E. Hitzer, A real explanation for imaginary eigenvalues and complex eigenvectors, (Nat. Symp. on Math. Sc., 2001.3, Nagpur, India.) Proc. Einst. Foundation Int. 1, pp. 1-26 (2001).

A real explanation for imaginary eigenvalues and complex eigenvectors

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March 1-5, 2001, Nagpur*, India

Abstract

This paper first reviews how anti-symmetric matrices in two dimensions yield imaginary eigenvalues and complex eigenvectors. It is shown how this carries on to rotations by means of the Cayley transformation. Then the necessary tools from real geometric algebra are introduced and a real geometric interpretation is given to the eigenvalues and eigenvectors. The latter are seen to be two component eigenspinors which can be further reduced to underlying vector duplets. The eigenvalues are interpreted as rotors, which rotate the underlying vector duplets. The second part of this paper extends and generalizes the treatment to three dimensions. The final part shows how all entities and relations can be obtained in a constructive way, purely assuming the geometric algebras of 2-space and 3-space.

I. Introduction

… for geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a little child.

William K. Clifford[4]

But the gate to life is narrow and the way that leads to it is hard, and there are few people who find it. …I assure you that unless you change and become like children, you will never enter the Kingdom of heaven.

Jesus Christ[9]

The motivation for this article appears somehow accidental. I had to make linear algebra problems for students about eigenvectors of matrices and their Cayley transformations. The textbook[1] already had the problem to show that the (real) eigenvector of a three-dimensional anti-symmetric matrix was also an eigenvector of its Cayley transformation. I thought somehow why restrict it to the one real eigenvector,

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* Accepted for publication for the *Proceedings of the National Symposium on Mathematical Sciences*, March 1-5, Nagpur, India.

why not let the students work on a problem with the two complex eigenvectors as well?

But somehow the question came back to me like a boomerang, and I asked myself, what does it really mean to have complex eigenvalues and complex eigenvectors. I thought there must somehow be some real geometric meaning to this and began to explore the simple two-dimensional case. I was already familiar with geometric algebra [3,4,5,7] and was somehow convinced to get an answer applying it. Geometric algebra in a way completes our knowledge about how to properly multiply vectors, by adding the inner product and a dimension independent outer product to one new invertible, associative and distributive geometric vector product. In two dimensions it contains an even sub-algebra isomorphic to complex numbers.

This paper therefore follows a line of argument first presenting the usual problem that occurs in matrix algebra and then showing how to shift the interpretation to a completely real interpretation in terms of geometric algebra. For this purpose I briefly introduce the basics of geometric algebra and show how it helps to fill out the gaps of our understanding.

Having achieved the task in two dimensions it is only natural to try it in three dimensions as well, learn thereby something about three-dimensional geometric algebra and with hindsight get even a better understanding of what happened in two dimensions. This will also show what is particular in two dimensions and needs to be refined for solving the three-dimensional problem.

After I mastered all this in two and three dimensions, using geometric algebra to provide a real geometric understanding for what I did in complex matrix algebra, I desired to turn the strategy around: I wanted to know if it was feasible to pretend not to know about anti-symmetric matrices, their imaginary eigenvalues and complex eigenvectors in the first place, but arrive at all these entities and their relationships in a synthetic way. In the last section of this article I therefore start with pure geometric algebra in two and three dimensions and show how in a natural way relationships arise which can be put into a form completely resembling the relationships of complex matrix algebra, yet equipped with clear and well defined real-geometric meanings.

The first quotation stems from Clifford himself, who initially was a theologian and then became an atheist. But somehow his view of science was strongly colored by what Jesus taught as the Gospel about the Kingdom of God. To agree or disagree on what Clifford believed is a matter of faith and not of science. But I think the point made by him, that geometry is like a gateway to a new understanding of science is quite worthwhile to reflect upon. I hope that the reading of this article may help the reader to appreciate Clifford's opinion to some degree.

II. Two real dimensions

II.1 Complex treatment

Any anti-symmetric matrix in two real dimensions is proportional to

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The characteristic polynomial equation of the matrix U is

$$|U - \lambda E| = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

or

$$\lambda^2 = -1. \qquad \text{(J)}$$

The classical way to solve this equation is to postulate an imaginary entity j to be the root of –1: $j = \sqrt{-1}$. This leads to many interesting consequences, yet any real geometric meaning of this imaginary quantity is left obscure.

The two eigenvalues are therefore the imaginary unit j and $-j$.

$$\lambda_1 = j, \ \lambda_2 = -j$$

The corresponding complex eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are

$$U\mathbf{x_1} = \lambda_1\mathbf{x_1} = j\mathbf{x_1} \rightarrow \mathbf{x_1} = \begin{pmatrix} 1 \\ -j \end{pmatrix} \text{(Evc)}$$

$$U\mathbf{x_2} = \lambda_2\mathbf{x_2} = -j\mathbf{x_2} \rightarrow \mathbf{x_2} = \begin{pmatrix} 1 \\ j \end{pmatrix}$$

The Cayley transformation[1] C(–kU), with $k = \dfrac{1-\cos\vartheta}{\sin\vartheta}$

$$C(-kU) = (E + (-kU))^{-1}(E - (-kU)) = E - \frac{2}{1+k^2}(-kU - (-kU)^2) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

(C)

allows to describe two dimensional rotations.

The third expression of equation (C) shows that U and C(–kU) must have the same eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. The corresponding eigenvalues of C(–kU) can now easily be calculated from (C) as

$$\lambda_{c1} = 1 + \frac{2k\lambda_1(1+k\lambda_1)}{1+k^2}, \ \lambda_{c2} = 1 + \frac{2k\lambda_2(1+k\lambda_2)}{1+k^2}.$$

Inserting $\lambda_1 = j$ and $\lambda_2 = -j$ we obtain the complex eigenvalues of the two-dimensional rotation C(–kU) as

$$\lambda_{c1} = \cos\vartheta + j\sin\vartheta\,,\;\; \lambda_{c2} = \cos\vartheta - j\sin\vartheta$$

We now face the question what the imaginary and complex eigenvalues and the complex eigenvectors of U and the rotation C(–kU) mean in terms of purely real geometry. In order to do this let us turn to the real geometric algebra of a real two-dimensional vector space.

II.2 Real two-dimensional geometric algebra

The theory developed in this section is not limited to two dimensions. In the case of higher dimensions we can always deal with the two-dimensional subspace spanned by two vectors involved, etc.

II.2.1 The geometric product

Let us start with the real two-dimensional vector space R^2. It is well known that vectors can be multiplied by the inner product which corresponds to a mutual projection of one vector **a** onto another vector **b** and yields a scalar: the projected length *a cosθ* times the vector length b, i.e.

$$\mathbf{a}\cdot\mathbf{b} = (a\cos\vartheta)b = ab\cos\vartheta\,.$$

The projected vector $\mathbf{a}_{\parallel}$ itself can then be written as

$$\mathbf{a}_{\parallel} = \mathbf{a}\cdot\mathbf{b}\mathbf{b}/b^2\,, \qquad \text{(P1)}$$

where I use the convention that inner and outer products have preference to geometric products.

In 1844 the German mathematician H. Grassmann [2] introduced another general (dimension independent) vector product: the anti-symmetric exterior product. This product yields the size of the area of the parallelogram spanned by the two vectors together with an orientation, depending on the sense of following the contour line (e.g. clockwise and anticlockwise),

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}\,.$$

Grassmann later on unified the inner product and the exterior product to yield the *extensive* product, or how it was later called by W. Clifford, the *geometric* product[7] of vectors:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,. \qquad \text{(GP)}$$

We now demand (nontrivial!) this geometric product to be associative, i.e. (**ab**)**c** = **a**(**bc**) and distributive, i.e. $\mathbf{a(b+c)} = \mathbf{ab} + \mathbf{ac}$.

