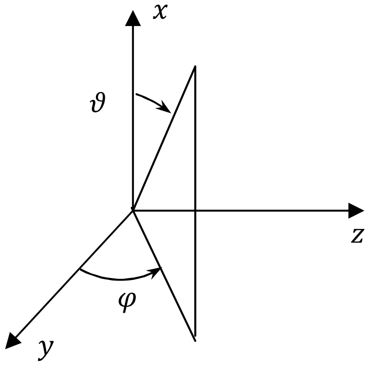
The 3D analogous is build up through the term analogous to Z (which gives the flow at ∞) and the term analogous to $\frac{1}{Z}$, which respectively are:

$$\begin{cases} Z \quad \rightarrow \left(x + i\frac{y}{2} + j\frac{z}{2}\right) = \frac{x + Z}{2} \\ \frac{a^2}{Z} \quad \rightarrow \qquad \frac{1}{2}a^3 \frac{\frac{Z^*}{r}}{r^2} \end{cases}$$

Both functions are analytical and so is her sum.
Assume as complex potential

$$A = \frac{x + Z}{2} + \frac{1}{2}a^3 \frac{Z^*}{r^3}$$

Passing to spherical coordinates with x as polar axis



$$\begin{cases} x = r cos\vartheta \\ y = r sin\vartheta cos\varphi \\ z = r sin\vartheta sin\varphi \end{cases}$$

we get:

$$A = \left(r + \frac{1}{2}\frac{a^3}{r^2}\right) cos\vartheta + i\left(\frac{r}{2} - \frac{1}{2}\frac{a^3}{r^2}\right) sin\vartheta cos\varphi + j\left(\frac{r}{2} - \frac{1}{2}\frac{a^3}{r^2}\right) sin\vartheta sin\varphi$$

The imaginary part (analogous to the 2D stream function Ψ) is zero for $r = a$, sphere surface. The derivative of the potential (which can be $\frac{1}{2}\partial A$ or $\partial Re(A)$ or $\frac{\partial A}{\partial x}$ as you want) is:

$$A = \left(x + i\frac{y}{2} + j\frac{z}{2}\right) + \frac{1}{2}a^3\frac{x + iy + jz}{r^3}$$

$$U = \frac{1}{2}\partial A = \partial Re(A)$$

$$U = \partial Re(A) = \partial(x + \frac{1}{2}a^3\frac{x}{r^3})$$

$$U = 1 + \frac{1}{2}a^3\frac{1 - \frac{3xz^*}{r^2}}{r^3}$$

The velocity field $\vec{v} = \left(v_x, v_y, v_z\right) = (U_1, -U_2, -U_3)$ is:

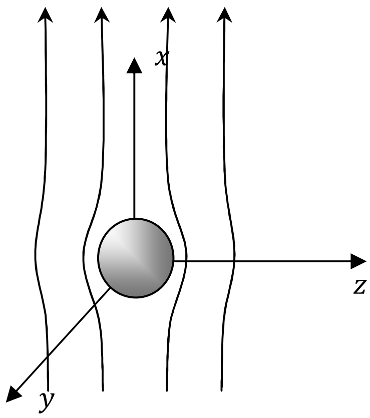
$$v_x = -\frac{3}{2}\frac{a^3x^2}{r^5} + \frac{1}{2}\frac{a^3}{r^3} + 1$$

$$v_y = -\frac{3}{2}\frac{a^3xy}{r^5}$$

$$v_z = -\frac{3}{2}\frac{a^3xz}{r^5}$$

This coincides (.... as it should be) with Ashley, [2].

Note a term 1 of speed parallel to the x axis The fluid moves in the direction of the positive x



If we assume for the analytic potential the provisional name

$$A = \Phi + i\Psi_1 + j\Psi_2$$

we have analogies with 2D, example:

$$\partial^* \Phi = -\partial^*(i\Psi_1 + j\Psi_2)$$

Note that here a doubt arises. The term $i\Psi_1 + j\Psi_2$ seems "similar" to the stream function Ψ of the 2D case. However a) it seems that I have just completed for the analyticity the harmonic function $\Phi = Re(A) = (x + \frac{1}{2}a^3 \frac{x}{r^3})$ with an imaginary part $Im(A) = i\Psi_1 + j\Psi_2$ which ….
b) do not use it (because I have not used, I only did $\partial\Phi$, ie practically $grad\Phi$ and …..
c) do not know how to interpret, I do not know what it is.
So it seems that $i\Psi_1 + j\Psi_2$ is good for nothing.
Obviously this is not true, but at the moment (…… 1994, n.d.r.) I cannot understand.
Sommerfeld [3] says: *"In general, no stream function Ψ can be associated with the velocity potential in the three-dimensional case"*.

Exercise 2 - Sphere under the effect of a constant electric field.

