A momentum space for Majorana spinors

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Abstract

In this work I will study the Dirac Gamma matrices in Majorana basis and Majorana spinors. A Fourier like transform is defined with Gamma matrices, defining a momentum space for Majorana spinors. It is shown that the Wheeler propagator has asymptotic states with well defined momentum.

Contents

1 Introduction

The Dirac matrices, γ^μ, in Majorana basis are purely imaginary. That means that $i\gamma^\mu$ are 4x4 real matrices.

An example of such matrices in a particular basis is:

$$\begin{aligned} i\gamma^1 &= \begin{bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \quad i\gamma^2 = \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{bmatrix} \quad i\gamma^3 = \begin{bmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ i\gamma^0 &= \begin{bmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad i\gamma^5 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad = -\gamma^0\gamma^1\gamma^2\gamma^3 \end{aligned} \tag{1.1}$$

The metric, given by the anti-commutator of the matrices, is the Minkowski space-time metric:

$$g^{\mu\nu} = -\{i\gamma^\mu, i\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = diag(1,-1,-1,-1),\ \mu,\nu = 0,1,2,3 \tag{1.2}$$

In fact, when working with 4x4 real matrices, we can only find a set of 5 anti-commuting matrices. This means that with 4x4 real matrices we can describe the Minkowski space-time, but we can not describe, for instance, a 4D euclidean space.

We define $\not{p} = \gamma^\mu p_\mu$. The Dirac equation for the free fermion can be written only with real matrices:

$$i\gamma^0(i\gamma^\mu\partial_\mu - m)\Psi(x) = i\gamma^0(i\not{\partial} - m)\Psi(x) = 0 \tag{1.3}$$

And we can express Lorentz transforms only with real matrices.

The spin operators are defined as:

$$\sigma^k = i\gamma^k i\gamma^5 \quad k = 1,2,3 \tag{1.4}$$

They verify:

$$[\sigma^i, \sigma^j] = i\gamma^0\epsilon^{ij}_k\sigma^k \tag{1.5}$$

Where ϵ^{ij}_k is the Levi-Civita symbol. Note that $i\gamma^0$ commutes with σ^k and squares to -1, so it can be thought of as the imaginary unit in the spin algebra.

We will use the following conventions:

If p is a Lorentz vector:

$$(\gamma^\mu p_\mu)(\gamma^\mu p_\mu) = (\not{p})(\not{p}) = p^\mu p_\mu = p\cdot p = p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 \tag{1.6}$$

Given a 3-vector $\vec{p}$ and a real number $m > 0$, we define:

$$\vec{p}^i = p^i,\ i = 1,2,3 \tag{1.7}$$

$$\vec{\not{p}} = \vec{\gamma}\cdot\vec{p} \tag{1.8}$$

$$E_p = \sqrt{\vec{p}^2 + m^2} \tag{1.9}$$

$$\not{m}_p = \gamma^0 E_p - \vec{\gamma}\cdot\vec{p} \tag{1.10}$$

$$\not{m}_{-p} = \gamma^0 E_p + \vec{\gamma}\cdot\vec{p} \tag{1.11}$$

Note that $(\not{m}_p)^2 = m^2$. We use the definition $\not{m}_p$ to distinguish from $\not{p}$ when $p^0 \neq E_p$.

A Majorana spinor is a real 4D vector on which the Dirac matrices act. A Dirac spinor is a complex 4D vector, on which the Dirac matrices act.

The references I most used were [2] and [3].

