Might Baryons be Yang-Mills Magnetic Monopoles?

Jay R. Yablon[*]
910 Northumberland Drive
Schenectady, New York, 12309-2814

Abstract:

We demonstrate how the baryons which constitute the vast preponderance of the material universe are no more and no less than Yang-Mills magnetic monopoles, with quarks and gluons confined, and only mesons permitted to net flux in and out. The confinement of color in Yang-Mills theory is fundamentally and inextricably tied to the absence of magnetic monopoles in Abelian Gauge theory.

1. Introduction

In this paper, we pose the following questions:

> Why, theoretically, do there exist in nature, naturally-occurring sources, namely baryons, consisting of exactly three strongly-interacting fermion constituents which we call "quarks"? Why, and by what mechanism, do the massless gauge particles of Quantum Chromodynamics (QCD), which we call gluons, cause these quarks to remain confined within the baryons? How, and why is it, that the interactions between baryons only occur via the exchange of mediating quark / antiquark pairs that we call "mesons," and not through any free gluon exchange? And how, despite the absence of any known symmetry breaking in QCD, and even with the gluons being massless, do these meson mediators obtain their mass?

These are questions of more than passing interest, because two most-common types of baryon, of course, are the proton and neutron, which account for the very vast preponderance of the material universe. It would be good to have a theoretical foundation for understanding what these baryons actually are.

We do know, because there are three quarks per baryon, that it is very helpful and can explain many things about the strong interactions, if we employ the Yang-Mills color group $SU(3)_C$ with a wavefunction $\psi^T = \begin{pmatrix} R & G & B \end{pmatrix}$ in the fundamental representation to ensure Fermi-Pauli-Dirac exclusion, i.e., to make sure that no two fermions in a given system have the exact same set of quantum numbers. But this merely descriptive, and does not explain the underlying question of why there are three quarks per baryon and not some different number, or the even more challenging questions about confinement and meson interactions. If nature were to provide 4 or 7 or 11, for example, then we would merely enforce Fermi-Dirac statistics with SU(4) or SU(7) or SU(11) instead, and would still be asking "why?" there were instead 4 or 7 or 11 quarks per baryon.

From an historical perspective, Rabi once quipped about the muon, "who ordered this?" Of course, there has been ample *experimental* evidence for the existence of nucleons since Rutherford and Chadwick respectively discovered the proton and neutron in 1917 and 1933. But for these baryons and others, from a *theoretical* viewpoint, it is still not really understood even to this day, "who ordered this?" Today, we know that baryons contain three quarks, but we don't know why this number is three. We just take that as a given and build around that. It is still a struggle to understand why and how these quarks remain stubbornly confined, and how an interaction such as $SU(3)_C$ of QCD which relies on massless gauge bosons (gluons) can still give rise to massive quark / antiquark pairs (mesons) which mediate nuclear interactions. Much research has been focused on finding clever ways to "glue" quarks together, but a fundamental understanding of baryons and quark and gluon confinement remains elusive. In fact, properly understanding baryons and confinement and massive meson exchange has proved to be so challenging, that it led the Clay Institute to in 2000 to offer a large prize for solving the so-called "mass-gap" problem of Yang-Mills Theory, [1]

[*]Email: jyablon@nycap.rr.com

which today remains unclaimed. And at bottom, the biggest barrier to cracking this puzzle emanates from the fact that to this day, nobody really knows, theoretically, what a baryon is. “Who ordered baryons?” is still very much a live question.

On a seemingly-different front – which this paper will endeavor to show is not at all a different front – almost as soon as James Clerk Maxwell published his 1873 *A Treatise on Electricity and Magnetism*, questions arose about magnetic monopoles: “Why is there not symmetry between electric and magnetic charges?” “Do magnetic monopoles exist?” “If so, where and how can they be found?” For almost 140 years, those questions have been asked, and many experiments have been done and continue to be done to detect magnetic monopoles. t’Hooft & Polyakov [2] and others [3], [4] have pointed out and tried to develop the theoretical observation that Yang-Mills field theory seems to give rise to magnetic monopoles, but to date, magnetic charges have never been conclusively detected and they remain one of the deepest and most elusive mysteries of the natural world.

The thesis of this paper is simple: that the magnetic monopoles characterized classically by $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ which come into existence in Yang-Mills theory are synonymous with baryons. <u>Baryons are Yang-Mills magnetic monopoles</u>. Yang-Mills magnetic monopoles contain exactly three confined quarks, interacting with one another via massless gluons, with interactions between these monopoles mediated by massive mesons. To the question what is a baryon? the answer is this: a Yang-Mills magnetic monopole. To the question do magnetic monopoles exist and if so where can we find them? the answer is this: yes, they exist, and they are everywhere. We ourselves and everything we see and touch and hear and smell and taste is built predominantly out of Yang-Mills magnetic monopoles. Whenever we talk about a proton or a neutron or any other baryon, we are talking about a Yang-Mills magnetic monopole. We just don’t realize that, yet. A theoretical oddity and orphan child for close to 140 years, magnetic monopoles are in fact the very heart of the material world, but have been hiding in plain sight from our theoretical understanding ever since the time of Maxwell. Nuclear physics, and the physics of confinement and mesons, is the physics of magnetic monopoles, governed classically by Maxwell’s equations plus Yang-Mills, and quantum mechanically by QCD. And to Rabi’s question who ordered this? the answer, for baryons, is this: James Clerk Maxwell, Chen Ning Yang and Robert Mills. They are the theorists who ordered what Rutherford and Chadwick found in their laboratories the better part of a century ago.

2. Maxwell’s Classical Field Equations in Yang-Mills Theory

Maxwell’s classical field equations of Abelian gauge theory are most often presented in the form of two separate equations for electric and magnetic charge densities:

$$\begin{aligned} J^\nu &= \partial_\mu F^{\mu\nu} \\ P^{\sigma\mu\nu} &= \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} \end{aligned} . \tag{2.1}$$

Taken as is, there is nothing in the above to prevent the existence of a magnetic charge density $P^{\sigma\mu\nu}$ a.k.a. magnetic monopole (which we endeavor to demonstrate is a baryon density when fully developed in Yang-Mills theory). However, as soon as one defines the field strength density $F^{\mu\nu}$ from the Abelian gauge vector potential A^μ (which in QED represents the photon) using:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu , \tag{2.2}$$

the latter equation (2.1) becomes $P^{\sigma\mu\nu} = 0$, by identity. Thus, the timeless mystery of Maxwell’s equations: no magnetic monopoles.

One might think to discard the vector potential A^μ in (2.2) entirely, and specify electrodynamics entirely in terms of the field strength $F^{\mu\nu}$. But as Witten points out: ([5] at page 28)

> “the vector potential is not just a convenience [but] is needed in 20th-century physics for three very good purposes:
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> - To write a Schrödinger equation for an electron in a magnetic field.

- To make it possible to derive Maxwell's equations from a Lagrangian.
- To write anything at all for non-Abelian gauge theory, which – in our modern understanding of elementary particle physics – is the starting point in describing the strong, weak and electromagnetic interactions."

Non-Abelian Yang-Mills gauge theory differs from the Abelian gauge theory in the simple respect that its gauge fields G^μ are non-commuting, i.e., in the fact that $[G^\mu, G^\nu] \neq 0$, in contrast to $[A^\mu, A^\nu] = 0$ which is taken to be the case in (2.2) above. Specifically, for any Yang-Mills gauge group SU(N) with group generators T^i related by the group structure $f^{ijk}T_i = -i[T^j, T^k]$, and where $F^{\mu\nu} \equiv T^i F_i^{\mu\nu}$ and $G^\mu \equiv T^i G_i^{\ \mu}$ are NxN matrices, the field strength (2.2) is simply replaced by:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i[G^\mu, G^\nu] = D^\mu G^\nu - D^\nu G^\mu = D^{[\mu} G^{\nu]}, \tag{2.3}$$

where we use the gauge-covariant derivative

$$D^\mu \equiv \partial^\mu - iG^\mu \tag{2.4}$$

to put $F^{\mu\nu}$ into a form that facilitates calculation and allows a very transparent comparison to Abelian gauge theory.

So, as soon as one substitutes the non-Abelian (2.3) into Maxwell's equation (2.1) for $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$, while the terms based on $\partial^\mu G^\nu - \partial^\nu G^\mu$ continue to zero out by identity in the usual way, one nonetheless arrives at a residual non-zero magnetic charge:

$$\begin{aligned} P^{\sigma\mu\nu} &= -i\left(\partial^\sigma [G^\mu, G^\nu] + \partial^\mu [G^\nu, G^\sigma] + \partial^\nu [G^\sigma, G^\mu]\right) \\ &= -i\left([\partial^\sigma G^\mu, G^\nu] + [G^\mu, \partial^\sigma G^\nu] + [\partial^\mu G^\nu, G^\sigma] + [G^\nu, \partial^\mu G^\sigma] + [\partial^\nu G^\sigma, G^\mu] + [G^\sigma, \partial^\nu G^\mu]\right) \end{aligned}. \tag{2.5}$$

This is all because of the fact that $[G^\mu, G^\nu] \neq 0$. *The thesis of this paper will be to show that these non-zero $P^{\sigma\mu\nu}$ objects are baryons, and that these $[G^\mu, G^\nu]$ objects are mesons which mediate nuclear and other baryon interactions*. In particular, as we shall later see in, for example, equation (4.24), the three cyclically-symmetric spacetime indexes μ, ν, σ in $P^{\sigma\mu\nu}$ are indicative of three fermion / anti-fermion propagators within $P^{\sigma\mu\nu}$. Meanwhile, the two antisymmetric indexes μ, ν in $[G^\mu, G^\nu]$ are indicative of fermion / anti-fermion propagators which flow across closed baryon surfaces as mesons. And, perhaps most fundamentally, the "zero" that signifies absence of magnetic monopoles in Abelian gauge theory translates directly into a "zero" that signifies the absence of color flux across any closed surface surrounding a non-Abelian magnetic monopole.

The question that naturally arises is whether this approach to Yang-Mills theory via the classical equations (2.1) is viable, and this question has two aspects: First, while $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ and $J^\nu = \partial_\mu F^{\mu\nu}$ are the classical field equations for Abelian (commuting field) gauge theory, will they remain the classical field equations in Yang-Mills theory? The answer to this first question is yes: Start with a path integral. Apply stationary phase (or steepest descent) approximation in the $\hbar \to 0$ limit, that is, in situations where the relevant action being considered is much greater than $\hbar$, i.e., $S(\varphi) = \int d^4x \mathfrak{L}(\varphi) >> \hbar$, so as to derive the Euler-Lagrange equation. Use this on the electric and magnetic Lagrangians of Yang-Mills theory. The resulting field equations will be:

$$\begin{aligned} J^\nu &= \partial_\mu F^{\mu\nu} \\ P^\nu &= \partial_\mu {}^*F^{\mu\nu} \end{aligned}. \tag{2.6}$$

Then, apply the "duality" formalism first developed by Reinich [6] and later elaborated by Wheeler, [7] which uses the Levi-Civita formalism (see [8] at pages 87-89) to $P^{\nu} = \partial_{\mu} {}^{*}F^{\mu\nu}$, to obtain $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu}$, which was used to obtain (2.5).

Second, if we can demonstrate that $P^{\sigma\mu\nu}$ does have all the requisite characteristics of a baryon using the classical field equations (2.1) for which $S(\varphi) >> \hbar$, will these results remain valid even for quantum field theory in which $S(\varphi) \sim \hbar$? Here too, the answer is yes. Why? If we can establish in the classical arena that $P^{\sigma\mu\nu}$ has all the properties of a baryon in circumstances where $S(\varphi) >> \hbar$, then there is no logic to suggest that $P^{\sigma\mu\nu}$ will cease to be a baryon once we consider quantum conditions where $S(\varphi) \sim \hbar$. Once a baryon, always a baryon! Just as in classical electrodynamics, a classical current density J^{μ} still remains a current density in quantum electrodynamics; it is just subject to a different set of (amplitude) equations (following path integration). What will happen, however, is that once we move into the low-action arena where $S(\varphi) \sim \hbar$, we will have to forego the use of all but the lowest-order terms that we develop to establish $P^{\sigma\mu\nu}$ as a baryon. Any of the higher-order terms will no longer correctly describe, mathematically, the behavior of these baryons in the low action arena. Indeed, using Maxwell's classical equations to try to understand confinement by using Gauss' law for surface integrals has a distinguished history, not the least of which includes the MIT Bag Model, see [9] at (18.45) and 426.

So in a very basic sense, using a "bicycle riding" metaphor, we will use the classical equations $J^{\nu} = \partial_{\mu} F^{\mu\nu}$ and $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu}$ as "training wheels" to demonstrate that $P^{\sigma\mu\nu}$ is in fact a baryon under classical, high-action conditions. In the process, we will establish the *lowest-order terms* – which will survive intact through path integration – to describe this baryon physics from the classical equations. With the basic connection established between Yang-Mills magnetic charges and baryons, we would remove the classical training wheel equations, and rely on the path integral formulation of quantum field theory to tell us how these baryons behave in the quantum arena in which the higher-order terms from our training wheel equations begin to break down or simply cease to work. But no matter what the action, high or low, the Yang-Mills magnetic monopole $P^{\sigma\mu\nu}$ will still be a baryon! It will just adhere to different mathematical equations in different action arenas.

In the development to follow, we will do no more and no less that than simply combine three equations together: The two classical Maxwell equations $J^{\nu} = \partial_{\mu} F^{\mu\nu}$ and $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu}$ of (2.1), and the field strength $F^{\mu\nu} = D^{[\mu} G^{\nu]}$ of (2.3) for non-commuting Yang-Mills fields. Nothing further is needed to show that $P^{\sigma\mu\nu}$ has all the essential features to be a baryon. We have already combined $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu}$ and $F^{\mu\nu} = D^{[\mu} G^{\nu]}$ to arrive at the non-zero magnetic monopole of (2.5). In section 3 we will deduce an inverse $I_{\sigma\nu}$ and inverse equation $G_{\nu} \equiv I_{\sigma\nu} J^{\sigma}$ for $J^{\nu} = \partial_{\mu} F^{\mu\nu}$ by combining $J^{\nu} = \partial_{\mu} F^{\mu\nu}$ with $F^{\mu\nu} = D^{[\mu} G^{\nu]}$. In section 4 we will employ this inverse in (2.5) to show that $P^{\sigma\mu\nu}$ is a natural system containing exactly three fermions which makes it a baryon candidate, and further, that the field commutator $\left[G^{\mu}, G^{\nu}\right]$ is a natural system consisting of a fermion and antifermion which makes this a meson candidate. In sections 5 and 6 and we will examine these classical equations in integral form, to examine what does and does not flow across any closed surface surrounding $P^{\sigma\mu\nu}$. This will show that $P^{\sigma\mu\nu}$ exhibits precisely the characteristics that are understood to characterize confinement phenomena, and so provide further support for thesis that Yang-Mills magnetic monopoles are natural baryons.