I change a sign in A

$$A = \frac{x+Z}{2} - \frac{1}{2}a^3\frac{Z^*}{r^3}$$
$$Re(A) = \left(x - \frac{1}{2}a^3\frac{x}{r^3}\right)$$

$Re(A)$ is zero if $\frac{1}{2}\frac{a^3}{r^3} = 1$. Let's take $Re(A)$ as potential and we have:

$$U = 1 - \frac{1}{2}a^3\frac{1 - \frac{3xZ^*}{r^2}}{r^3}$$

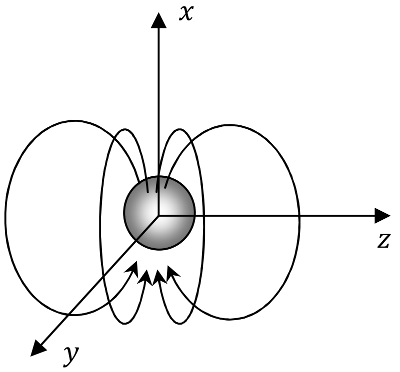
Assume $\vec{E} = \left(E_x, E_y, E_z\right) = (U_1, -U_2, -U_3)$ as an electric field.

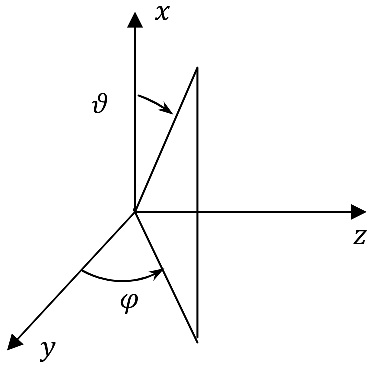
$$E_x = \frac{3}{2}\frac{a^3x^2}{r^5} - \frac{1}{2}\frac{a^3}{r^3} + 1$$
$$E_y = \frac{3}{2}\frac{a^3xy}{r^5}$$
$$E_z = \frac{3}{2}\frac{a^3xz}{r^5}$$

If $\frac{1}{2}\frac{a^3}{r^3} = 1$ (surface of sphere with radius "so that $\frac{1}{2}\frac{a^3}{r^3} = 1$") the field is:

$$E_x = 3\frac{x}{r}\frac{x}{r}$$
$$E_y = 3\frac{x}{r}\frac{y}{r}$$
$$E_z = 3\frac{x}{r}\frac{z}{r}$$

Summarizing, the $\vec{E}$ field on the surface is $\vec{E} = 3\ cos\vartheta\ \hat{e}_r$ ie a radial electric field weighted by $cos\vartheta$.
The sphere is under the effect of a constant electric field in the direction of the real axis x.





Exercise 3 – 3D flow into a trihedron (analogous to the 2D flow into a 90 degree corner).

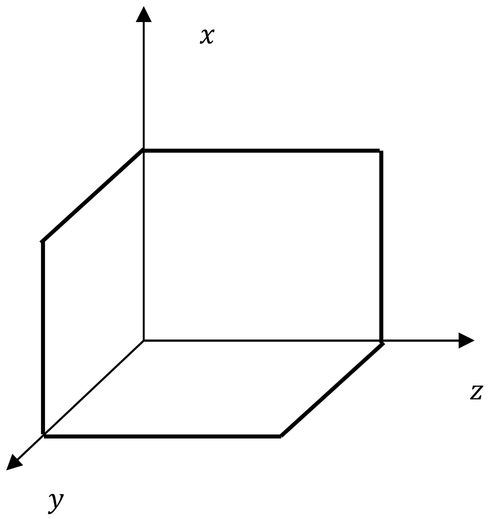
The analytic function

$$A = (2x^2 - y^2 - z^2) + 2ixy + 2jxz$$

corresponds to

$$\begin{cases} \Phi = 2x^2 - y^2 - z^2 \\ \Psi = 2ixy + 2jxz = i\Psi_1 + j\Psi_2 \end{cases}$$

You get $\Psi = 0$ on $x = 0$ or $(y = 0$ and $z = 0)$ which means a trihedron.



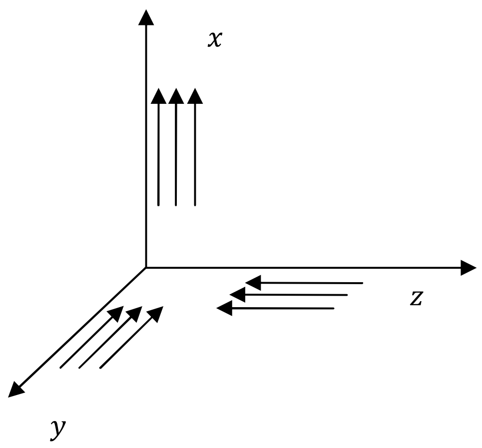
If we delete the region $x < 0$, y< 0, $z < 0$, we have the internal of the trihedron.
The fluid velocity (analytic) is obtained as

$$v = \frac{\partial}{\partial x} A = 2(2x + iy + jz)$$

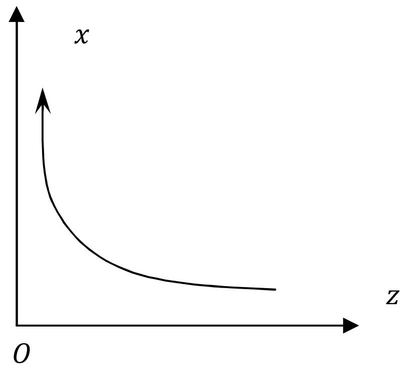
The velocity field $\vec{v} = (v_x, v_y, v_z) = (U_1, -U_2, -U_3)$ is:

$$v_x = 4x$$
$$v_y = -2y$$
$$v_z = -2z$$

The fluid velocity $\vec{v}_\infty$ is very strong and proportional to x for $x > 0$, going in the positive x direction. Instead it is coming from $y \to \infty$ and $z \to \infty$.



