Pauli Matrices and Dirac Matrices in Geometric Algebra of Quarks

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Abstract **It is a historical accident that we describe Pauli matrices as (2 x 2) matrices and Dirac matrices as (4 x 4) matrices. As it will be shown in this paper we can use (3 x 3) matrices or (9 x 9) matrices for this purpose as well. This hopefully will enable us one day to construct a unified geometric algebra picture which includes Gell-Mann matrices in an appropriate manner.**

Keywords: Geometric algebra; S_3 permutation algebra; Pauli matrices; Dirac matrices.

I. Introduction

In mathematical physics we have to deal with two different mathematical concepts which describe physical phenomena. First, the physics of quarks is described by the matrices of Gell-Mann. These matrices are (3 x 3) matrices.

But quarks exist (as everything else we have to deal with in our world) in spacetime. This is the second concept. Euclidean space is described by Pauli matrices in geometric algebra. These matrices are (2 x 2) matrices. And spacetime of special relativity is described by Dirac matrices [1], [2], [3], [4]. These matrices can be identified with space-like and time-like base vectors [5] and are (4 x 4) matrices.

To find a unified geometric and algebraic picture which embraces both mathematical concepts, it makes sense to formulate both concepts on an equal footing. But to do this two different strategies can be followed: First, it can be tried to transfer Gell-Mann matrices into the mathematical language of standard geometric algebra, which uses (4 x 4) matrices, see [6], [7]. Or Pauli and Dirac matrices can be tried to translate into ternary (3 x 3) or (9 x 9) matrices. In this paper, this last strategy is followed.

II. Pauli Matrices and Dirac Matrices

According to [5, p. 44] Pauli matrices are usually expressed in standard notation as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

With the help of the direct product of Zehfuss and Kronecker [8], [9, chap. 11] the Dirac matrices [4, p. 278, eq. 8.67] can be constructed in the following way:

$$\gamma_t = \sigma_z \otimes \mathbf{1} = -(\sigma_x\sigma_y) \otimes (\sigma_x\sigma_y\sigma_z) \tag{4}$$

$$\gamma_v = \sigma_x \otimes \mathbf{1} = -(\sigma_y\sigma_z) \otimes (\sigma_x\sigma_y\sigma_z) \tag{5}$$

$$\gamma_x = -(\sigma_z\sigma_x) \otimes \sigma_x = -\sigma_y \otimes (\sigma_y\sigma_z) \tag{6}$$

$$\gamma_y = -(\sigma_z\sigma_x) \otimes \sigma_y = -\sigma_y \otimes (\sigma_z\sigma_x) \tag{7}$$

$$\gamma_z = -(\sigma_z\sigma_x) \otimes \sigma_z = -\sigma_y \otimes (\sigma_x\sigma_y) \tag{8}$$

Please note that the time-like base vector γ_v of eq. (5) can be interpreted as a vector pointing into a fifth independent dimension. For example this dimension can be thought of being the velocity dimension [10], [11] of Carmeli's spacetimevelocity world of cosmological special relativity [12], [13], [14].

Thus these orthogonal base vectors are in standard matrix notation:

$$\gamma_t = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{9}$$

$$\gamma_v = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{10}$$

$$\gamma_x = \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

$$\gamma_y = \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

$$\gamma_z = \begin{pmatrix} 0 & -\sigma_z \\ \sigma_z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{13}$$

III. Geometric Algebra of Quarks

Geometric algebra of quarks is a geometric algebra version of S_3 permutation algebra [15], [16], [17]. According to this geometric algebra version the following (3 x 3) matrices can be identified with unit vectors:

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{14}$$

$$e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{15}$$

$$e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

$$e_4 = \frac{1}{\sqrt{3}} i(e_0 + 2e_{21}) = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 2i & 0 \\ 0 & i & 2i \\ 2i & 0 & i \end{pmatrix} \tag{17}$$

They span a three-dimensional Euclidean space. e_0 is the (3 x 3) unit scalar:

$$e_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{18}$$

And the geometric products e_{12} and e_{21} of the first three unit vectors e_1, e_2, and e_3 can be identified with parallelograms:

$$e_{12} = e_1 e_2 = e_2 e_3 = e_3 e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{19}$$

$$e_{21} = e_2 e_1 = e_3 e_2 = e_1 e_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{20}$$