3. A Classical Yang Mills Inverse, with a "Revealed" Vector Boson Pseudo-Mass

With the non-zero magnetic monopole of (2.5) already specified, we begin the next stage of development by using $F^{\mu\nu} = D^{[\mu} G^{\nu]}$ of (2.3) in Maxwell's charge equation $J^{\nu} = \partial_{\mu} F^{\mu\nu}$ of (2.1) to obtain:

$$J^{\nu} = \partial_{\mu} F^{\mu\nu} = \partial_{\mu} D^{[\mu} G^{\nu]} = \partial_{\mu} D^{\mu} G^{\nu} - \partial_{\mu} D^{\nu} G^{\mu} = \left(g^{\mu\nu} \partial_{\sigma} D^{\sigma} - \partial^{\mu} D^{\nu}\right) G_{\mu} . \tag{3.1}$$

We now want to obtain the inverse expression for G_μ in terms of J^ν. That is, we now seek the inverse $I_{\mu\nu}$ of the configuration space operator $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$, defined such that $G_\nu \equiv I_{\sigma\nu} J^\sigma$, i.e., we now wish to obtain:

$$I_{\nu\lambda}\left(g^{\mu\nu}\partial^\sigma D_\sigma - \partial^\mu D^\nu\right)e^{ik^\alpha x_\alpha} = I_{\nu\lambda}\left(g^{\mu\nu}\left(\partial^\sigma\partial_\sigma - \partial^\sigma G_\sigma\right) - \left(\partial^\mu\partial^\nu - \partial^\mu G^\nu\right)\right)e^{ik^\alpha x_\alpha} = \delta^\mu{}_\lambda e^{ik^\alpha x_\alpha}. \qquad (3.2)$$

The presence in the above of the terms such as $\partial^\mu G^\nu$ which are derivatives of fields, introduces a complexity that is not encountered in U(1) Abelian gauge theory. This added complexity occurs because these derivatives in $\partial^\mu G^\nu$ do not directly operate on the Fourier kernel $e^{ik^\alpha x_\alpha}$ but instead operate on the gauge field G^ν. Because this field $G^\nu = G^\nu\left(x^\sigma\right)$ is a function of spacetime, we may make use of the commutator relationship:

$$\partial^\sigma G^\mu = i\left[k^\sigma, G^\mu\right] \qquad (3.3)$$

to replace then various $\partial^\sigma G^\mu$ which appear in (3.2). The space components of this relationship, $\partial^a A^b = i\left[k^a, A^b\right]$ for the photon field are used in Dirac theory to derive the electron magnetic moment, see, for example, [10], just after equation (2.964).* The time component of the above, $\partial^0 G^\mu = i\left[k^0, G^\mu\right]$ is a variant of Heisenberg's equation of motion, see for example [11], equation (3.61), which also uses this four-dimensional expression.

So, we substitute (3.3) into (3.2), and with some renaming of indexes to get a $\delta^\mu{}_\nu$ on the right, we obtain:

$$I_{\sigma\nu}\left(-g^{\mu\sigma}\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right) + k^\mu k^\sigma + i\left[k^\mu, G^\sigma\right]\right) = \delta^\mu{}_\nu. \qquad (3.4)$$

Before we try to calculate this inverse, knowing that this might have no inverse (see, e.g., [12], chapter III.4), let us add a square mass Proca term m^2 by hand in the usual way. Also, let us require that this configuration space operator be symmetric under $\mu \leftrightarrow \sigma$ interchange by symmetrizing the above expression using an index anticommutator $\tfrac{1}{2}k^{\{\mu}, G^{\sigma\}}$. Thus, we re-specify (3.4) as:

$$I_{\sigma\nu}\left(-g^{\mu\sigma}\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2\right) + k^\mu k^\sigma + \tfrac{1}{2}i\left[k^{\{\mu}, G^{\sigma\}}\right]\right) = \delta^\mu{}_\nu \qquad (3.5)$$

Finally, let us also require that $I_{\sigma\nu}$ be symmetric under $\sigma \leftrightarrow \nu$ interchange, by writing this in general form for three unknowns *A*, *B* and *C* as:

$$I_{\sigma\nu} \equiv A g_{\sigma\nu} + B k_\sigma k_\nu + \tfrac{1}{2}Ci\left[k_{\{\sigma}, G_{\nu\}}\right]. \qquad (3.6)$$

Finally, we plug this into (3.5). *We now need to solve the expression*:

$$\left(A g_{\sigma\nu} + B k_\sigma k_\nu + \tfrac{1}{2}Ci\left[k_{\{\sigma}, G_{\nu\}}\right]\right)\left(-g^{\mu\sigma}\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right] - m^2\right) + k^\mu k^\sigma + \tfrac{1}{2}i\left[k^{\{\mu}, G^{\sigma\}}\right]\right) = \delta^\mu{}_\nu. \qquad (3.7)$$

It is *very important* as we proceed, to keep in mind that the G^σ is an NxN matrix for the Yang-Mills gauge group SU(N). Thus, any expressions which put G^σ into a denominator have to be understood as requiring the

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* One can see how this operates as a derivative by considering the very simple example $(\partial/\partial x)x^2 = 2x$. The canonical Heisenberg commutator in the space dimensions is $\left[x^i, p^j\right] = i\hbar g^{ij}$. If we apply this to $\left[x^i x^k, p^j\right]$, we find that $\left[x^i x^k, p^j\right] = 2i\hbar g^{ij} x^k$, which we can write as $\partial^j\left(x^i x^k\right) = 2\hbar g^{ij} x^k = -i\left[x^i x^k, p^j\right]$. This is just a fancy way of writing $(\partial/\partial x)x^2 = 2x$. But it turns that this works like a derivative for any polynomial containing any order in *x*, i.e., $(\partial/\partial x)x^n = nx^{n-1}$, etc., so that any time we have a field $A^i(x)$, we can apply $\partial^j A^i(x) = i\left[p^j, A^i(x)\right]$.

formation of a *Yang-Mills matrix inverse*. So that the expressions we develop have a similar "look" to familiar expressions from QED, we will generally use a "quoted denominator" notation $1/"M" \equiv M^{-1}$ to designate a Yang-Mills matrix inverse. Thus, $G^{\sigma^{-1}} = 1/"G^{\sigma}"$, etc.

As we start to solve (3.7) in the usual way, we first determine that:

$$A = -\frac{1}{"k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^2"} = -\left(k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^2\right)^{-1}, \tag{3.8}$$

where as stated we use the quotes to denote a matrix inverse. Putting this back into (3.7), and after absorbing out the metric tensor, we find ourselves left with the expression:

$$\begin{aligned}
&\frac{k^{\mu}k_{\nu} + \frac{1}{2}i\left[k^{\{\mu}, G_{\nu\}}\right]}{"k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^2"} \\
&= -\left(Bk^{\mu}k_{\nu} + \tfrac{1}{2}Ci\left[k^{\{\mu}, G_{\nu\}}\right]\right)\left(k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^2\right) \\
&\quad + \tfrac{1}{2}Bk_{\sigma}k_{\nu}i\left[k^{\{\mu}, G^{\sigma\}}\right] - \tfrac{1}{4}C\left[k_{\{\sigma}, G_{\nu\}}\right]\left[k^{\{\mu}, G^{\sigma\}}\right] + \tfrac{1}{2}Ci\left[k_{\{\sigma}, G_{\nu\}}\right]k^{\mu}k^{\sigma} + Bk_{\sigma}k_{\nu}k^{\mu}k^{\sigma}
\end{aligned}, \tag{3.9}$$

Observing that the top line term has a numerator $k^{\mu}k_{\nu} + \frac{1}{2}i\left[k^{\{\mu}, G_{\nu\}}\right]$ and the second line term contains $Bk^{\mu}k_{\nu} + \frac{1}{2}Ci\left[k^{\{\mu}, G_{\nu\}}\right]$, we see that these numerators can be cancelled out if we set B=C, and if the terms on the third line can somehow be zeroed out. In fact, to be able to form this inverse at all, that is exactly what we are *required* to do. So, we now set B=C, and we also set the entire third line to zero, which as we shall momentarily review, amounts to a gauge fixing condition. We then do some reduction and consolidation to obtain:

$$B = C = -\frac{\dfrac{1}{"k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^2"}}{"k^{\alpha}k_{\alpha} + i\left[k^{\alpha}, G_{\alpha}\right] - m^2"}, \tag{3.10}$$

subject to the gauge condition:

$$\left(k_{\sigma}k_{\nu} + \tfrac{1}{2}i\left[k_{\{\sigma}, G_{\nu\}}\right]\right)\left(k^{\mu}k^{\sigma} + \tfrac{1}{2}i\left[k^{\{\mu}, G^{\sigma\}}\right]\right) = 0. \tag{3.11}$$

Again, these result from setting B=C and then setting the third line of (3.9) to zero, which were <u>required</u> in order to form an inverse.

So we now plug (3.8) and (3.10) with B=C into (3.6) in the gauge (3.11), to obtain the inverse:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_{\sigma}k_{\nu} + \frac{1}{2}i\left[k_{\{\sigma}, G_{\nu\}}\right]}{"m^2 - k^{\alpha}k_{\alpha} - i\left[k^{\alpha}, G_{\alpha}\right]"}}{"k^{\alpha}k_{\alpha} - m^2 + i\left[k^{\alpha}, G_{\alpha}\right]"}. \tag{3.12}$$

We may also use (3.3) and $k_{\sigma}k_{\nu} \to -\partial_{\sigma}\partial_{\nu}$ to convert this inverse fully back into configuration space, thus:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{-\partial_{\sigma}\partial_{\nu} + \frac{1}{2}\partial_{\{\sigma}G_{\nu\}}}{"m^2 + \partial^{\alpha}\partial_{\alpha} - \partial^{\alpha}G_{\alpha}"}}{"-\partial^{\alpha}\partial_{\alpha} - m^2 + \partial^{\alpha}G_{\alpha}"}. \tag{3.13}$$

Note that the term $i[k^\alpha, G_\alpha] = \partial^\alpha G_\alpha$ appears in two places in the above, but we do not set this to zero here because we are using different gauge fixing conditions, namely, those of (3.11), to be further reviewed shortly.

Now, we look at some special cases of (3.13). First, we compare (3.12) to the usual, well-known propagator for a massive vector boson in QED, which is reproduced below for convenience:

$$D_{\mu\nu} = \frac{-g_{\mu\nu} + \dfrac{k_\mu k_\nu}{m^2}}{k^\alpha k_\alpha - m^2 + i\varepsilon}. \tag{3.14}$$

In the case where $i[k_\nu, G_\sigma] = \partial_\nu G_\sigma \to 0$, we no longer need to take any matrix inverses, and (3.12) reduces to:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_\sigma k_\nu}{m^2 - k^\alpha k_\alpha}}{k^\alpha k_\alpha - m^2}. \tag{3.15}$$

This closely resembles (3.14), sans the $+i\varepsilon$, and also with $m^2 - k^\alpha k_\alpha$ rather than just m^2 appearing in the denominator of the right hand term in the numerator. Were we to wish to make use of (3.15), this does have a pole, and so we would need to add $+i\varepsilon$ in the usual way.

But, in comparing the classical Yang-Mills inverse developed in (3.12) to the usual massive propagator (3.14), we see two substantial differences. First, the denominators in (3.12) are actually matrix inverses because they include the NxN Yang-Mills matrices G^σ for SU(N). Second, and this is an absolutely fundamental point, consider what happens to (3.12) and (3.14) when we set the mass term $m^2 = 0$. In (3.14) for the usual propagator, the term $k_\sigma k_\nu / m^2 \to \infty$ because of the m^2 in that denominator. This originates in the fact that the QED configuration space operator $g^{\mu\nu}\partial_\sigma\partial^\sigma - \partial^\mu\partial^\nu$ has no inverse. So the massless propagator becomes infinite! This is what leads to the need for gauge fixing techniques such as Faddeev-Popov, whereby we end up with the massless propagator:

$$D_{\mu\nu}(k) = \frac{-g_{\mu\nu} + (1-\xi)\dfrac{k_\mu k_\nu}{k^\sigma k_\sigma}}{k^\sigma k_\sigma + i\varepsilon}, \tag{3.16}$$

which has been also reproduced above for the reader's convenience. We cannot just set $m^2 = 0$ in (3.14) and keep a finite expression.

But in (3.12) or (3.15) derived here via the classical Yang-Mills inverse, we can set $m^2 = 0$ with impunity. That is, we can make the gauge boson G^σ mass $m = 0$ without causing the inverse to become infinite. In fact, if we do set $m^2 = 0$, (3.12) simply becomes a Yang-Mills massless particle propagator:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} - \dfrac{k_\sigma k_\nu + \frac{1}{2} i [k_{\{\sigma}, G_{\nu\}}]}{"k^\alpha k_\alpha + i[k^\alpha, G_\alpha]"}}{"k^\alpha k_\alpha + i[k^\alpha, G_\alpha]"}. \tag{3.17}$$

This a perfectly finite expression! We do not set $i[k^\alpha, G_\alpha] = \partial^\alpha G_\alpha$ because in (3.11), setting $\left(k_\sigma k_\nu + \frac{1}{2} i [k_{\{\sigma}, G_{\nu\}}]\right)\left(k^\mu k^\sigma + \frac{1}{2} i [k^{\{\mu}, G^{\sigma\}}]\right) = 0$ in (3.11) to obtain the inverse (3.12) implicitly did all the required gauge fixing. But, most importantly: *we have revealed a vector boson "mass" without having ever engaged in*

<u>spontaneous symmetry breaking. This is a new mechanism for generating a vector boson mass, even with massless m=0 gauge bosons G^σ!</u>

Specifically, comparing the bottom "denominator" of the usual massive propagator (3.14) with the bottom "denominator" of the Yang-Mills inverse (3.17), which denominators are where we expect to find the mass of a vector boson, we find the correspondence:

$$\frac{1}{k^\alpha k_\alpha - m^2 + i\varepsilon} \leftrightarrow \frac{1}{"k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]"} = \left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}. \tag{3.18}$$

Most precisely – and this is very important to fully understand – if the interaction under consideration, say QCD, contains massless gauge bosons because we have not broken any symmetry to give rise to gauge boson masses as we do, for example, in electroweak $SU(2)_W \times U(1)_Y$, one will be "expecting" a massless propagator / inverse of the usual form that is used for the massless photon of QED, namely (3.16). But in fact, as seen in (3.18), when observing vector particles, one will be "observing" masses which originate from the massless propagator denominator / inverse $\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}$ for a massless gauge boson in Yang-Mills theory. Not knowing about this $\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}$ denominator / inverse, one will be expecting a massless propagator (3.16), but upon observing a mass, one will compare one's observations to the denominator $k^\alpha k_\alpha - m^2 + i\varepsilon$ which is known for a massive vector boson, and will conclude that the (inverted) $k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]$ is actually the (inverted) $k^\alpha k_\alpha - m^2 + i\varepsilon$ term that is expected from the known massive boson propagator (3.14). And so, the observer will conclude that there are massive vector bosons, despite the fact that all the gauge bosons are massless, and will wonder how this can occur and maybe even call this a "mass gap" and offer a reward for figuring out how this can happen.