Let us now work out the consequences of these definitions in the two-dimensional real vector space R^2. We choose an orthonormal basis $\{\sigma_1,\sigma_2\}$. This means that

$$\boldsymbol{\sigma}_1^2 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_1 = \mathbf{1}\,,\;\; \boldsymbol{\sigma}_2^2 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_2 = \mathbf{1}\,,\;\; \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1 = \mathbf{0}\;\;. \qquad \text{(Unit)}$$

Please note that e.g. in (Unit) we don't simply multiply the coordinate representations of the basis vectors, we multiply the vectors themselves. We are therefore still free to make a certain choice of the basis vectors, i.e. we work coordinate free! The product of the two basis vectors gives

$$\sigma_1\sigma_2 = \sigma_1 \cdot \sigma_2 + \sigma_1 \wedge \sigma_2 = \sigma_1 \wedge \sigma_2 = -\sigma_2 \wedge \sigma_1 = -\sigma_2 \cdot \sigma_1 - \sigma_2 \wedge \sigma_1 = -\sigma_2\sigma_1 \equiv \mathbf{i} \quad \text{(I)}$$

the real oriented area element, which I call **i**. It is important that you beware of confusing this real area element **i** with the imaginary unit *j* mentioned in the last section!
But what is then the square of **i**?

$$\mathbf{i}^2 = \mathbf{ii} = (\sigma_1\sigma_2)(\sigma_1\sigma_2) = -(\sigma_1\sigma_2)(\sigma_2\sigma_1) = -\sigma_1(\sigma_2\sigma_2)\sigma_1 = -\sigma_1\sigma_1 = -1 \quad \text{(II)}$$

The square of the oriented real unit area element of **i** is therefore $\mathbf{i}^2 = -1$! This is the same value as the square of the imaginary unit *j*. The big difference however is, that *j* is postulated just so that the equation (J) can be solved, whereas for **i** we followed a constructive approach: We just performed the geometric product repeatedly on the basis vectors of a real two-dimensional vector space!

So far we have geometrically multiplied vectors with vectors and area elements with area elements. But what happens when we multiply vectors and area elements geometrically?

II.2.2 Rotations, vector inverse and spinors

We demonstrate this by calculating both $\sigma_1\mathbf{i}$ and $\sigma_2\mathbf{i}$:
$\sigma_1\mathbf{i} = \sigma_1(\sigma_1\sigma_2) = (\sigma_1\sigma_1)\,\sigma_2 = \sigma_2$
$\sigma_2\mathbf{i} = \sigma_2(-\sigma_2\sigma_1) = -\,(\sigma_2\sigma_2)\,\sigma_1 = -\sigma_1$.
This is precisely a 90 degree anticlockwise (mathematically positive) rotation of the two basis vectors and therefore of all vectors by linearity. From this we immediately conclude that multiplying a vector twice with the oriented unit area element **i** constitutes a rotation by 180 degree. Consequently, the square $\mathbf{i}^2 = -1$ geometrically means just to rotate vectors by 180 degree. I emphasize again that *j* and **i** need to be thoroughly kept apart. *j* also generates a rotation by 90 degree, but this is in the plane of complex numbers commonly referred to as the Gaussian plane. It is not to be confused with the 90 degree real rotation **i** of real vectors in the two-dimensional real vector space.

i also generates all real rotations with arbitrary angles. To see this let **a** and **b** be unit vectors. Then I calculate:

$$\mathbf{a}(\mathbf{ab}) = (\mathbf{aa})\mathbf{b} = \mathbf{b} \quad \text{(r)}$$

Multiplying **a** with the product **ab** therefore rotates **a** into **b**. $R_{ab}=\mathbf{ab}$ is therefore the "rotor" that rotates (even all!) vectors by the angle between **a** and **b**. What this has to do with **i**? Performing the geometric product **ab** explicitely yields:

$$\mathbf{ab} = \cos\theta + \sin\theta\,\mathbf{i}$$

(Please keep in mind that here $\mathbf{a}^2 = \mathbf{b}^2 = 1$ and that the area of the parallelogram spanned by **a** and **b** is precisely $\sin\theta_{ab}$, which explains the second term.) This can formally be written by using the exponential function as:

$$R_{ab} = \mathbf{ab} = \exp(\mathbf{i}\,\theta_{ab}) = \cos\theta_{ab} + \sin\theta_{ab}\,\mathbf{i} \qquad \text{(R)}$$

We can therefore conclude that the oriented unit area element **i** generates indeed all rotations of vectors in the real two-dimensional vector space.

Another important facet of the geometric product is that it allows to universally define the inverse of a vector with respect to (geometric) multiplication as:

$$\mathbf{x}^{-1} = \frac{1}{\mathbf{x}} \overset{def}{=} \frac{\mathbf{x}}{\mathbf{x}^2}, \quad \mathbf{x}^2 = \mathbf{xx} = \mathbf{x}\cdot\mathbf{x}.$$

That this is indeed the inverse can be seen by calculating

$$\mathbf{xx}^{-1} = \mathbf{x}^{-1}\mathbf{x} = \frac{\mathbf{xx}}{\mathbf{x}^2} = 1.$$

Using the inverse $\mathbf{b}^{-1}$ of the vector **b**, we can rewrite the projection of **a** unto **b** simply as

$$\mathbf{a}_{\|} = \mathbf{a}\cdot\mathbf{b}\mathbf{b}^{-1}. \qquad \text{(P2)}$$

It proves sometimes useful to also define an inverse for area elements $A = \pm|A|\mathbf{i}$:

$$A^{-1} = A/A^2 = A/(-|A|^2) = -A/|A|^2,$$

where $|A|$ is the scalar size of the area and one of the signs stands for the orientation of A relative to **i**. We can see that this is really the inverse by calculating

$$AA^{-1} = A^{-1}A = AA/A^2 = A^2/A^2 = -|A|^2/(-|A|^2) = 1.$$

By now we also know that performing the geometric product of vectors of a real two-dimensional vector space will only lead to (real) scalar multiples and linear combinations of scalars (grade 0), vectors (grade 1) and oriented area elements (grade 2). In algebraic theory one assigns grades to each of these. All these entities which are generated such form the real geometric algebra of a real two-dimensional vector space, designated with R_2 (note that the index is now a lower index). R_2 can be generated through (real scalar) linear combinations of the following list of 2^2=4 elements

$$\{1, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \mathbf{i}\}.$$

This list is said to form the basis of R_2. When analyzing any algebra it is always very interesting to know if there are any subsets of an algebra which stay closed when performing both linear combinations and the geometric product. Indeed it is not difficult to see that the subset {1, **i**} is closed, because 1**i** = **i** and **ii** = –1. This sub-algebra is in one-to-one correspondence with the complex numbers C. We thus see that we can "transfer" all relationships of complex numbers, etc. to the real two-dimensional geometric algebra R_2. We suffer therefore no disadvantage by refraining from the use of complex numbers altogether. The important operation of complex conjugation (replacing *j* by *–j* in a complex number) corresponds to *reversion* in geometric algebra, that is the order of all vectors in a product is reversed:

$(\mathbf{ab})^{\dagger} = \mathbf{ba}$ and therefore $\mathbf{i}^{\dagger} = (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)^{\dagger} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = -\mathbf{i}$.

In mathematics the geometric product of two vectors (compare e.g. (GP),(Unit),(II),(R)) is also termed a *spinor*. In physics use of spinors is frequently considered to be confined to quantum mechanics, but as we have just seen in (R), spinors describe every elementary rotation in two dimensions. (Spinors describe rotations in higher dimensions as well, since rotations are always performed in plane two-dimensional subspaces, e.g. in three dimensions the planes perpendicular to the axis of rotation.)

By now we have accumulated enough real geometric tools in order to work out the real explanation for the imaginary and complex eigenvalues and –vectors of section 1.