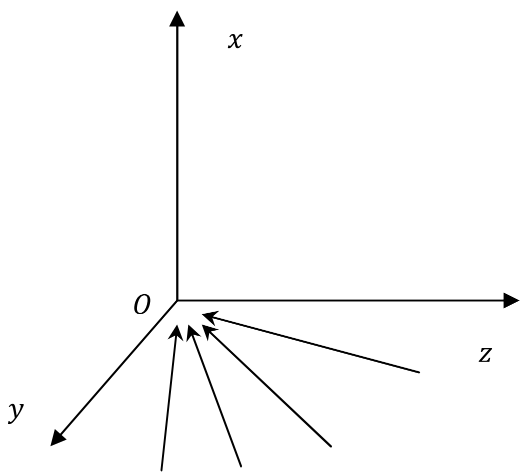
On the x, z plane ($y = 0$) the velocity is $\vec{v} = 2(2x, -z)$, see figure below.
In figure, a streamline is plotted.

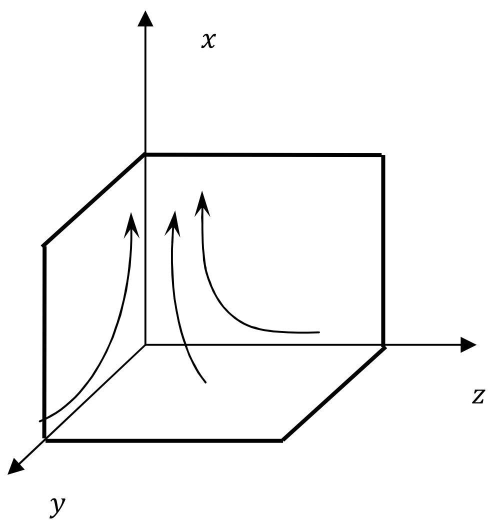


On the $x = 0$ plane the velocity is $\vec{v} = 2(-y, -z)$, see figure below.
In that plane the streamlines correspond to

$$v_y = -2y$$
$$v_z = -2z$$

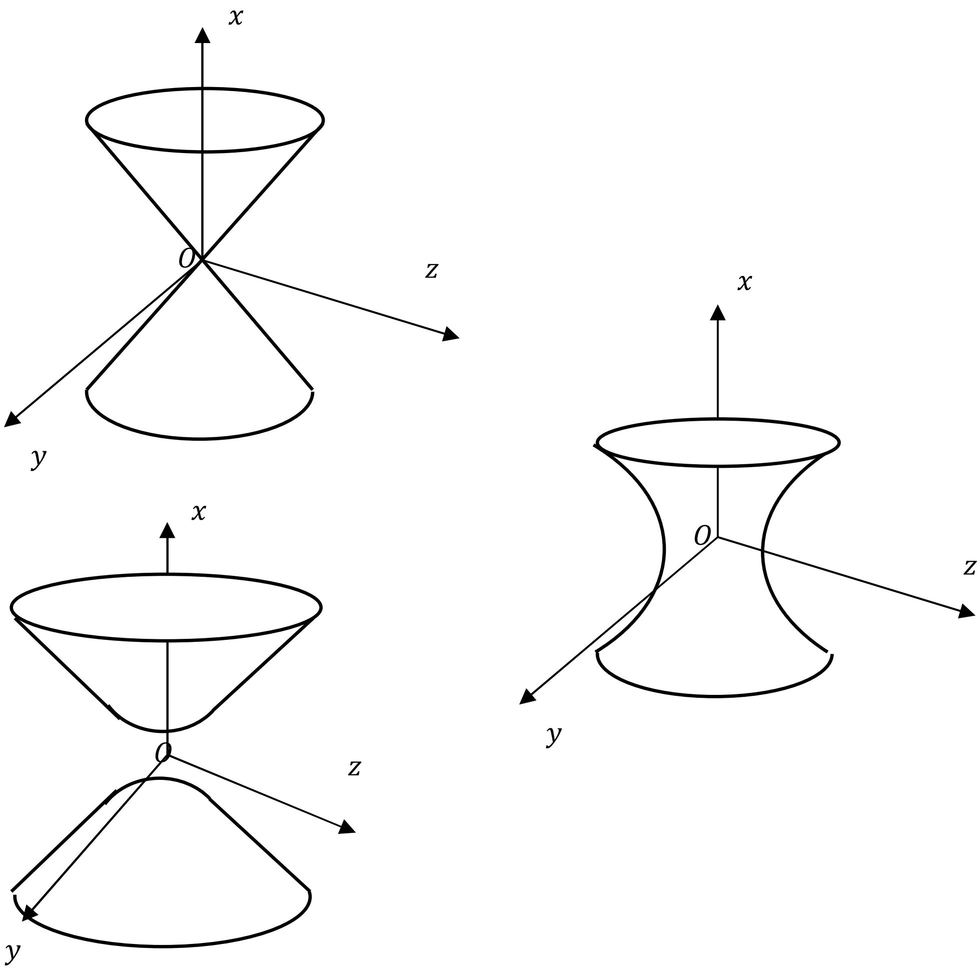
The lines $\frac{v_y}{v_z} = \frac{y}{z} = const$ are straight lines passing from the origin O with velocity towards O.