2 Lorentz transformations

A Lorentz transformation can be represented by a tensor ${a_\rho}^\mu$ which leaves the metric invariant:

$$g^{\rho\sigma}{a_\rho}^\mu{a_\sigma}^\nu = g^{\mu\nu} \tag{2.1}$$

Let S be a Majorana matrix that verifies:

$$\gamma^\rho{a_\rho}^\mu = S^{-1}\gamma^\mu S \tag{2.2}$$

Then it verifies $\gamma^0 S^{-1} = S^T\gamma^0$ and $\gamma^5 S = S\gamma^5$. In the particular case of a Lorentz boost, the S matrix is given by:

$$S_L = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2m}} \tag{2.3}$$

$$S_L^{-1} = -\alpha^0 S_L^T\alpha^0 = \frac{\gamma^0\not{p} + m}{\sqrt{E_p + m}\sqrt{2m}} \tag{2.4}$$

where $\frac{\vec{p}}{m} = \frac{\vec{v}}{\sqrt{1-\vec{v}^2}}$,$\vec{v}$ is the boost velocity. In the particular case of a rotation, the S matrix is given by:

$$S_R = exp(i\gamma^5\gamma^0\gamma^i\varphi_i),\ i = 1, 2, 3 \tag{2.5}$$

$$S_R^{-1} = S_R^T = -\gamma^0 S_R^T\gamma^0 \tag{2.6}$$

In general, the S matrix is the product of a Lorentz boost and a rotation.

3 Fourier-Majorana transform (in space)

Given a 4x4 matrix $M(\vec{x})$, the Fourier-Majorana transform (in space) is defined as:

$$M(\vec{p}) = \int d^3\vec{x}O(\vec{p}, \vec{x})M(\vec{x}) \tag{3.1}$$

Where O is the real 4x4 matrix given by:

$$O(\vec{p}, \vec{x}) = e^{-i\gamma^0\vec{p}\cdot\vec{x}}\frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} \tag{3.2}$$

The inverse Fourier-Majorana transform is given by:

$$M(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}O^T(\vec{p}, \vec{x})M(\vec{p}) \tag{3.3}$$

Where O^T is the transpose of O, given by:

$$O^T(\vec{p}, \vec{x}) = \frac{\not{p}\gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}}e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{3.4}$$

To prove it:

$$\int \frac{d^3\vec{p}}{(2\pi)^3} O^T(\vec{p},\vec{y}) O(\vec{p},\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 E_p} \frac{\not{h}_p\gamma^0 + m}{\sqrt{2(E_p+m)}} e^{i\gamma^0 \vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{h}_p\gamma^0 + m}{\sqrt{2(E_p+m)}} \tag{3.5}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\frac{\not{h}_p}{m}\vec{p}\cdot(\vec{y}-\vec{x})} \frac{\not{h}_p\gamma^0}{E_p} \tag{3.6}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} cos(\vec{p}\cdot(\vec{y}-\vec{x})) \tag{3.7}$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} (-cos(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{\vec{\not{p}}\gamma^0}{E_p} + sin(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{mi\gamma^0}{E_p} \tag{3.8}$$

$$= \delta^3(\vec{y}-\vec{x}) \tag{3.9}$$

Note that both $cos(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{\vec{\not{p}}\alpha^0}{E_p}$ and $sin(\vec{p}\cdot(\vec{y}-\vec{x}))\frac{mi\gamma^0}{E_p}$ are odd in $\vec{p}$ and therefore do not contribute to the integral.

$$\int d^3\vec{x} O(\vec{q},\vec{x}) O^T(\vec{p},\vec{x}) = \int d^3\vec{x} e^{-i\gamma^0\vec{q}\cdot\vec{x}} \frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{i\gamma^0\vec{p}\cdot\vec{x}} \tag{3.10}$$

$$= \frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \int d^3\vec{x} cos(\vec{q}\cdot\vec{x}) cos(\vec{p}\cdot\vec{x}) + \tag{3.11}$$

$$- i\gamma^0 \frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} i\gamma^0 \int d^3\vec{x} sin(\vec{q}\cdot\vec{x}) sin(\vec{p}\cdot\vec{x}) \tag{3.12}$$

$$= (2\pi)^3 \frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \frac{\delta^3(\vec{q}-\vec{p}) + \delta^3(\vec{q}+\vec{p})}{2} + \tag{3.13}$$

$$- (2\pi)^3 i\gamma^0 \frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} i\gamma^0 \frac{\delta^3(\vec{q}-\vec{p}) - \delta^3(\vec{q}+\vec{p})}{2} \tag{3.14}$$