II.3 Real explanation

Let us skip back to the characteristic polynomial equation of the matrix U in section 1:

$$\lambda^2=-1.$$

Instead of postulating the imaginary unit *j* we now turn to the real two-dimensional algebra R_2 and set the eigenvalues λ_1 and λ_2 to simply be:

$$\lambda_1 = \mathbf{i}, \quad \lambda_2= -\mathbf{i}.$$

The corresponding "eigenvectors" $\mathbf{x}_1$ and $\mathbf{x}_2$ will then be:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix}.$$

As in section 1, the eigenvectors of the Cayley transformation C(–kU) will be the same. And the eigenvalues of C(–kU) now become:

$$\lambda_{c1} = \cos\vartheta + \mathbf{i}\sin\vartheta, \quad \lambda_{c2} = \cos\vartheta - \mathbf{i}\sin\vartheta. \qquad \text{(LC)}$$

We can now take the first step in our real explanation and identify λ_1 and λ_2 as the real oriented unit area element with both orientations (+,–). We can further identify the two "eigenvectors" $\mathbf{x}_1$ and $\mathbf{x}_2$ as *two-component spinors* with the entries: $x_{11}=1$, $x_{12}= -\mathbf{i}$ and $x_{21}=1$, $x_{22}= \mathbf{i}$. Finally the eigenvalues λ_{c1} and λ_{c2} of the Cayley transformation C(–kU) are seen to simply be rotors (compare (R)), i.e. operators which rotate vectors by θ and $-\theta$, respectively.

Now we want to better understand what the real-oriented-unit-area-element eigenvalues $\lambda_1 = \mathbf{i}$, $\lambda_2= -\mathbf{i}$ as well as $\lambda_{c1} = \cos\vartheta + \mathbf{i}\sin\vartheta$ and $\lambda_{c2} = \cos\vartheta - \mathbf{i}\sin\vartheta$ do when multiplied with the two-component *eigen-spinors* (previously termed "eigenvectors") $\mathbf{x}_1$ and $\mathbf{x}_2$. In the last section on the real two-dimensional geometric algebra, we already learnt that every spinor can be understood to be the geometric product of two vectors. We therefore choose an arbitrary, but fixed reference vector $\mathbf{z}$ from the vector space R^2. For simplicity let us take $\mathbf{z}$ to be $\mathbf{z} =\sigma_1$. We can then factorize the spinor components of the eigen-spinors $\mathbf{x}_1$ and $\mathbf{x}_2$ to:

$$x_{11}=1= \sigma_1 \sigma_1,\ x_{12}= -\mathbf{i} = -\sigma_1 \sigma_2 = \sigma_2 \sigma_1$$
$$\text{and } x_{21}=1= \sigma_1 \sigma_1,\ x_{22}= \mathbf{i} = \sigma_1 \sigma_2 = -\sigma_2 \sigma_1.$$

The eigen-spinor $\mathbf{x}_1$ is thus seen to correspond (modulus the geometric multiplication from the right with $\mathbf{z} =\sigma_1$) to the real vector pair (σ_1, σ_2), whereas $\mathbf{x}_2$ corresponds to the real vector pair (σ_1, $-\sigma_2$). Multiplication with λ_1 from the left as in

$$U\mathbf{x}_1 = \lambda_1\mathbf{x}_1 = \mathbf{i}\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} \mathbf{i}x_{11} \\ \mathbf{i}x_{12} \end{pmatrix}$$

results in

$$\mathbf{x}_{11} \to \mathbf{i}\mathbf{x}_{11} = \mathbf{i}1 = (\sigma_1\sigma_2)(\sigma_1\sigma_1) = -\sigma_2(\sigma_1\sigma_1)\sigma_1 = (-\sigma_2)\sigma_1 .$$

That is the multiplication with $\lambda_1 = \mathbf{i}$ from the left transforms the first entry in the vector pair (σ_1, σ_2), which corresponds to $\mathbf{x}_1$, to $-\sigma_2$. Performing the same calculation for the second entry σ_2 in (σ_1, σ_2) yields σ_1. So the whole vector pair (σ_1, σ_2) is transformed to the new pair ($-\sigma_2$, σ_1). Which upon a close look is seen to be a simple rotation by –90 degree. Here the mathematically speaking *non-commutative* nature of the geometric product comes into play. That is the order of the factors in a geometric product does really matter. The multiplication of a vector with **i** from the right gives a rotation by +90 degree, whilst the multiplication with **i** from the left yields a rotation by –90 degree.

Analogous calculations for $\lambda_2\,\mathbf{x}_2 = -\mathbf{i}\mathbf{x}_2$ show that the pair (σ_1, $-\sigma_2$), which corresponds to $\mathbf{x}_2$ is transformed to (σ_2, σ_1), i.e. it is rotated by +90 degree.

Let me summarize therefore that the multiplications $\lambda_1\,\mathbf{x}_1$ and $\lambda_2\,\mathbf{x}_2$ are thus understood to rotate the underlying vector pairs (corresponding to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively) by –90 and +90 degrees, respectively!

In analogy to the treatment of $\lambda_1\,\mathbf{x}_1$ and $\lambda_2\,\mathbf{x}_2$, I will now treat $\lambda_{c1}\,\mathbf{x}_1$ and $\lambda_{c2}\,\mathbf{x}_2$. Comparing (LC) and (R) one may already suspect that λ_{c1} and λ_{c2} may simply rotate the vector pairs corresponding to $\mathbf{x}_1$ and $\mathbf{x}_2$ by $-\theta$ and $+\theta$, respectively. (As for the sign of θ, the non-commutative nature of the geometric product needs again to be taken into account.) But let us prove this now explicitly.

$$\mathbf{x}_{11} \to \lambda_{c1}\mathbf{x}_{11} = (\cos\vartheta + \mathbf{i}\sin\vartheta)(\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1) = (\cos\vartheta\boldsymbol{\sigma}_1 + \mathbf{i}\boldsymbol{\sigma}_1\sin\vartheta)\boldsymbol{\sigma}_1 = (\cos\vartheta\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_1\mathbf{i}\sin\vartheta)\boldsymbol{\sigma}_1 =$$

$$\boldsymbol{\sigma}_1(\cos\vartheta - \mathbf{i}\sin\vartheta)\boldsymbol{\sigma}_1 = \boldsymbol{\sigma}_1(\cos(-\vartheta) + \mathbf{i}\sin(-\vartheta))\boldsymbol{\sigma}_1 = (\boldsymbol{\sigma}_1 R(-\vartheta))\boldsymbol{\sigma}_1 .$$

After the third equation sign we have used the fact that the oriented unit area element **i** anti-commutes with all vectors in the plane characterized by **i**. I will show this explicitly for σ_1:

$$\mathbf{i}\sigma_1 = (\sigma_1\sigma_2)\sigma_1 = \sigma_1(\sigma_2\sigma_1) = \sigma_1(-\sigma_1\sigma_2) = -\sigma_1\mathbf{i} . \text{ (ac)}$$

The multiplication $\lambda_{c1}\,\mathbf{x}_1$ is therefore shown to rotate the first vector σ_1, in the vector pair that corresponds to the eigen-spinor $\mathbf{x}_1$, into $\sigma_1 \to \sigma_1 R(-\theta)$. According to (r) and (R) this is a rotation of σ_1 by $-\theta$. Performing the same calculation for $\lambda_{c1}\,x_{12}$ we find that the second vector σ_2 in the vector pair that corresponds to $\mathbf{x}_1$ is rotated likewise: $\sigma_2 \to \sigma_2 R(-\theta)$. In the very same way it can be proven explicitly that $\lambda_{c2}\,\mathbf{x}_2$ rotates the vector pair (σ_1, $-\sigma_2$), which corresponds to $\mathbf{x}_2$ into ($\sigma_1 R(\theta)$, $-\sigma_2 R(\theta)$).

We have therefore confirmed that the multiplications $\lambda_{c1}\,\mathbf{x}_1$ and $\lambda_{c2}\,\mathbf{x}_2$ geometrically mean to rotate the vector pairs corresponding to $\mathbf{x}_1$ and $\mathbf{x}_2$ by $-\theta$ and $+\theta$, respectively.

