Scientia Magna
Vol. 2 (2006), No. 1, 13-19

Smarandache Quasigroups

Arun S. Muktibodh

Mohota Science College
Nagpur

Abstract In this paper, we have introduced Smarandache quasigroups which are Smarandache non-associative structures. W.B.Kandasamy [2] has studied Smarandache groupoids and Smarandache semigroups etc. Substructure of Smarandache quasigroups are also studied.

Keywords Quasigroup; Smarandache Quasigroup.

1. Introduction

W.B.Kandasamy has already defined and studied Smarandache groupoids, Smarandache semigroups etc. A quasigroup is a groupoid whose composition table is LATIN SQUARE. We define Smarandache quasigroup as a quasigroup which contains a group properly.

2. Preliminaries

Definition 2.1. A groupoid S such that for all a, $b \in S$ there exist unique x, $y \in S$ such that $ax = b$ and $ya = b$ is called a quasigroup.

Thus a quasigroup does not have an identity element and it is also non-associative.

Example 2.1. Here is a quasigroup that is not a loop.

*	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

We note that the definition of quasigroup Q forces it to have a property that every element of Q appears exactly once in every row and column of its operation tables. Such a table is called a LATIN SQUARE. Thus, quasigroup is precisely a groupoid whose multiplication table is a LATIN SQUARE.

Definition 2.2. If a quasigroup $(Q, *)$ contains a group $(G, *)$ properly then the quasigroup is said to be Smarandache quasigroup.

A Smarandache quasigroup is also denoted by S-quasigroup.

Example 2.2. Let Q be a quasigroup defined by the following table;

$*$	a_0	a_1	a_2	a_3	a_4
a_0	a_0	a_1	a_3	a_4	a_2
a_1	a_1	a_0	a_2	a_3	a_4
a_2	a_3	a_4	a_1	a_2	a_0
a_3	a_4	a_2	a_0	a_1	a_3
a_4	a_2	a_3	a_4	a_0	a_1

Clearly, $A = \{a_0, a_1\}$ is a group *w.r.t.* $*$ which is a proper subset of Q. Therefore Q is a Smarandache quasigroup.

Definition 2.3. A quasigroup Q is idempotent if every element x in Q satisfies $x * x = x$.

Theorem 2.1. If a quasigroup contains a Smarandache quasigroup then the quasigroup is a Smarandache quasigroup.

Proof. Follows from definition of Smarandache quasigroup.

Example 2.3. $(Q, *)$ defined by the following table is a quasigroup.

$*$	1	2	3	4
1	1	3	4	2
2	4	2	1	3
3	2	4	3	1
4	3	1	2	4

$(Q, *)$ is an idempotent quasigroup.

Definition 2.4. An element x in a quasigroup Q is called idempotent if $x.x = x$.

Consider a quasigroup;

$*$	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Here 2 is an idempotent element.

Example 2.4. The smallest quasigroup which is neither a group nor a loop is a quasigroup of order 3 as given by the following table;

$*$	q_1	q_2	q_3
q_1	q_1	q_2	q_3
q_2	q_3	q_1	q_2
q_3	q_2	q_3	q_1

3. A new class of Quasigroups

V.B.Kandasamy [2] has defined a new class of groupoids as follows;

Definition 3.1. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$. For a, $b \in Z_n$ define a binary operation $*$ on Z_n as: $a * b = ta + ub$ $(mod\ n)$ where t, u are two distinct element in $Z_n \setminus \{0\}$ and $(t, u) = 1$. Here $+$ is the usual addition of two integers and ta means the product of two integers t and a. We denote this groupoid by $Z_n(t, u)$.

Theorem 3.1. Let $Z_n(t, u)$ be a groupoid. If $n = t + u$ where both t and u are primes then $Z_n(t, u)$ is a quasigroup.

Proof. When t and u are primes every row and column in the composition table will have distinct n element. As a result $Z_n(t, u)$ is a quasigroup.