Both the three unit vectors e_1, e_2, and e_3 as well as the unit scalar and the two geometric products e_{12} and e_{21} add to the nihilation matrix $\boldsymbol{N}$:

$$e_1 + e_2 + e_3 = e_0 + e_{12} + e_{21} = \boldsymbol{N} \tag{21}$$

Walking successively a step into the e_1-direction, into the e_2-direction, and into the e_3-direction results in walking a step of zero length. Therefore the nihilation matrix $\boldsymbol{N}$, which is the matrix of ones at every position (sometimes called democratic matrix) has to be identified with the null matrix $\boldsymbol{O}$. And every multiple of $\boldsymbol{N}$ equals $\boldsymbol{O}$ too, of course.

$$\boldsymbol{N} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boldsymbol{O} \qquad (22)$$

Because of these strange relations (21) and (22) we do not necessarily need a minus sign in this mathematical framework. Instead all we usually do with minus signs can now be done with the \ominus matrix Θ:

$$\Theta = e_{12} + e_{21} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad (23)$$

In geometric algebra of quarks we live in a mathematical purely positive world, enriched by imaginary numbers

$$\mathbf{I} = \frac{1}{\sqrt{3}} e_1 (e_1 + 2e_2)\, e_4 = i\, e_0 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \qquad (24)$$

with (see warning in [15, sec.1, eq. 2]):

$$\mathbf{I}^2 = \begin{pmatrix} 1+i & 1 & 1 \\ 1 & 1+i & 1 \\ 1 & 1 & 1+i \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \Theta \qquad (25)$$

And as it is common in geometric algebra this pseudoscalar or imaginary unit scalar **I** can of course be identified with a three-dimensional oriented unit volume element.

IV. CONSTRUCTION OF (9 X 9) DIRAC MATRICES (PART I)

(3 x 3) Pauli matrices can be imagined and thought as a representation of three orthogonal unit vectors. As there are indefinitely many possibilities to do this, only the two of them presented in [17, chap. 6] will be discussed in this paper.

According to eq. (6.63), (6.64), and (6.65) of [17] (3 x 3) Pauli matrices can be written as:

$$\sigma_x = e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (26)$$

$$\sigma_y = \frac{1}{\sqrt{3}} (e_1 + 2e_2) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix} \qquad (27)$$

$$\sigma_z = e_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 2i & 0 \\ 0 & i & 2i \\ 2i & 0 & i \end{pmatrix} \qquad (28)$$

Using eq. (26), (27), and (28), the (3 x 3) Pauli bivectors and the (3 x 3) Pauli trivector can be found:

$$\sigma_x\sigma_y = \frac{1}{\sqrt{3}} (e_0 + 2e_{12}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \qquad (29)$$

$$\sigma_y\sigma_z = i\, e_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \qquad (30)$$

$$\sigma_z\sigma_x = \frac{1}{\sqrt{3}}\, i(e_1 + 2e_2) = \frac{1}{\sqrt{3}} \begin{pmatrix} i & 0 & 2i \\ 0 & 2i & i \\ 2i & i & 0 \end{pmatrix} \qquad (31)$$

$$\sigma_x\sigma_y\sigma_z = i\, e_0 = \begin{pmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \qquad (32)$$

As expected eq. (32) is identical to eq. (24). According to eq. (4), (5), (6), (7), and (8) the five Dirac matrices of geometric algebra of quarks can now be constructed as the following (9 x 9) matrices:

$$\gamma_t = e_4 \otimes e_0 = \frac{i}{\sqrt{3}} \begin{pmatrix} e_0 & 2e_0 & \boldsymbol{O} \\ \boldsymbol{O} & e_0 & 2e_0 \\ 2e_0 & \boldsymbol{O} & e_0 \end{pmatrix} \qquad (33)$$

$$\gamma_v = e_1 \otimes e_0 = \begin{pmatrix} e_0 & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & e_0 \\ \boldsymbol{O} & e_0 & \boldsymbol{O} \end{pmatrix} \qquad (34)$$

$$\gamma_x = \Theta \frac{i}{\sqrt{3}} (e_1 + 2e_2) \otimes e_1 = \frac{i}{\sqrt{3}} (e_1 + 2e_3) \otimes e_1$$

$$= \frac{i}{\sqrt{3}} \begin{pmatrix} e_1 & 2e_1 & \boldsymbol{O} \\ 2e_1 & \boldsymbol{O} & e_1 \\ \boldsymbol{O} & e_1 & 2e_1 \end{pmatrix} \tag{35}$$