Note: the equipotential surfaces $\Phi = 2x^2 - y^2 - z^2 = const$ are respectively

$$\begin{cases} \dfrac{y^2+z^2}{a^2} - \dfrac{x^2}{b^2} = 1 & \text{Hyperboloid of one sheet} \\ \dfrac{y^2+z^2}{a^2} - \dfrac{x^2}{b^2} = -1 & \text{Hyperboloid of two sheets} \\ \dfrac{y^2+z^2}{a^2} - \dfrac{x^2}{b^2} = 0 & \text{Circular cone} \end{cases}$$



More, the analytic function $A = (2x^2 - y^2 - z^2) + 2ixy + 2jxz$ simply is $(x+iy)^2 + (x+jz)^2$, so it could be interesting to study the more general family of analytic function

$$a(x+iy)^p + b(x+jz)^q$$

which presumably gives a lot of shapes.

Exercise 4 – A source/sink, or a point charge.

Note that the aforementioned analytic function

$$U = \frac{\frac{Z^*}{r}}{r^2}$$

has to as "primitive"

$$A = \frac{1 + \frac{Z^*}{r}}{x + r} = \frac{1}{r} - i\,\frac{\frac{y}{r}}{x + r} - j\,\frac{\frac{z}{r}}{x + r}$$

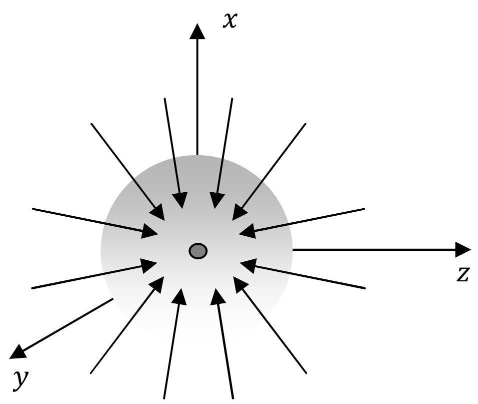
which in turn is analytic.
In other words is $U = \frac{\frac{Z^*}{r}}{r^2}$ obtained by the analytic potential $A = \frac{1+\frac{Z^*}{r}}{x+r} = \frac{1}{r} - i\frac{\frac{y}{r}}{x+r} - j\frac{\frac{z}{r}}{x+r}$ performing the derivative $U = \frac{1}{2}\partial A$.
The interpretation of this potential as a "source/sink" can be very easy. It can be interpreted as the potential of a fluid flow, and taking $U = \frac{1}{2}\partial A$ in order to obtain the velocity field $\vec{v} = \left(v_x, v_y, v_z\right)$.
We can simply perform

$$U = \frac{1}{2}\partial A = \partial Re(A) = \partial\frac{1}{r}$$

from which we get:

$$U = -\frac{\frac{Z^*}{r}}{r^2}$$

As usual the velocity field is $\vec{v} = \left(v_x, v_y, v_z\right) = (U_1, -U_2, -U_3)$.
The streamlines are <u>radial</u> and <u>inward</u> (sink).



The same can represent a point charge.

Exercise 5 – Line of charge on negative x-axis.

The analytic function $\frac{1+\frac{Z^*}{r}}{x+r}=\frac{1}{r}-i\,\frac{\frac{y}{r}}{x+r}-j\,\frac{\frac{z}{r}}{x+r}$ represents an electric field $\vec{E}$ whose components are $E_x=\frac{1}{r}, E_y=\frac{\frac{y}{r}}{x+r}, E_z=\frac{\frac{z}{r}}{x+r}$. The aforementioned analytic function can be obtained by the harmonic potential $\Phi=\ln(x+r)$ if we remember the general rule "if Φ is harmonic, $\partial\Phi$ is analytic". Verify.

$$\partial\Phi=(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}-j\frac{\partial}{\partial z})\ \ln(x+\sqrt{x^2+y^2+z^2})$$

$$\frac{\partial}{\partial x}=\frac{1}{\sqrt{x^2+y^2+z^2}}=\frac{1}{r}$$

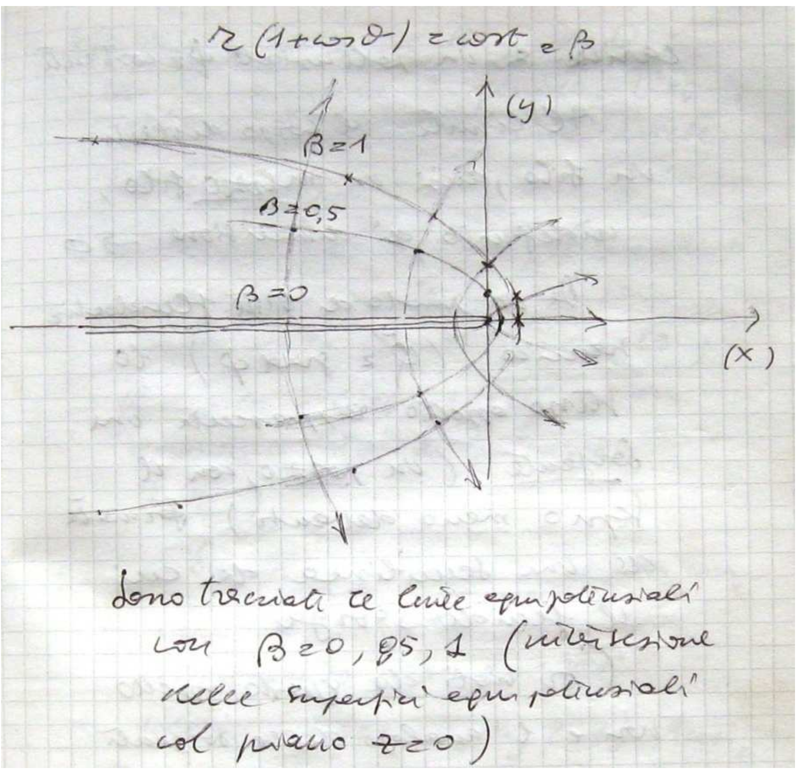
$$\frac{\partial}{\partial y}=\frac{y}{\left(x+\sqrt{x^2+y^2+z^2}\right)(\sqrt{x^2+y^2+z^2})}=\frac{y}{(x+r)r}$$