Corollary 3.1. If $Z_p(t, u)$ is a groupoid and $t + u = p$, $(t, u) = 1$ then $Z_p(t, u)$ is a quasigroup.

Proof. Follows from the theorem.

Example 3.1. Consider $Z_5 = \{0, 1, 2, 3, 4\}$. Let $t = 2$ and $u = 3$. Then $5 = 2 + 3$, $(2, 3) = 1$ and the composition table is:

*	0	1	2	3	4
0	0	3	1	4	2
1	2	0	3	1	4
2	4	2	0	3	1
3	1	4	2	0	3
4	3	1	4	2	0

Thus $Z_5(2, 3)$ is a quasigroup.

Definition 3.2. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub$ $(mod\ n)$ where t and $u \in Z_n \setminus \{0\}$ and $t = u$. For a fixed integer n and varying t and u we get a class of quasigroups of order n.

Example 3.2. Consider $Z_5 = \{0, 1, 2, 3, 4\}$. Then $Z_5(3, 3)$ is a quasigroup as given by the following table:

*	0	1	2	3	4
0	0	3	1	4	2
1	3	1	4	2	0
2	1	4	2	0	3
3	4	2	0	3	1
4	2	0	3	1	4

Definition 3.3. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub$ $(mod\ n)$ where t and $u \in Z_n \setminus \{0\}$ and $t = 1$ and $u = n - 1$. For a fixed integer n and varying t and u we get a class of quasigroups of order n.

Example 3.3. Consider $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $Z_8(1, 7)$ is a quasigroup as given by the following table:

*	0	1	2	3	4	5	6	7
0	0	7	6	5	4	3	2	1
1	1	0	7	6	5	4	3	2
2	2	1	0	7	6	5	4	3
3	3	2	1	0	7	6	5	4
4	4	3	2	1	0	7	6	5
5	5	4	3	2	1	0	7	6
6	6	5	4	3	2	1	0	7
7	7	6	5	4	3	2	1	0

Definition 3.4. Let $Z_n = \{0, 1, 2, \cdots, n-1\}$, $n \geq 3$, $n < \infty$. Define $*$ on Z_n as $a * b = ta + ub$ $(mod\ n)$ where t and $u \in Z_n \setminus \{0\}$ and $(t, u) = 1$, $t + u = n$ and $|t - u|$ is a minimum. For a fixed integer n and varying t and u we get a class of quasigroups of order n.

Example 3.4. Consider $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then $Z_8(3, 5)$ is a quasigroup as given by the following table:

*	0	1	2	3	4	5	6	7
0	0	5	2	7	4	1	6	3
1	3	0	5	2	7	4	1	6
2	6	3	0	5	2	7	4	1
3	1	6	3	0	5	2	7	4
4	4	1	6	3	0	5	2	7
5	7	4	1	6	3	0	5	2
6	2	7	4	1	6	3	0	5
7	5	2	7	4	1	6	3	0

Definition 3.5. Let $(Q, *)$ be a quasigroup. A proper subset V of Q is called a subquaisgroup of Q if V itself is a quasigroup under $*$.

Definition 3.6. Let Q be a quasigroup. A subquaisgroup V of Q is said to be normal subquaisgroup of Q if:

1. $aV = Va$
2. $(Vx)y = V(xy)$
3. $y(xV) = (yx)V$

for all a, x, $y \in Q$.

Example 3.5. Let Q be a quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	4	3	2	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	2	1	4	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Here $V = \{1, 2, 3, 4\}$ is a normal subquasigroup of Q.

Definition 3.7. A subquasigroup is said to be simple if it has no proper nontrivial normal subgroup.

4. Substructures of Smarandache Quasigroups

Definition 4.1. Let $(Q, *)$ be a Smarandache quasigroup. A nonempty subset H of Q is said to be a Smarandache subquasigroup if H contains a proper subset K such that k is a group under $*$.

