An algebraic approach to systems with dynamical constraints

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June 2012

Abstract

Constraints imposed directly on accelerations of the system leading to the relation of constants of motion with appropriate local projectors occurring in the derived equations are considered. In this way a generalization of the Noether's theorem is obtained and constraints are also considered in the phase space.

Key words:

Dynamical and canonical constraints, reaction forces, virtual work, projectors

Contents

1 Introduction

We consider equations describing discrete or continuous systems with constraints. If no constraints are present, we will assume that the unconstrained system is described by the 'field' equation:

$$L[\tilde{x};\varphi] + \lambda N[\tilde{x};\varphi] + G(\tilde{x}) = 0 \tag{1}$$

with the main linear functional L depending on the unknown 'field' (function) $\varphi(\tilde{x})$, which necessarily includes differential operations, for example

$$L[\tilde{x};\varphi] = \left(\Box + m^2\right)\varphi(\tilde{x})$$

the N a functional, **usually** nonlinear (although may also contain additional linear terms), depending on the field φ, for example

$$N[\tilde{x};\varphi] = \varphi^3(\tilde{x})$$

and the given function G usually describing external forces acting on the system. Here and further square brackets mean that a given quantity, except that it is a function, it is also a functional. For discrete systems, such as N material points, $\varphi = (q_1, ..., q_{3N})$ can be a 3N dimensional vector and $\tilde{x} = (t, i)$ besides the time t describes the component indexes $i = 1, ..., 3N$. In this case, we can choose $L[t, i;\varphi] = \ddot{q}_i(t)$. In general, the 'vector' $\tilde{x} \in \tilde{M}$ has time-'space' components describing 'points', components characterizing the field φ as its tensor type, and the time t. Usually, we will distiguish the time and 'space' components by writing $\tilde{x} = (t, \bar{x})$. We will assume that all components of the vector $\tilde{x}$ are discrete variables. In other words,$\tilde{M}$ is a set defined by a specific properties of the considered system, see also App.7.

We are in good company. Even the space can be described by means of the field $\varphi(\tilde{x})$.

The functionals L, N are also functions depending on the vectors $\tilde{x} \in \tilde{M}$. The set of functions dependent on the fixed φ will be denoted by $\tilde{F}_\varphi$.

As usual, we will assume that a freedom of the theory described by Eq.1 is such as the freedom of the theory descibed by the main linear part:

$$L[\tilde{x};\varphi] = 0 \tag{2}$$

It means that in both cases the same type of initial and boundary conditions can be used to get an unique solution.

We also assume, following an analogy with the classical mechanics, see also [1] and App.1, that the system represented by the field φ - subjects to the restrictions of the following type:

$$\int Q[\tilde{x},\tilde{y};\varphi]L[\tilde{y};\varphi]d\tilde{y} = f[\tilde{x};\varphi] \tag{3}$$

where $Q[\tilde{x},\tilde{y};\varphi]$ is a given projector (kernel of a projector $Q := Q^2$) acting in the linear space of functions $\tilde{F}_\varphi$ and $f[\tilde{x};\varphi]$ is a given function. They both are depending in the linear or nonlinear way on the field φ. The restrictions (3) together with additional assumption imposed on the 'reaction forces' $R[\tilde{x};\varphi]$, (7), are called here the ***dynamical constraints*** (DC). They can be **ideal** (IC) or ***non-ideal constraints*** (NIC).

In the paper we show how Eq.1 is changing in case of NIDC, Sec.2, and how Eq.3 can be interpreted, Sec.3. In Sec.3 we also show how all formulas and equations are changing in the case of ideal constraints, see also App.3.

In the paper the concept of virtual displacements, typical tool when discussing systems with constraints, is replace by the algebraic concepts such as the projection operators (projectors), see: [3, 4]. We hope that it gives a new look at the classic problems.