This is how it happens: The non-linear interactions of Yang-Mills theory give rise to a "pseudo mass" (and also a finite lifetime because a complex mass value simply indicates a massive particle with a defined half-life while an imaginary mass indicates a massless particle of defined half-life, see [13] at 150) which arises from the observables of mass dimension -2 on the right hand side of the $\leftrightarrow$ in (3.18) being mistaken for observables of mass dimension -2 on the left hand side. A person who is "confused" in this way will wonder why there appear to be non-zero rest masses and finite lifetimes when in fact the gauge bosons have zero mass and the symmetry of the Yang-Mills theory has never been broken. Thus, there will appear to be particles with masses and defined half-lives such as the spin 1 π mesons, *even if the gauge bosons are massless* (which they are because we have set $m = 0$ to get to (3.17) / (3.18)). *<u>We have therefore "revealed" a "mass" even while the Yang-Mills gauge bosons have remained massless.</u>* This is similar to how after ordinary spontaneous symmetry breaking such as that used in electroweak theory, one finds terms of the form $\frac{1}{2}\left(\frac{1}{2}vg\right)^2 B^\sigma B_\sigma$ in the Lagrangian where one expects to see $\frac{1}{2}m^2 B^\sigma B_\sigma$, and so associates $m = \frac{1}{2}vg$ with the mass of the boson B^σ, that is $\frac{1}{2}m^2 B^\sigma B_\sigma \leftrightarrow \frac{1}{2}\left(\frac{1}{2}vg\right)^2 B^\sigma B_\sigma$. But in the present situation, no scalar degrees of freedom have been transferred into the observed boson, so this is just a pseudo mass in which the interaction energy generated by $\partial^\alpha G_\alpha = i\left[k^\alpha, G_\alpha\right]$ (which we do not set to zero because here we have a different gauge condition) *appears* to give rise to a mass but in fact does not, because it sits in a propagator "denominator" in the place where one ordinarily expects to find a mass. This is the approach that one uses to fill the so-called "mass gap"!

Note that when it comes to actually calculating masses, the correspondence (3.18) will yield some rich mass spectra, particularly because any calculation will require taking SU(N) matrix inverses first. That is, the NxN matrix $k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]$ for SU(N) must first be inverted, *and then and only then* will the reciprocals of the numeric results that emerge correspond to an observed boson mass. Imagine calculating $\left(k^\alpha k_\alpha + i\left[k^\alpha, G_\alpha\right]\right)^{-1}$ in SU(3) for example, and all of the complicated real and imaginary and complex terms that will emerge, and then using a transition amplitude to pick off masses from the denominators of the resultant expressions. That, in effect, is how these masses are generated to fill the mass gap, and how in principle, a detailed calculation of meson mass spectra would occur.

Keep in mind also, that this is all based on *classical* high-action field equation $J^{\nu} = \partial_{\mu} D^{[\mu} G^{\nu]}$ of (3.1), so in fact (3.18) will be modified once quantum fields are accounted for. In fact, the limitations of using this particular expression $\partial^{\alpha} G_{\alpha} = i[k^{\alpha}, G_{\alpha}]$ to generate a pseudo mass here can be seen by realizing that $\partial^{\alpha} G_{\alpha}$ is *not* invariant under a Yang-Mills gauge transformation $G_{\nu} \to G_{\nu} + \partial_{\nu}\theta + i[\theta, G_{\nu}]$. Because any mass – pseudo or not – ought not depend on the chosen gauge, in a complete theory we require a term that is gauge invariant. The term which is properly gauge invariant is in fact a "perturbation" $-V = (\partial^{\sigma} G_{\sigma} + G_{\sigma}\partial^{\sigma}) + G^{\sigma} G_{\sigma}$ rather than merely $\partial^{\alpha} G_{\alpha}$, but the development of this term $-V$ from a quantum Yang-Mills field theory is an entirely separate matter from showing that Yang-Mills magnetic monopoles are baryons and so is beyond the scope of this paper. But the basic idea imparted by (3.18) will remain intact despite the fact that the expression emerging from the fully-quantum version of the foregoing will be different from that shown above and will contain $-V = (\partial^{\sigma} G_{\sigma} + G_{\sigma}\partial^{\sigma}) + G^{\sigma} G_{\sigma}$ in place of $\partial^{\alpha} G_{\alpha}$.

Having looked at m=0, let us move on to consider the special case where both m=0 and $G^{\sigma} \to 0$. Here there is no longer the need to take any matrix inverses, so we remove the inversion quotes, and (3.17) becomes:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} - \dfrac{k_{\sigma} k_{\nu}}{k^{\alpha} k_{\alpha}}}{k^{\alpha} k_{\alpha}} . \tag{3.19}$$

This is just the massless vector boson propagator (3.16) sans $+i\varepsilon$, forced into the gauge $\xi = 2$, and bypassing entirely Faddeev-Popov and the usual approaches to gauge fixing.

Finally, let us return to (3.12), and consider a particle that is "on mass shell," with either $k^{\alpha} k_{\alpha} - m^2 = 0$ for a massive particle or $k^{\alpha} k_{\alpha} = 0$ for a massless particle. For an on-shell particle, the usual propagators (3.14) and (3.16) become ((3.14) becomes (3.16) in Landau gauge $\xi = 0$):

$$D_{\mu\nu} = \frac{-g_{\mu\nu} + \dfrac{k_{\mu} k_{\nu}}{k^{\alpha} k_{\alpha}}}{+i\varepsilon} \text{ (massive 3.14) and } D_{\mu\nu}(k) = \frac{-g_{\mu\nu} + (1-\xi)\dfrac{k_{\mu} k_{\nu}}{k^{\sigma} k_{\sigma}}}{+i\varepsilon} \text{ (massless 3.16)} \tag{3.20}$$

But from (3.12), with either $k^{\alpha} k_{\alpha} - m^2 = 0$ for a massive particle or $k^{\alpha} k_{\alpha} = 0$ for one that is massless, we obtain:

$$I_{\sigma\nu} = \frac{-g_{\sigma\nu} + \dfrac{k_{\sigma} k_{\nu} + \frac{1}{2} i [k_{\{\sigma}, G_{\nu\}}]}{"{-i}[k^{\alpha}, G_{\alpha}]"}}{"{+i}[k^{\alpha}, G_{\alpha}]"} = \frac{i g_{\sigma\nu}}{"[k^{\alpha}, G_{\alpha}]"} + \frac{k_{\sigma} k_{\nu} + \frac{1}{2} i [k_{\{\sigma}, G_{\nu\}}]}{"[k^{\alpha}, G_{\alpha}][k^{\beta}, G_{\beta}]"} . \tag{3.21}$$

This is a "naturally-occurring" form of $+i\varepsilon$ based on Yang-Mills interactions, with the term $+i[k^{\alpha}, G_{\alpha}] = \partial^{\alpha} G_{\alpha}$ (again, which we do <u>not</u> set to zero here because we are using different gauge fixing conditions here) playing a role identical to $+i\varepsilon$ to avoid poles for on shell particles. The "confused" observer, who is "expecting" a $+i\varepsilon$ term and instead observes a $+i[k^{\alpha}, G_{\alpha}]$ term, will simply calculate the lifetime parameter $1/\varepsilon$ based on what is produced by $[k^{\alpha}, G_{\alpha}]^{-1}$.

So, the Yang-Mills inverse (3.12) steers around all the usual problems with propagators and inverses. Not only does it explain how vector "pseudo masses" will come into existence even if the gauge bosons of the underlying theory remain massless, but it has no problem with becoming undefined (infinite) for a massless boson, and it does not require using the $+i\varepsilon$ prescription to avoid infinite poles, because it produces fully finite results under all the usual scenarios.

Now, one may ask, how did we get to a massless vector particle inverse (3.19) forced into the $\xi = 2$ gauge of (3.16) without any apparent gauge fixing? Keeping in mind that gauge-fixing methods such as Faddeev-Popov are about understanding the conditions required to obtain defined inverses, the answer is that we did in fact fix a gauge back in (3.11). Equation (3.11) is to be regarded as the gauge condition which, in Yang-Mills theory, is required to be able to form a matrix inverse for the (classical, high action, $S(\varphi) >> \hbar$) configuration space operator $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$ in the Yang-Mills-Maxwell field equation (3.1). And, in the process of this, we have been forced to fix the Faddeev-Popov gauge to $\xi = 2$, see (3.19), and to forego the usual covariant gauge condition $\partial^\alpha G_\alpha = 0$. That (3.11) is in the nature of a covariant gauge condition which becomes most striking if we also convert (3.11) back into configuration space, as we did with the inverse in (3.13). Doing so yields the rather fascinating operator equation:

$$\begin{aligned} &\left(\partial_\sigma \partial_\nu - \tfrac{1}{2}\partial_{\{\sigma} G_{\nu\}}\right)\left(\partial^\mu \partial^\sigma - \tfrac{1}{2}\partial^{\{\mu} G^{\sigma\}}\right) \\ &= \partial_\sigma \partial_\nu \partial^\mu \partial^\sigma - \tfrac{1}{2}\partial_\sigma \partial_\nu \partial^{\{\mu} G^{\sigma\}} - \tfrac{1}{2}\partial_{\{\sigma} G_{\nu\}} \partial^\mu \partial^\sigma + \tfrac{1}{4}\partial_{\{\sigma} G_{\nu\}} \partial^{\{\mu} G^{\sigma\}} = 0 \end{aligned} . \tag{3.22}$$

This is the spacetime equivalent of the gauge fixing condition that is required to form an inverse for the Yang-Mills configuration space operator $g^{\mu\nu}\partial_\sigma D^\sigma - \partial^\mu D^\nu$ in (3.1), derived from obtaining the inverse of the *classical* equation $J^\nu = \partial_\mu F^{\mu\nu}$ for a non-commuting Yang-Mills field. This is a sixteen component mixed equation in ${}^\mu{}_\nu$ indexes, and when raised or lowered into contravariant or covariant form it is not symmetric under $\mu \leftrightarrow \nu$ transposition unless one takes additional steps to symmetrize this relationship. While most physics usually stops at two derivatives from the fields (or three if one counts the conservation of sources, $\partial^\mu J_\mu = 0$ and $\partial^\mu T_{\mu\nu} = 0$), this relationship contains <u>fourth</u> derivatives $\partial_\sigma \partial_\nu \partial^\mu \partial^\sigma$, as well as third derivatives including a $\partial^{\{\mu} G^{\sigma\}}$, and finally the term $\partial_{\{\sigma} G_{\nu\}} \partial^{\{\mu} G^{\sigma\}}$ which is second order in symmetric field derivatives (contrast the antisymmetric term $\partial_{[\sigma} G_{\tau]} \partial^{[\tau} G^{\sigma]}$ that appears in Lagrangians). The above (3.22) replaces any and all of the usual gauge conditions that are used in QED, and all those other gauge conditions, most notably $\partial^\alpha G_\alpha = 0$, must <u>not</u> be used here. And as noted, it forces the Fadeev-Popov gauge to be $\xi = 2$.

Now that we know the Yang-Mills inverse of the classical field equation (3.1) and the gauge conditions required to produce that inverse, let us return to where we started, and make use of this inverse in $G_\nu = I_{\sigma\nu} J^\sigma$ to specify G_ν as a function of J^σ. Using (3.12) in $G_\nu = I_{\sigma\nu} J^\sigma$ we first obtain:

$$G_\nu = I_{\sigma\nu} J^\sigma = \frac{-g_{\sigma\nu} + \dfrac{k_\sigma k_\nu + \tfrac{1}{2} i\left[k_{\{\sigma}, G_{\nu\}}\right]}{\text{"}m^2 - k^\alpha k_\alpha - i\left[k^\alpha, G_\alpha\right]\text{"}}}{\text{"}k^\alpha k_\alpha - m^2 + i\left[k^\alpha, G_\alpha\right]\text{"}} J^\sigma . \tag{3.23}$$

However, in momentum space, the current conservation equation $\partial_\mu J^\mu(x) = 0$ becomes $k_\mu J^\mu(k) = 0$ (see [12] after I.5(4)). This modifies (3.23) in two respects. First, the term $k_\sigma k_\nu J^\sigma = 0$. Secondly, and of special interest because it breaks a symmetry, the term $\tfrac{1}{2} i\left[k_{\{\sigma}, G_{\nu\}}\right] J^\sigma = \tfrac{1}{2} i\left[k_\nu, G_\sigma\right] J^\sigma$. That is, one of the two terms in the anticommutator zeros out, but the second term does not. Given that $I_{\sigma\nu}$ was designed to be symmetric under transposition of the $\sigma \leftrightarrow \nu$ indexes, that symmetry is broken in (3.23). So with those reductions, (3.23) becomes:

$$G_\nu = I_{\sigma\nu} J^\sigma = \frac{-g_{\sigma\nu} + \dfrac{\frac{1}{2} i[k_\nu, G_\sigma]}{"m^2 - k^\alpha k_\alpha - i[k^\alpha, G_\alpha]"}}{"k^\alpha k_\alpha - m^2 + i[k^\alpha, G_\alpha]"} J^\sigma . \tag{3.24}$$

One can follow the same path outlined above, to derive this inverse for the various special cases already explored: low-perturbation where $i[k_\nu, G_\sigma] = \partial_\nu G_\sigma \to 0$ (3.15); massless boson $m = 0$ (3.17); both $G^\sigma \to 0$ and $m = 0$ (3.19); and on shell $k^\alpha k_\alpha - m^2 = 0$ for a massive or $k^\alpha k_\alpha = 0$ for a massless particle (3.21). Note that (3.24) is a "recursive" expression, i.e., that G_ν is defined partly in terms of itself, reflecting the non-linear nature of Yang-Mills theory. This is helpful to keep in mind when thinking about path integration, because normally one uses $\varphi = (\delta / \delta J) J \cdot \varphi$ to turn the field of integration φ into an operator $\delta / \delta J$ that is independent of the variable of integration φ to facilitate applying the Gaussian integral $\int dx \exp\left(\frac{1}{2} A x^2 + Jx\right) = (-2\pi / A)^{.5} \exp\left(-J^2 / 2A\right)$. Equation (3.24) enables this also, but in a different way: one may, recursively, replace G_ν with $I_{\sigma\nu} J^\sigma$ to as many orders of recursive "nesting" as desired, thereby bypassing the field of integration in a different way, with an exact path integral thereby emerging from the limit of infinite recursive nesting.