$$= (2\pi)^3 \frac{\delta^3(\vec{q}-\vec{p})}{2} (\frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} + \tag{3.15}$$

$$- i\gamma^0 \frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} i\gamma^0) + \tag{3.16}$$

$$+ (2\pi)^3 \frac{\delta^3(\vec{q}+\vec{p})}{2} (\frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} + \tag{3.17}$$

$$+ i\gamma^0 \frac{\not{h}_q\gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_p\gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} i\gamma^0) \tag{3.18}$$

$$= (2\pi)^3 \frac{\delta^3(\vec{q}-\vec{p})}{2} \Big(\frac{\not{h}_q \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_q \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} + \tag{3.19}$$

$$- i\gamma^0 \frac{\not{h}_q \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_q \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} i\gamma^0 \Big) + \tag{3.20}$$

$$+ (2\pi)^3 \frac{\delta^3(\vec{q}+\vec{p})}{2} \Big(\frac{\not{h}_q \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_{-q} \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} + \tag{3.21}$$

$$+ i\gamma^0 \frac{\not{h}_q \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} \frac{\not{h}_{-q} \gamma^0 + m}{\sqrt{E_q+m}\sqrt{2E_q}} i\gamma^0 \Big) \tag{3.22}$$

$$= (2\pi)^3 \frac{\not{h}_q \gamma^0 + \gamma^0 \not{h}_q}{2E_q} \delta^3(\vec{q}-\vec{p}) \tag{3.23}$$

$$= (2\pi)^3 \delta^3(\vec{q}-\vec{p}) \tag{3.24}$$

4 Fourier-Majorana transform (in space-time)

Given a 4x4 matrix $M(x)$, the Fourier-Majorana transform (in space-time) is defined as:

$$M(p) = \int d^4x O(p,x) M(x) \tag{4.1}$$

Where O is the real 4x4 matrix given by:

$$O(p,x) = e^{i\gamma^0 p^0 x^0} O(\vec{p},\vec{x}) = e^{i\gamma^0 p\cdot x} \frac{\not{h}_p \gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} \tag{4.2}$$

Note that E_p and $\not{h}_p$ don't depend on p^0. The inverse Fourier-Majorana transform is given by:

$$M(x) = \int \frac{d^4p}{(2\pi)^4} O^T(p,x) M(p) \tag{4.3}$$

Where O^T is the transpose of O,given by:

$$O^T(p,x) = O^T(\vec{p},\vec{x}) e^{-i\gamma^0 p^0 \cdot x^0} = \frac{\not{h}_p \gamma^0 + m}{\sqrt{E_p+m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x} \tag{4.4}$$

To prove it:

$$\int \frac{d^4p}{(2\pi)^4} O^T(p,y) O(p,x) = \int \frac{d^3\vec{p}}{(2\pi)^3} O^T(\vec{p},\vec{y}) \Big(\int \frac{dp^0}{2\pi} e^{-i\gamma^0 p^0 (y^0 - x^0)} \Big) O(\vec{p},\vec{x}) \tag{4.5}$$

$$= \delta(y^0 - x^0) \int \frac{d^3\vec{p}}{(2\pi)^3} O^T(\vec{p},\vec{y}) O(\vec{p},\vec{x}) \tag{4.6}$$

$$= \delta^4(y-x) \tag{4.7}$$

$$\int d^4x O(q,x) O^T(p,x) = \int dx^0 e^{i\gamma^0 q^0 x^0} (\int d^3\vec{x} O(\vec{q},\vec{x}) O^T(\vec{p},\vec{x})) e^{-i\gamma^0 p^0 x^0} \quad (4.8)$$

$$= (2\pi)^3 \delta^3(\vec{q}-\vec{p}) \int dx^0 e^{i\gamma^0 (q^0-p^0) x^0} \quad (4.9)$$

$$= (2\pi)^4 \delta^4(q-p) \quad (4.10)$$

In what follows we will call just Fourier-Majorana transform to both Fourier-Majorana transforms in space and space-time. It will be clear from the context to which we are referring to.