Summarizing we see that the complex eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ may rightfully be interpreted as two-component eigen-spinors with underlying vector pairs. The multiplication of these eigen-spinors with the unit-oriented-area-element eigenvalues λ_1 and λ_2 means a real rotation of the underlying vector pairs by –90 and +90 degrees, respectively. Whereas the multiplication with λ_{c1} and λ_{c2} means a real rotation of the underlying vector pairs by $-\theta$ and $+\theta$, respectively.

I concede that despite of the strong case for a real explanation of both imaginary (and complex) eigenvalues and complex eigenvectors, one may at first sight wonder
(1) how to extend this explanation to higher dimensions and
(2) whether the treatment of higher dimensions might not become too complicated.
In order to show that the extension to higher dimensions is fairly easy, straight forward and not at all complicated, I will now discuss the same problem for the case of three dimensions.

III. Three real dimensions

III.1 Complex treatment of three dimensions

Any anti-symmetric matrix in 3 dimensions is proportional to a matrix of the form

$$U = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

with $a^2+b^2+c^2=1$. The characteristic polynomial equation of the matrix U is

$$|U - \lambda E| = \det\begin{pmatrix} -\lambda & -c & a \\ c & -\lambda & -b \\ -a & b & -\lambda \end{pmatrix} = -\lambda(\lambda^2 + a^2 + b^2 + c^2) = 0\,.$$

If we use the condition that $a^2+b^2+c^2=1$, this simplifies and breaks up into the two equations

$$\lambda_{1,2}^2 = -1 \qquad \text{(J3)}$$

and $\lambda_3 = 0$.

That means we have one eigenvalue λ_3 equal to zero and for the other two eigenvalues λ_1, λ_2 we have the same condition as in the two-dimensional case for the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

It is therefore clear that in the conventional treatment one would again assign $\lambda_1=j$ and $\lambda_2 = -j$. The corresponding eigenvectors are:

$$\mathbf{x_1} = \begin{pmatrix} 1-a^2 \\ -ab-jc \\ -ac+jb \end{pmatrix} \cong \begin{pmatrix} -ab+jc \\ 1-b^2 \\ -bc-ja \end{pmatrix} \cong \begin{pmatrix} -ac-jb \\ -bc+ja \\ 1-c^2 \end{pmatrix}$$

$$\mathbf{x_2} = \begin{pmatrix} 1-a^2 \\ -ab+jc \\ -ac-jb \end{pmatrix} \cong \begin{pmatrix} -ab-jc \\ 1-b^2 \\ -bc+ja \end{pmatrix} \cong \begin{pmatrix} -ac+jb \\ -bc-ja \\ 1-c^2 \end{pmatrix}. \text{(CEV)}$$

The ≑ sign expresses that all three given forms are equivalent up to the multiplication with a scalar (complex) constant. The eigenvector that corresponds to λ_3 simply is:

$$\mathbf{x_3} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The fact that $\lambda_3 = 0$ simply means that the matrix U projects out any component of a vector parallel to $\mathbf{x_3}$. U maps the three-dimensional vector space therefore to a plane perpendicular to $\mathbf{x_3}$ containing the origin.

The Cayley transformation[1] C(–kU) with $k = \frac{1-\cos\vartheta}{\sin\vartheta}$ now describes rotations in three dimensions:

$$C(-kU) = (E+(-kU))^{-1}(E-(-kU)) = E - \frac{2}{1+k^2(a^2+b^2+c^2)}(-kU-(-kU)^2) =$$

$$= \begin{pmatrix} 1+(1-\cos\vartheta)(a^2-1) & -c\sin\vartheta+ab(1-\cos\vartheta) & b\sin\vartheta+ac(1-\cos\vartheta) \\ c\sin\vartheta+ab(1-\cos\vartheta) & 1+(1-\cos\vartheta)(b^2-1) & -a\sin\vartheta+bc(1-\cos\vartheta) \\ -b\sin\vartheta+ac(1-\cos\vartheta) & a\sin\vartheta+bc(1-\cos\vartheta) & 1+(1-\cos\vartheta)(c^2-1) \end{pmatrix}. \text{(C3)}$$

The vector $\mathbf{x_3}$ plays here the role of the rotation axis. If e.g. we set a=b=0, c=1, we get the usual rotation around the z-axis:

$$C(-kU)[\mathbf{x}_3 = (0,0,1)] = \begin{pmatrix} \cos\vartheta & -\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The expression for C(–kU) after the second equal sign clearly shows that the eigenvectors of U and C(–kU) agree in three dimensions as well. The general formula for calculating the eigenvalues λ_c of C(–kU) from the eigenvalues λ of U reads as follows:

$$\lambda_c = 1 + \frac{2k\lambda(1+k\lambda)}{1+k^2(a^2+b^2+c^2)} \overset{a^2+b^2+c^2=1}{=} 1 + \frac{2k\lambda(1+k\lambda)}{1+k^2}$$

Inserting λ_1, λ_2 and λ_3 in this formula yields:

$$\lambda_{c1} = \cos\vartheta + j\sin\vartheta,\ \lambda_{c2} = \cos\vartheta - j\sin\vartheta,\ \lambda_{c3} = 1.$$

We now already see that the most interesting differences to the two-dimensional case lie in the (complex) eigenvectors, besides the fact that the expression for the rotation matrix C(–kU) looks rather more complicated.

III.2 Real three-dimensional geometric algebra

We begin with a real three-dimensional vector space R^3. In R^3 we introduce an orthonormal set of basis vectors $\{\sigma_1, \sigma_2, \sigma_3\}$, that is $\boldsymbol{\sigma}_m \cdot \boldsymbol{\sigma}_n = 1$ for n=m and $\boldsymbol{\sigma}_m \cdot \boldsymbol{\sigma}_n = 0$ for $n \neq m$, {n,m=1,2,3}. The basic $2^3 = 8$ geometric entities we can form with these basis vectors are:

1,	$\{\sigma_1, \sigma_2, \sigma_3\}$,	$\{\mathbf{i}_3 = \sigma_1 \sigma_2, \mathbf{i}_1 = \sigma_2 \sigma_3, \mathbf{i}_2 = \sigma_3 \sigma_1\}$,	$i = \sigma_1 \sigma_2 \sigma_3$
scalar	vectors	oriented unit real area	oriented real volume
(grade 0)	(grade 1)	elements (grade 2)	element (grade 3)

We now have three real oriented unit area elements $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ corresponding to the three plane area elements of a cube oriented with its edges along σ_1, σ_2, and σ_3. This set of eight elements $\{1, \sigma_1, \sigma_2, \sigma_3, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3, i\}$ forms the real geometric algebra R_3 of the three dimensional vector space R^3. By looking at the subsets $\{1, \sigma_1, \sigma_2, \mathbf{i}_3\}$, $\{1, \sigma_2, \sigma_3, \mathbf{i}_1\}$ and $\{1, \sigma_3, \sigma_1, \mathbf{i}_2\}$ we see that R_3 comprises three plane geometric sub-algebras, as we have studied them in section I.2. In general, by taking any two unit vectors {**u**, **v**} which are perpendicular to each other, we can generate new two-dimensional plane geometric sub-algebras of R_3 with the unit area element **i** = **uv**.

As in the two-dimensional case we have $\mathbf{i}_1^2 = \mathbf{i}_2^2 = \mathbf{i}_3^2 = \mathbf{i}^2 = -1$. And we have

$$i^2 = i\,i = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3 = -\sigma_1 \sigma_2 \sigma_3 \sigma_1 (\sigma_3 \sigma_2) = \sigma_1 \sigma_2 \sigma_3 (\sigma_3 \sigma_1) \sigma_2 = -\sigma_1 \sigma_2 \sigma_3 (\sigma_3 \sigma_2) \sigma_1$$
$$= -\sigma_1 \sigma_2 (\sigma_3 \sigma_3) \sigma_1 \sigma_2 = \ldots = -1.$$

Each permutation after the third, fourth and fifth equal sign introduced a factor of –1 as in (I). The square of the oriented three-dimensional volume element is therefore also $i^2 = -1$.

