The generalizations of the First Noether theorem.

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Abstract.

This paper deals with the generalizations of the First Noether theorem. It takes into account not only the first derivatives of the fields by the coordinates in Lagrangian, but also the second. And this theorem is generalized on the curved spaces. And also it's generalized on asymmetric metric tensors.

Content

1) Action function.

It is said in [1] that if g is the determinant of the metric tensor and S – the scalar function constructed from a system of fields, then magnitude

$$A = \int S \cdot \sqrt{-g} \cdot d^4 x \qquad (1.1)$$

is invariant relative to coordinate transformations. And so it is possible to take A as the action function for this system of fields. That can be generalized and on N – dimensional spaces.

$$L = S \cdot \sqrt{-g} \qquad (1.2)$$ L - Lagrangian

2) Field equations.

Let $L = L(u_i; u_{i,k}; u_{i,kl})$

u_i here is the symbol for any field. Let us vary A by the u_i :

$$\delta A=\int\left(\frac{\partial L}{\partial u_i}\cdot\delta u_i+\frac{\partial L}{\partial u_{i,k}}\cdot\delta u_{i,k}+\frac{\partial L}{\partial u_{i,kl}}\cdot\delta u_{i,kl}\right)\cdot d^N x=$$

$$=\int\left[\frac{\partial L}{\partial u_i}-\partial_k\left(\frac{\partial L}{\partial u_{i,k}}\right)+\partial_k\left(\partial_{\mathbf{l}}\left(\frac{\partial L}{\partial u_{i,k\mathbf{l}}}\right)\right)\right]\cdot\delta u_i\cdot d^N x \qquad (2.1)$$

If at any δu_i variation δA= 0, then we get the equations for u_i :

$$\frac{\partial L}{\partial u_i}-\partial_k\left(\frac{\partial L}{\partial u_{i,k}}\right)+\partial_k\left(\partial_{\mathbf{l}}\left(\frac{\partial L}{\partial u_{i,k\mathbf{l}}}\right)\right)=0 \qquad (2.2)$$

3). The First Noether theorem.

We will follow [2], and in some places will add new formule.
:

Let us consider an infinitesimal transformation of coordinates and field functions:

$$x'^k = x^k + \delta x^k \qquad (3.1)$$

$$u_i'(x') = u_i(x) + \delta u_i(x) \qquad (3.2)$$

The variations δx^k and δu_i can be expressed in terms of the infinitesimal linearly independent transformation parameters $\delta\omega_n$ as follows:

$$\delta x^k = \sum_{1\le n\le s} \mathrm{X}^k{}_{(n)}\cdot\delta\omega_n \qquad \delta u_i(x) = \sum_{1\le n\le s} \Psi_{i(n)}\cdot\delta\omega_n \qquad (3.3)$$

The indices i and n of the field functions and the transformation parameters may (or may not) have a simple tensorial significance. We shall not specify it, and will agree to interpret repeated indices as indicating summation.

We note that the transformation law for the field functions

$$u'_{i,k}(x') = u_{i,k}(x) + \delta u_{i,k}(x)$$

contains the variations $\delta u_{i,k}$ that are nonderivatives of δu_i. In other words, the operations δ and ∂/∂x do not commute. The point is that δu_i is the variation of the field function due to both the change in its form and the change in its argument. The variation due to the change in the form of the function is defined by

$$\bar{\delta} u_i(x) = u_i'(x) - u_i(x)$$

which to within second-order terms can be written in the form

$$\bar{\delta} u_i(x) = \delta u_i(x) - u_{i;k}\cdot\delta x^k = (\Psi_{i(n)} - u_{i;k}\cdot\mathrm{X}^k{}_{(n)})\cdot\delta\omega_n \qquad (3.4)$$

Covariant derivative in (3.4) **takes into account the space curvature:**

$$(\vec{e}_i \cdot u^i{}_{;k}) = \partial_k(\vec{e}_i \cdot u^i) = (\partial_k \vec{e}_i) \cdot u^i + \vec{e}_i \cdot u^i{}_{,k} = \vec{e}_n \cdot \Gamma^n{}_{ik} \cdot u^i + \vec{e}_n \cdot u^n{}_{,k} \quad (3.5)$$

By definition, the operation $\overline{\delta}$ commutes with $\partial/\partial x$.