5 Dirac equation

The Dirac equation for the free fermion is:

$$i\gamma^0(i\partial\!\!\!/ - m)\Psi(x) = 0 \quad (5.1)$$

Where Ψ is a spinor, a vector of the 4D space, on which the Dirac matrices act. Note that the equation contains only real matrices.

We can make a Fourier-Majorana transform and go to momentum space:

$$i\gamma^0(i\partial\!\!\!/ - m)\Psi(x) = \int \frac{d^4p}{(2\pi)^4} (i\frac{\not{m}_p}{m} p^0 - i\gamma^0 \vec{p\!\!\!/} \frac{\not{m}_p}{m} - i\gamma^0 m) O^T(p,x)\Psi(p) \quad (5.2)$$

$$= \int \frac{d^4p}{(2\pi)^4} (i\frac{\not{m}_p}{m} p^0 + i\vec{p\!\!\!/} \frac{E_p}{m} - i\gamma^0 \frac{E_p^2}{m}) O^T(p,x)\Psi(p) \quad (5.3)$$

$$= \int \frac{d^4p}{(2\pi)^4} (i\frac{\not{m}_p}{m} p^0 - i\frac{\not{m}_p}{m} E_p) O^T(p,x)\Psi(p) \quad (5.4)$$

$$= \int \frac{d^4p}{(2\pi)^4} O^T(p,x) i\gamma^0 (p^0 - E_p)\Psi(p) \quad (5.5)$$

The Dirac equation in momentum space is then:

$$i\gamma^0(p^0 - E_p)\Psi(p) = 0 \quad (5.6)$$

The solution is:

$$\Psi(p) = (2\pi)\delta(p^0 - E_p)\psi(\vec{p}) \quad (5.7)$$

Making an inverse Fourier-Majorana transform we get:

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{m}_p \gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{-i\gamma^0 p\cdot x} \psi(\vec{p}) \quad (5.8)$$

Where $p^0 = E_p$ and $\psi(\vec{p})$ is a spinor. If $\psi(\vec{p})$ is a real spinor, then the solution $\Psi(x)$ is real.

6 Spin

A real spinor has 4 degrees of freedom. When we want a real spinor to satisfy the Dirac equation, we are left with 2 degrees of freedom, because we are rejecting spinors of the type:

$$\Psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\not{n}_p \gamma^0 + m}{\sqrt{E_p + m}\sqrt{2E_p}} e^{i\gamma^0 p\cdot x} \psi(\vec{p}) \tag{6.1}$$

Which would satisfy the Dirac equation with negative mass. The 2 degrees of freedom that left correspond to the spin up/down property of the spinor.

The spin vector s verifies $s^\mu s_\mu = -1$ and $s^0 = 0$. The spin operator $\not{s}\gamma^5$ commutes with $i\gamma^0$ and squares to 1. Therefore, it has eigenvalues 1 (up) and -1 (down). The eigen-vectors of $\not{s}\gamma^5$, in momentum space, can be defined as:

$$\psi(\vec{p}, s) = \frac{1 + \not{s}\gamma^5}{2} \psi(\vec{p}) \tag{6.2}$$

$$\psi(\vec{p}, -s) = \frac{1 - \not{s}\gamma^5}{2} \psi(\vec{p}) \tag{6.3}$$

And the Majorana spinor in momentum space with a defined spin and that satisfies the Dirac equation is:

$$\Psi(x^0, \vec{p}, s) = e^{-i\gamma^0 E_p x^0} \psi(\vec{p}, s) \tag{6.4}$$

The spin operators are defined as:

$$\sigma^k = \gamma^k \gamma^5 \quad k = 1, 2, 3 \tag{6.5}$$

They verify:

$$[\sigma^i, \sigma^j] = i\gamma^0 \epsilon^{ij}_k \sigma^k \tag{6.6}$$

Where ϵ^{ij}_k is the Levi-Civita symbol. Note that $i\gamma^0$ commutes with σ^k and squares to -1, so it can be thought of as the imaginary unit in the spin algebra.