What I found interesting in the present study is a connection of the certain constants of motion with the presence of certain projection operators in the considered equations, see Eq.10. It's like combining constants of motion with a certain symmetry of considered equations resulting from the Noether's theorem. Equations with projectors as in Eq.10 or Eq.26 mean that certain changes of functionals describing these equations do not change the whole equations.We see in this actually a generalization of symmetries of the equations.

In the case of restrictions (3), Eq.1 has to be changed by

$$L[\tilde{x};\varphi] + \lambda N[\tilde{x};\varphi] + G(\tilde{x}) = R[\tilde{x};\varphi] \tag{4}$$

with temporarily unknown 'reaction force' R (a generalization of Lagrange's equations of the first kind). In addition, I believe that the emphasis placed here on Lagrange's equations of the first kind is an expression of a broader approach to the description of the nature including space, see [5], - the opposite of any kind of reductionist approache - inspite of this that they may be acceptable in certain cases, see all arguments behind of Lagrange's equations of the second kind.

For the systems with the constraints, we can look back in such a way that we want to modify the theory determined by measuring of local entities as the position of the various parts, taking into account certain global (non-local) entities, for example energy of the system. In this and only this sense, the presented approach to classical mechanics contains some elements of quantum mechanics.

In the case of economics system the local and global entities are important ingredients of many theories. In this case, Adam Smith's the invisible hand of the market would solve all the problems of capitalism if the constrains imposed by theory would be a result of the primary Eq.1. Otherwise, the global rules (constraints) can be used to modify the interaction (reaction forces) between the various actors in the market.

As in other papers, author is using integration sign even in the case of discrete variables.

2 The 'reaction force' R and a principle of virtual works surrogate (PVW(S)); non-ideal constraints

Introducing the complementary projector P:

$$P[\tilde{x},\tilde{y};\varphi]+Q[\tilde{x},\tilde{y};\varphi]=\delta(\tilde{x},\tilde{y}) \tag{5}$$

where δ is Kronecker or Dirac's delta, we can express the general solution to Eq.3 as follows:

$$L[\tilde{x};\varphi]=f[\tilde{x};\varphi]+g[\tilde{x};\varphi] \tag{6}$$

where $f[\varphi]=Q[\varphi]f[\varphi]\in Q\tilde{F}_{\varphi}$ and $g=Pg$ is an arbitrary function from $P\tilde{F}_{\varphi}$, see (29). Here and elsewhere, for example:

$$f[\varphi]=Q[\varphi]f[\varphi]\Leftrightarrow\int d\tilde{y}Q[\tilde{x},\tilde{y};\varphi]f[\tilde{y};\varphi]$$

Assuming that the 'reaction forces' are such that

$$\int d\tilde{y}P[\tilde{x},\tilde{y};\varphi]R[\tilde{y};\varphi]=0 \tag{7}$$

we get from Eq.4 that

$$\int d\tilde{y}P[\tilde{x},\tilde{y};\varphi]\left\{L[\tilde{y};\varphi]+\lambda N[\tilde{y};\varphi]+G(\tilde{y})\right\}=0 \tag{8}$$

see (29). From that, the arbitrary element g in the expression (6):

$$g[\tilde{x};\varphi]=-\int d\tilde{y}P[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi]+G(\tilde{y})\right\} \tag{9}$$

and the formula (6) can be described as this:

$$L[\tilde{x};\varphi]-f[\tilde{x};\varphi]+\int d\tilde{y}P[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi]+G(\tilde{y})\right\}=0 \tag{10}$$

Here, $f[\tilde{x};\varphi]\in Q[\varphi]\tilde{F}_{\varphi}$. Eq.10 substituts Eq.1 in the case of GCs (3), which are satisfied by any solution to Eq.10.