In three dimensions the vector **a** unto **b** projection formula (P2) does not change, since it relates only entities in the **a**,**b** plane. But beyond that we can also project vectors **a** onto **i** planes, by characterizing a plane by its oriented unit area element **i**. In this context it proves useful to generalize the definition of scalar product to elements of higher grades [3]:

$$\mathbf{a} \cdot B_r = \frac{1}{2}\left(\mathbf{a}B_r + (-1)^{1+r} B_r \mathbf{a}\right),$$

where r denotes the grade of the algebraic element B_r. For $B_r = \mathbf{b}$ (r=1) we have as usual $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$, but for $B_r = \mathbf{i}$ (example with grade r=2) we have

$$\mathbf{a} \cdot \mathbf{i} = \frac{1}{2}(\mathbf{ai} - \mathbf{ia}).$$

We can calculate for example

$$\sigma_1 \cdot \mathbf{i}_1 = \frac{1}{2}(\sigma_1\sigma_2\sigma_3 - \sigma_2\sigma_3\sigma_1) = 0$$

$$\sigma_2 \cdot \mathbf{i}_1 = \frac{1}{2}(\sigma_2\sigma_2\sigma_3 - \sigma_2\sigma_3\sigma_2) = \sigma_3$$

$$\sigma_3 \cdot \mathbf{i}_1 = \frac{1}{2}(\sigma_3\sigma_2\sigma_3 - \sigma_2\sigma_3\sigma_3) = -\sigma_2 .$$

If we now rotate σ_3 (and $-\sigma_2$) with $\mathbf{i}_1^{-1} = -\mathbf{i}_1$ from the right by –90 degree in the σ_2, σ_3 plane, we obtain

$$\sigma_2 \cdot \mathbf{i}_1\mathbf{i}_1^{-1} = -\sigma_3\mathbf{i}_1 = -\sigma_3\sigma_2\sigma_3 = \sigma_2, \;\; \sigma_3 \cdot \mathbf{i}_1\mathbf{i}_1^{-1} = \sigma_2\mathbf{i}_1 = \sigma_2\sigma_2\sigma_3 = \sigma_3 ,$$

respectively.
The projection of any vector **a** unto the σ_3, σ_2 plane is therefore given by

$$\mathbf{a}_{||} = \mathbf{a} \cdot \mathbf{i}_1\mathbf{i}_1^{-1} .$$

We say therefore instead of σ_2, σ_3 plane also simply $\mathbf{i}_1$-plane. And in general the projection of a vector **a** unto any **i**-plane is then given by

$$\mathbf{a}_{||} = \mathbf{a} \cdot \mathbf{i}\ \mathbf{i}^{-1} ,$$

which is in perfect analogy to the vector unto vector projection in formula (P2).

There is more[4,5] to be said about R_3, but the above may suffice for our present purposes.

III.3 Real explanations for three dimensions

If we follow the treatment of the two-dimensional case given in section II.3, then we need to replace the imaginary unit *j* in the eigenvalues λ_1, λ_2 and in the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$ by an element of the real three-dimensional geometric algebra R_3. In principle there are two different choices: The volume element *i* or any two-dimensional unit area element like e.g. $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$.

Let me argue for the second possibility: We have seen in section III.1 that the multiplication of U with a vector always projects out the component of this vector parallel to $\mathbf{x}_3$ so that the **y** on the right hand side of equations like U**x** = **y** is necessarily a vector in the two-dimensional plane perpendicular to $\mathbf{x}_3$ containing the origin. $\lambda_1 \mathbf{x}_1$ and $\lambda_2\mathbf{x}_2$ are precisely such vectors, since they arise from $U\mathbf{x}_1$ and $U\mathbf{x}_2$, respectively. $\lambda_1 \mathbf{x}_1$ and $\lambda_2\mathbf{x}_2$ are therefore recognized as vectors belonging to the two-dimensional plane perpendicular to $\mathbf{x}_3$ containing the origin. Thus it seems only natural to interpret the squareroot of –1 in the solution of equation (J3) to be the oriented unit area element **i** characteristic for the plane perpendicular to $\mathbf{x}_3$ containing the origin as opposed to the volume element element *i* or any other two-dimensional unit area element.

The oriented unit area elements $\mathbf{i}_1$ of the σ_2–σ_3 plane, $\mathbf{i}_2$ of the σ_3–σ_1 plane, and $\mathbf{i}_3$ of the σ_2–σ_1 plane, are perpendicular to the vectors σ_1, σ_2, and σ_3, respectively. The unit area element $\mathbf{i}$ of the two-dimensional plane perpendicular to $\mathbf{x}_3$ will therefore simply be:

$$\mathbf{i} = a\,\mathbf{i}_1 + b\,\mathbf{i}_2 + c\,\mathbf{i}_3 = a\;\sigma_2\sigma_3 + b\;\sigma_3\sigma_1 + c\;\sigma_1\sigma_2 .$$

I therefore consequently set $\lambda_1 = \mathbf{i}$, $\lambda_2 = -\mathbf{i}$ and

$$\mathbf{x}_1 = \begin{pmatrix} 1-a^2 \\ -ab-\mathbf{i}c \\ -ac+\mathbf{i}b \end{pmatrix} \cong \begin{pmatrix} -ab+\mathbf{i}c \\ 1-b^2 \\ -bc-\mathbf{i}a \end{pmatrix} \cong \begin{pmatrix} -ac-\mathbf{i}b \\ -bc+\mathbf{i}a \\ 1-c^2 \end{pmatrix}$$

$$\mathbf{x}_2 = \begin{pmatrix} 1-a^2 \\ -ab+\mathbf{i}c \\ -ac-\mathbf{i}b \end{pmatrix} \cong \begin{pmatrix} -ab-\mathbf{i}c \\ 1-b^2 \\ -bc+\mathbf{i}a \end{pmatrix} \cong \begin{pmatrix} -ac+\mathbf{i}b \\ -bc-\mathbf{i}a \\ 1-c^2 \end{pmatrix} \qquad \text{(EV3)}$$

As in the two-dimensional case I interpret the three components of each "eigenvector" as spinorial components, i.e. elementary geometric products of two vectors. (In the following we will therefore use the expression *three component eigenspinor* instead of "eigenvector".) I again arbitrarily fix one vector from the $\mathbf{i}$ plane (the plane perpendicular to $\mathbf{x}_3$) as a reference vector $\mathbf{z}$ with respect to which I will factorize the three component eigenspinors $\mathbf{x}_1$ and $\mathbf{x}_2$. With respect to the first representation of the eigenspinors $\mathbf{x}_1$ and $\mathbf{x}_2$ we choose to set $\mathbf{z} = \sigma_{1\|} = \sigma_1 \cdot \mathbf{i}\mathbf{i}^{-1} = (1-a^2)\sigma_1 - ab\sigma_2 - ac\sigma_3$.

The square of $\mathbf{z}$ gives the first component spinor of $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{z}^2 = \mathbf{z}\mathbf{z} = \mathbf{z}\cdot\mathbf{z} = (1-a^2)^2 + a^2b^2 + a^2c^2 = 1-a^2 ,$$

where we make use of the condition $a^2 + b^2 + c^2 = 1$.