$$\gamma_y = \Theta \frac{i}{3} (e_1 + 2e_2) \otimes (e_1 + 2e_2) = \frac{1}{\sqrt{3}} (e_1 + 2e_3) \otimes (e_4 e_1)$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} e_4 e_1 & 2e_4 e_1 & \boldsymbol{O} \\ 2e_4 e_1 & \boldsymbol{O} & e_4 e_1 \\ \boldsymbol{O} & e_4 e_1 & 2e_4 e_1 \end{pmatrix} \tag{36}$$

$$\gamma_z = \Theta \frac{i}{\sqrt{3}} (e_1 + 2e_2) \otimes e_4 = \frac{i}{\sqrt{3}} (e_1 + 2e_3) \otimes e_4$$

$$= \frac{i}{\sqrt{3}} \begin{pmatrix} e_4 & 2e_4 & \boldsymbol{O} \\ 2e_4 & \boldsymbol{O} & e_4 \\ \boldsymbol{O} & e_4 & 2e_4 \end{pmatrix} \tag{37}$$

These (9 x 9) Dirac matrices should meet the basic relations of Dirac algebra: Base vectors are normalized and anti-commutative. This will be shown in section VI.

V. Multiple Representation of Mathematical Objects

In standard (4 x 4) Dirac algebra we do not notice any differences in different constructions of the unit scalar. Both ways of constructing the (4 x 4) unit scalar $\mathbf{1}_4$

$$\mathbf{1}_4 = \mathbf{1} \otimes \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{38}$$

and

$$\mathbf{1}_4 = (-\mathbf{1}) \otimes (-\mathbf{1}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & -\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{39}$$

result in the same matrix representation of the (4 x 4) unit scalar.

In (9 x 9) geometric algebra of quarks they are different however: We find two clearly distinguishable matrix representations of the (9 x 9) unit scalar. In analogy to eq. (38) the unit scalar is represented by

$$\mathbf{1}_9 = e_0 \otimes e_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{40}$$

And in analogy to eq. (39) the (9 x 9) unit scalar is represented by

$$\mathbf{1}_9 = \Theta \otimes \Theta = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{41}$$

Both (9 x 9) matrices of eq. (40) and (41) have the same, identical mathematical meaning: They represent the unit scalar. Thus they are equal:

$$\mathbf{1}_9 = e_0 \otimes e_0 = \Theta \otimes \Theta \tag{42}$$

The same strange multiple representation or bifocalisation can the seen when we construct the /ominus matrix representing negative entities or pseudoscalars or oriented volume elements. In standard (4 x 4) geometric algebra we always get one and only one matrix representation:

$$(-\mathbf{1})_4 \;=\; (-\mathbf{1}) \otimes \mathbf{1} \;=\; \mathbf{1} \otimes (-\mathbf{1}) \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (43)$$

In (9 x 9) geometric algebra of quarks there are two different and clearly distinguishable matrix representations:

$$\Theta \otimes e_0 \;=\; \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad (44)$$

And there is

$$e_0 \otimes \Theta \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad (45)$$

Both (9 x 9) matrices of eq. (44) and (45) have the same, identical mathematical meaning: They represent the \ominus matrix as negative of the unit scalar. Thus they are equal.

$$(-\mathbf{1})_9 \;=\; \Theta \otimes e_0 \;=\; e_0 \otimes \Theta \quad (46)$$

And the same bifocalisation applies to every other matrix. Therefore it is possible to find a second matrix representation for the (9 x 9) Dirac matrices of eq. (33) to (37):

$$\gamma_t \;=\; (\Theta\, e_4) \otimes \Theta \;=\; \frac{i}{\sqrt{3}} \begin{pmatrix} \Theta & \boldsymbol{O} & 2\Theta \\ 2\Theta & \Theta & \boldsymbol{O} \\ \boldsymbol{O} & 2\Theta & \Theta \end{pmatrix} \quad (47)$$

$$\gamma_v \;=\; (\Theta\, e_1) \otimes \Theta \;=\; \begin{pmatrix} \boldsymbol{O} & \Theta & \Theta \\ \Theta & \Theta & \boldsymbol{O} \\ \Theta & \boldsymbol{O} & \Theta \end{pmatrix} \quad (48)$$