We now define the variation of the action by

$$\delta A = \delta \int L(x) \cdot dx = \int L'(x') \cdot dx' - \int L(x) \cdot dx$$

where

$$L'(x') = L(u_i'(x'), u_{i,k}'(x'), u_{i,kl}'(x')) = L(x) + \delta L(x)$$

and

$$\delta L(x) = \frac{\partial L}{\partial u_i} \cdot \delta u_i + \frac{\partial L}{\partial u_{i,k}} \cdot \delta u_{i,k} + \frac{\partial L}{\partial u_{i,kl}} \cdot \delta u_{i,kl} = \overline{\delta} L(x) + L_{;k} \cdot \delta x^k$$

Covariant derivative here **takes into account the space curvature.**

In these expressions, $\overline{\delta}$L is the variation of L due to variations in the form of $u_i, u_{i,k}, u_{i,kl}$:

$$\overline{\delta} L(x) = \frac{\partial L}{\partial u_i} \cdot \overline{\delta} u_i + \frac{\partial L}{\partial u_{i,k}} \cdot \overline{\delta} u_{i,k} + \frac{\partial L}{\partial u_{i,kl}} \cdot \overline{\delta} u_{i,kl}$$

and the second term describes the total variation due to variations in the coordinates.

Thus:

$$\delta A = \int (\overline{\delta} L(x) + L(x)_{;k} \cdot \delta x^k) \cdot dx + \int L(x) \cdot dx' - \int L(x) \cdot dx$$

We shell now consider the difference between the last two terms, which describes the variation in the volume of integration.

We have

$$dx' \equiv dx_1' \cdot dx_2' \cdot dx_3' \cdot dx_4' = \frac{\partial(x_1', x_2', x_3', x_4')}{\partial(x_1, x_2, x_3, x_4)} \cdot dx \approx (1 + \delta x^k{}_{;k}) \cdot dx$$

Covariant derivative here **takes into account the space curvature.**

And therefore

$$\int L(x) \cdot dx' - \int L(x) \cdot dx = \int L(x) \cdot \delta x^k{}_{;k} \cdot dx$$

and

$$\delta A = \int [\overline{\delta} L(x) + (L(x) \cdot \delta x^k)_{;k}] \cdot dx$$

3.1). The asymmetric metric tensors.

The definitions for asymmetric metric tensor are in [3].

Let us find $g_{\mu\nu;k}$ **for asymmetric** $g_{\mu\nu}$**.**

$$\vec{e}_\mu \otimes \vec{e}_\nu \cdot g^{\mu\nu}{}_{;k} = (\vec{e}_\mu \otimes \vec{e}_\nu \cdot g^{\mu\nu})_{,k} \qquad \vec{e}_{\mu,k} = \vec{e}_\sigma \cdot \Gamma^\sigma{}_{\mu k}$$

$$g^{\mu\nu}{}_{;k} = \Gamma^{\mu}{}_{\sigma k} \cdot g^{\sigma\nu} + \Gamma^{\nu}{}_{\rho k} \cdot g^{\mu\rho} + g^{\mu\nu}{}_{,k}$$

$$g^{\mu\nu} \cdot g_{\nu\lambda} = \delta^{\mu}{}_{\lambda} \qquad g^{\mu s}{}_{,k} = -g^{qs} \cdot g^{\mu\nu} \cdot g_{\nu q,k}$$

$$g_{s\lambda;k} = -g_{s\mu} \cdot g^{\mu\nu}{}_{;k} \cdot g_{\nu\lambda} = g_{s\lambda,k} - g_{s\sigma} \cdot \Gamma^{\sigma}{}_{\lambda k} - \Gamma^{\rho}{}_{sk} \cdot g_{\rho\lambda}$$

$$g_{s\lambda,k} = (\vec{e}_s, \vec{e}_\lambda)_{,k} = \Gamma^{\rho}{}_{sk} \cdot g_{\rho\lambda} + g_{s\sigma} \cdot \Gamma^{\sigma}{}_{\lambda k}$$

$$g_{s\lambda;k} = 0 \tag{3.1.1}$$

From $g_{\mu\nu;k} = 0$ it follows $\sqrt{-g}_{;k} = 0$

Taking into account

$$L = S \cdot \sqrt{-g} \qquad S_{;k} = S_{,k} \quad \text{и} \quad \delta x^{k}{}_{;k} = \delta x^{k}{}_{,k} + \Gamma^{k}{}_{nk} \cdot \delta x^{n}$$