By comparison with Eq.4, the 'reaction force'

$$R[\tilde{x};\varphi] = f[\tilde{x};\varphi] + \int d\tilde{y} Q[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} \tag{11}$$

Following the analogy with classical mechanics one can say that Eq.7 resembles some surrogate of the **virtual work principle (VWP)** - a surrogate because the 'reaction forces', at the moment t, can not be perpendicular to the surface of the constraints (DC), Eq.10 resembles **Lagrange's equations of the first kind**, and Eq.11 is a formula for the **'reaction forces'** of DC (3). In this analogy, instead of the **virtual displacements**, we have used appropriate linear projectors depending on the field φ. The 'field' φ in the simplest case may represent the radius vector. But the main difference of presented approach to constraints and canonical approach lies in the fact that there are explicitly described rather acceleration restrictions caused by the presence of constraints than constraint surfaces. See also **[2].**

3 Constraints, constants of motion and projectors. Ideal constraints

We ask now when the restrictions (3) can be interpreted as contraints of the dynamical system (1)? To answer this question let us consider classical mechanics with the general form of constraints:

$$\sum a_{ij}(q,t)\dot{q}_j + g_i(q,t) = 0 \tag{12}$$

where $\dot{q}_j$ is the j-th component of the vector $\dot{q}$. Holonomic constraints can be differentiated once with respect to time to get Eq.12. Differentiating once more with respect to time, in both cases we get equations which, in the matrix-vector form, are:

$$B(q,t)\ddot{q} = b(\dot{q},q,t) \tag{13}$$

The matrix B in this equation has to be a singular. Otherwise, it would be a dynamic equation, which for given initial conditions would describe the problem in an unique way. If we assume that B is a right invertible matrix, then such a right inverse exists that

$$B(q,t)B_R^{-1}(q,t) = I \tag{14}$$

and

$$B_R^{-1}(q,t)B(q,t) = Q(q,t) \tag{15}$$

After multiplication of Eq.13 by the inverse $B_R^{-1}(q,t)$ we get analoge of Eq.3.

In fact, constraints equations obtained in the above way can have the following structure:

$$Q'B(q,t)\ddot{q} = b(\dot{q},q,t) \tag{16}$$

with projected right invertible or invertible operator B, which actually corresponds to a situation in which there are fewer constraints than degrees of freedom, see App.3. Then, the equivalent equation:

$$Q\ddot{q} \equiv B_R^{-1} Q' B(q,t)\ddot{q} = B_R^{-1} b(\dot{q},q,t) \equiv f \tag{17}$$

has the form (3) with projector $Q = B_R^{-1} Q' B(q,t)$, $L = \ddot{q}$ and the functional $f = B_R^{-1} b(\dot{q},q,t)$. Q indeed is a projector because: $B_R^{-1} Q' B(q,t) \cdot B_R^{-1} Q' B(q,t) = B_R^{-1} Q' B(q,t) \Longleftrightarrow Q^2 = Q$. See also App.3.

Multiplying Eq.3 with an operator depending on the field φ:

$$A[\varphi] \Longleftrightarrow A[\tilde{x},\tilde{y};\varphi] \tag{18}$$

we get equation:

$$A[\varphi]Q[\varphi]L[\varphi] = A[\varphi]f[\varphi] \tag{19}$$

where $A[\varphi]$ and $Q[\varphi]$ operate in the space of functions $\tilde{F}_\varphi \ni L[\varphi], f[\varphi]$. This equation is equivalent to Eq.3 if, for example, we assume that operator $A[\varphi]$ is a right invertible:

$$A[\varphi]A[\varphi]_R^{-1} = I \tag{20}$$

where I is the unit operator in space $\tilde{F}_\varphi$, and that

$$A_R^{-1}[\varphi]A[\varphi] = Q'' \supseteq Q \tag{21}$$

where Q'', Q are projectors.