$$\frac{\partial}{\partial z}=\frac{z}{(x+r)r}$$

QED.
It may therefore be interesting the following interpretation.
This field of which I spoke has equipotential surfaces $\Phi=\ln(x+r)=const.$
Study, for example, in the x, y plane ($z=0$) for which we have:

$$x+\sqrt{x^2+y^2}=const=\beta$$
$$x+r=const=\beta$$
$$r\cos\vartheta+r=const=\beta$$
$$r(1+\cos\vartheta)=const=\beta$$

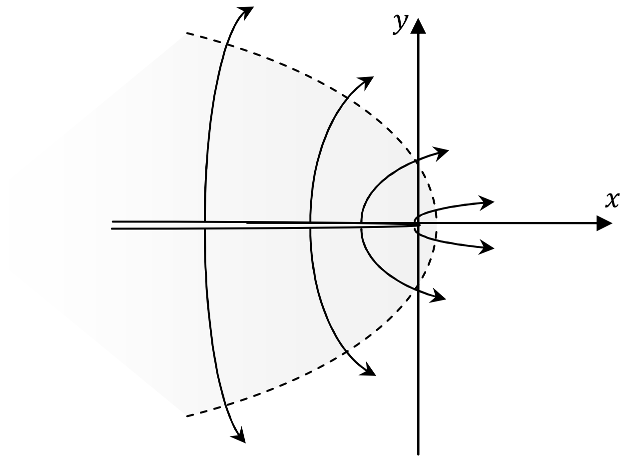
Here I've plotted the equipotential lines $\beta=0$; $\beta=0,5$; $\beta=1$ (intersection of equipotential surfaces with the $z=0$ plane). See figure below.



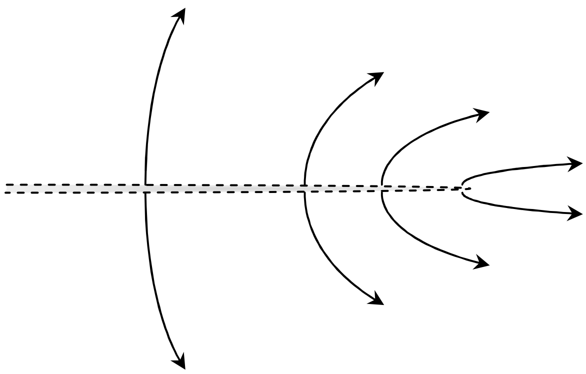
Each equipotential surface is a paraboloid of revolution.
In fact, from $x + \sqrt{x^2 + y^2} = const = \beta$ ($z = 0$ plane) squaring and simplifying is obtained:

$$y^2 = \beta^2 - 2\beta x$$

ie a parabola passing on the x axis in the point $x = \frac{\beta}{2}$.



The paraboloid of revolution degenerates in a semi infinite straight line on the negative x axis ie a line charge.



Exercise 6 – Charged ellipsoid.

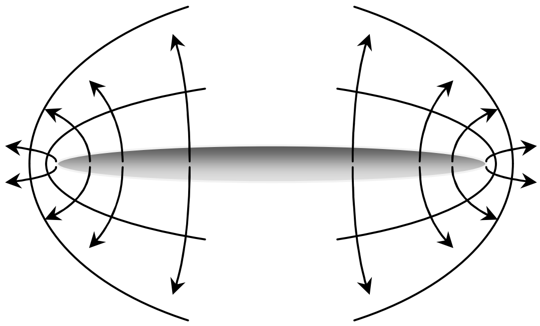
It is possible to combine the previous potential $\Phi = ln(x + \sqrt{x^2 + y^2 + z^2})$ shifted in $x = \pm c$ in the following way:

$$\Phi = ln(x + c + \sqrt{(x + c)^2 + y^2 + z^2}) - ln(x - c + \sqrt{(x - c)^2 + y^2 + z^2})$$

This is obviously harmonic because $ln(x + \sqrt{x^2 + y^2 + z^2})$ is harmonic.
So the derivative $\partial\Phi$ is analytic and represents an electric field $\vec{E}$.
Rewrite

$$\Phi = ln\frac{(x + c + \sqrt{(x + c)^2 + y^2 + z^2}}{(x - c + \sqrt{(x - c)^2 + y^2 + z^2})}$$

and in this form it’s quite easy to show that (see Sommerfeld [4]) this expression assumes a constant value on a family of confocal ellipsoids with the separation of focal points $2c$. So this field is the field of a charged ellipsoid.