7 The Partition Function

In the second quantization, a fermion must obey to the Fermi-statistics. This is achieved by considering that the fields $\Psi_a(x)$ are Grassmann numbers (anti-commuting). The Lagrangian for the free Majorana fermion (real spinors) is:

$$\mathcal{L}(x) = \frac{1}{2} \Psi^T(x) \gamma^0 (i\not{\partial} - m) \Psi(x) \tag{7.1}$$

The Lagrangian is invariant under Lorentz transformations:

$$\Psi \to S\Psi \tag{7.2}$$

$$x^\mu \to x^\rho a_\rho{}^\mu \tag{7.3}$$

$$\mathcal{L} \to \mathcal{L} \tag{7.4}$$

Where S is the matrix that makes Lorentz transformations and ${a_\rho}^\mu$ is tensor of the Lorentz transform, verifying:

$$\gamma^\rho {a_\rho}^\mu = S^{-1}\gamma^\mu S \tag{7.5}$$

The action is given by:

$$\mathcal{S} = \frac{1}{2}\int d^4x \Psi^T(x)\gamma^0(i\partial\!\!\!/ - m)\Psi(x) \tag{7.6}$$

In momentum space, the action is given by:

$$\mathcal{S} = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\Psi^T(p)\gamma^0(p^0 - E_p)\Psi(p) \tag{7.7}$$

$$= -\frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\Psi^T(p)\nabla(p,q)\Psi(q) \tag{7.8}$$

Where the matrix ∇ is:

$$\nabla(p,q) = \delta^4(p-q)i\gamma^0(p^0 - E_p) \tag{7.9}$$

Returning to the space of the coordinates, we get:

$$\nabla(x,y) = \int \frac{d^4p}{(2\pi)^4}O^T(p,x)\alpha^0(p^0 - E_p)O(p,y) \tag{7.10}$$

$$= \int \frac{d^4p}{(2\pi)^4}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}mi\gamma^0\frac{p^0 - E_p}{E_p} \tag{7.11}$$

And:

$$\mathcal{S} = -\frac{i}{2}\int d^4x d^4y \Psi^T(x)\nabla(x,y)\Psi(y) \tag{7.12}$$

The matrix ∇ is real and antisymmetric, that is $\nabla(x,y)_{ab} = -\nabla(y,x)_{ba}$. This property is very important, because we are working with Grassmann variables and a symmetric part of $\nabla(x,y)_{ab}$ would not contribute to the action.

In what follows it will be convenient to represent the variables in the same way we treat the spinor indexes and represent both by the same index. This way, the action can be written as:

$$\mathcal{S} = -\frac{i}{2}\Psi_x\nabla_{xy}\Psi_y = -\frac{i}{2}\Psi^T\nabla\Psi \tag{7.13}$$

The Partition Function is defined by the Feynman path integral:

$$Z[\eta] = \int \mathcal{D}\Psi e^{i\mathcal{S}[\Psi]+\eta^T\Psi} \tag{7.14}$$

Note that $i\mathcal{S}$ is real. The functional integral is done in the Ψ_x space: $\mathcal{D}\Psi = \prod_x d\Psi_x$. Where η is the external source of the field, composed by Grassmann numbers. We can express the Partition Function as:

$$
\begin{aligned}
Z[\eta] &= \int \mathcal{D}\Psi exp(i\mathcal{S} + \eta^T \Psi) && (7.15)\\
&= \int \mathcal{D}\Psi exp(\frac{1}{2}(\Psi - \nabla^{-1}\eta)^T \nabla (\Psi - \nabla^{-1}\eta) + \frac{1}{2}\eta^T \nabla^{-1}\eta) && (7.16)\\
&= exp(\frac{1}{2}\eta_x \nabla^{-1}_{xy}\eta_y) \int \mathcal{D}\Psi exp(i\mathcal{S}) && (7.17)\\
&= exp(\frac{1}{2}\eta_x \Delta_{xy}\eta_y) \int \mathcal{D}\Psi exp(i\mathcal{S}) && (7.18)
\end{aligned}
$$

Where the relation $\nabla^{-1T} = -\nabla^{-1}$ was used. $\Delta \equiv \nabla^{-1}$ is the propagator. We can now use the partition function to obtain information about the system.