In the case of ideal constraints in which the reaction forces $R_{ideal}[\tilde{x};\varphi]$ are perpendicular to the constraint surfaces and projectors $P_{ideal}[\tilde{x},\tilde{y};\varphi]$ projecting on the tangent surfaces at 'points' $\varphi(\tilde{x})$ are known, then we have, of course:

$$\int P_{ideal}[\tilde{x},\tilde{y};\varphi]R_{ideal}[\tilde{y};\varphi]d\tilde{y} = 0 \tag{22}$$

In this case all derived formulas above will not be changed if

$$Q_{ideal}Q = Q_{ideal} \tag{23}$$

and $P_{ideal} = I - Q_{ideal}$but then, of course, $Q \to Q_{ideal}$. If (23) is not satisfied, then, starting from the formula (7), we have changes: so that (8) is modified by

$$\int d\tilde{y}P_{ideal}[\tilde{x},\tilde{y};\varphi]\left\{L[\tilde{y};\varphi] + \lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} = 0 \tag{24}$$

(9) is substituted by:

$$P_{ideal}g[\tilde{x};\varphi] = -P_{ideal}f[\tilde{x};\varphi] - \int d\tilde{y}P_{ideal}[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} \tag{25}$$

and (10) is substituted by:

$$L[\tilde{x};\varphi] - Q_{ideal} f[\tilde{x};\varphi] + \int d\tilde{y} P_{ideal}[\tilde{x},\tilde{y};\varphi] \{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\} = Q_{ideal} g[\tilde{x};\varphi] \tag{26}$$

with arbitrary element $Q_{ideal} g[\tilde{x};\varphi]$. In this case it is also clear that 'reaction forces' are also not unique:

$$R[\tilde{x};\varphi] = Q_{ideal} f[\tilde{x};\varphi] + \int d\tilde{y} Q_{ideal}[\tilde{x},\tilde{y};\varphi] \{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\} + Q_{ideal} g[\tilde{x};\varphi] \tag{27}$$

if

$$Q_{ideal} g[\tilde{x};\varphi] \neq 0 \tag{28}$$

Compare with (11).

We hope however that even in the case of (28) we get an unique result, if the initial conditions are chosen in accordance with the used constraints.

Constants of motion principle ((CM)P)?

At this point we remind the obvious fact that the constraints in classical mechanics are the constants of motion (CM) of equations in which the reaction forces appear explicitly (e.g., in Lagrange equations of the first kind). Since, for the restrictions (3), the above regularity is not always satisfied - the *constats of motion principle* is not true - and we use the name of DC. We would like to point out that, for the same reason, DC considered are more general than the non-ideal constraints. If (3) corresponds to the CM, then, because the projector P occurring in the condition (7) generally does not project on virtual diplacements, the presented formalizm describes rather non-ideal constraints, see App.3.

In the situation encountered in astrophysics when the energy of the system changes for no apparent reason, which is expressed in phrases such as dark matter or energy, such information can be considered as the presence of DC which can be obtained by the time differentation of the changing in the time the integral of energy.

But the problem of determining reaction forces has to be solved with some additional principle or by the trial and error method. For example, the reaction forces can be determined by the requiring that for constraints which are CM of the original Eq.1 , disappeared.

4 Examples of linear dynamical constraints (LDC)

Let us collect the main results:

Eq.10 is

$$L[\tilde{x};\varphi] - f[\tilde{x};\varphi] + \int d\tilde{y} P[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} = 0$$

DC (3) are

$$f[\tilde{x};\varphi] = QL[\tilde{x};\varphi] \equiv \int Q[\tilde{x},\tilde{y};\varphi] L[\tilde{y};\varphi] d\tilde{y}$$

with fixed functionals L, f. P, Q - conjugate projectors (idempotent operators) satisfying Eq.5:

$$P[\tilde{x},\tilde{y};\varphi] + Q[\tilde{x},\tilde{y};\varphi] = \delta(\tilde{x},\tilde{y}) \Longleftrightarrow P + Q = I$$

Because

$$PQ = QP = 0,\ P = P^2,\ Q = Q^2 \tag{29}$$

we see that DC (3) result immediately from Eq.10.