The ellipsoid which degenerates to a straight line of length $2c$ also belongs to that family.
Note that the corresponding 2D problem is already in Maxwell “Treatise” [5] under the name “confocal ellipses and hyperbolas”, where it was solved with the method of conjugate functions ie analytic functions.

Exercise 7 – Point charge in presence of a grounded conducting flat plate.

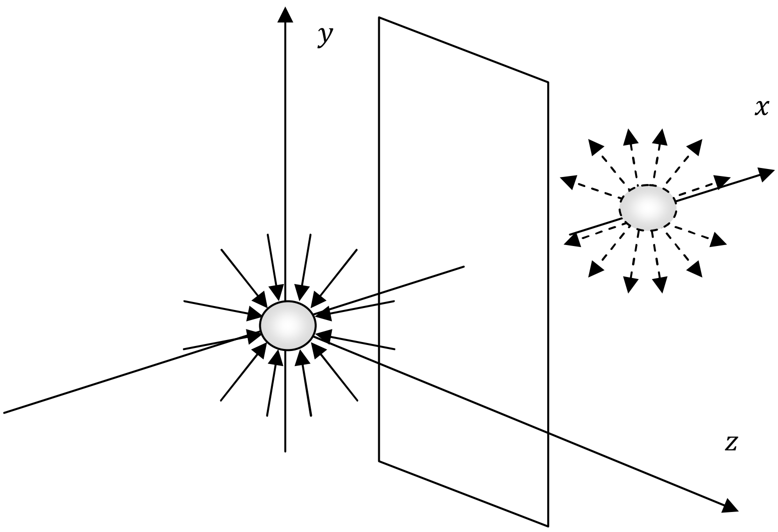
Consider the analytic potential $A = \frac{1+\frac{Z^*}{r}}{x+r} = \frac{1}{r} - i\frac{\frac{y}{r}}{x+r} - j\frac{\frac{z}{r}}{x+r}$, from which we get the analytic function:

$$U = -\frac{\frac{(x+iy+jz)^*}{r}}{r^2}$$

We assume $\vec{E} = (E_x, E_y, E_z) = (U_1, -U_2, -U_3)$ as electric field.
The electric field $\vec{E}$ is radial and inward.
Consider a 2nd analytic function with opposite sign, shifted in $x = d$. Analytic potential:

$$A_1 = -\frac{1}{r_1} + i\frac{\frac{y}{r_1}}{x-d+r_1} + j\frac{\frac{z}{r_1}}{x-d+r_1}$$

$$r_1 = \sqrt{(x-d)^2 + y^2 + z^2}$$



Take the sum of the potentials, which is analytic.
The real part

$$\Phi = \frac{1}{r} - \frac{1}{r_1} = (r_1 - r)\frac{1}{r_1 r}$$

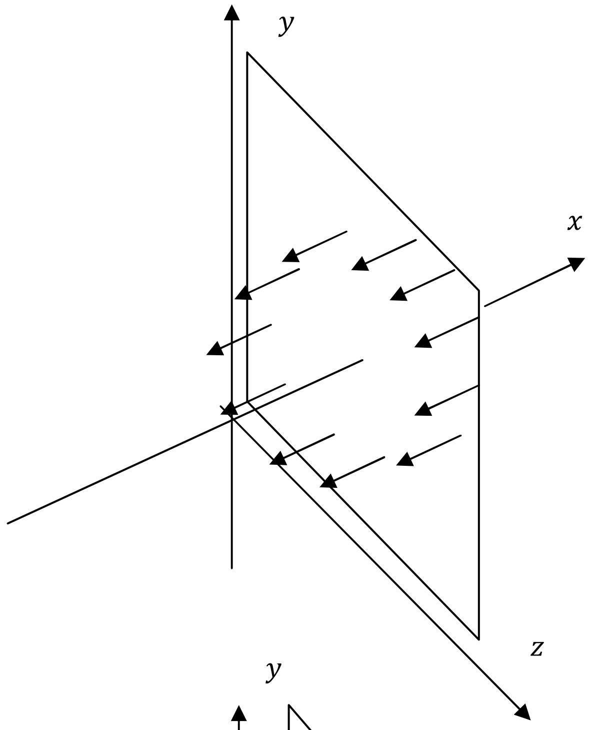
is zero on $r_1 = r$, which means the plane $x = \frac{d}{2}$.
We obtain the total field from the analytic function

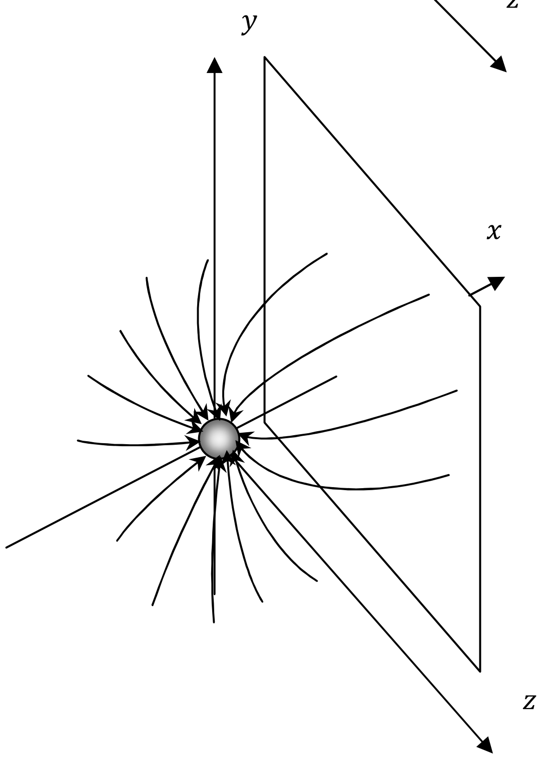
$$U_{tot} = -\frac{\frac{(x+iy+jz)^*}{r}}{r^2} + \frac{\frac{((x-d)+iy+jz)^*}{r_1}}{{r_1}^2}$$