8 Propagator

The Propagator is the solution to the linear differential equation:

$$i\gamma^0(i\partial\!\!\!/_x - m)\Delta(x,y) = \delta^4(x-y) \tag{8.1}$$

The general solution to the linear differential equation is the sum of the general solution of the related homogeneous equation $\Delta^H(x,y)$ and a particular solution Δ^W.

Going to momentum space we get:

$$i\gamma^0(p^0 - E_p)\Delta(p,q) = 1 \tag{8.2}$$

With solution:

$$\Delta(p,q) = \Big(-\frac{i\gamma^0}{p^0 - E_p} + \delta(p^0 - E_p)M(\vec{p})\Big)(2\pi)^4\delta^4(p-q) \tag{8.3}$$

Where $M(\vec{p})$ is an arbitrary matrix and $\frac{1}{p^0 - E_p}$ should be understood as the Cauchy Principle Value. The arbitrariness of $M(\vec{p})$ comes from the fact that the inverse of $(p^0 - E_p)$ must be done in the sense of the distributions and $(p^0 - E_p)\delta(p^0 - E_p) = 0$.

Now we define a particular solution Δ^W:

$$\Delta^W(p,q) = -\frac{i\gamma^0}{p^0 - E_p}(2\pi)^4\delta^4(p-q) \tag{8.4}$$

And the general solution of the related homogeneous equation Δ^H.

$$\Delta^H(p,q) = \delta(p^0 - E_p)M(\vec{p})(2\pi)^4\delta^4(p-q) \tag{8.5}$$

The general solution to the linear differential equation is the sum of a particular solution Δ^W and the general solution of the related homogeneous equation Δ^H.

Making an inverse Fourier-Majorana transform in time we get:

$$\Delta^W(x^0, \vec{p}, y^0, \vec{q}) = -(2\pi)^3\delta^3(\vec{p} - \vec{q}) \int \frac{dp^0}{2\pi} \frac{i\gamma^0 e^{-i\gamma^0 p^0 (x^0 - y^0)}}{p^0 - E_p} \tag{8.6}$$

We make the change of variables $\xi = p^0 - E_p$, obtaining:

$$\Delta^W(x^0, \vec{q}, y^0, \vec{p}) = -(2\pi)^3\delta^3(\vec{p} - \vec{q})e^{-i\gamma^0 E_p(y^0 - z^0)}\left(\int \frac{d\xi}{2\pi}\frac{i\gamma^0 e^{-i\gamma^0\xi(x^0-y^0)}}{\xi}\right) \tag{8.7}$$

$$= -(2\pi)^3\delta^3(\vec{p} - \vec{q})e^{-i\gamma^0 E_p(y^0 - z^0)}\frac{sign(x^0 - y^0)}{2} \tag{8.8}$$

Where ϑ is half the sign function. We can recover the propagator in coordinate space as:

$$\Delta^W(x, y) = -\int \frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}\frac{\not{h}_p\gamma^0}{E_p}\vartheta(x^0 - y^0) \tag{8.9}$$

With $p^0 = E_p$. This particular solution is called the Wheeler propagator. [4].