This inverse expression (3.24) is what we set out to derive at the start of this section, and it will play a very central role in helping us to establish that the Yang-Mills magnetic charge $P^{\sigma\mu\nu}$ is in fact a baryon. With all of the preliminary groundwork now laid, and with the understanding discussed in section 2 that by using field equations such as $J^\nu = \partial_\mu F^{\mu\nu}$ in (2.8) and $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ we are exploring the high-action realm in which $S(\varphi) = \int d^4 x \mathfrak{L}(\varphi) >> \hbar$, it is time to discover the underlying theoretical basis for the baryons that constitute the very nuclear heart of the material universe.

4. The Baryon and Meson Structure of Yang Mills Magnetic Monopoles, and "Revealed" Fermion Masses

In section 2 we observed that the Maxwell equation $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ for a magnetic monopole carries over intact to Yang-Mills theory for high-action arenas where the action $S(\varphi) >> \hbar$. Therefore, we can now carry forward on the basis of our earlier equation (2.5), which was derived by the simple substitution the Yang Mills field density $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i[G^\mu, G^\nu]$ of (2.3) into the $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ of (2.1), which is the "classical" magnetic charge equation for Yang-Mills theory as well as Abelian gauge theory.

The first thing we do is substitute $\partial^\sigma G^\mu = i[k^\sigma, G^\mu]$ from (3.3) into (2.5) to yield:

$$P^{\sigma\mu\nu} = \left([[k^\sigma, G^\mu], G^\nu] + [G^\mu, [k^\sigma, G^\nu]] + [[k^\mu, G^\nu], G^\sigma] + [G^\nu, [k^\mu, G^\sigma]] + [[k^\nu, G^\sigma], G^\mu] + [G^\sigma, [k^\nu, G^\mu]]\right) . \tag{4.1}$$

If we expand the commutators in the above, terms of the form $G^\mu k^\sigma G^\nu - G^\mu k^\sigma G^\nu$ appear throughout, so that all terms with k^σ sandwiched between the two G^μ drop out. Then, re-consolidating the commutators, (4.1) reduces to:

$$P^{\sigma\mu\nu} = -\left([[G^\mu, G^\nu], k^\sigma] + [[G^\nu, G^\sigma], k^\mu] + [[G^\sigma, G^\mu], k^\nu]\right). \tag{4.2}$$

This will be our starting point for exploring the baryonic properties of $P^{\sigma\mu\nu}$.

First, we insert the hard-won Yang-Mills inverse (3.24) for G_ν into (4.2). Keep in mind that we have done nothing to break the symmetry of the Yang-Mills theory and so the gauge bosons must be presumed to be massless. Nonetheless, we will carry the Proca mass term in these equations, so whatever we derive is perfectly general. If we want to explore the special case for $m = 0$ we can always do so by zeroing out the mass at the time, but at the outset, we ought not limit ourselves in this way.

Also, to maintain full generality at the outset, because there are six different appearances of G_ν in (4.2), there will be six independent substitutions of (3.24) into (4.2). To track this, we will use the first six letters of the Greek alphabet $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta$ to carry out the internal index summations within each of the six substitutions of the inverse (3.24). While it ordinarily does not matter what letters one chooses to do summations, the summation index will in this case double as a label so we can quickly ascertain where any term originated from as we progress. And more importantly, while $k^\alpha k_\alpha = k^\beta k_\beta$ where the momenta are equal, $k^\alpha = k^\beta$, in the event that $k^\alpha \neq k^\beta$ – for example if these are momentum vectors for two different particles – then $k^\alpha k_\alpha \neq k^\beta k_\beta$. So we are using this index convention to simultaneously label the momenta and to avoid making any *a priori* assumptions about the actual physical values and meanings of the k^α in each of the six inverse substitutions we are making. Similarly, substituting (3.24) into each of the G_ν in (4.2) introduces six Proca mass numbers *m*. Here too, we wish to avoid assuming anything *a priori*. So, we similarly label each mass with one of the $\alpha,\beta,\gamma,\delta,\varepsilon,\zeta$, and so regard these at least at the outset, as six different, independent mass numbers. Thus, the expression below in (4.3) will contain six momenta $k^\alpha,k^\beta,k^\gamma,k^\delta,k^\varepsilon,k^\zeta$ which may or may not be different form one another, as well as six labeled masses $m_{(\alpha)},m_{(\beta)},m_{(\gamma)},m_{(\lambda)},m_{(\varepsilon)},m_{(\zeta)}$ which also may or may not be different from each other and may also be zero or non-zero. This provides complete generality and maximum flexibility to explore.

Finally, prior to substituting this inverse (3.24) into (4.2), for the three G^μ,G^ν,G^σ in the left hand side of the commutators in (4.2) we have arranged for the free indexes μ,ν,σ to be in the right hand position of metric tensor $g^{\alpha\mu}=g^{\mu\alpha}$ of (3.24). Conversely, for the three G^μ,G^ν,G^σ from the right hand side of the commutators, we have arranged for the free indexes μ,ν,σ to be in the left hand position in $g^{\alpha\mu}=g^{\mu\alpha}$ (3.24). We may do this because $g^{\alpha\mu}=g^{\mu\alpha}$ is a symmetric tensor and the indexes can thus be disposed in either order, and the order makes no mathematical difference. While not needed for this paper, this choice of index placement will be helpful when drawing Feynman diagrams.

With all of the foregoing, finally substituting the inverse equation (3.24) for G_ν into (4.2) yields:

$$P^{\sigma\mu\nu} = -\left(\begin{array}{l} \left[\left[\dfrac{-g^{\alpha\mu}+\dfrac{\frac{1}{2}i\left[k^\mu,G^\alpha\right]}{"m_{(\alpha)}{}^2-k^\alpha k_\alpha - i\left[k^\alpha,G_\alpha\right]"}}{"k^\alpha k_\alpha - m_{(\alpha)}{}^2+i\left[k^\alpha,G_\alpha\right]"} J_\alpha , \dfrac{-g^{\nu\beta}+\dfrac{\frac{1}{2}i\left[k^\nu,G^\beta\right]}{"m_{(\beta)}{}^2-k^\beta k_\beta - i\left[k^\beta,G_\beta\right]"}}{"k^\beta k_\beta - m_{(\beta)}{}^2+i\left[k^\beta,G_\beta\right]"} J_\beta \right], k^\sigma \right] \\ + \left[\left[\dfrac{-g^{\gamma\nu}+\dfrac{\frac{1}{2}i\left[k^\nu,G^\gamma\right]}{"m_{(\gamma)}{}^2-k^\gamma k_\gamma - i\left[k^\gamma,G_\gamma\right]"}}{"k^\gamma k_\gamma - m_{(\gamma)}{}^2+i\left[k^\gamma,G_\gamma\right]"} J_\gamma , \dfrac{-g^{\sigma\delta}+\dfrac{\frac{1}{2}i\left[k^\sigma,G^\delta\right]}{"m_{(\delta)}{}^2-k^\delta k_\delta - i\left[k^\delta,G_\delta\right]"}}{"k^\delta k_\delta - m_{(\delta)}{}^2+i\left[k^\delta,G_\delta\right]"} J_\delta \right], k^\mu \right] \\ + \left[\left[\dfrac{-g^{\varepsilon\sigma}+\dfrac{\frac{1}{2}i\left[k^\sigma,G^\varepsilon\right]}{"m_{(\varepsilon)}{}^2-k^\varepsilon k_\varepsilon - i\left[k^\varepsilon,G_\varepsilon\right]"}}{"k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2+i\left[k^\varepsilon,G_\varepsilon\right]"} J_\varepsilon , \dfrac{-g^{\mu\zeta}+\dfrac{\frac{1}{2}i\left[k^\mu,G^\zeta\right]}{"m_{(\zeta)}{}^2-k^\zeta k_\zeta - i\left[k^\zeta,G_\zeta\right]"}}{"k^\zeta k_\zeta - m_{(\zeta)}{}^2+i\left[k^\zeta,G_\zeta\right]"} J_\zeta \right], k^\nu \right] \end{array} \right). \qquad (4.3)$$

To start, we review lowest order terms for which $\partial^\sigma G^\mu = i\left[k^\sigma,G^\mu\right] \to 0$. This allows us to remove all the "quoted" inverses and many other terms, and (4.3) simplifies to:

$$P^{\sigma\mu\nu} = -\left(\begin{array}{l} \left[\left[\dfrac{g^{\alpha\mu}}{k^\alpha k_\alpha - m_{(\alpha)}{}^2} J_\alpha , \dfrac{g^{\nu\beta}}{k^\beta k_\beta - m_{(\beta)}{}^2} J_\beta \right], k^\sigma \right] \\ + \left[\left[\dfrac{g^{\gamma\nu}}{k^\gamma k_\gamma - m_{(\gamma)}{}^2} J_\gamma , \dfrac{g^{\sigma\delta}}{k^\delta k_\delta - m_{(\delta)}{}^2} J_\delta \right], k^\mu \right] \\ + \left[\left[\dfrac{g^{\varepsilon\sigma}}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2} J_\varepsilon , \dfrac{g^{\mu\zeta}}{k^\zeta k_\zeta - m_{(\zeta)}{}^2} J_\zeta \right], k^\nu \right] \end{array} \right). \qquad (4.4)$$

While both (4.3) and (4.4) are classical insofar as they depend upon an action $S(\varphi) >> \hbar$, (4.4) lays out the basic structure of $P^{\sigma\mu\nu}$, while (4.3) shows what happens then the G^μ dynamics come into play and start to exert a dominant role. The terms in (4.3) involving $\left[k^\nu, G^\tau\right]$ will generate higher-order interactions via (4.3), but because they emerged from the classical equations (2.1), they actually need to be modified once we consider quantum field theory (and as noted following (3.18) will become a gauge-invariant "perturbation" $-V = \left(\partial^\sigma G_\sigma + G_\sigma \partial^\sigma\right) + G^\sigma G_\sigma$). Not so for (4.4). This is the "skeleton" of $P^{\sigma\mu\nu}$ which reveals the underlying structural characteristics of $P^{\sigma\mu\nu}$ in the lowest order, and it will survive intact to establish the Yang-Mills path integral in lowest order. So we will now explore this structural equation (4.4) in earnest, to see what it tells us about what is going on inside of the magnetic charges $P^{\sigma\mu\nu}$, at the lowest, classical order.

We shall at times refer to (4.4) as a "chameleon equation," because depending on how one manipulates this equation, one may highlight the currents / fermions, one may highlight the gauge bosons, and one may explore both currents and gauge bosons in a mixed view. In the gauge boson view, one leaves the $g^{\mu\nu}$ showing explicitly in the above, which thereby displays complete boson propagators. In the current view, the $g^{\mu\nu}$ are absorbed into the currents via $J^\nu = g^{\nu\beta} J_\beta$. In "mixed" view, we have little of each. We start with the current / fermion view, by applying $J^\mu = g^{\alpha\mu} J_\alpha$ to (4.4) thus:

$$P^{\sigma\mu\nu} = -\left(\begin{array}{l} \left[\left[\dfrac{J^\mu}{k^\alpha k_\alpha - m_{(\alpha)}{}^2} , \dfrac{J^\nu}{k^\beta k_\beta - m_{(\beta)}{}^2} \right], k^\sigma \right] \\ + \left[\left[\dfrac{J^\nu}{k^\gamma k_\gamma - m_{(\gamma)}{}^2} , \dfrac{J^\sigma}{k^\delta k_\delta - m_{(\delta)}{}^2} \right], k^\mu \right] \\ + \left[\left[\dfrac{J^\sigma}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2} , \dfrac{J^\mu}{k^\zeta k_\zeta - m_{(\zeta)}{}^2} \right], k^\nu \right] \end{array} \right). \qquad (4.5)$$

The reader should pause at this point to compare this closely, term by term, with (4.2). While there are other benefits that emerges from having the inverse (3.12) developed in section 3, it was to get from (4.2) to (4.5) that we went through all the work in section 3 to develop the inverse $I_{\sigma\nu}$ of (3.12) and inverse relation $G_\nu = I_{\sigma\nu} J^\sigma$ of (3.24).

Next, the J^μ above are all NxN matrices for SU(N), and the internal symmetries of these groups are hidden inside just to keep notation compact and easy for the calculations we have done thus far. Now, however, it is time to start bringing the internal symmetry explicitly into the picture, so we use $J^\mu = T^i J_i{}^\mu$, $i = 1,2,3 \ldots N^2 - 1$, and similar expressions for the other five currents in (4.5). With some renaming of summed internal symmetry indexes, we obtain:

$$P^{\sigma\mu\nu} = -\begin{pmatrix} \left[\left[\dfrac{T^i J_i^{\ \mu}}{k^\alpha k_\alpha - m_{(\alpha)}{}^2}, \dfrac{T^j J_j^{\ \nu}}{k^\beta k_\beta - m_{(\beta)}{}^2}\right], k^\sigma\right] \\ + \left[\left[\dfrac{T^i J_i^{\ \nu}}{k^\gamma k_\gamma - m_{(\gamma)}{}^2}, \dfrac{T^j J_j^{\ \sigma}}{k^\delta k_\delta - m_{(\delta)}{}^2}\right], k^\mu\right] \\ + \left[\left[\dfrac{T^i J_i^{\ \sigma}}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2}, \dfrac{T^j J_j^{\ \mu}}{k^\zeta k_\zeta - m_{(\zeta)}{}^2}\right], k^\nu\right] \end{pmatrix}. \tag{4.6}$$

The group structure matrices T^i and their associated commutator may then be factored out of this entire expression (the reader can check this by expanding all commutators, factoring these out, and then reconsolidating), so as to write:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right]\begin{pmatrix} \left[\left(\dfrac{J_i^{\ \mu}}{k^\alpha k_\alpha - m_{(\alpha)}{}^2} \dfrac{J_j^{\ \nu}}{k^\beta k_\beta - m_{(\beta)}{}^2}\right), k^\sigma\right] \\ + \left[\left(\dfrac{J_i^{\ \nu}}{k^\gamma k_\gamma - m_{(\gamma)}{}^2} \dfrac{J_j^{\ \sigma}}{k^\delta k_\delta - m_{(\delta)}{}^2}\right), k^\mu\right] \\ + \left[\left(\dfrac{J_i^{\ \sigma}}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2} \dfrac{J_j^{\ \mu}}{k^\zeta k_\zeta - m_{(\zeta)}{}^2}\right), k^\nu\right] \end{pmatrix}. \tag{4.7}$$

The group structure constants f^{ijk} in $if^{ijk}T_k = \left[T^i, T^j\right]$ maintain the commutation position of each of the $J_i^{\ \mu}$, that is, $\left[T^i, T^j\right]J_i^{\ \mu} J_j^{\ \nu} = \left[J^\mu, J^\nu\right]$. This expression is perfectly symmetrical in appearance as between currents $J_i^{\ \mu}$, but now we will take a simple step to break this symmetry: we will simply move both currents into the right hand numerators, and rewrite the above as:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right]\begin{pmatrix} \left[\left(\dfrac{1}{k^\alpha k_\alpha - m_{(\alpha)}{}^2} \dfrac{J_i^{\ \mu} J_j^{\ \nu}}{k^\beta k_\beta - m_{(\beta)}{}^2}\right), k^\sigma\right] \\ + \left[\left(\dfrac{1}{k^\gamma k_\gamma - m_{(\gamma)}{}^2} \dfrac{J_i^{\ \nu} J_j^{\ \sigma}}{k^\delta k_\delta - m_{(\delta)}{}^2}\right), k^\mu\right] \\ + \left[\left(\dfrac{1}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2} \dfrac{J_i^{\ \sigma} J_j^{\ \mu}}{k^\zeta k_\zeta - m_{(\zeta)}{}^2}\right), k^\nu\right] \end{pmatrix}. \tag{4.8}$$

It is worth noting by the way, that the six currents may be referred to and distinguished by 6=3x2 combinations of the spacetime indexes μ, ν, σ and internal symmetry indexes i, j.