On $r_1 = r$, which means the plane $x = \frac{d}{2}$, this reduces to

$$U_{tot} = -\frac{\frac{(\frac{d}{2} + iy + jz)^*}{r_1}}{{r_1}^2} + \frac{\frac{((\frac{d}{2} - d) + iy + jz)^*}{r_1}}{{r_1}^2} = -\frac{d}{\left(\sqrt{(\frac{d}{2})^2 + y^2 + z^2}\right)^3}$$

The electric field $\vec{E}$ on that plane is proportional to $\frac{1}{{r_1}^3}$, constant for $(y^2 + z^2) = const$ and is directed in the negative x direction.





This exercise has of course to do with the method of electrical images.

Exercise 8 – Point charge in the presence of a grounded conducting sphere

Let's start from the analytic potential $\frac{1+\frac{Z^*}{r}}{x+r} = \frac{1}{r} - i\frac{\frac{y}{r}}{x+r} - j\frac{\frac{z}{r}}{x+r}$, from which, by $\frac{1}{2}\partial$ or $\frac{\partial}{\partial x}$ as you will, you get the analytic function:

$$U = -\frac{\frac{(x+iy+jz)^*}{r}}{r^2}$$

with minus sign.

The electric field $\vec{E} = \left(E_x, E_y, E_z\right) = (U_1, -U_2, -U_3)$ is <u>radial</u> and <u>inward.</u>

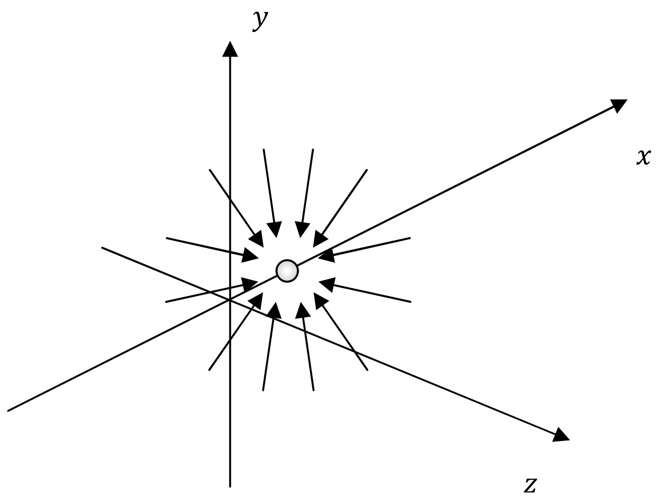
Shifted in a point A with $x = x_1$ and weighted by $+a$, we have the analytic potential:

$$+a\left(\frac{1}{r_1} - i\frac{\frac{y}{r_1}}{x - x_1 + r_1} - j\frac{\frac{z}{r_1}}{x - x_1 + r_1}\right)$$

$$r_1 = \sqrt{(x-x_1)^2 + y^2 + z^2}$$

and the analytic field:

$$U_A = -a\frac{\frac{((x-x_1)+iy+jz)^*}{r_1}}{{r_1}^2}$$



Consider a 2nd analytic function with opposite sign, shifted in a point B with $x = x_2$ and weighted by b:

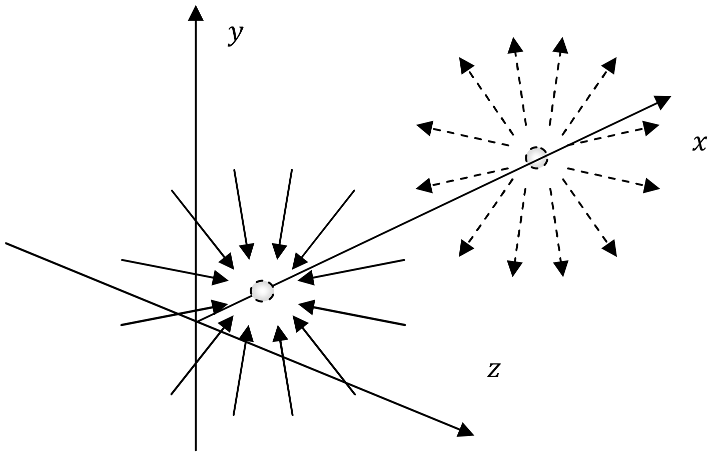
$$-b\left(\frac{1}{r_2} - i\frac{\frac{y}{r_2}}{x - x_2 + r_1} - j\frac{\frac{z}{r_2}}{x - x_2 + r_1}\right)$$

$$r_2 = \sqrt{(x-x_2)^2 + y^2 + z^2}$$

The analytic field is:

$$U_B = b\frac{\frac{((x-x_2)+iy+jz)^*}{r_2}}{{r_2}^2}$$

The analytic field everywhere is of course $U_A + U_B$.