The general solution to the homogeneous equation:

$$i\gamma^0(i\not{\partial} - m)\Delta^H(x, y) = 0 \tag{8.10}$$

Is given by:

$$\Delta^H(x, y) = \int \frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}M(\vec{p}) \tag{8.11}$$

One can now obtain other particular solutions, for particular $M(p)$. The Retarded propagator is:

$$\Delta^R(x, y) = \Delta^W(x, y) - \int \frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}\frac{\not{h}_p\gamma^0}{2E_p} \tag{8.12}$$

$$= -\int \frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}\frac{\not{h}_p\gamma^0}{E_p}\theta(x^0 - y^0) \tag{8.13}$$

Where $\theta(x^0 - y^0)$ is the Heaviside step function. The Advanced propagator is:

$$\Delta^A(x, y) = \Delta^W(x, y) + \int \frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}\frac{\not{h}_p\gamma^0}{2E_p} \tag{8.14}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}\frac{\not{h}_p\gamma^0}{E_p}\theta(y^0 - x^0) \tag{8.15}$$

The Feynman propagator is:

$$\Delta^F(x, y) = \Delta^W(x, y) - \int \frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)}\frac{m\gamma^0}{2E_p} \tag{8.16}$$

$$= -\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p}((\not{h}_p + m)e^{-ip\cdot(x-y)}\theta(x^0 - y^0) + (-\not{h}_p + m)e^{ip\cdot(x-y)}\theta(y^0 - x^0))\gamma^0 \tag{8.17}$$

9 Causality and anti-symmetry lead to the Wheeler propagator

We will show that a propagator is null for $x-y$ space-like ($(x^0-y^0)^2 < (\vec{x}-\vec{y})^2$) if it is of the form:

$$\Delta(x,y) = \int \frac{d^3\vec{p}}{(2\pi)^3 E_p} e^{-i\frac{\not{h}_p}{m}p\cdot(x-y)} \not{h}_p \gamma^0 f(x^0-y^0) \tag{9.1}$$

Where $f(x^0-y^0)$ is a scalar verifying $-\partial_{x^0} f(x^0-y^0) = \delta(x^0-y^0)$. Since $x-y$ is space-like, by making a Lorentz transform we can go to a referential in which $x^{0'}-y^{0'}=0$ and $(\vec{x}'-\vec{y}')^2 > 0$. Now we have:

$$\Delta(x,y) = S\Delta'(x',y')S^T \tag{9.2}$$

With

$$\Delta'(x',y') = \int \frac{d^3\vec{p}}{(2\pi)^3 E_p} e^{-i\frac{\not{h}_p}{m}p\cdot(x'-y')} \not{h}_p \gamma^0 f'(x^{0'}-y^{0'}) \tag{9.3}$$

Where $f'(x^{0'}-y^{0'}) = f(x^0-y^0)$. Since $x^{0'}=y^{0'}$, we get:

$$\begin{aligned}\Delta'(x',y') &= \int \frac{d^3\vec{p}}{(2\pi)^3 E_p} e^{-i\frac{\not{h}_p}{m}\vec{p}\cdot(\vec{x}'-\vec{y}')} \not{h}_p \gamma^0 f'(0) &(9.4)\\ &= \int \frac{d^3\vec{p}}{(2\pi)^3} cos(\vec{p}\cdot(\vec{x}'-\vec{y}'))f'(0) &(9.5)\\ &= \delta^3(\vec{x'}-\vec{y'})f'(0) &(9.6)\end{aligned}$$

Which is null for $(\vec{x}'-\vec{y}')^2 > 0$.

If we impose that the propagator is causal, then the Feynman propagator is excluded. If we want the propagator to be anti-symmetric, to be consistent with the definition from the partition function, then the Advanced and Retarded propagators are excluded. A good candidate is the Wheeler propagator.

10 Transition Probability

Let's consider two different points x_i and x_f, with $x_f^0 > x_i^0$ and two fixed spinor indexes i and f. Let's suppose we have:

$$\eta_a(x) = \delta^4(x-x_f)\delta_{af}\eta_f + \delta^4(x-x_i)\delta_{ai}\eta_i \tag{10.1}$$

Where η_f and η_i are Grassmann numbers. The source η_i at $x^0 = x_i^0$ is called the initial state, the source η_f at $x^0 = x_f^0$ is called the final state.