Let us take DC (3) with

$$f[\tilde{x};\varphi] = \mu QL[\tilde{x};\varphi] \tag{30}$$

Hence and from Eq.10

$$(I - \mu Q)L[\tilde{x};\varphi] + \int d\tilde{y} P[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} = 0 \tag{31}$$

We can tell immediately that, at $\mu = 1$, DC (30), (3) lead only to weakning of the original Eq.1.What happens, for $\mu \neq 1$? In this case, by inverting the operator $I - \mu Q$, we get the following equation:

$$\begin{aligned} L[\tilde{x};\varphi] + (I - \mu Q)^{-1} \textstyle\int d\tilde{y} P[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} = \\ L[\tilde{x};\varphi] + \textstyle\int d\tilde{y} P[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} = 0 \end{aligned} \tag{32}$$

One can understand this result if we take into account that now $\int Q[\tilde{x},\tilde{y};\varphi]L[\tilde{y};\varphi]d\tilde{y} = 0$.

Another example of the linear DC (3) is given by:

$$f[\tilde{x};\varphi] = -QM\varphi(\tilde{x}) \equiv -\int d\tilde{y} d\tilde{z} Q(\tilde{x},\tilde{y}) M(\tilde{y},\tilde{z}) \varphi(\tilde{z}) \tag{33}$$

where M is a given constant matrix and Q a projector, both independent of φ. Now, Eq.10 is given by

$$L_M[\tilde{x};\varphi] + \int d\tilde{y} P(\tilde{x},\tilde{y})\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} = 0 \tag{34}$$

with the linear functional: $L_M[\tilde{x};\varphi] = L[\tilde{x};\varphi] + QM\varphi(\tilde{x})$. The term $QM\varphi$ can describe parameters which do not appear in the first term because, e.g., of symmetry in a certain area of considered equations.

One can finally say that **any knowledge about the main linear term of Eq.1,** expressed in the form of (3), allows us to change this equation to the form of Eq.10 if the analoge of the virtual work principle is assumed. In this way, relying more on observation than on the proliferation of some ideas, you can try to understand some phenomena.

5 Examples of nonlinear dynamical constraints (NDC)

We assume

$$f[\tilde{x};\varphi] = \mu QN[\tilde{x};\varphi] \Longleftrightarrow f[\tilde{x};\varphi] = \mu \int Q[\tilde{x},\tilde{y};\varphi]N[\tilde{y};\varphi]d\tilde{y} \tag{35}$$

This effectively means that, for $G = 0$, we modify the nonlinear part of Eq.1. In this case, Eq.10 takes the form:

$$L[\tilde{x};\varphi] - \mu \int Q[\tilde{x},\tilde{y};\varphi]N[\tilde{y};\varphi]d\tilde{y} + \int d\tilde{y}P[\tilde{x},\tilde{y};\varphi]\left\{\lambda N[\tilde{y};\varphi] + G(\tilde{y})\right\} = 0 \tag{36}$$

or in short as

$$L + (\lambda P - \mu Q)\, N + PG = 0 \tag{37}$$

Hence, the equivalent,

$$(\lambda P - \mu Q)^{-1} (L + PG) + N = 0 \tag{38}$$

where $(\lambda P - \mu Q)^{-1} = \lambda^{-1}P - \mu^{-1}Q$, see (29). In other words, all the modification of the theory can be transferred to linear terms, although without non-linear terms the above modification disappears!

In all these examples one can treat some constant of motions as constatraints related to a kind of material surfaces and some, as energy, as purely dynamical quantities, see also App.1.

These examples also show how 'beautiful' theory can be changed when the facts clearly demonstrate its partial falsity:(

6 App.1 about analogy with classical mechanics and the essence of the constraints

In classical mechanics the main quantity around which everything revolves is an accerelation of objects either extended or point like particles. The accelerations in the dynamical equations appear in the linear way. Moreover, if the constrains are proper times differentiated with respect to time (once or twice), then accelerations also appear in the linear way, see [1]. Such quantities, which describe changes, or changes of changes as in the case of acceleration, also appear in a linear way in the case of 'physical' fields describing extended systems. They are responsible for the additional conditions as the initial and boundary conditions, which must be taken into account to get an unique solution to the considered equations.