For a next step, we drill down even further, by employing $J_i^{\ \mu} = \overline{\psi} T_i \gamma^\mu \psi$ and the like to introduce fermion wavefunctions. So (4.8) now becomes:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right]\left(\begin{array}{l}\left[\left(\dfrac{1}{k^\alpha k_\alpha - m_{(\alpha)}{}^2}\dfrac{\overline{\psi} T_i \gamma^\mu \psi \overline{\psi} T_j \gamma^\nu \psi}{k^\beta k_\beta - m_{(\beta)}{}^2}\right), k^\sigma\right] \\ + \left[\left(\dfrac{1}{k^\gamma k_\gamma - m_{(\gamma)}{}^2}\dfrac{\overline{\psi} T_i \gamma^\nu \psi \overline{\psi} T_j \gamma^\sigma \psi}{k^\delta k_\delta - m_{(\delta)}{}^2}\right), k^\mu\right] \\ + \left[\left(\dfrac{1}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2}\dfrac{\overline{\psi} T_i \gamma^\sigma \psi \overline{\psi} T_j \gamma^\mu \psi}{k^\zeta k_\zeta - m_{(\zeta)}{}^2}\right), k^\nu\right]\end{array}\right). \tag{4.9}$$

Now, the next steps are very important, so let's walk through them carefully. We first write the two back-to-back wavefunctions $\psi\overline{\psi}$ using $\psi = u(p)e^{-ip_1{}^\alpha x_\alpha}$ and $\overline{\psi} = \overline{u}(p)e^{p_2{}^\alpha x_\alpha}$. But because these are back to back, they represent a single, unbroken, unmediated fermion propagator line, such that $p_2{}^\alpha = p_1{}^\alpha$, and so $\psi\overline{\psi} = u\overline{u}$. Keep in mind, because we are working with SU(N) in Yang-Mills theory, that $u\overline{u}$ is an NxN SU(N) matrix, in addition to having the usual 4x4 Dirac structure. So if some variant of $u\overline{u}$ finds its way into any denominators as it momentarily will, we have to take an SU(N) *matrix inverse*, and not just write an ordinary denominator.

Now, in the sum over spins is often written as $\sum_{spins} u\overline{u} = \not{p} + m$ (see [13], section 5.5). But there is an implied normalization in this expression, and to be explicit, this should really be written as:

$$\sum_{spins} u\overline{u} = \frac{\left|N^2\right|}{E+m}\left(\not{p} + m\right) = \not{p} + m\,, \tag{4.10}$$

where one get to the final term using the covariant normalization in which $\left|N^2\right| = E + m$ (see [13], problem solution 5.9). What is attractive about this normalization, is that this does yield the covariant expression $\sum_{spins} u\overline{u} = \not{p} + m$. But there is one problem, which has to do with mass dimensionality. Specifically, in (4.10), each of $u, \overline{u}$ has a mass dimension of +3/2, so that $u\overline{u}$ has mass dimension of +3. But in the final term of (4.10), $\not{p} + m$ only has a mass dimension of +1. We will wish, therefore, to choose a different $\left|N^2\right|$ normalization in (4.10) a) is still covariant, but also b) retains a mass dimension of +3. So for the moment, we leave open our choice of normalization and employ the middle term of (4.10) which is more general.

In addition to (4.10) introducing $\not{p} + m$, we also know that $\left(\not{p} + m\right)/\left(p^\beta p_\beta - m^2\right) = 1/\left(\not{p} - m\right)$. So suddenly, we find that terms which started as vector boson propagators interacting with currents in (4.4) are turning, chameleon-like in (4.9), into a fermion propagator, complete with a "revealed" mass for the fermion. For example, in the top line of (4.9), we make the following progression of substitutions:

$$\begin{aligned}&\frac{\overline{\psi} T_i \gamma^\mu \psi \overline{\psi} T_j \gamma^\nu \psi}{k^\beta k_\beta - m_{(\beta)}{}^2} = \frac{\overline{\psi} T_i \gamma^\mu \sum u\overline{u} T_j \gamma^\nu \psi}{k^\beta k_\beta - m_{(\beta)}{}^2} = \frac{\left|N^2\right|}{E+m}\frac{\overline{\psi} T_i \gamma^\mu \left(\not{p} + m\right) T_j \gamma^\nu \psi}{k^\beta k_\beta - m_{(\beta)}{}^2} \\ &= \frac{\left|N_{(\beta)}{}^2\right|}{E_{(\beta)} + m_{(\beta)}}\frac{\overline{\psi} T_i \gamma^\mu \left(\not{p}_{(\beta)} + m_{(\beta)}\right) T_j \gamma^\nu \psi}{p^\beta p_\beta - m_{(\beta)}{}^2} = \frac{\left|N_{(\beta)}{}^2\right|}{E_{(\beta)} + m_{(\beta)}}\frac{\overline{\psi} T_i \gamma^\mu T_j \gamma^\nu \psi}{"\not{p}_{(\beta)} - m_{(\beta)}"} \\ &= \frac{\left|N_{(\beta)}{}^2\right|}{E_{(\beta)} + m_{(\beta)}}\overline{\psi} T_i \gamma^\mu T_j \gamma^\nu \psi \times \left(\not{p}_{(\beta)} - m_{(\beta)}\right)^{-1}\end{aligned} \quad . \tag{4.11}$$

Specifically, first, we use $\psi\bar{\psi} = u\bar{u}$ and sum over all spin states. Then we set $\sum u\bar{u} = \left(\left|N^2\right|/(E+m)\right)(\not{p}+m)$ as in (4.10). Next, we take the *affirmative step* (which as we will discuss shortly requires some accounting for degrees of freedom and so will render the gauge bosons massless) of setting the rest mass in the resultant $\not{p}+m$ to be equal to the labeled mass $m_{(\beta)}$ in the denominator, that is, we now set $m = m_{(\beta)}$, identifying $m_{(\beta)}$ with the fermion rest mass in $\sum_{spins} u\bar{u}$. (This $m_{(\beta)}$, of course, started out in (4.4) as a gauge boson mass in a gauge boson propagator denominator, and has now turned into a fermion rest mass – more chameleon-like behavior! In a moment, we see how to account for degrees of freedom to make this all work properly.) And we simultaneously promote $k^\beta \to p^\beta$ into the momentum four-vector p^β for an actual fermion, and label $E = E_{(\beta)}$ and $N = N_{(\beta)}$. Finally, we set:

$$\frac{\not{p}_{(\beta)} + m_{(\beta)}}{p^\beta p_\beta - m_{(\beta)}{}^2} = \frac{1}{"\not{p}_{(\beta)} - m_{(\beta)}"} = \left(\not{p}_{(\beta)} - m_{(\beta)}\right)^{-1} \tag{4.12}$$

in recognition of the fact, which was discussed in section 3, that whenever an SU(N) matrix (including the $\sum u\bar{u} = \not{p}+m$) needs to go into a "denominator," we must form its inverse. So, these fermion rest masses $m_{(\beta)}$, etc., such as they are, will be obtained via SU(N) matrix inversion. To maintain a clear visual comparison with familiar equation forms, we will continue to use the "quoted denominators" to designate inverses.

So, we now use (4.11) to rewrite all three terms in (4.9), yielding:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right] \left(\begin{array}{l} \left[\left(\dfrac{g^{\alpha\mu}}{k^\alpha k_\alpha - m_{(\alpha)}{}^2} \dfrac{\left|N_{(\beta)}{}^2\right|}{E_{(\beta)} + m_{(\beta)}} \dfrac{\bar{\psi} T_i \gamma_\alpha T_j \gamma^\nu \psi}{"\not{p}_{(\beta)} - m_{(\beta)}"}\right), k^\sigma\right] \\ + \left[\left(\dfrac{g^{\alpha\nu}}{k^\gamma k_\gamma - m_{(\gamma)}{}^2} \dfrac{\left|N_{(\delta)}{}^2\right|}{E_{(\delta)} + m_{(\delta)}} \dfrac{\bar{\psi} T_i \gamma_\alpha T_j \gamma^\sigma \psi}{"\not{p}_{(\delta)} - m_{(\delta)}"}\right), k^\mu\right] \\ + \left[\left(\dfrac{g^{\alpha\sigma}}{k^\varepsilon k_\varepsilon - m_{(\varepsilon)}{}^2} \dfrac{\left|N_{(\zeta)}{}^2\right|}{E_{(\zeta)} + m_{(\zeta)}} \dfrac{\bar{\psi} T_i \gamma_\alpha T_j \gamma^\mu \psi}{"\not{p}_{(\zeta)} - m_{(\zeta)}"}\right), k^\nu\right] \end{array} \right). \tag{4.13}$$

Here, we have also lowered the index on the left-hand vertices in order to reintroduce the $g^{\alpha\mu}$ to the left-hand terms which once again display explicitly, the appearance of a gauge boson propagator. This "chameleon equation" is now in a fully-mixed fermion / boson view, because we now see three fermion propagators and three gauge boson propagators. And, we see how simply moving both currents into the right hand numerators in (4.8) broke the initial symmetry, led to both fermion and boson propagators in each term of (4.13), and turned three of the six masses $m_{(\beta)}, m_{(\delta)}, m_{(\zeta)}$ into fermion masses while leaving the other three masses $m_{(\alpha)}, m_{(\gamma)}, m_{(\varepsilon)}$ intact as boson masses. What we have done here, is break a mass symmetry that started out with all boson masses, into a mass asymmetry containing both boson and fermion masses.

But there is one final piece of the puzzle that is required to make this all work properly, which is to account for the degrees of freedom in what we just did to turn (4.9) into (4.13). In going from (4.9) to (4.13), (or from (4.4) to (4.13) where this is even more evident), we started with six vector bosons with presumed Proca masses $m_{(\alpha)}, m_{(\beta)}, m_{(\gamma)}, m_{(\lambda)}, m_{(\varepsilon)}, m_{(\zeta)}$. A massive vector boson has three degrees of freedom, so the six bosons we started with in (4.4) brought 3x6=18 degrees of freedom into $P^{\sigma\mu\nu}$. But then between (4.9) and (4.13) we took three of those boson masses and turned them into fermion masses. Massive fermions, however, have <u>four</u> degrees of freedom, not three. So for us to promote a massive boson mass into a fermion mass, we must transfer one degree of freedom over from the boson to the fermion. So, by associating $m_{(\beta)}, m_{(\lambda)}, m_{(\zeta)}$ in (4.13) with fermion masses, we are required to steal one degree of freedom from each remaining vector gauge bosons. So, now these bosons must drop down to <u>two</u> degrees of freedom apiece and must become massless, which means that all of $m_{(\alpha)}, m_{(\gamma)}, m_{(\varepsilon)}$ now must be set to zero. Now, the 18 degrees of freedom that initially belonged three apiece to six massive vector bosons have been redistributed: 12 of these now belong to the 3 fermions, and only 6 belong to the 3 remaining bosons. This should seem very familiar, as this is the same way in which massless gauge bosons first become massive by swallowing a degree of freedom from a scalar field via the Goldstone mechanism. So, to balance the degrees of freedom to account for what we just did, we must now set all of the remaining $m_{(\alpha)}, m_{(\gamma)}, m_{(\varepsilon)} = 0$. Raising the index on the currents once again, (4.13) now becomes:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right]\left(\begin{array}{l} \left[\left(\dfrac{1}{k^\alpha k_\alpha}\dfrac{\left|N_{(\beta)}{}^2\right|}{E_{(\beta)}+m_{(\beta)}}\dfrac{\bar{\psi}T_i\gamma^\mu T_j\gamma^\nu\psi}{"\not{p}_{(\beta)}-m_{(\beta)}"}\right), k^\sigma\right] \\ +\left[\left(\dfrac{1}{k^\gamma k_\gamma}\dfrac{\left|N_{(\delta)}{}^2\right|}{E_{(\delta)}+m_{(\delta)}}\dfrac{\bar{\psi}T_i\gamma^\nu T_j\gamma^\sigma\psi}{"\not{p}_{(\delta)}-m_{(\delta)}"}\right), k^\mu\right] \\ +\left[\left(\dfrac{1}{k^\varepsilon k_\varepsilon}\dfrac{\left|N_{(\zeta)}{}^2\right|}{E_{(\zeta)}+m_{(\zeta)}}\dfrac{\bar{\psi}T_i\gamma^\sigma T_j\gamma^\mu\psi}{"\not{p}_{(\zeta)}-m_{(\zeta)}"}\right), k^\nu\right] \end{array}\right). \tag{4.14}$$

The above, (4.14), can now be said to be equal to (4.9) in all respects, including a proper degrees of freedom accounting. (While we shall not explore this here, this seems to suggest a "cascade" to generate fermion masses: First, start with scalar fields in the usual way and apply gauge symmetry. Next, break the symmetry using the Higgs mechanism to give mass to some of the vector bosons, by transferring Goldstone scalar degrees of freedom into the gauge bosons. Finally, pass along those extra degrees of freedom to from the massive gauge bosons to the fermions, rendering the fermions massive, while reverting the gauge bosons back to massless status. The scalar degrees of freedom that originate in the scalar fields, finally reside in the fermions to give them a rest mass!)