The inverse of $\mathbf{z}$ will therefore be:

$$\mathbf{z}^{-1} = \mathbf{z}/\mathbf{z}\mathbf{z} = \sigma_1 - ab/(1-a^2)\,\sigma_2 - ac/(1-a^2)\,\sigma_3 ..$$

In order to now solve the equations for the two other vectors underlying the eigenspinor components $-ab-\mathbf{i}c$ and $-ac-\mathbf{i}b$ of $\mathbf{x}_1$:

$$\mathbf{n}_2\mathbf{z} = -ab-\mathbf{i}c \quad \text{and} \quad \mathbf{n}_3\mathbf{z} = -ac-\mathbf{i}b,$$

we can simply multiply with $\mathbf{z}^{-1}$ from the right:

$\mathbf{n}_2 = \mathbf{n}_2\mathbf{z}\mathbf{z}^{-1} = (-ab-\mathbf{i}c)\,\mathbf{z}^{-1} = (-ab - c\,\{a\,\mathbf{i}_1 + b\,\mathbf{i}_2 + c\;\mathbf{i}_3\})(\sigma_1 - ab/(1-a^2)\,\sigma_2 - ac/(1-a^2)\,\sigma_3)$

$= (-ab\,\sigma_2 + a^2b^2/(1-a^2)\sigma_2 + a^2bc/(1-a^2)\,\sigma_3 - ca\,\mathbf{i}_1\sigma_1 - bc\,\mathbf{i}_2\sigma_1 - c\,\mathbf{i}_3\sigma_1 +$

$a^2bc/(1-a^2)\,\mathbf{i}_1\sigma_2 + ab^2c/(1-a^2)\,\mathbf{i}_2\sigma_2 + abc^2/(1-a^2)\;\mathbf{i}_3\sigma_2 +$

$a^2c^2/(1-a^2)\,\mathbf{i}_1\sigma_3 + abc^2/(1-a^2)\,\mathbf{i}_2\sigma_3 + ac^3/(1-a^2)\;\mathbf{i}_3\sigma_3)$.

In order to simplify the above equation we first calculate

$\mathbf{i}_1\sigma_1 = \sigma_2\sigma_3\sigma_1 = i$

$\mathbf{i}_2\sigma_1 = \sigma_3\sigma_1\sigma_1 = \sigma_3$

$\mathbf{i}_3\sigma_1 = \sigma_1\sigma_2\sigma_1 = -\,\sigma_1\sigma_1\sigma_2 = -\sigma_2$

$\mathbf{i}_1\sigma_2 = \sigma_2\sigma_3\sigma_2 = -\,\sigma_2\sigma_2\sigma_3 = -\,\sigma_3$

$\mathbf{i}_2\sigma_2 = \sigma_3\sigma_1\sigma_2 = (-1)^2\,\sigma_1\sigma_2\sigma_3 = i$

$\mathbf{i}_3\sigma_2 = \sigma_1\sigma_2\sigma_2 = \sigma_1$

$\mathbf{i}_1\sigma_3 = \sigma_2\sigma_3\sigma_3 = \sigma_2$

$\mathbf{i}_2\sigma_3 = \sigma_3\sigma_1\sigma_3 = -\sigma_1\sigma_3\sigma_3 = -\sigma_1$

$\mathbf{i}_3\sigma_3 = \sigma_1\sigma_2\sigma_3 = i$.

After reordering everything we get:
$\mathbf{n}_2 = (1-b^2)\sigma_2 - ab\,\sigma_1 - bc\,\sigma_3 = \sigma_{2\,||}$.
In the very same way we can now calculate
$\mathbf{n}_3 = \mathbf{n}_3\mathbf{z}\mathbf{z}^{-1} = (1-c^2)\sigma_3 - ac\,\sigma_1 - bc\,\sigma_2 = \sigma_{3\,||}$.
Summarizing these calculations we have (setting $\mathbf{n}_1 = \mathbf{z} = \sigma_{1\,||}$):

$$\mathbf{x}_1 = \begin{pmatrix} 1-a^2 \\ -ab-\mathbf{i}c \\ -ac+\mathbf{i}b \end{pmatrix} = \begin{pmatrix} \mathbf{n}_1\mathbf{z} \\ \mathbf{n}_2\mathbf{z} \\ \mathbf{n}_3\mathbf{z} \end{pmatrix} = \begin{pmatrix} \sigma_{1||}\sigma_{1||} \\ \sigma_{2||}\sigma_{1||} \\ \sigma_{3||}\sigma_{1||} \end{pmatrix}$$

So all we need to give a real geometric interpretation for the three-component eigenspinor x_1 are geometric products of the projections of the three basis vectors σ_1, σ_2 and σ_3 onto the $\mathbf{i}$ plane. The other two equivalent representations of $\mathbf{x}_1$ given in (EV3) can be written as:

$$\mathbf{x}_1 \cong \begin{pmatrix} -ab+\mathbf{i}c \\ 1-b^2 \\ -bc-\mathbf{i}a \end{pmatrix} = \begin{pmatrix} \sigma_{1||}\sigma_{2||} \\ \sigma_{2||}\sigma_{2||} \\ \sigma_{3||}\sigma_{2||} \end{pmatrix} \quad \text{and} \quad \mathbf{x}_1 \cong \begin{pmatrix} -ac-\mathbf{i}b \\ -bc+\mathbf{i}a \\ 1-c^2 \end{pmatrix} = \begin{pmatrix} \sigma_{1||}\sigma_{3||} \\ \sigma_{2||}\sigma_{3||} \\ \sigma_{3||}\sigma_{3||} \end{pmatrix}.$$

We see that this simply corresponds to a different choice of the reference vector $\mathbf{z}$, as $\mathbf{z}=\sigma_{2\,||}$ and as $\mathbf{z}=\sigma_{3\,||}$, respectively. In general all possible ways to write $\mathbf{x}_1$ correspond to different choices of $\mathbf{z}$ from the $\mathbf{i}$ plane. The geometric product $R_{zz'} = \mathbf{z}\mathbf{z}'$ of any two such reference vectors $\mathbf{z}$ and $\mathbf{z}'$ gives the rotation operation to rotate one $\mathbf{x}_1(\mathbf{z})$ choice into the other $\mathbf{x}_1(\mathbf{z}') = \mathbf{x}_1(\mathbf{z})\, R_{zz'}$.

Comparing the (complex) "eigenvectors" $\mathbf{x}_1$ and $\mathbf{x}_2$ in (CEV) we see that they are related to each other by complex conjugation. Since the corresponding operation in geometric algebra is the reversion of the order of vectors it is no wonder that the components of the eigenspinor $\mathbf{x}_2$ are formed by taking the reverse order of vector factors appearing in the factorization of $\mathbf{x}_1$:

$$\mathbf{x}_2 = \begin{pmatrix} 1-a^2 \\ -ab+\mathbf{i}c \\ -ac-\mathbf{i}b \end{pmatrix} = \begin{pmatrix} \mathbf{z}\mathbf{n}_1 \\ \mathbf{z}\mathbf{n}_2 \\ \mathbf{z}\mathbf{n}_3 \end{pmatrix} = \begin{pmatrix} \sigma_{1||}\sigma_{1||} \\ \sigma_{1||}\sigma_{2||} \\ \sigma_{1||}\sigma_{3||} \end{pmatrix} \cong \begin{pmatrix} \sigma_{2||}\sigma_{1||} \\ \sigma_{2||}\sigma_{2||} \\ \sigma_{2||}\sigma_{3||} \end{pmatrix} \cong \begin{pmatrix} \sigma_{3||}\sigma_{1||} \\ \sigma_{3||}\sigma_{2||} \\ \sigma_{3||}\sigma_{3||} \end{pmatrix}.$$

Naturally nothing can hinder us to factorize out the reference vector $\mathbf{z}$ in for $\mathbf{x}_2$ also to the right, we just end up with somewhat less handy expressions.