[The Christoffel symbols for the for **the asymmetric metric tensor** must be taken from [3] - (2.35)]

$$\Gamma^{k}{}_{nk} = \tfrac{1}{2} \cdot g^{k\nu} \cdot g_{\nu k,n} + \tfrac{1}{2} \cdot g^{k\nu} \cdot b_{k\nu,n} = \frac{1}{2 \cdot g} \cdot g_{,n} + n_n = \frac{\sqrt{-g},_{n}}{\sqrt{-g}} + n_n$$

$$n_n = \tfrac{1}{2} \cdot g^{m\nu} \cdot b_{m\nu,n}$$

we obtain :

$$\delta A = \int [\bar{\delta} L(x) + (L(x) \cdot \delta x^k)_{,k} + n_k \cdot \delta x^k\ ] \cdot dx$$

Using the equation of motion (2.2) :

$$\frac{\partial L}{\partial u_i} = \partial_k \Big(\frac{\partial L}{\partial u_{i,k}} - \partial_{\mathbf{l}} \Big(\frac{\partial L}{\partial u_{i,k\mathbf{l}}} \Big) \Big)$$

we obtain :

$$\delta A = \int [\partial_k \Big(\frac{\partial L}{\partial u_{i,k}} - \partial_{\mathbf{l}} \Big(\frac{\partial L}{\partial u_{i,k\mathbf{l}}} \Big) \Big) \cdot \bar{\delta} u_i + \frac{\partial L}{\partial u_{i,k}} \cdot \partial_k \bar{\delta} u_i +$$

$$+ \partial_{\mathbf{l}} \Big(\frac{\partial L}{\partial u_{i,k\mathbf{l}}} \cdot \bar{\delta} u_{i,k} \Big) -$$

$$- \partial_{\mathbf{l}} \Big(\frac{\partial L}{\partial u_{i,k\mathbf{l}}} \Big) \cdot \partial_k \bar{\delta} u_i + \partial_k (L(x) \cdot \delta x^k) + n_k \cdot \delta x^k\ ] \cdot dx =$$

$$= \int \{ \partial_k [(\frac{\partial L}{\partial u_{i,k}} - \partial_{\mathbf{l}} (\frac{\partial L}{\partial u_{i,k\mathbf{l}}})) \cdot \bar{\delta} u_i + \frac{\partial L}{\partial u_{i,\mathbf{l}k}} \cdot \bar{\delta} u_{i,\mathbf{l}} +$$

$$+ L \cdot \delta x^k] + n_k \cdot \delta x^k \} \cdot dx$$

Taking into account (3.3) и (3.4) we have :

$$\delta A = - \sum_{1 \le n \le s} \int (\partial_k [\theta^k{}_{(n)}(x)] - j_{(n)}) \cdot dx \cdot \delta\omega_n$$

where

$$j_{(n)} = n_k \cdot \mathrm{X}^k{}_{(n)} = \tfrac{1}{2} \cdot g^{m\nu} \cdot b_{m\nu,k} \cdot \mathrm{X}^k{}_{(n)} \qquad (3.1.2)$$

$$\theta^k{}_{(n)}(x) = -[\frac{\partial L}{\partial u_{i,k}} - \partial_{\mathbf{l}} (\frac{\partial L}{\partial u_{i,k\mathbf{l}}})] \cdot (\Psi_{i\,(n)} - u_{i;m} \cdot \mathrm{X}^m{}_{(n)}) -$$

$$- \frac{\partial L}{\partial u_{i,\mathbf{l}k}} \cdot (\Psi_{i\,(n),\mathbf{l}} - \partial_{\mathbf{l}} [u_{i;m} \cdot \mathrm{X}^m{}_{(n)}]) - L(x) \cdot \mathrm{X}^k{}_{(n)} \qquad (3.1.3)$$