Now we see that $P^{\sigma\mu\nu}$ contains three fermions, with terms $\bar{\psi}T_i\gamma_\alpha T_j\gamma^\nu\psi\times\left(\not{p}-m_{(\beta)}\right)^{-1}$ that look exactly like the expressions for the Compton scattering of a fermion with a gauge boson, such as $\gamma e \to \gamma e$ of an electron with a photon in QED. (e.g., [13] at 141) As we now see clearly from (4.14), <u>$P^{\sigma\mu\nu}$ naturally contains three fermions, just like a baryon, along with all the machinery for fermion propagation, right alongside of propagators for associated, now massless, gauge bosons</u>. As a result, for the first time, we will stop referring to this as a Yang-Mills magnetic charge, and think of it as a true "baryon candidate." Now we need to show that this really has all the required formal characteristics to be a real, physical baryon.

Looking closely at (4.14), we now also see a path to choosing a normalization $\left|N^2\right|$ which simultaneously a) is covariant b) retains the original mass dimensionality of +3 for $u\bar{u}$, and c) greatly simplifies (4.14). Specifically, we now choose the covariant normalizations:

$$\left|N_{(\beta)}{}^2\right| = \left(E_{(\beta)}+m_{(\beta)}\right)k^\alpha k_\alpha; \quad \left|N_{(\delta)}{}^2\right| = \left(E_{(\delta)}+m_{(\delta)}\right)k^\gamma k_\gamma; \quad \left|N_{(\zeta)}{}^2\right| = \left(E_{(\zeta)}+m_{(\zeta)}\right)k^\varepsilon k_\varepsilon, \tag{4.15}$$

making use of the $k^\alpha k_\alpha$ to supply the two mass dimensions lost in the usual covariant normalization. This greatly simplifies (4.14) to:

$$P^{\sigma\mu\nu} = -\left[T^i, T^j\right]\left(\left[\frac{\bar{\psi}T_i\gamma^\mu T_j\gamma^\nu\psi}{"\not{p}_{(\beta)}-m_{(\beta)}"}, k^\sigma\right] + \left[\frac{\bar{\psi}T_i\gamma^\nu T_j\gamma^\sigma\psi}{"\not{p}_{(\delta)}-m_{(\delta)}"}, k^\mu\right] + \left[\frac{\bar{\psi}T_i\gamma^\sigma T_j\gamma^\mu\psi}{"\not{p}_{(\zeta)}-m_{(\zeta)}"}, k^\nu\right]\right). \tag{4.16}$$

In this variation of the "chameleon," all we see are propagating fermions in Compton scattering form.

Proceeding apace, the commutator $\left[T^i, T^j\right]$ is still sitting out front of (4.16), so let's now work with that. The $\left[T^i, T^j\right]$ operates to commute the vertices $\left(T_i\gamma^\mu\right)\left(T_j\gamma^\nu\right)$, and in particular, the operation it now performs on each current / fermion term in (4.14) is:

$$\left[T^i, T^j\right]\overline{\psi}\left(T_i\gamma^\mu\right)\left(T_j\gamma^\nu\right)\psi = \overline{\psi}\left[\gamma^\mu, \gamma^\nu\right]\psi\,, \tag{4.17}$$

which is the same commutation $\left[G^\mu, G^\nu\right]$ of free indexes μ, ν with which everything started back in (4.2), and even further back, in the underlying field density $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i\left[G^\mu, G^\nu\right]$ of (2.3) which is the heart of Yang-Mills theory. So, using the above in (4.16) now yields:

$$P^{\sigma\mu\nu} = -\left(\left[\frac{\overline{\psi}\left[\gamma^\mu, \gamma^\nu\right]\psi}{\text{"}\,\not{p}_{(\beta)} - m_{(\beta)}\text{"}}, k^\sigma\right] + \left[\frac{\overline{\psi}\left[\gamma^\nu, \gamma^\sigma\right]\psi}{\text{"}\,\not{p}_{(\delta)} - m_{(\delta)}\text{"}}, k^\mu\right] + \left[\frac{\overline{\psi}\left[\gamma^\sigma, \gamma^\mu\right]\psi}{\text{"}\,\not{p}_{(\zeta)} - m_{(\zeta)}\text{"}}, k^\nu\right]\right). \tag{4.18}$$

Now it is time for mesons to make their first appearance. Using the first term of (4.18) for an example, let us first expand the commutator:

$$\overline{\psi}\left[\gamma^\mu, \gamma^\nu\right]\psi = \overline{\psi}\gamma^\mu\gamma^\nu\psi - \overline{\psi}\gamma^\nu\gamma^\mu\psi\,. \tag{4.19}$$

Now let's look at the charge conjugates (antiparticles) of the above. Using the Dirac conjugation relationships $\psi_C = C\overline{\psi}^T$, $\overline{\psi}_C = -\psi^T C^{-1}$, $\gamma^\nu C = C\left(-\gamma^\nu\right)^T$, and $C^{-1}\gamma^\mu C = \left(-\gamma^\mu\right)^T$, we obtain:

$$\overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C = -\psi^T C^{-1}\gamma^\mu\gamma^\nu C\overline{\psi}^T = -\psi^T\left(-\gamma^\mu\right)^T\left(-\gamma^\nu\right)^T\overline{\psi}^T = -\overline{\psi}\gamma^\nu\gamma^\mu\psi\,. \tag{4.20}$$

This means that (4.19) may be rewritten as:

$$\overline{\psi}\left[\gamma^\mu, \gamma^\nu\right]\psi = \overline{\psi}\gamma^\mu\gamma^\nu\psi - \overline{\psi}\gamma^\nu\gamma^\mu\psi = \overline{\psi}\gamma^\mu\gamma^\nu\psi + \overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C\,. \tag{4.21}$$

The commutator – which is central to Yang-Mills theory – naturally pairs a particle wavefunction with an antiparticle wavefunction to produce a meson! So we go back to (4.18), and now write:

$$P^{\sigma\mu\nu} = -\left[\frac{\overline{\psi}\gamma^\mu\gamma^\nu\psi + \overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C}{\text{"}\,\not{p}_{(\beta)} - m_{(\beta)}\text{"}}, k^\sigma\right] - \left[\frac{\overline{\psi}\gamma^\nu\gamma^\sigma\psi + \overline{\psi}_C\gamma^\nu\gamma^\sigma\psi_C}{\text{"}\,\not{p}_{(\delta)} - m_{(\delta)}\text{"}}, k^\mu\right] - \left[\frac{\overline{\psi}\gamma^\sigma\gamma^\mu\psi + \overline{\psi}_C\gamma^\sigma\gamma^\mu\psi_C}{\text{"}\,\not{p}_{(\zeta)} - m_{(\zeta)}\text{"}}, k^\nu\right]. \tag{4.22}$$

The above also tells us that the antifermions have the same masses as the fermions, because they are all over a common propagator denominator / inverse.

All that now remains in (4.22) is the final commutator with momentum terms such as k^σ. Going back to (3.3), which tells us that commuting a spacetime field with k^σ is just a clever way to take its derivatives, we can write that in general, for a second rank tensor field $M^{\mu\nu}(x^\sigma)$:

$$\partial^\sigma M^{\mu\nu} = i\left[k^\sigma, M^{\mu\nu}\right]. \tag{4.23}$$

With this, (4.22) above may finally be expressed without any commutators, as:

$$P^{\sigma\mu\nu} = -i\partial^\sigma\left(\frac{\overline{\psi}\gamma^\mu\gamma^\nu\psi + \overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C}{"\not{p}_{(\beta)} - m_{(\beta)}"} + \ldots\right) - i\partial^\mu\left(\frac{\overline{\psi}\gamma^\nu\gamma^\sigma\psi + \overline{\psi}_C\gamma^\nu\gamma^\sigma\psi_C}{"\not{p}_{(\delta)} - m_{(\delta)}"} + \ldots\right) - i\partial^\nu\left(\frac{\overline{\psi}\gamma^\sigma\gamma^\mu\psi + \overline{\psi}_C\gamma^\sigma\gamma^\mu\psi_C}{"\not{p}_{(\zeta)} - m_{(\zeta)}"} + \ldots\right). \quad (4.24)$$

In the above, we have now also added a $+\ldots$, because going back to (4.3), we see that these are the lowest order terms in this candidate baryon. *No matter what other interactions may take place, and even as we start to consider quantum fields where the classical field equations no longer apply, these basic, zero-order terms will always remain.* Different conditions and special cases may and will change the higher order terms, but what appears in (4.24) will always remain the fundamental backbone of a baryon. (Notably and importantly, at lowest order (4.24) is free of any appearance of G^μ, which is especially helpful when it comes to having terms which are quadratic-only in the fields for path integration.)

Comparing the first term in (4.22) with the like term in (4.2) also yields one other very important result, which will be used momentarily to formally show that mesons are the only particles allowed to leave a baryon, thus confining quarks and gluons. Specifically, this comparison yields:

$$\left[G^\mu, G^\nu\right] = \frac{\overline{\psi}\gamma^\mu\gamma^\nu\psi + \overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C}{"\not{p}_{(\beta)} - m_{(\beta)}"} + \ldots. \quad (4.25)$$

Before concluding, there is one final point to note, dealing generally with Yang-Mills theory, and not specifically with baryons or QCD. The commutator $\left[G^\mu, G^\nu\right]$ in (4.25) above is central to Yang-Mills theory. In fact, it appears in the very foundational equation of Yang-Mills theory, namely, (2.3). So this often-seen equation can be written in a totally novel form, as:

$$F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i\left[G^\mu, G^\nu\right] = \partial^\mu G^\nu - \partial^\nu G^\mu + i\left(\frac{\overline{\psi}\gamma^\mu\gamma^\nu\psi + \overline{\psi}_C\gamma^\mu\gamma^\nu\psi_C}{"\not{p}_{(\beta)} - m_{(\beta)}"} + \ldots\right). \quad (4.26)$$

This tells us at the lowest order, what the non-commuting field term $\left[G^\mu, G^\nu\right]$ brings to the field strength $F^{\mu\nu}$ that does not appear in U(1) Abelian (commuting field) gauge theory: a paired fermion and anti-fermion wavefunction, which means that this meson-like character is endemic to Yang-Mills gauge theories. One may use this to go back to all the equations of Yang-Mills theory, make use of the field strength in the form of (4.26), and see what sorts of new insights emerge. This includes the path integral, because (4.26) is quadratic (not cubic or quartic!) in G^ν and the associated Gaussian can be solved exactly. Keep in mind also, that one can exercise this chameleon-like expression for $\left[G^\mu, G^\nu\right]$ into a variety of other forms as well, including backtracking through the development in this section. Those chameleon exercises are also very helpful if one wishes to draw Feynman diagrams for baryons and mesons, and they lead to term combinations we have not elaborated here because they were not essential to the main line of development. Now, we turn to confinement.

5. Confinement Part 1: Integral Equation Symmetries and Color confinement

The so-called "MIT Bag Model" [14], [15] [9] Chapter 18, which was one of the first efforts to understand confinement, pays close attention – very properly so – to what does and does not flow across a closed surface around a baryon. As such, analogies to Maxwell's charge equation in integral form, $\iiint * J = \iiint d * F$, are very apt. But it is important to explain quark and gluon confinement without any backpressure or other ad-hoc contrivances, and also explain why the nuclear interaction is mediated by mesons. That will be the subject of these next two sections.

Let's therefore start to use the language of differential forms to examine issues pertaining to confinement, which helps to establish our "candidate" baryons and mesons as true, physical baryons and mesons. For the field strength $F^{\mu\nu} = \partial^\mu G^\nu - \partial^\nu G^\mu - i\left[G^\mu, G^\nu\right]$, we multiply through by $dx_\mu dx_\nu$, and use the differential forms $G = G^\mu dx_\mu$, $F = F^{\mu\nu} dx_\mu dx_\nu$, $G^2 = \left[G^\mu, G^\nu\right] dx_\mu dx_\nu$, and $dG = \left(\partial^\mu G^\nu - \partial^\nu G^\mu\right) dx_\mu dx_\nu$, in a well-known fashion, to compact this to (see [12], Chapter (4.5)):

$$F = dG - iG^2 . \tag{5.1}$$

For $P^{\sigma\mu\nu}$ we use the magnetic three-form $P = P^{\sigma\mu\nu} dx_\sigma dx_\mu dx_\nu$, as well as $dF = \left(\partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}\right) dx_\sigma dx_\mu dx_\nu$ and $dG^2 = \left(\partial^\sigma \left[G^\mu, G^\nu\right] + \partial^\mu \left[G^\nu, G^\sigma\right] + \partial^\nu \left[G^\sigma, G^\mu\right]\right) dx_\sigma dx_\mu dx_\nu$ to multiply $P^{\sigma\mu\nu} = \partial^\sigma F^{\mu\nu} + \partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu}$ through by $dx_\sigma dx_\mu dx_\nu$ and then express this in the compacted form:

$$P = dF = d\left(dG - iG^2\right) = -idG^2 . \tag{5.2}$$

This includes the well-known application of $dd = 0$: the exterior derivative of an exterior derivative is zero. This is what made the QED magnetic charge vanish back in (2.1) and (2.2), while $-idG^2$ compacts the residual Yang-Mills terms of (2.5).