When we look at constraints as constants of motion, the question naturally arises, what is the difference? The answer is as follows: the equations describing systems with constraints, as, for instance Eq.(10), is obtained by a suitably modified equations that describe systems without constraints, as in this case the Eq. (1). Modifications to be made can be done in many ways. In mechanics, this is done by introducing the constraint forces also called the reaction forces. In the case of the ideal constraints, however, which generalization we considered in the submitted work, unambiguous results are obtained due to the principle of virtual work. In the paper, this principle is replaced, or generalized, by Eq.7.

In practice, constants of motion describing constraints can be so easily obtained that to find them we do not even need to know the explicit form of Eq.10! This is it if, for instance, Eq.12 is enough to find appropriate constants of motion.

7 App.2 about the canonical and dynamical constraints ((CC) and(DC))

By CC we understand mathematical or physical restrictions which descriptions does not require 'accelerations' or their analogues. In classical mechanics they are called holonomic and nonholonomic constraints. From their definitions results that for systems with CC the initial and boundary conditions can not be arbitrary. It is result of fact that CC eliminate some number of degrees of freedom like in the case of pendulum or incompressible liquid.

Main difference with DC is such that CC are automatically realized by there equations: 'surfaces' which realizes such constraints. **This is not the case of DC which are realized by the extra forces calculated with the help of dynamical equations!**

8 App.3 An example of CC described in the form of DC (see Eq.3). Spherical constraints

Let us assume that we have the following spherical constraint:

$$\int d\bar{y}\varphi^2(\bar{y},t) = constant \tag{39}$$

Hence,

$$\int d\bar{y}\varphi(\bar{y},t)\dot{\varphi}(\bar{y},t) = 0 \tag{40}$$

and

$$\int d\bar{y}\varphi(\bar{y},t)\ddot{\varphi}(\bar{y},t) + \int d\bar{y}\dot{\varphi}(\bar{y},t)\dot{\varphi}(\bar{y},t) = 0 \tag{41}$$

In this case, to get an analoge of formula (13), or rather (16, we can choose:

$$B[\bar{x},\bar{y},t;\varphi] = \delta(\bar{x}-\bar{y})\varphi(\bar{x},t) \tag{42}$$

where B is a nonsingular operator at least for t for which $\varphi \neq 0$:

$$B_R^{-1}[\bar{y},\bar{z},t;\varphi] = B^{-1}[\bar{y},\bar{z},t;\varphi] = \delta(\bar{y}-\bar{z})\frac{1}{\varphi(\bar{y},t)} \tag{43}$$

$$Q'(\bar{x},\bar{y}) = \frac{1}{V}\int d\bar{x}\delta(\bar{x}-\bar{y}) = \frac{1}{V} \tag{44}$$

and

$$b[t;\varphi] = -\int d\bar{y}\dot{\varphi}(\bar{y},t)\dot{\varphi}(\bar{y},t) \tag{45}$$

Here V denotes the volume of an integration region, $\bar{x} \in V$. Of course, (44) is a projector, which action on a function is reduced to integration and multiplication by the factor $1/V$ to get in result a constant. Now, we can use the formula (16) and (17) to describe CC (39) in the form of Eq.3) of DC with

$$Q[\bar{x},\bar{y},t;\varphi] = B_R^{-1}Q'B[\bar{x},\bar{y},t;\varphi] = \int B_R^{-1}[\bar{x},\bar{z},t;\varphi]Q'(\bar{z},\bar{w})B[\bar{w},\bar{y},t;\varphi] =$$
$$\int d\bar{z}d\bar{w}\delta(\bar{x}-\bar{z})\frac{1}{\varphi(\bar{x},t)}\frac{1}{V}\int d\bar{z}'\delta(\bar{z}'-\bar{w})\delta(\bar{w}-\bar{y})\varphi(\bar{w},t) = \frac{1}{V}\frac{\varphi(\bar{y},t)}{\varphi(\bar{x},t)} \tag{46}$$