$$\gamma_x \;=\; \frac{i}{\sqrt{3}}(e_1 + 2e_2) \otimes (\Theta\, e_1) \;=\; \frac{i}{\sqrt{3}}(e_1 + 2e_2) \otimes (e_2 + e_3)$$
$$= \frac{i}{\sqrt{3}} \begin{pmatrix} e_2 + e_3 & \boldsymbol{O} & 2(e_2 + e_3) \\ \boldsymbol{O} & 2(e_2 + e_3) & e_2 + e_3 \\ 2(e_2 + e_3) & e_2 + e_3 & \boldsymbol{O} \end{pmatrix} \quad (49)$$

$$\gamma_y \;=\; \frac{i}{3}(e_1 + 2e_2) \otimes (\Theta\,(e_1 + 2e_2)) \;=\; \frac{i}{3}(e_1 + 2e_2) \otimes (e_1 + 2e_3)$$
$$= \frac{i}{3} \begin{pmatrix} e_1 + 2e_3 & \boldsymbol{O} & 2e_1 + 4e_3 \\ \boldsymbol{O} & 2e_1 + 4e_3 & e_1 + 2e_3 \\ 2e_1 + 4e_3 & e_1 + 2e_3 & \boldsymbol{O} \end{pmatrix} \quad (50)$$

$$\gamma_z \;=\; \frac{i}{\sqrt{3}}(e_1 + 2e_2) \otimes (\Theta\, e_4)$$
$$= \frac{i}{\sqrt{3}} \begin{pmatrix} \Theta e_4 & \boldsymbol{O} & 2\Theta e_4 \\ \boldsymbol{O} & 2\Theta e_4 & \Theta e_4 \\ 2\Theta e_4 & \Theta e_4 & \boldsymbol{O} \end{pmatrix} \quad (51)$$

As these different explicit matrix representations are a little bit confusing, in the following explicit (9 x 9) matrix representations are avoid and only the algebraic relations are given. As the /ominus matrix and the unit pseudoscalar **I** commute with every element of (9 x 9) geometric algebra of quarks, it is possible to shift them without problems inside these algebraic formulae at every position.

VI. Construction of (9 x 9) Dirac Matrices (Part II)

As an alternative, (9 x 9) Dirac matrices can be constructed in an aesthetical more appealing way. This time, Pauli matrices are identified with the following expressions according to [15, eq. 57, eq. 58, eq. 59] or [17, eq. 6.71, eq. 6.72, eq. 6.73]:

$$\sigma_x \;=\; \frac{1}{3}\left(\sqrt{2}(e_1 + 2e_3) + \sqrt{3}e_4\right)$$
$$= \frac{1}{3}\left(\sqrt{2}e_0 + ie_1\right)(e_1 + 2e_3) = \frac{1}{\sqrt{3}}(e_4 + \sqrt{2}\; i\, e_4 e_1) \quad (52)$$

$$\sigma_y = \frac{1}{3}\left(\sqrt{2}(e_2 + 2e_1) + \sqrt{3}e_4\right)$$

$$= \frac{1}{3}\left(\sqrt{2}e_0 + ie_2\right)(e_2 + 2e_1) = \frac{1}{\sqrt{3}}(e_4 + \sqrt{2}\ i\ e_4e_2) \quad (53)$$

$$\sigma_z = \frac{1}{3}\left(\sqrt{2}(e_3 + 2e_2) + \sqrt{3}e_4\right)$$

$$= \frac{1}{3}\left(\sqrt{2}e_0 + ie_3\right)(e_3 + 2e_2) = \frac{1}{\sqrt{3}}(e_4 + \sqrt{2}\ i\ e_4e_3) \quad (54)$$

The first expressions of these vectors σ_x, σ_y, and σ_z show the fundamental representations with positive coefficients only as a minimal linear combination of three of the five unit vectors e_1, e_2, e_3, e_4, and Θe_4. The first expressions in the second line of these vectors show the paravector-like representations with complex coefficients as a minimal linear combination of only two of the three unit vectors e_1, e_2, and e_3.