The wave function $\Psi_{fi}(x_f, x_i)$, at point x_f and spinor index f, which is solution to the Dirac equation and has an initial source at point x_i and spinor index i is obtained by:

$$\Psi_{fi}(x_f, x_i) = 2\frac{\delta^2}{\delta\eta_f\delta\eta_i}\frac{Z[\eta]}{Z[0]}|_{\eta_f,\eta_i=0} \tag{10.2}$$

$$= 2\frac{\delta^2}{\delta\eta_f\delta\eta_i}e^{\frac{1}{2}\int d^4xd^4y\eta^T(x)\Delta^W(x,y)\eta(y)}|_{\eta_f,\eta_i=0} \tag{10.3}$$

$$= 2\frac{\delta^2}{\delta\eta_f\delta\eta_i}e^{-\eta_f[\int\frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{n}_p}{m}p\cdot(x_f-x_i)}\frac{\not{n}_p\gamma^0}{2E_p}]_{fi}\eta_i}|_{\eta_f,\eta_i=0} \tag{10.4}$$

$$= [\int\frac{d^3\vec{p}}{(2\pi)^3}e^{-i\frac{\not{n}_p}{m}p\cdot(x_f-x_i)}\frac{\not{n}_p\gamma^0}{E_p}]_{fi} \tag{10.5}$$

With $x_f^0 > x_i^0$. The causality is guaranteed by the fact that given two sources, we choose for initial source the one with $x_i^0 < x_f^0$. Note that for fixed i, Ψ_{fi} is a spinor in the index f.

The density of probability of having the state f, given the source i, is:

$$dP(f) = \frac{\Psi_f^2(x_f)d^3\vec{x}_f}{\int\Psi^T(x_f^0,\vec{x})\Psi(x_f^0,\vec{x})d^3\vec{x}} \tag{10.6}$$

The normalization is given by:

$$\int\Psi^T(x_f^0,\vec{x})\Psi(x_f^0,\vec{x})d^3\vec{x} = \sum_f\int\Psi_{fi}(x_f,x_i)\Psi_{fi}(x_f,x_i)d^3\vec{x}_f \tag{10.7}$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}1 = 1\delta_x^3(0) \tag{10.8}$$

Instead of sources in coordinate space, we can have sources in momentum space:

$$\eta_a(x^0,\vec{p}) = +\,\delta(x^0-x_f^0)(2\pi)^3\delta^3(\vec{p}-\vec{p}_f)[e^{i\gamma^0E_px^0}]_{af}\eta_f \tag{10.9}$$

$$+\,\delta(x^0-x_i^0)(2\pi)^3\delta^3(\vec{p}-\vec{p}_i)[e^{i\gamma^0E_px^0}]_{ai}\eta_i \tag{10.10}$$

In that case:

$$\Psi_{fi}(x_f^0,\vec{p}_f,x_i^0,\vec{p}_i) = (2\pi)^3\delta^3(\vec{p}_f-\vec{p}_i)\delta_{fi},\ (x_f^0 > x_i^0) \tag{10.11}$$

The density of probability of having the state f, given the source i, is:

$$dP(f) = \frac{\Psi_f^2(x_f^0,\vec{p}_f)\frac{d^3\vec{p}_f}{(2\pi)^3}}{\int\Psi^T(x_f^0,\vec{p})\Psi(x_f^0,\vec{p})\frac{d^3\vec{p}}{(2\pi)^3}} = \delta^3(\vec{p}_f-\vec{p}_i)\delta_{fi}d^3\vec{p}_f \tag{10.12}$$

The normalization is given by:

$$\int\Psi^T(x_f^0,\vec{p})\Psi(x_f^0,\vec{p})\frac{d^3\vec{p}}{(2\pi)^3} = \sum_f\int\Psi_{fi}(x_f^0,\vec{p})\Psi_{fi}(x_f^0,\vec{p})\frac{d^3\vec{p}}{(2\pi)^3} \tag{10.13}$$

$$= 1(2\pi)^3\delta_p^3(0) \tag{10.14}$$

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