Since the first variation of action must vanish, and if we equate to zero the coefficients of the independent transformation parameters $\delta\omega_n$, we obtain

$$\frac{\partial A}{\partial \omega_n} = - \int (\partial_k [\theta^k{}_{(n)}(x)] - j_{(n)}) \cdot dx = 0 \qquad (3.1.4)$$

Since the region of integration is arbitrary, we obtain the continuity equation :

$$\frac{d}{dx^k} \theta^k{}_{(n)}(x) = j_{(n)} \qquad (3.1.5)$$

3.2). The conservation laws for the symmetric **metric tensor in curved space and second derivatives in Lagrangian.**

If $j_{(n)} = 0$, then :

Transforming the right-hand side of (3.1.4) by Gauss' theorem, we obtain the conservation laws for the corresponding surface integrals. If we further suppose that the integral in (3.1.4) is valuated over a volume that expands without limit in space-like directions, but is bounded in time-like directions by space-like three-dimensional surfaces σ_1 and σ_2 , we find that if the field is practically zero on the boundaries of the spatial volume,

$$\int_{\sigma_1} d\sigma_k \cdot \theta^k{}_{(n)} - \int_{\sigma_2} d\sigma_k \cdot \theta^k{}_{(n)} = 0$$

In this expression, $d\sigma_k$ is the projection of the surface area element σ onto the three-plane perpendicular to the x^k axis. The above equation shows that the surface integrals

$$C_n(\sigma) = \int_{\sigma} d\sigma_k \cdot \theta^k{}_{(n)}$$

are in fact independent of the surface σ. In the special case where the surfaces are the three-planes x^1 = t = const, the integral is evaluated over the three-dimensional configuration space, and the integrals

$$C_{(n)}(x^1) = \int d^3x \cdot \theta^1{}_{(n)} = const \qquad (3.2.1)$$

are independent of time.

We have thus shown that to each continuous s – parameter transformation of coordinates (3.1) and field functions (3.2), there correspond s time-independent invariants (3.2.1) C_n (n = 1, …, s). That is the first Noether theorem (at $j_{(n)} = 0$).

The quantities $\theta^k_{(n)}$ are not unique. Expressions of the form

$$\frac{\partial}{\partial x^m} f^{km}{}_{(n)}$$

can be added to them if

$$f^{km}{}_{(n)} = -f^{mk}{}_{(n)}$$

This ambiguity does not, however, affect the value of the conserved integrals (3.2.1).

3.3). The conservation laws for the asymmetric metric tensor in curved space and second derivatives in Lagrangian.

If $j_{(n)} \neq 0$ then we can get from (3.1.4)

$$\int_{\sigma_2} d\sigma_k \cdot \theta^k{}_{(n)} - \iiint d^3x \cdot \int_{t_1}^{t_2} j_{(n)} \cdot dt = \int_{\sigma_1} d\sigma_k \cdot \theta^k{}_{(n)} \qquad (3.3.1)$$

Let us define $\int dt \cdot j_{(n)} = D_{(n)}$ (3.3.2) then

$$\int_{t_1}^{t_2} dt \cdot j_{(n)} = D_{(n)}(t = t_2) - D_{(n)}(t = t_1) \qquad (3.3.3)$$

Hence we have :

$$\int_{\sigma_2} d\sigma_k \cdot \theta^k{}_{(n)} - \iiint d^3x \cdot D_{(n)}(t = t_2) = \int_{\sigma_1} d\sigma_k \cdot \theta^k{}_{(n)} - \iiint d^3x \cdot D_{(n)}(t = t_1) \qquad (3.3.4)$$

From here we see that the values

$$B_{(n)}(x^1) = \iiint d^3x \cdot (\theta^1{}_{(n)} - \int dx^1 \cdot j_{(n)}) = const \qquad (3.3.5)$$

are independent of time. That is the first Noether theorem for the asymmetric metric tensors in curved spaces with taking into account the second derivatives in Lagrangian.

4). Applications of the generalized First Noether theorem.

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1. P. A. M. Dirac “General Theory of Relativity.”
2. N. N. Bogoliubov, D. V. Shirkov “Introduction to the theory of quantized fields. 3-d edition. 1980.
3 V. Telnin “Christoffel symbols for asymmetric metric tensors.”
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