Using eq. (52), (53), and (54), the (3 x 3) Pauli bivectors and the (3 x 3) Pauli trivector can be found:

$$\sigma_x\sigma_y = \frac{1}{3}(e_1 + e_2 + \sqrt{2}\ ie_0)(e_3 + 2e_2) = i\ \sigma_z \quad (55)$$

$$\sigma_y\sigma_z = \frac{1}{3}(e_2 + e_3 + \sqrt{2}\ ie_0)(e_1 + 2e_3) = i\ \sigma_x \quad (56)$$

$$\sigma_z\sigma_x = \frac{1}{3}(e_3 + e_1 + \sqrt{2}\ ie_0)(e_2 + 2e_1) = i\ \sigma_y \quad (57)$$

$$\sigma_x\sigma_y\sigma_z = i\ e_0 = \mathbf{I} \quad (58)$$

As again expected eq. (58) is identical to eq. (24). According to eq. (4), (5), (6), (7), and (8) the five Dirac matrices of geometric algebra of quarks can now be constructed as the following (9 x 9) matrices:

$$\gamma_t = \sigma_z \otimes e_0 = \frac{1}{\sqrt{3}}(e_4 + \sqrt{2}\ i\ e_4e_3) \otimes e_0 \quad (59)$$

$$\gamma_v = \sigma_x \otimes e_0 = \frac{1}{\sqrt{3}}(e_4 + \sqrt{2}\ i\ e_4e_1) \otimes e_0 \quad (60)$$

$$\gamma_x = (\sigma_x\sigma_z) \otimes \sigma_x = (\Theta\ i\sigma_y) \otimes \sigma_x$$

$$= \frac{1}{3}(\Theta\ ie_4 + \sqrt{2}\ e_4e_2) \otimes (e_4 + \sqrt{2}\ i\ e_4e_1) \quad (61)$$

$$\gamma_y = (\sigma_x\sigma_z) \otimes \sigma_y = (\Theta\ i\sigma_y) \otimes \sigma_y$$

$$= \frac{1}{3}(\Theta\ ie_4 + \sqrt{2}\ e_4e_2) \otimes (e_4 + \sqrt{2}\ i\ e_4e_2) \quad (62)$$

$$\gamma_z = (\sigma_x\sigma_z) \otimes \sigma_z = (\Theta\ i\sigma_y) \otimes \sigma_z$$

$$= \frac{1}{3}(\Theta\ ie_4 + \sqrt{2}\ e_4e_2) \otimes (e_4 + \sqrt{2}\ i\ e_4e_3) \quad (63)$$

These (9 x 9) Dirac matrices should meet again the basic relations of Dirac algebra, which will be shown in the next section.

VII. (9 X 9) DIRAC ALGEBRA

To show normalization and orthogonality of (9 x 9) Dirac matrices the multiplication rule for direct Zehfuss-Kronecker products of four matrices A, B, C, and D [8, p. 16] will be helpful.

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \quad (64)$$

Now the normalization of the Dirac matrices can be evaluated:

$$\gamma_t^2 = (\sigma_z \otimes e_0)^2 = e_0 \otimes e_0 \quad (65)$$

$$\gamma_v^2 = (\sigma_x \otimes e_0)^2 = e_0 \otimes e_0 \quad (66)$$

$$\gamma_x^2 = ((\sigma_x\sigma_z) \otimes \sigma_x)^2 = (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes e_0 = \Theta \otimes e_0 \quad (67)$$

$$\gamma_y^2 = ((\sigma_x\sigma_z) \otimes \sigma_y)^2 = (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes e_0 = \Theta \otimes e_0 \quad (68)$$

$$\gamma_z^2 = ((\sigma_x\sigma_z) \otimes \sigma_z)^2 = (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes e_0 = \Theta \otimes e_0 \quad (69)$$

Thus γ_t and γ_v are time-like unit vectors as they square to the (9 x 9) unit scalar $(e_0 \otimes e_0)$. And γ_x, γ_y, and γ_z are space-like unit vectors, as they square to the (9 x 9) \ominus matrix $(\Theta \otimes e_0)$. In a similar way anti-commutativity can be shown:

$$\gamma_t\gamma_v = (\sigma_z \otimes e_0)(\sigma_x \otimes e_0) = (\sigma_z\sigma_x) \otimes e_0$$

$$= (\Theta\ \sigma_x\sigma_z) \otimes e_0 = (\Theta \otimes e_0)(\sigma_x \otimes e_0)(\sigma_z \otimes e_0)$$

$$= (\Theta \otimes e_0)\ \gamma_v\gamma_t \quad (70)$$

$$\gamma_t\gamma_x = (\sigma_z \otimes e_0)((\sigma_x\sigma_z) \otimes \sigma_x) = (\sigma_z\sigma_x\sigma_z) \otimes \sigma_x$$