Hence, in the DC (3),

$$f[\bar{x},t;\varphi] = -\frac{1}{\varphi(\bar{x},t)}\int d\bar{y}\dot{\varphi}(\bar{y},t)\dot{\varphi}(\bar{y},t) \tag{47}$$

It is worth noting here that Q is a projector, but it is a symmetric projector only for all field variables equal to each other:

$$\varphi(\bar{x},t) = \varphi(\bar{y},t),\ for\ \bar{x},\bar{y} \in V \tag{48}$$

In other cases, (3) and (7), with (46), can describe the **non-ideal constraints** described by a **surrogate of virtual work principle**:

$$PR[\bar{x},t;\varphi] = R[\bar{x},t;\varphi] - \frac{1}{V\varphi(\bar{x},t)}\int d\bar{y}\varphi(\bar{y},t)R[\bar{y},t;\varphi] = 0 \tag{49}$$

where the projector P was chosen as:

$$P[\bar{x}, \bar{y}, t; \varphi] = \delta(\bar{x} - \bar{y}) - Q[\bar{x}, \bar{y}, t; \varphi] = \delta(\bar{x} - \bar{y}) - \frac{1}{V} \frac{\varphi(\bar{y}, t)}{\varphi(\bar{x}, t)} \tag{50}$$

This projector reflects circular symmetry in the case of non-ideal costraints (39).

From (49),

$$R[\bar{y}, t; \varphi] = \frac{G[t; \varphi]}{\varphi(\bar{y}, t)} \tag{51}$$

where a functional G does not depend on the variable $\bar{y}$. The values of 'field' in the denominator should not necessarily worry us, because the infinity of the expression $1/\varphi(\bar{y}, t)$, for $t \to t'$, can be simultaneosly neutralized by $G \to 0$.

Once more, for ideal constraints, we should have:

$$R[\bar{y}, t; \varphi] = H[\varphi]\varphi(\bar{y}, t) \tag{52}$$

with a scalar $H[\varphi]$. From (51) we get

$$\varphi(\bar{y}, t)^2 = \frac{G[\varphi]}{H[\varphi]}$$

but this would mean that φ does not depend on $\bar{y}$ in a continuous way. It also means that the conditions (49) and (3) can not describe ideal constraints.

Spherical constraints describe the simplest nonlinear, holonomic constraints in physics. They contain the symmetry of the circle, which throughout human history has been synonymous with - excellence. So would not be strange if they would be found in some basic field theory describing the Universe. The sphere in the configuration space of such a system as the universe allows for a situation in which the individual subsystems run on a sphere of smaller area, the favorite model in cosmology, as well as a situation reminding the pendulum-like model in which the kinetic energy takes maximum, and, there are moments in which the kinetic energy of subsystems completely disappear. In this way the cyclical evolution takes place.

In the case of ideal constraints (52) we have another projection operator than (50):

$$P_{ideal} = \frac{\dot{\varphi}(\bar{x}, t)\dot{\varphi}(\bar{y}, t)}{\int \dot{\varphi}(\bar{z}, t)\dot{\varphi}(\bar{z}, t) d\bar{z}} \tag{53}$$

In this case Eq.22 is satisfied, see (52) and (40).

9 App.4 about one-sided constraints (CC) in classical mechanics; short-range forces

On this subject I speak of the following reasons: First, in the Internet, I found the discussion of such constraints by means of advanced means or complicated cases including solid mechanics. Secondly, as previously discussed, I am focusing not on the elimination of redundant degrees of freedom, but on the forces that are doing it.

In the case of n material points, the one-sided constraints are characterized not by equations but by inequalities. Thus, in the case of holonomic constraints we hawe:

$$f_i(\vec{r}_1, ..., \vec{r}_n; t) \leq 0, \ for \ i = 1, ..., k < 3n$$

where $\vec{r}_i$ means the radius vector of the i-th particle. Inequalities mean a drastic loosening of restrictions: only if there is 'threat' of their failure, the system 'suffers' of reaction forces. Such situation can be described by short-range forces, whose centers satisfy the equations

$$f_i(\vec{r}_1, ..., \vec{r}_n; t) = 0, \ for \ i = 1, ..., k < 3n$$

Usually, the surfaces satisfying the above equations are calle the *walls*. Short-range reaction forces should be a priori chosen in such a way that an energy, which is available for individual particles is not enough to cross the walls. In this way we avoid tracking, when the particles are hiting in to the walls, nor the need for discontinuous changes in their momenta. Everything is encoded in the dynamical equations.