Example 4.1. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Consider $S = \{1, 2, 3, 4\}$ then S is a subquasigroup which contains a group $G = \{1, 2\}$. Therefore S is a Smarandache subquasigroup.

Example 4.2. There do exist Smarandache quasigroup which do not posses any Smarandache subquasigroup. Consider the quasigroup Q defined by the following table:

*	1	2	3	4	5
1	3	1	4	2	5
2	5	2	3	1	4
3	1	4	2	5	3
4	4	5	1	3	2
5	2	3	5	4	1

Clearly, Q is Smarandache quasigroup as it contains a group $G = \{2\}$. But there is no subquasigroup, not to talk of Smarandache subquasigroup.

Definition 4.2. Let Q be a S-quasigroup. If $A \subset Q$ is a proper subset of Q and A is a subgroup which can not be contained in any proper subquasigroup of Q we say A is the largest subgroup of Q.

Example 4.3. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Clearly, $A = \{1, 2, 3, 4\}$ is the largest subgroup of Q.

Definition 4.3. Let Q be a S-quasigroup. If A is a proper subset of Q which is subquasigroup of Q and A contains the largest group of Q then we say A to be the Smarandache hyper subquasigroup of Q.

Example 4.4. Let Q be a quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	4	3	6	5	8	7
2	2	1	3	4	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Here $A = \{1, 2, 3, 4\}$ is the subquasigroup of Q which contains the largest group $\{1, 2\}$ of Q. A is a Smarandache hyper subquasigroup of Q.

Definition 4.4. Let Q be a finite S-quasigroup. If the order of every subgroup of Q divides the order of the S-quasigroup Q then we say Q is a Smarandache Lagrange quasigroup.

Example 4.5. In the above example 4.4, Q is a S-quasigroup whose only subgroup are $\{1\}$ and $\{1, 2\}$. Clearly, order of these subgroups divide the order of the quasigroup Q. Thus Q is the Smarandache Lagrange quasigroup.

Definition 4.5. Let Q be a finite S-quasigroup. p is the prime such that p divides the order of Q. If there exist a subgroup A of Q of order p or p^l, $(l > 1)$ we say Q has a Smarandache p-Sylow subgroup.

Example 4.6. Let $Q = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the quasigroup defined by the following table:

*	1	2	3	4	5	6	7	8
1	1	2	3	4	6	5	8	7
2	2	1	4	3	5	6	7	8
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	7	8	1	2	3	4
6	5	6	8	7	2	3	4	1
7	8	7	6	5	3	4	1	2
8	7	8	5	6	4	1	2	3

Consider $A = \{1, 2, 3, 4\}$ then A is a subgroup of Q whose order 2^2 divides order of Q. Therefore Q has a Smarandache 2-Sylow subgroup.

Definition 4.6. Let Q be a finite S-quasigroup. An element $a \in A$, $a \subset Q$ (A a proper subset of Q and A is the subgroup under the operation of Q) is said to be a Smarandache Cauchy element of Q if $a^r = 1$, $(r > 1)$ and 1 is the unit element of A and r divides the order of Q otherwise a is not a Smarandache Cauchy element of Q.

Definition 4.7. Let Q be a finite S-quasigroup if every element in every subgroup of Q is a Smarandache Cauchy element of Q then we say that Q is a Smarandache Cauchy quasigroup.

Example 4.6. In the above example 4.6 there are three subgroup of Q. They are $\{1\}$, $\{1, 2\}$ and $\{1, 2, 3, 4\}$. Each element in each subgroup is a Smarandache Cauchy element as $1^2 = 2^2 = 3^2 = 4^2 = 1$ in each respective subgroup. Thus Q is a Smarandache Cauchy group.

References

[1] R. H. Bruck, A survey of binary system, Springer-Verlag, New York, 1958.

[2] W. B. Kandasamy, Smarandache Semigroup, American Research Press, 2002.

[3] Robinson Derek J. S., A course in the theory of Groups, Springer-verlag, New York, 1996.