$$= (\Theta\ \sigma_x\sigma_z\sigma_z) \otimes \sigma_x$$

$$= (\Theta \otimes e_0)((\sigma_x\sigma_z) \otimes \sigma_x)(\sigma_z \otimes e_0)$$

$$= (\Theta \otimes e_0)\ \gamma_x\gamma_t \quad (71)$$

$$\gamma_t\gamma_y = (\sigma_z \otimes e_0)((\sigma_x\sigma_z) \otimes \sigma_y) = (\sigma_z\sigma_x\sigma_z) \otimes \sigma_y$$

$$= (\Theta\ \sigma_x\sigma_z\sigma_z) \otimes \sigma_y$$

$$= (\Theta \otimes e_0)((\sigma_x\sigma_z) \otimes \sigma_y)(\sigma_z \otimes e_0)$$

$$= (\Theta \otimes e_0)\ \gamma_y\gamma_t \quad (72)$$

$$\gamma_t\gamma_z = (\sigma_z \otimes e_0)((\sigma_x\sigma_z) \otimes \sigma_z) = (\sigma_z\sigma_x\sigma_z) \otimes \sigma_z$$

$$= (\Theta\ \sigma_x\sigma_z\sigma_z) \otimes \sigma_z$$

$$= (\Theta \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_z)\,(\sigma_z \otimes e_0)$$
$$= (\Theta \otimes e_0)\,\gamma_z\gamma_t \qquad (73)$$

$$\gamma_v\gamma_x = (\sigma_x \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_x) = (\sigma_x\sigma_x\sigma_z) \otimes \sigma_x$$
$$= (\Theta\, \sigma_x\sigma_z\sigma_x) \otimes \sigma_x$$
$$= (\Theta \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_x)\,(\sigma_x \otimes e_0)$$
$$= (\Theta \otimes e_0)\,\gamma_x\gamma_v \qquad (74)$$

$$\gamma_v\gamma_y = (\sigma_x \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_y) = (\sigma_x\sigma_x\sigma_z) \otimes \sigma_y$$
$$= (\Theta\, \sigma_x\sigma_z\sigma_x) \otimes \sigma_y$$
$$= (\Theta \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_y)\,(\sigma_x \otimes e_0)$$
$$= (\Theta \otimes e_0)\,\gamma_y\gamma_v \qquad (75)$$

$$\gamma_v\gamma_z = (\sigma_x \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_z) = (\sigma_x\sigma_x\sigma_z) \otimes \sigma_z$$
$$= (\Theta\, \sigma_x\sigma_z\sigma_x) \otimes \sigma_z$$
$$= (\Theta \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_z)\,(\sigma_x \otimes e_0)$$
$$= (\Theta \otimes e_0)\,\gamma_z\gamma_v \qquad (76)$$

$$\gamma_x\gamma_y = ((\sigma_x\sigma_z) \otimes \sigma_x)\,((\sigma_x\sigma_z) \otimes \sigma_y) = (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes (\sigma_x\sigma_y)$$
$$= (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes (\Theta\, \sigma_y\sigma_x)$$
$$= (e_0 \otimes \Theta)\,((\sigma_x\sigma_z) \otimes \sigma_y)\,((\sigma_x\sigma_z) \otimes \sigma_x)$$
$$= (\Theta \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_y)\,((\sigma_x\sigma_z) \otimes \sigma_x)$$
$$= (\Theta \otimes e_0)\,\gamma_y\gamma_x \qquad (77)$$

$$\gamma_x\gamma_z = ((\sigma_x\sigma_z) \otimes \sigma_x)\,((\sigma_x\sigma_z) \otimes \sigma_z) = (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes (\sigma_x\sigma_z)$$
$$= (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes (\Theta\, \sigma_z\sigma_x)$$
$$= (e_0 \otimes \Theta)\,((\sigma_x\sigma_z) \otimes \sigma_z)\,((\sigma_x\sigma_z) \otimes \sigma_x)$$
$$= (\Theta \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_z)\,((\sigma_x\sigma_z) \otimes \sigma_x)$$
$$= (\Theta \otimes e_0)\,\gamma_z\gamma_x \qquad (78)$$

$$\gamma_y\gamma_z = ((\sigma_x\sigma_z) \otimes \sigma_y)\,((\sigma_x\sigma_z) \otimes \sigma_z) = (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes (\sigma_y\sigma_z)$$
$$= (\sigma_x\sigma_z\sigma_x\sigma_z) \otimes (\Theta\, \sigma_z\sigma_y)$$
$$= (e_0 \otimes \Theta)\,((\sigma_x\sigma_z) \otimes \sigma_z)\,((\sigma_x\sigma_z) \otimes \sigma_y)$$
$$= (\Theta \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_z)\,((\sigma_x\sigma_z) \otimes \sigma_y)$$
$$= (\Theta \otimes e_0)\,\gamma_z\gamma_y \qquad (79)$$