10 App.5 about projectors and one-sided invertible operators

A *right invertible operator* A is defined as an operator for which one can write the following equation:

$$AA_R^{-1} = I \tag{54}$$

with not uniquely chosen a right inverse operator A_R^{-1} and the unite operator I in a considered linear space. For a left invertible operator, we would have a similar definition, but the operator $A_R^{-1}$is substituted by an operator A_l^{-1} standing at the l.h.s. of the operator A:

$$A_l^{-1}A = I \tag{55}$$

Occurring here operators A_R^{-1}, A_l^{-1}satisfy the first two demands of the Moore-Penrose definition of the generalized inverse (pseudoinverse) denoted by A^+:

$$(1) \ AA^+A = A \tag{56}$$

$$(2) \ A^+AA^+ = A^+ \tag{57}$$

and often, in considered examples, are satisfied the second two demands of the Moore-Penrose definition,:

$$(3)\ \left(AA^{+}\right)^{*}=AA^{+} \tag{58}$$

$$(4)\ \left(A^{+}A\right)^{*}=A^{+}A \tag{59}$$

see [4], what guarantees of getting a least squares solution to the considered system of equations.

We think however that one-sided invertible operators in the sense of Eqs(54,55), are more simple and therefore more useful for basic description of nature, and except that, the request: 'least squares solution' is not always necessary. see [3, 4] and other author's 'recent' papers.

11 App.6 about strange behavior of some objects

Let us assume that we consider a discrete system such that from Eq.4 R, the reaction force, has to be a vector. In this case Eq.51 means that G is not a scalar but must behave so that R , at the transformation of the coordinate system, is the vector. Taking, however, the scalar product of the two vectors φ, R :

$$(\varphi(\cdot,t), R[\cdot,t;\varphi]) = VG[t;\varphi] \tag{60}$$

we should get, in the r.h.s., the scalar. This explicit contradiction, we can probably explained by the fact that G behaves as a scalar on the subset of vectors φ satisfying Eq.39.

12 App.7 abot space $\tilde{M}$, Cantor's theorem and evolution theory

In Sec.1 we said that the set $\tilde{M}$ consists of elements (vectors) reflecting specific properties of the considered system. This is only partly true because in these elements are also included certain properties of the observer as the experience of one, two or three dimensional spaces. As we know from the Cantor's theorem, there is 1-1 correspondence between the points of the plane or of n-dimensional space and of the stright line. It seems, however, that the identification of objects with a higher dimensional space is **much simpler and effective** than using the one-dimensional, and this was used at least by some organisms, see also [5], page 20, where other opinions are presented.

Higher dimensional spaces particularly preferred by quantum field theory to get meaningful theory appear to be evidence of the fact that even in the field of logic a similar phenomenon can be observed. By means of constraints certain dimensions can be roll up. By means of them also some constants can be introduced into considered equations.

References

[1] Edwadia, F. and R. Kalaba. 1993. *On Motion.* University of Southern California preprint?

[2] Edwadia, F. and R. Kalaba. 2000. *Nonideal Constraints and Lagrangian Dynamics.* J. of Aerospace Engimieeering/January, 2000

[3] Przeworska-Rolewicz, D. 1988. *Algebraic Analysis.* PWN-Polish Scientific Publishers, Warsaw, 1988

[4] Hanckowiak, J. 2012. *Mtaphysics of the free Fock space with local and global information.* vikra.org./abs/1206.0026, or http://arxiv.org/abs/1206.4589v1

[5] Nerlich, G. 1994. *The shape of Sspace.* Cambridge University Press