Similarly, the chromoelectric charge equation is:

$$*J = d*F = d*\left(dG - iG^2\right). \tag{5.3}$$

Now, we apply Gauss' law to (5.3), to write:

$$\iiint *J = \iiint d*F = \iiint d*\left(dG - iG^2\right) = \oiint *F = \oiint *\left(dG - iG^2\right). \tag{5.4}$$

and most importantly, to (5.2) to write:

$$\iiint P = \iiint dF = \iiint d\left(dG - iG^2\right) = -i\iiint dG^2 = \oiint F = \oiint dG - i\oiint G^2 = -i\oiint G^2 . \tag{5.5}$$

These are the "Maxwell's equations" in integral form for "classical," i.e., high action $S >> \hbar$ <u>chromodynamics</u>, (and indeed, for any "classical" Yang-Mills theory) and they mirror the usual Maxwell equations of electrodynamics:

$$\iiint *J = \iiint d*F = \oiint *F = \oiint *dA . \tag{5.6}$$

and

$$\iiint P = \iiint dF = \iiint ddG = \oiint F = \oiint dA = 0 . \tag{5.7}$$

In (5.5), $\iiint P$ describes a three dimensional volume which contains the three-fermion / antifermion object $P^{\sigma\mu\nu}$ of (4.24) which is our candidate baryon. But while Maxwell's (5.7), particularly $\oiint F = 0$, tells us that nothing flows out of a volume which contains a magnetic charge, equation (5.5) for Yang-Mills theory says something very different. The crux of (5.5) is the part that reads:

$$\iiint P = \oiint F = -i\oiint G^2 . \tag{5.8}$$

This says that across any closed two-dimensional surface surrounding a three-dimensional volume which contains a magnetic charge P as developed in (4.24), <u>there is a net field flux</u>, and it is a net flux $-i\oiint G^2$ of $G^2 = \left[G^\mu, G^\nu\right] dx_\mu dx_\nu$ objects. But what are these objects? From (4.25), we learn that these $\left[G^\mu, G^\nu\right]$ objects contain fermion and antifermion wavefunctions $\left[G^{\mu},G^{\nu}\right]=-\dfrac{\overline{\psi}\gamma^{\mu}\gamma^{\nu}\psi+\overline{\psi}_{C}\gamma^{\mu}\gamma^{\nu}\psi_{C}}{"\,p_{(\beta)}-m_{(\beta)}\,"}+\ldots$, and so appear to be mesons! So, we have our first glimpse of why mesons are enabled to flow across a closed surface surrounding a baryon.

What does <u>not</u> flow across this surface are the gauge bosons (gluons). This confinement of gluons and the "colorless" nature of nuclear interactions are highlighted by contrasting the final two terms of (5.5), from which one may then deduce the companion equation identical to the "no-magnetic monopoles" equation $\oiint dA=0$ of QED:

$$\oiint dG=0\,. \tag{5.9}$$

Therefore, in integral form, we find that $\oiint F$ is invariant under the local gauge-like transformation

$$\oiint F\rightarrow\oiint F'=\oiint\left(F-dG\right)=\oiint F\,. \tag{5.10}$$

Expanded in terms of the field density tensor, this transformation is $F^{\mu\nu}\rightarrow F^{\mu\nu}{}'=F^{\mu\nu}-\partial^{[\nu}G^{\mu]}$. This means that the flow of the field strength $\oiint F=-i\oiint G^{2}$ (in the form of mesons) across a two dimensional surface is invariant under the local gauge-like transformation $F^{\mu\nu}\rightarrow F^{\mu\nu}{}'=F^{\mu\nu}-\partial^{[\nu}G^{\mu]}$. Now, we know that the invariance of the QED Lagrangian under the similar transformation $A^{\mu}\rightarrow A^{\mu}{}'=A^{\mu}+\partial^{\mu}\Lambda$ means that the gauge parameter Λ is not a physical observable. Similarly, the invariance of the gravitational Lagrangian under $g^{\mu\nu}\rightarrow g^{\mu\nu}{}'=g^{\mu\nu}+\partial^{\{\mu}\Lambda^{\nu\}}$ means that the gauge vector Λ^{ν} is not a physical observable (and we know Λ^{ν} is in fact connected merely with a coordinate transformation $x^{\mu}\rightarrow x'^{\mu}=x^{\mu}-\Lambda^{\mu}(x^{\nu})$). In this case, the invariance of $\oiint F$ under the transformation $F^{\mu\nu}\rightarrow F^{\mu\nu}{}'=F^{\mu\nu}-\partial^{[\nu}G^{\mu]}$ similarly tells us that the gauge field G^{μ} is not an observable over the surface through which the meson field $\oiint F=-i\oiint G^{2}$ is flowing. But G^{μ} is simply the gauge field, that is, the gluon candidate. So, simply put: <u>the gauge fields G^{μ} (gluon candidates) are not observables across any closed surface surrounding a baryon</u>. Whatever goes on inside a baryon, the nuclear interaction is colorless and gluons remain confined inside the baryon! Only mesons cross this boundary.

Taking this a step further, we see that the origins of confinement in fact lie in the 140-year old mystery as to why there are no magnetic monopoles in Abelian gauge theory. In differential forms language, the statement of this is $ddG=0$. But in integral form, that becomes $\oiint dG=0$, equation (5.9). And, it is precisely this same "zero" which renders $\oiint F\rightarrow\oiint F'=\oiint F$ invariant under $F^{\mu\nu}\rightarrow F^{\mu\nu}{}'=F^{\mu\nu}-\partial^{[\nu}G^{\mu]}$ in (5.10). So the physical observation that there are no magnetic monopoles in Abelian gauge theory becomes translated into a symmetry condition in non-Abelian gauge theory that gauge boson flow is not an observable over the surface of a magnetic charge. Again: In Abelian gauge theory there are no magnetic monopoles. In non-Abelian theory, this Abelian absence of magnetic monopoles translates into there being no flow of gauge bosons (e.g., gluons) across any closed surface surrounding a Yang-Mills magnetic monopole. Consequently, <u>*the absence of Abelian magnetic monopoles is fundamentally, organically equivalent to the absence of gluon flux, hence color, across surfaces surrounding non-Abelian chromo-magnetic monopoles.*</u> And, because this is turn originates in $dd=0$, we see that this confinement is geometrically mandated. Had nothing else been developed in this paper, this alone, would make Yang-Mills magnetic monopoles extremely compelling baryon candidates! The very same "zero" which in Abelian gauge theory says that there are no magnetic monopoles, in non-Abelian gauge theory says that there is no observable flux of Yang-Mills gauge fields across a closed surface surrounding a Yang-Mills magnetic monopole. We do not find a free gluon (hence free color charge) in Yang-Mills theory any more than we find an Abelian magnetic monopole in electrodynamics, for identical geometric reasons. And as Close makes clear in [9] at 426: "quark confinement arises out of colour confinement . . . a boundary condition that confines the coloured gluons has, by Gauss, confined the coloured quarks." Here, unlike in the MIT Bag Model, the boundary conditions are not *ad hoc*. They are naturally endemic to magnetic monopoles.

Let us now examine all of this more specifically, in terms of the baryon candidate we earlier derived in (4.24).

6. Confinement Part II: Quark Confinement and Baryon Flux in a Form Analogous to Maxwell's Equations

We now wish to explicitly examine quark confinement and meson flux by applying (5.5) in the form $\iiint P = -i\iiint dG^2$ to the results developed here. This equation is more explicitly written as:

$$\iiint P^{\sigma\mu\nu} dx_\sigma dx_\mu dx_\nu = -i\iiint\left(\partial^\sigma\left[G^\mu, G^\nu\right] + \partial^\mu\left[G^\nu, G^\sigma\right] + \partial^\nu\left[G^\sigma, G^\mu\right]\right) dx_\sigma dx_\mu dx_\nu \,. \tag{6.1}$$

Now, although we have designated the expression on the right hand side of (4.24) as $P^{\sigma\mu\nu}$, the manner in which this term was developed in section 4 should make clear that this is really equal to the cyclic expression $-i\left(\partial^\sigma\left[G^\mu, G^\nu\right] + \partial^\mu\left[G^\nu, G^\sigma\right] + \partial^\nu\left[G^\sigma, G^\mu\right]\right)$. We now wish to find a different $P^{\sigma\mu\nu}$ to use on the left hand side of (6.1) so that (6.1) is not merely an identity restatement of Gauss' law for integration.

To achieve this objective, we will wish to obtain then use the first rank dual $*P^\mu = \frac{1}{3!}\varepsilon^{\mu\nu\sigma\tau} P_{\nu\sigma\tau}$ of the expression for $P^{\sigma\mu\nu}$ in (4.24) which eliminates the derivatives ∂^σ that appear in the right hand side of (6.1), and which allows us to use (6.1) somewhat analogously to how we use Maxwell's equation $J^\nu = \partial_\mu F^{\mu\nu}$ of (2.1) with Gaussian integration. Starting with (4.24) we may use $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$ and $\varepsilon^{\alpha\sigma\mu\nu} g_{\mu\nu} = 0$ and some renaming of summation indexes to specify this dual as:

$$*P^\alpha = \frac{1}{3!}\varepsilon^{\alpha\sigma\mu\nu} P_{\sigma\mu\nu} = -\frac{1}{3!}\varepsilon^{\alpha\sigma\mu\nu}\partial_\sigma\left(\sum_{n=1,2,3}\frac{\overline{\psi}_n\sigma_{\mu\nu}\psi_n + \overline{\psi}_{Cn}\sigma_{\mu\nu}\psi_{Cn}}{"\not{p}_n - m_n"} + \ldots\right), \tag{6.2}$$

where we now use the labels "1, 2, 3" to label the three fermion masses and momenta, rather than β, δ, ζ. Similarly, we label the associated fermion wavefunctions themselves with "1, 2, 3," and to save space, we include a summation $\sum_{n=1,2,3}$ over the three fermions which naturally subsist in $P_{\sigma\mu\nu}$.

The calculation of $*P^\alpha$ is rather involved, so we do not show the derivation explicitly in this paper. But this calculation makes use of the following ingredients: a) expressing the relationship $1 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5$ among all of the Dirac gamma matrices as $\sigma^{\alpha\sigma}\gamma^5 = -\frac{1}{2}\varepsilon^{\alpha\sigma\mu\nu}\sigma_{\mu\nu}$; b) writing the wavefunctions in (6.2) more generally in the chiral form $\psi_n = \left(c_{Vn} - c_{An}\gamma^5\right)\psi_n$ and $\overline{\psi}_n = \overline{\psi}_n\left(c_{Vn} + c_{An}\gamma^5\right)$ and similarly for the conjugate wavefunctions; c) making use of the Dirac spinors $\psi_n \equiv q_n(p)e^{-ip_n{}^\alpha x_\alpha}$ and $\overline{\psi}_n \equiv \overline{q}_n(p)e^{ip_n'{}^\alpha x_\alpha}$ and similarly for conjugates; d) as a central step, making use of the Gordon decomposition in the form of: (see, e.g., [10] at 343-345)

$$\overline{q}_n i\sigma^{\alpha\sigma}\left(p' - p\right)_\sigma q_n = \frac{2m_n}{g_n}\overline{q}_n\left(2\gamma^\alpha - \frac{1}{m_n}\left(p' + p\right)^\alpha{}_n\right)q_n \tag{6.3}$$

to *recompose* a vector current $\overline{q}_n\gamma^\alpha q_n$ and also shed the derivatives ∂^σ from (6.2) (which derivatives become represented in $\left(p' - p\right)_\sigma$), where g_n are the gyromagnetic g-factors of each fermion (these are equal to 2 for a structureless, non-interacting point particle such as the Dirac electron) and m_n are the fermion masses; e) reducing and solving to determine that $c_{Vn} = \pm c_{An} = \frac{1}{2}$; f) applying these two solutions to the overall set of equations developed;

and g) forming linear combinations of these two solutions. The result of all of this is that:

$$*P^{\alpha} = \tfrac{1}{3!}\varepsilon^{\alpha\sigma\mu\nu}P_{\sigma\mu\nu} = \frac{4}{3}\left(\sum\nolimits_{n=1,2,3}\frac{m_n}{g_n}\frac{\overline{\psi}_n\gamma^{\alpha}\psi_n + \overline{\psi}_{Cn}\gamma^{\alpha}\psi_{Cn}}{"\not{p}_n - m_n"} + \ldots\right). \tag{6.4}$$

Then, from (6.4), we form the desired alternative expression for $P^{\sigma\mu\nu}$, which is:

$$P^{\sigma\mu\nu} = \varepsilon^{\alpha\sigma\mu\nu} * P_{\alpha} = \frac{4}{3}\varepsilon^{\alpha\sigma\mu\nu}\left(\sum\nolimits_{n=1,2,3}\frac{m_n}{g_n}\frac{\overline{\psi}_n\gamma_{\alpha}\psi_n + \overline{\psi}_{Cn}\gamma_{\alpha}\psi_{Cn}}{"\not{p}_n - m_n"} + \ldots\right). \tag{6.5}$$

We then use (6.5) in the left hand side of (6.1), and (4.24) in the right hand side, thus writing:

$$\boxed{\begin{aligned}&\iiint P = \iiint dF = \iiint d\left(dG - iG^2\right) = -i\iiint dG^2 = \oiint F = \oiint dG - i\oiint G^2 = -i\oiint G^2\\ &= \iiint \frac{4}{3}\varepsilon^{\alpha\sigma\mu\nu}\left(\sum\nolimits_{n=1,2,3}\frac{m_n}{g_n}\frac{\overline{\psi}_n\gamma_{\alpha}\psi_n + \overline{\psi}_{Cn}\gamma_{\alpha}\psi_{Cn}}{"\not{p}_n - m_n"} + \ldots\right)dx_{\sigma}dx_{\mu}dx_{\nu}\\ &= \iiint \begin{bmatrix} -\partial^{\sigma}\left(\dfrac{\overline{\psi}_1\sigma^{\mu\nu}\psi_1 + \overline{\psi}_{C1}\sigma^{\mu\nu}\psi_{C1}}{"\not{p}_1 - m_1"} + \ldots\right)\\ -\partial^{\mu}\left(\dfrac{\overline{\psi}_2\sigma^{\nu\sigma}\psi_2 + \overline{\psi}_{C2}\sigma^{\nu\sigma}\psi_{C2}}{"\not{p}_2 - m_2"} + \ldots\right)\\ -\partial^{\nu}\left(\dfrac{\overline{\psi}_3\sigma^{\sigma\mu}\psi_3 + \overline{\psi}_{C3}\sigma^{\sigma\mu}\gamma^{\mu}\psi_{C3}}{"\not{p}_3 - m_3"} + \ldots\right)\end{bmatrix} dx_{\sigma}dx_{\mu}dx_{\nu}\end{aligned}}. \tag{6.6}$$

This equation, which also displays (5.5) (which in turn embeds gluon confinement via $\oiint dG = 0$, see (5.9) and (5.10)), summarizes the entire thesis of this paper in a nutshell. In the above, we have again used $\gamma^{\mu}\gamma^{\nu} = g^{\mu\nu} - i\sigma^{\mu\nu}$ and the "1, 2, 3" labels for the fermions, together with the fact that $g^{\mu\nu}dx_{\mu}dx_{\nu} = 0$ given the antisymmetric nature of $\{dx_{\mu}, dx_{\nu}\} = 0$ in differential forms. The numerator of the first term contains the meson-like four current density $J_{\alpha n} + J_{\alpha Cn} = \overline{\psi}_n\gamma_{\alpha}\psi_n + \overline{\psi}_{Cn}\gamma_{\alpha}\psi_{Cn}$. The numerators of the terms after the equal sign contain meson-like magnetization and polarization bivectors: (see, e.g. [16])

$$-\overline{\psi}\sigma^{\mu\nu}\psi - \overline{\psi}_C\sigma^{\mu\nu}\psi_C + \ldots = \begin{pmatrix} 0 & -P_x - P_{xC} & -P_y - P_{yC} & -P_z - P_{zC}\\ P_x + P_{xC} & 0 & -M_z - M_{zC} & M_y + M_{yC}\\ P_y + P_{yC} & M_z + M_{zC} & 0 & -M_x - M_{xC}\\ P_z + P_{zC} & -M_y - M_{yC} & M_x + M_{xC} & 0\end{pmatrix} + \ldots, \tag{6.7}$$

Thus, in (6.6), the magnetization three-vector $\mathbf{M} + \mathbf{M}_C$ takes the place of the electric field vector $\mathbf{E}$ in Maxwell's theory while the polarization three-vector $\mathbf{P} + \mathbf{P}_C$ takes the place of the magnetic field vector $\mathbf{B}$. And in general, $\overline{\psi}\sigma^{\mu\nu}\psi + \overline{\psi}_C\sigma^{\mu\nu}\psi_C$ takes the place of the dual field strength tensor $*F^{\mu\nu}$. Keep in mind that the SU(N) group generators are embedded though not shown explicitly in the foregoing, see the development in section 4.