And the handedness can be checked by

$$\gamma_v\gamma_t\gamma_x\gamma_y\gamma_z$$
$$= (\sigma_x \otimes e_0)\,(\sigma_z \otimes e_0)\,((\sigma_x\sigma_z) \otimes \sigma_x)\,((\sigma_x\sigma_z) \otimes \sigma_y)\,((\sigma_x\sigma_z) \otimes \sigma_z)$$
$$= (\sigma_x\sigma_z\sigma_x\sigma_z\sigma_x\sigma_z\sigma_x\sigma_z) \otimes (\sigma_x\sigma_y\sigma_z)$$
$$= e_0 \otimes (i\, e_0) = i\,(e_0 \otimes e_0) \qquad (80)$$

Of course these equations look like the equations of (2 x 2) Pauli algebra and of (4 x 4) Dirac algebra. But these equations are equations which contain (3 x 3) Pauli matrices and (9 x 9) Dirac matrices.

VIII. Outlook

(9 x 9) Dirac matrices give us the time-like and space-like base vectors of the four- or five-dimensional world we live in. The next step should be to include the nonion algebra or ternary Clifford algebra of Kerner [18] and Suzuki [19] into this picture to represent quarks mathematically in this world.

It seems that at least the two sums

$$j + j^2 + 1 = 0 \qquad (81)$$

(see [18, p. 154], [20, slide 21 & 42]) and

$$j^2 + j = -1 \qquad (82)$$

(see [18, p. 159], [20, slide 83]) can be identified with the nihilation matrix ***N*** of eq. (22) and the \ominus matrix Θ of eq. (23) in this paper.

But the obvious philosophical differences between the ideas followed in [18], [19] or [20] and the ideas of this paper should not be underestimated. In this paper all matrices are seen as geometrical objects according the basic concepts of geometric algebra. They act not only as operators (representing generators of reflections, rotations or linear combinations of reflections and rotations), but they are seen here as operands (representing always scalars, vectors, bivectors, trivectors, quadrovectors, pentavectors or linear combinations of these objects) as well.

While Kerner sees spacetime emerging from quark algebra in [20], I urgently like to see quark algebra as emerging from spacetime algebra. Not only we as human beings live in space-time; quarks live in spacetime too.

Please note that this view follows the philosophy of John Snygg, when he describes the history of the electron and its algebra: “It was necessary to attribute to the electron a spin of ½ and a periodicity of 4π. In recent years, it has become more widely recognized that objects larger than electrons also have 4π periodicities” [9, p. 11]. In the same way the Dirac belt trick demonstrates that extended macroscopic objects “in some sense loosely attached to its surroundings” [9, p. 12] show the 4π symmetry of electrons, I am convinced another belt-like trick will show us quark symmetry one day.

It should be only a matter of time to find a way demon-strating quark symmetry with extended macroscopic objects, revealing the geometrical simplicity of quark algebra.

IX. Literature

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X. Additional Remarks

This paper was uploaded at the internet preprint arxive of Cornell University (www.arxiv.org) at 2. Dec. 2012 with the temporary submission identifier arXiv:submit/0605626 after being endorsed by a colleague engaged in Clifford algebra research.

At 4. Dec. 2012 I was informed by the arxiv administration that this submission "has been removed upon a notice from our moderators, who determined it inappropriate for arXiv." Therefore I now decided to upload this paper at www.vixra.org.

Meanwhile I have written another paper about geometric algebra of quarks [21, second part] in which I prefer yet another Dirac matrix construction based on some ideas presented in [22] as "quarks should be regarded as entities having absolutely no rectangular symmetry" [21, footnote 2]. Nevertheless the present paper about Pauli matrices and Dirac matrices in geometric algebra of quarks is surely of some interest, as the discussion of multiple representations of mathematical objects in chapter 5 is important and cannot be found in my other papers.

I myself still ponder over the consequences of these multiple representations for physics, and there even is a slight possibility that different representations might indeed represent different physical objects being not distinguishable in standard matrix algebra or standard geometric algebra.