Take the sum of the potentials, which is analytic. The real part $\Phi = a\frac{1}{r_1} - b\frac{1}{r_2} = \frac{ar_2 - br_1}{r_2 r_1}$ is zero if:

$$\frac{r_1}{r_2} = \frac{a}{b}$$

Squared and written explicitly:

$$(x - x_1)^2 + y^2 + z^2 - \frac{a^2}{b^2}[\,(x - x_2)^2 + y^2 + z^2]$$
$$= \,(x^2 - 2xx_1 + x_1{}^2) - \frac{a^2}{b^2}(x^2 - 2xx_2 + x_2{}^2) + (1 - \frac{a^2}{b^2})(y^2 + z^2)$$
$$= 2x(-x_1 + \frac{a^2}{b^2}x_2) + x_1{}^2 - x_2{}^2\frac{a^2}{b^2} + (1 - \frac{a^2}{b^2})(x^2 + y^2 + z^2)$$

Let now

$$R^2 = x_1 x_2 \quad and \quad \frac{a}{b} = \frac{R}{x_2}(= \frac{x_1}{R})$$

and finally we get

$$(1 - \frac{a^2}{b^2})(x^2 + y^2 + z^2) = (1 - \frac{a^2}{b^2})R^2$$

ie the equation of a sphere of radius R around the center O, where $R^2 = x_1 x_2$, and with the other relation $\frac{a}{b} = \frac{R}{x_2}(= \frac{x_1}{R})$.

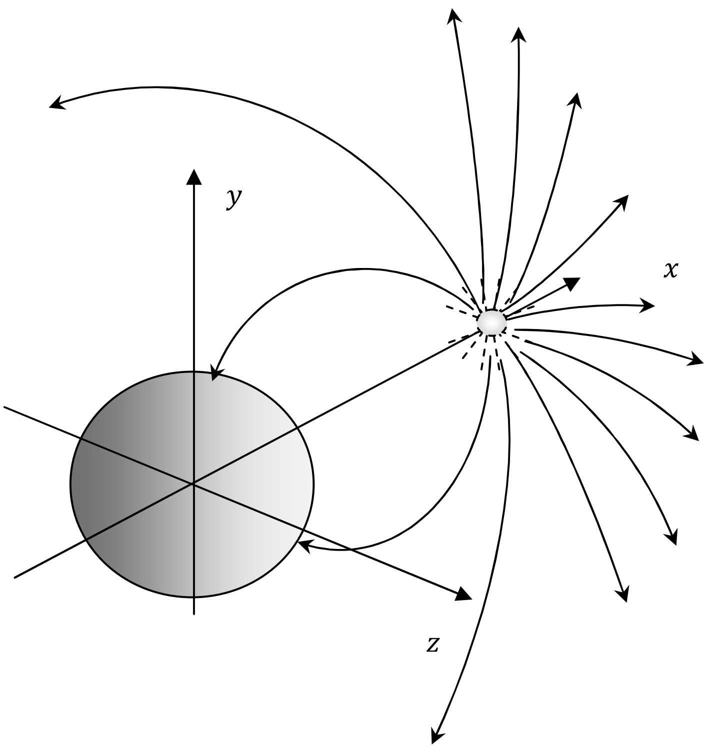
What does this mean? We can quote the brilliant words of Maxwell [5], Chap. XI, "Theory of electric images and electric inversion":

"*Since this spherical surface is at potential zero, if we suppose it constructed of thin metal and connected with the earth, there will be no alteration of the potential at any point either outside or inside, but the electrical action everywhere will remain that due to the two electrified points A and B.*

If we now keep the metallic shell in connection with the earth and remove the point B, the potential within the sphere will become everywhere zero, but outside it will remain the same as before. For the surface of the sphere still remains at the same potential, and no changes has be made in the exterior electrification.

Hence, if an electrified point A be placed outside a spherical conductor which is at potential zero, the electrical action at all points outside the sphere will be that due to the point A together with another point B within the sphere, which we may call the electrical image of A.
In the same way we may shew that if B is a point placed inside the spherical shell, the electrical action within the sphere is that due to B, together with its image A".

In short, this is the method of reciprocal radii, developed trough the pioneering work of William Thomson , Cambridge and Dublin Math. Journ., (1848).
Sommerfeld [6] says:
"The term "reciprocal" arises from the (bad) habit of setting R=1, in which case $x_1 = \frac{1}{x_2}$*. For reasons of dimensionality we consider it better to retain the radius R as a length".*
Summing up, the (analytic) field outside the sphere is $U_A + U_B$.and the shape of the electric field $\vec{E}$ is like that:



Conclusion

I've presented here some examples of applications of 3-dimensional analytic functions to electrostatics and fluid flow, mainly devoted to engineers and physicists.
Aiming of this work were:
1 st: show how easy is the complex notation I've adopted elsewhere for Clifford algebra. In this case it reduces to real and imaginary i , j, squared -1:
2 nd: suggest the idea that "the powerful tool of the theory of complex functions can be used in three-dimensional potential theory".
In fact each analytic function means a problem already solved (applied to a certain shape). So it is my opinion that it would be very interesting to pursue studies on this topic.

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[5] James Clerk Maxwell, "A Treatise On Electricity And Magnetism - Volume One" (Dover)

[6] Arnold Sommerfeld, "Partial Differential Equations in Physics" - Lectures on Theoretical Physics Volume VI (Academic Press, 1949)