Using $g^{\mu\nu} = \eta^{\mu\nu}$ to raise and lower indexes, as well as $J^0 + J^0{}_C = \rho + \rho_C$ to represent the meson charge density, and also using (6.7), the *time component* equation in (6.6) is seen to be as follows:

$$\iiint\left(\sum_{n=1,2,3}\frac{\nabla\cdot(\mathbf{M}_n+\mathbf{M}_{Cn})}{"\not{p}_n-m_n"}+\dots\right)dV=\iiint\frac{4}{3}\left(\sum_{n=1,2,3}\frac{m_n}{g_n}\frac{\rho_n+\rho_{Cn}}{"\not{p}_n-m_n"}+\dots\right)dV, \tag{6.8}$$

where $dV=dx_1dx_2dx_3$. The terms inside the integral should be compared to the analogous Maxwell equation $\nabla\cdot\mathbf{E}=\rho$, which is Gauss' law for electricity, in which the $\mathbf{E}\to\mathbf{M}$ correspondence is apparent. In integral form, applying Gauss' law, (6.8) becomes:

$$\boxed{\oiint\left(\sum_{n=1,2,3}\frac{\mathbf{M}_n+\mathbf{M}_{Cn}}{"\not{p}_n-m_n"}+\dots\right)\cdot\mathbf{dA}=\frac{4}{3}\sum_{n=1,2,3}\frac{m_n}{g_n}\frac{Q_n+Q_{Cn}}{"\not{p}_n-m_n"}+\dots}, \tag{6.9}$$

where we have defined the total meson (quark plus anti-quark) charge enclosed within the volume as $Q_n+Q_{Cn}=\iiint(\rho_n+\rho_{Cn})dV$. Equation (6.9) should be contrasted with Maxwell's $\oiint\mathbf{E}\cdot\mathbf{dA}=Q$. In electrodynamics, an electric field flows across the surface in proportion to the enclosed electric charge. Here, a *magnetization field* flows across the surface in proportion to the enclosed chromoelectric charge and, due to its $\mathbf{M}_n+\mathbf{M}_{Cn}$ character combining a field with the conjugate field, this may also be thought of as a meson field which is classically responsible for nuclear interactions in the same manner that the electric field is classically responsible for electrodynamic interactions. In sum: the total flow of the meson magnetization field $\mathbf{M}_n+\mathbf{M}_{Cn}$ across a closed surface $\mathbf{dA}$ is proportional to the total meson-like quark charge Q_n+Q_{Cn} contained within the volume enclosed by that surface. However, individual fermion magnetizations do not flow across this surface by themselves, but only in particle / antiparticle pairs.

Turing again to (6.6), and again using (6.7) and $g^{\mu\nu}=\eta^{\mu\nu}$, we now extract the *space components* equation:

$$\begin{aligned}&\iiint\left(\sum_{n=1,2,3}\nabla\times\frac{(\mathbf{P}_n+\mathbf{P}_{Cn})}{"\not{p}_n-m_n"}+\dots\right)dx_0\cdot\mathbf{dA}\\&=-\iiint\frac{4}{3}\left(\sum_{n=1,2,3}\frac{m_n}{g_n}\frac{\mathbf{J}_n+\mathbf{J}_{Cn}}{"\not{p}_n-m_n"}+\dots\right)dx_0\cdot\mathbf{dA}+\iiint\left(\sum_{n=1,2,3}\frac{\partial}{\partial t}\frac{(\mathbf{M}_n+\mathbf{M}_{Cn})}{"\not{p}_n-m_n"}+\dots\right)dx_0\cdot\mathbf{dA}\end{aligned}, \tag{6.10}$$

where we also employ $\mathbf{J}+\mathbf{J}_C=\overline{\psi}_n\boldsymbol{\gamma}\psi_n+\overline{\psi}_{nC}\boldsymbol{\gamma}\,\psi_{nC}$. The terms inside the integral should be compared to the analogous Maxwell equation $\nabla\times\mathbf{B}=\mathbf{J}+\partial\mathbf{E}/\partial t$, which is Ampere's law. As noted after (6.7), we see the magnetization $\mathbf{M}$ behaving similarly to the electric field $\mathbf{E}$ and the polarization $\mathbf{P}$ behaving similarly to the magnetic field $\mathbf{B}$. Now converted into integral form using Gauss' law, (6.10) becomes:

$$\boxed{\begin{aligned}&\oint\left(\sum_{n=1,2,3}\frac{\mathbf{P}_n+\mathbf{P}_{Cn}}{"\not{p}_n-m_n"}+\dots\right)\cdot\mathbf{dL}\\&=-\frac{4}{3}\left(\sum_{n=1,2,3}\frac{m_n}{g_n}\frac{\mathbf{I}_n+\mathbf{I}_{Cn}}{"\not{p}_n-m_n"}+\dots\right)+\frac{\partial}{\partial t}\oiint\left(\sum_{n=1,2,3}\frac{\mathbf{M}_n+\mathbf{M}_{Cn}}{"\not{p}_n-m_n"}+\dots\right)\cdot\mathbf{dA}\end{aligned}}, \tag{6.11}$$

with the total current flow specified as $\mathbf{I}_n+\mathbf{I}_{Cn}=\iiint(\mathbf{J}_n+\mathbf{J}_{Cn})dt\cdot\mathbf{dA}$. This should be contrasted with Ampere's law in integral form $\oint\mathbf{B}\cdot\mathbf{dL}=\mathbf{I}+\frac{\partial}{\partial t}\oiint\mathbf{E}\cdot\mathbf{dA}$. In electrodynamics, for a time-independent electric field $\mathbf{E}$, the line integral $\mathbf{dL}$ of the magnetic field $\mathbf{B}$ around any closed loop is proportional to the total electric $\mathbf{I}$ current flowing through the loop. Here, for a time-independent meson magnetization field $\mathbf{M}_n + \mathbf{M}_{Cn}$, the line integral of the *meson polarization* field $\mathbf{P}_n + \mathbf{P}_{C_n}$ around any closed loop is proportional to the total chromoelectric *meson current* $\mathbf{I}_n + \mathbf{I}_{Cn}$ flowing through that loop. What we do not see anywhere, is a free flow of individual quarks. They always travel through any surface in pairs, and so as individuals, are confined.

7. Conclusion

At the end of section 5, we showed on the basis of formal symmetry considerations that the gauge fields G^{μ} are confined and do not flow across a closed surface surrounding P by virtue of the invariance of $\oiint F \to \oiint F' = \oiint (F - dG)$ under the gauge-like transformation $F^{\mu\nu} \to F^{\mu\nu}{}' = F^{\mu\nu} - \partial^{[\nu} G^{\mu]}$. In section 6 we showed that only mesons flows are permitted to flow in and out of *P*, and that quark currents never appear separately from conjugate (antiparticle) quark currents.

So in sum, all that is permitted to net flow across a closed two-dimensional surface are the quark / anti-quark objects we call mesons, and their associated magnetizations. Gluons, and individual quarks not paired with an antiquark, can never in isolation show a net flux over any closed surface. Interactions between baryons thus occur only via meson exchange. Because of the foregoing, not only does *P* resemble a baryon insofar as it naturally contains three fermions, but it also exhibits all of the hallmarks of a baryon in terms of quark and gluon confinement and meson flow. Because of the foregoing, we now promote the *P* to actual physical baryons, the G^2 to actual physical mesons, and the *J* to actual physical quark currents!

Because *P* is now a three-quark system, we must because of Fermi-Dirac statistics make certain that no two quarks in this system have the same quantum numbers. So now, for the first time, we formally may select the gauge group SU(3) as our Yang-Mills gauge group, set each of fermion wavefunctions to $\psi^{T}{}_{n} = \begin{pmatrix} R & G & B \end{pmatrix}_n$, and thereby enable each of the three quarks inside of *P* quark to occupy three distinct color eigenstates and so enforce a Fermi-Dirac exclusion principle. And in the process, we have answered the very first question we posed: "Why, theoretically, do there exist in nature, naturally-occurring sources, namely baryons, consisting of exactly three strongly-interacting fermion constituents which we call 'quarks'?" The answer: because the Yang-Mills magnetic monopoles – which are indeed baryons – naturally contain three quarks. Baryons do not contain three quarks because SU(3) is the QCD gauge group. The causal arrow is reversed: SU(3) is the QCD gauge group because each Yang-Mills magnetic monopole – now a baryon – contains three quarks and we need to enforce exclusion. And so, the horse is properly before the cart. And, for reasons developed to go from (4.13) to (4.14), we do not break any symmetries for this group, now formally $SU(3)_C$, but maintain the eight gauge bosons G^{μ} – now gluons – as massless.

Having fully developed the baryon and quarks and mesons according to (6.6), another point should now be made, which brings us back to the very beginning of this paper. Equation (6.6) which is the upshot of the thesis developed in this paper is no more and no less than the logical result of combining the two classical Maxwell equations $J^{\nu} = \partial_{\mu} F^{\mu\nu}$ and $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu}$ of (2.1) in the context of Yang-Mills gauge theory with the field strength $F^{\mu\nu} = \partial^{\mu} G^{\nu} - \partial^{\nu} G^{\mu} - i\left[G^{\mu}, G^{\nu}\right]$ of (2.3) for non-commuting gauge fields. (To be perfectly fair, we did also use the Gordon decomposition (6.3) which has its roots in Dirac's equation, so there is really a fourth fundamental equation involved as well, namely that named for Dirac.) Just find the inverse of Maxwell's charge equation $J^{\nu} = \partial_{\mu} F^{\mu\nu}$, plug it into $P^{\sigma\mu\nu} = \partial^{\sigma} F^{\mu\nu} + \partial^{\mu} F^{\nu\sigma} + \partial^{\nu} F^{\sigma\mu}$, do the calculations, and arrive at (6.6). *In short, (6.6) is what one obtains when Maxwell's two equations in the context of Yang-Mills theory are merged together into a single equation.* Think about this again: both of Maxwell's equations are embedded in (6.6), i.e., (6.6) is what one inexorably gets from joining together both of Maxwell's equations in Yang-Mills theory. No more, no less. That simple! For anyone who has ever wondered what Maxwell equations would look like if they were all one equation rather than two, (6.6) is the answer! Maxwell's equations, for non-commuting fields, when combined into one, are the classical equations of baryon nuclear physics!

One final, overarching point, which returns us to section 3. As made clear throughout, (6.6) is a classical equation, valid for high-action $S(\varphi) >> \hbar$. This means that (6.6) (and even the more general equation developed from

(4.3) with $\partial^\sigma G^\mu = i[k^\sigma, G^\mu] \neq 0$) will become inexact in the quantum arena. Does this mean that $P^{\sigma\mu\nu}$ will stop being a baryon? Of course not. No more than a classical current density J^μ stops being a current density in quantum electrodynamics. It merely means that we will be using different (quantum amplitude) equations, derived via path integration, in order to describe the behaviors of these baryons in the low-action arena. It merely means that the higher order terms will change from what we have seen here. But the lowest-order, baryon structural terms in (6.6) will always remain intact.

So to conclude: the long-sought and pursued and ever-elusive magnetic monopole, in Yang-Mills theory, is a baryon, and it exists everywhere and anywhere that there is matter in the universe, hiding in plain sight!

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References

[1] Jaffe, A., and Witten, E., *Quantum Yang Mills Theory*, Clay Mathematics Institute, (2000).

[2] t' Hooft, G., *On the future of quantum field theory for particle physics*, Nucl. Phys. B190 (1981) 455.

[3] Prasad, M. K., and Sommerfield, C. M., *Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*, Phys. Rev. Lett. 35, 760–762 (1975).

[4] Di Giacomo, A., *Confinement of color and geometry*, http://arxiv.org/abs/0910.5097v1, Presented at the X workshop on perturbative QCD, Paris 8-12 June 2009

[5] Witten, E., *Duality, Spacetime and Quantum Mechanics*, Physics Today, May 1997, pp. 28 to 33.

[6] Reinich, G.Y., *Electrodynamics in the General Relativity Theory*, Trans. Am. Math. Soc., Vol, **27**, pp. 106-136 (1925).

[7] Wheeler, J. A., *Geometrodynamics*, Academic Press, pp. 225-253 (1962).

[8] Misner, C. W., Thorne, K. S., and Wheeler, J. A., *Gravitation*, W. H. Freeman & Co. (1973).

[9] Close, F. E., *An Introduction to Quarks and Partons*, Academic Press (1979).

[10] Ryder, L., *Quantum Field Theory*, Cambridge (1996).

[11] Crewther, R. J., *Introduction to Quantum Field Theory*, http://lanl.arxiv.org/abs/hep-th/9505152 (1995)

[12] Zee, A., *Quantum Field Theory in a Nutshell*, Princeton (2003).

[13] Halzen, F., and Martin A. D., *Quarks and Leptons: An Introductory Course in Modern Particle Physics*, J. Wiley & Sons (1984).

[14] Chodos, A., Jaffe, R. L., Johnson, K., Thorn, C.B., Weisskopf, V. F., *New extended model of hadrons*, Phys. Rev. D 9, 3471 - 3495 (1974).

[15] Chodos, A., Jaffe, R. L., Johnson, K., Thorn, C.B., *Baryon structure in the bag theory*, Phys. Rev. D 10, 2599 - 2604 (1974).

[16] DeVries, H., *Understanding Relativistic Quantum Field Theory*, http://physics-quest.org/Book_Chapter_Dirac_Fields.pdf.