Let us now turn to the interpretation of the eigenvalues λ_{c1} and λ_{c2} of the Cayley transformation C(–kU). (We will see that the eigenvalues λ_1 and λ_2 may indeed be understood as special cases of λ_{c1} and λ_{c2} by simply setting θ=90 degree.) We now write them – replacing j by $\mathbf{i}$ – as:

$$\lambda_{c1} = \cos\theta + \mathbf{i}\,\sin\theta \quad \text{and} \quad \lambda_{c2} = \cos\theta - \mathbf{i}\,\sin\theta\,.$$

The action of λ_{c1} on the three-component eigenspinor $\mathbf{x}_1$ is:

$$C(-kU)\mathbf{x}_1 = \lambda_{c1}\mathbf{x}_1 = \lambda_{c1}\begin{pmatrix}\mathbf{n}_1\mathbf{z}\\ \mathbf{n}_2\mathbf{z}\\ \mathbf{n}_3\mathbf{z}\end{pmatrix} = \begin{pmatrix}\lambda_{c1}\mathbf{n}_1\mathbf{z}\\ \lambda_{c1}\mathbf{n}_2\mathbf{z}\\ \lambda_{c1}\mathbf{n}_3\mathbf{z}\end{pmatrix}. \quad \text{(Lc3.1)}$$

As in the two-dimensional treatment of section II.2, the products $\lambda_{c1}\mathbf{n}_1$, $\lambda_{c1}\mathbf{n}_2$, $\lambda_{c1}\mathbf{n}_3$ can now be understood as a real two-dimensional rotation of the underlying vector triplet $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ by the angle $-\theta$ in the plane perpendicular to $\mathbf{x}_3$. Since all triplet vectors are elements of the plane perpendicular to $\mathbf{x}_3$, and $\mathbf{i}$ is the oriented unit area element of precisely this plane, the two-dimensional treatment fully applies.

In line with this, the action of λ_{c2} on the three-component eigenspinor $\mathbf{x}_2$:

$$C(-kU)\mathbf{x}_2 = \lambda_{c2}\mathbf{x}_2 = \lambda_{c2}\begin{pmatrix}\mathbf{z}\mathbf{n}_1\\ \mathbf{z}\mathbf{n}_2\\ \mathbf{z}\mathbf{n}_3\end{pmatrix} = \begin{pmatrix}\lambda_{c2}\mathbf{z}\mathbf{n}_1\\ \lambda_{c2}\mathbf{z}\mathbf{n}_2\\ \lambda_{c2}\mathbf{z}\mathbf{n}_3\end{pmatrix} = \begin{pmatrix}\mathbf{z}\mathbf{n}_1\lambda_{c2}\\ \mathbf{z}\mathbf{n}_2\lambda_{c2}\\ \mathbf{z}\mathbf{n}_3\lambda_{c2}\end{pmatrix} \quad \text{(Lc3.2)}$$

may now be interpreted as the same two-dimensional rotation of the vector triplet $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ by the angle $-\theta$. Note that here we have λ_{c2} to the right of each triplet vector, and not to the left as in (Lc3.1), which explains the same sign of the angle. For the third equation sign we have used the fact that spinors commute, since 1 (one) and $\mathbf{i}$ commute with each other. Note also that the resulting spinor components continue to be different, because the different order of vector factors in the spinorial components of $\mathbf{x}_1$ and $\mathbf{x}_2$.

As for λ_1 and λ_2 we now set θ=90 degree and we conclude from the above discussion that λ_1 rotates the vector triplet $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ belonging to $\mathbf{x}_1$ by –90 degree. If we look at the last expression in (Lc3.2) λ_2 can be interpreted to also rotate the vector triplet $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ by –90 degree. (Note again the order of vector factors in (Lc3.2)!)

As already mentioned the fact that λ_3=0 means a projecting out of any component parallel to $\mathbf{x}_3$. The third eigenvalue of the Cayley transformation C(–kU) is λ_{c3}=1, which means that any component parallel to $\mathbf{x}_3$ will be invariant under multiplication with C(–kU).

Let me also remark that instead of interpreting the action of the eigenvalues λ_1, λ_2, λ_{c1} and λ_{c2} as triplet rotations, we may just as well interpret them as rotating the second factor in the vector factorization of the eigenspinor components, i.e. the reference vector $\mathbf{z}$ itself. Yet this interpretation has the drawback that we would need to abandon the fact that $\mathbf{z}$ did not change so far, which makes comparisons in the plane perpendicular to $\mathbf{x}_3$ more straight forward.

Summarizing the real situation in three dimensions, we see that the projecting out of the components parallel to $\mathbf{x}_3$ lead to the use of the oriented unit area element $\mathbf{i}$, which characterizes the plane perpendicular to $\mathbf{x}_3$. $\mathbf{i}$ thus replaces the imaginary unit j, which lacked any real geometric interpretation. Subsequently λ_1 and λ_2 (λ_{c1} and λ_{c2}) were seen to act as rotation operators on the vector triplet $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ underlying the three-component eigenspinors $\mathbf{x}_1$ and $\mathbf{x}_2$. The underlying triplet vectors themselves (and the reference vector **z**) are all elements of the plane perpendicular to $\mathbf{x}_3$ and continue to be so, when being rotated.

IV. Bottom up real explanation

It is interesting to consider whether geometric algebra can only serve as a tool of real analysis of the complex situation, or if we may even completely forget about matrices and complex numbers and let geometric algebra generate expressions that are finally interpretable in terms of matrices and complex numbers.

IV.1 Bottom up for two dimensions

We therefore now only assume the real two-dimensional vector space R^2 and its geometric algebra R_2 as described in section II.2. In section II.3 we learnt that the geometric multiplications of the two basis vectors σ_1 and σ_2 with the plane oriented unit area element **i** from the left yield:

$$\mathbf{i}\,\sigma_1 = -\sigma_2$$
$$\mathbf{i}\,\sigma_2 = \sigma_1$$

If we simply write this in matrix form we obtain

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} -\sigma_2 \\ \sigma_1 \end{pmatrix}. \quad \text{(B2)}$$

And we therefore see that the matrix corresponding to the geometric multiplication with **i** from the left is exactly the two-dimensional anti-symmetric matrix U with which we started off in section II.1. Multiplying both sides of (B2) with σ_1, from the right, we obtain the two-component spinorial form of the equation:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_1\sigma_1 \\ \sigma_2\sigma_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix} = \begin{pmatrix} -\sigma_2\sigma_1 \\ \sigma_1\sigma_1 \end{pmatrix} = \begin{pmatrix} \mathbf{i} \\ 1 \end{pmatrix}. \quad \text{(B2s)}$$

Please note that, because of the existence of the vector inverse in section II.2.2, this is an invertible operation! The second and fourth expressions in equation (B2s) correspond exactly to the equation $U\mathbf{x}_1=\lambda_1\mathbf{x}_1= \mathbf{i}\mathbf{x}_1$ of section II.3. By applying the reversion operation of the order of vectors in all geometric products involved, as defined in section II.2.2, we get:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_1\sigma_1 \\ \sigma_1\sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} = \begin{pmatrix} -\sigma_1\sigma_2 \\ \sigma_1\sigma_1 \end{pmatrix} = \begin{pmatrix} -\mathbf{i} \\ 1 \end{pmatrix} \quad \text{(B2r)},$$

which corresponds exactly to the equation $U\mathbf{x}_2=\lambda_2\mathbf{x}_2= -\mathbf{i}\mathbf{x}_2$ explained in section II.3.

What we just did with the unit oriented area element **i**, we will now do with the rotor $R(\theta)=\cos\theta + \mathbf{i}\sin\theta$. It transforms two basis vectors σ_1 and σ_2 to:

$$R(\theta)\sigma_1 = (\cos\theta + \mathbf{i}\sin\theta)\,\sigma_1 = \cos\theta\,\sigma_1 + \mathbf{i}\,\sigma_1\sin\theta = \cos\theta\,\sigma_1 - \sin\theta\,\sigma_2$$
$$R(\theta)\sigma_2 = (\cos\theta + \mathbf{i}\sin\theta)\,\sigma_2 = \cos\theta\,\sigma_2 + \mathbf{i}\,\sigma_2\sin\theta = \sin\theta\,\sigma_1 + \cos\theta\,\sigma_2.$$

Rewriting this in Matrix form we obtain:

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{pmatrix} = \begin{pmatrix} \cos\vartheta\boldsymbol{\sigma}_1 - \sin\vartheta\boldsymbol{\sigma}_2 \\ \sin\vartheta\boldsymbol{\sigma}_1 + \cos\vartheta\boldsymbol{\sigma}_2 \end{pmatrix}. \quad \text{(BR2)}$$

The matrix that corresponds to the rotor R(θ) operating on the basis vectors $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ is therefore exactly the Cayley transformation C(–kU). Geometrically multiplying both sides of equation (BR2) with $\boldsymbol{\sigma}_1$ from the right we obtain:

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}_1\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix} = (\cos\vartheta + \mathbf{i}\sin\vartheta)\begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix} =$$

$$= \begin{pmatrix} \boldsymbol{\sigma}_1(\cos(-\vartheta) + \mathbf{i}\sin(-\vartheta))\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2(\cos(-\vartheta) + \mathbf{i}\sin(-\vartheta))\boldsymbol{\sigma}_1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_1 R(-\vartheta)\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 R(-\vartheta)\boldsymbol{\sigma}_1 \end{pmatrix}. \quad \text{(BR2s)}$$

We now see a complete correspondence of equation (BR2s) with the eigenspinor equation C(–kU)$\mathbf{x}_1 = \lambda_{c1}\mathbf{x}_1$=(cosθ + isinθ) $\mathbf{x}_1$, with $\mathbf{x}_1 = (\boldsymbol{\sigma}_1\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1) = (1, -\mathbf{i})$, as explained in section II.3. Reversing the order of all elementary geometric vector products in equation (BR2s) yields:

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} \boldsymbol{\sigma}_1\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} = (\cos\vartheta - \mathbf{i}\sin\vartheta)\begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} =$$

$$= \begin{pmatrix} \boldsymbol{\sigma}_1(\cos\vartheta + \mathbf{i}\sin\vartheta)\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1(\cos\vartheta + \mathbf{i}\sin\vartheta)\boldsymbol{\sigma}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma}_1 R(\vartheta)\boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 R(\vartheta)\boldsymbol{\sigma}_2 \end{pmatrix}.$$

This reversed form of (BR2s) is therefore seen to perfectly correspond to the second eigenspinor equation of the Cayley transformation C(–kU)$\mathbf{x}_2 = \lambda_{c2}\mathbf{x}_2$=(cosθ – isinθ) $\mathbf{x}_2$.

It is therefore demonstrated that by purely assuming the real two-dimensional geometric algebra R_2 all expressions which are known from complex matrix algebra are generated automatically. The invertible transformation of the vector equations into spinor equations through the geometric multiplication with $\boldsymbol{\sigma}_1$ from the right makes the correspondence explicite.

IV.2 Bottom up for three dimensions

That what we have shown in two dimensions is not only due to the simplicity of two-dimensional algebra, will now be explicitly demonstrated in three dimensions as well.

Here we start in the real three-dimensional geometric algebra R_3 by considering the multiplication of the plane area element $\mathbf{i} = a\,\mathbf{i}_1 + b\,\mathbf{i}_2 + c\,\mathbf{i}_3$ with the components of the three basis vectors $\boldsymbol{\sigma}_{1\|}, \boldsymbol{\sigma}_{2\|}$, and $\boldsymbol{\sigma}_{3\|}$, parallel to the plane characterized by $\mathbf{i}$, i.e. the plane perpendicular to (a,b,c):

$$\mathbf{i}\boldsymbol{\sigma}_{1\|} = -c\,\boldsymbol{\sigma}_2 + b\,\boldsymbol{\sigma}_3$$
$$\mathbf{i}\boldsymbol{\sigma}_{2\|} = c\,\boldsymbol{\sigma}_1 - a\,\boldsymbol{\sigma}_3$$
$$\mathbf{i}\boldsymbol{\sigma}_{3\|} = -b\,\boldsymbol{\sigma}_1 + a\,\boldsymbol{\sigma}_2. \quad \text{(iB3)}$$

It is made use of the condition that $a^2+b^2+c^2=1$. These calculations can either be performed by hand as in the previous sections, or one may simply use geometric algebra capable mathematical software, like as MAPLE V with the Cambridge GA package[6], etc. Rewriting the three equations in matrix form yields:

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} \sigma_{1\|} \\ \sigma_{2\|} \\ \sigma_{3\|} \end{pmatrix} = \begin{pmatrix} -c\sigma_{2\|} + b\sigma_{3\|} \\ c\sigma_{1\|} - a\sigma_{3\|} \\ -b\sigma_{1\|} + a\sigma_{2\|} \end{pmatrix} = \begin{pmatrix} -c\sigma_2 + b\sigma_3 \\ c\sigma_1 - a\sigma_3 \\ -b\sigma_1 + a\sigma_2 \end{pmatrix} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}. \text{ (B3)}$$

We now recognize the three-dimensional anti-symmetric matrix U of section III.1. The third equation (B3) may simply be calculated from the second one, the fourth expression is just rewriting the third in matrix form as well. The distinction between the basis vectors and their **i**-plane projections seems at first sight unnecessary, but the operation of **i** on the basis vectors themselves does not yield the equations (iB3). Geometrically multiplying both sides of (B3) with $\sigma_{1\|}$ from the right, we obtain the three-component spinorial form:

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} \sigma_{1\|}\sigma_{1\|} \\ \sigma_{2\|}\sigma_{1\|} \\ \sigma_{3\|}\sigma_{1\|} \end{pmatrix} = \begin{pmatrix} \mathbf{i}\sigma_{1\|}\sigma_{1\|} \\ \mathbf{i}\sigma_{2\|}\sigma_{1\|} \\ \mathbf{i}\sigma_{3\|}\sigma_{1\|} \end{pmatrix} = \begin{pmatrix} \mathbf{i}(1-a^2) \\ \mathbf{i}(-ab-c\mathbf{i}) \\ \mathbf{i}(-ac+b\mathbf{i}) \end{pmatrix} = \mathbf{i}\begin{pmatrix} 1-a^2 \\ -ab-c\mathbf{i} \\ -ac+b\mathbf{i} \end{pmatrix}. \text{ (B3s)}$$

This exactly corresponds to the equation $U\mathbf{x}_1=\lambda_1\mathbf{x}_1= \mathbf{i}\mathbf{x}_1$ of section III.3. Note that the multiplication with $\sigma_{1\|}$ is invertible. Multiplying with $\sigma_{2\|}$ or $\sigma_{3\|}$ instead yields the equivalent forms of $\mathbf{x}_1$ also mentioned in section III.3. Reversing all elementary geometric vector products involved in (B3s) yields:

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}\begin{pmatrix} \sigma_{1\|}\sigma_{1\|} \\ \sigma_{1\|}\sigma_{2\|} \\ \sigma_{1\|}\sigma_{3\|} \end{pmatrix} = \begin{pmatrix} \mathbf{i}\sigma_{1\|}\sigma_{1\|} \\ \mathbf{i}\sigma_{1\|}\sigma_{2\|} \\ \mathbf{i}\sigma_{1\|}\sigma_{2\|} \end{pmatrix} = \begin{pmatrix} -\mathbf{i}(1-a^2) \\ -\mathbf{i}(-ab+c\mathbf{i}) \\ -\mathbf{i}(-ac-b\mathbf{i}) \end{pmatrix} = -\mathbf{i}\begin{pmatrix} 1-a^2 \\ -ab+c\mathbf{i} \\ -ac-b\mathbf{i} \end{pmatrix}. \text{ (B3r)}$$

This is seen to be $U\mathbf{x}_2=\lambda_2\mathbf{x}_2= -\mathbf{i}\mathbf{x}_2$ of section III.3.

As in the bottom up explanation for two dimensions, we will now look at the action of the rotor $R(\theta)=\cos\theta + \mathbf{i}\sin\theta$, with $\mathbf{i}=a\,\mathbf{i}_1 + b\,\mathbf{i}_2 + c\,\mathbf{i}_3$. As before, we could look straight away for what $R(\theta)$ does with the three basis vectors projected onto the **i**-plane, i.e. with $\sigma_{1\|}$,$\sigma_{2\|}$, and $\sigma_{3\|}$. But a better way is to use the fact that **i** anti-commutes with all vectors in its plane [see section II.3, equ. (ac)] and commutes with vectors perpendicular to its plane:

$$\mathbf{i}(a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3) = (a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3)\,\mathbf{i}. \text{ (com)}$$

In three dimensions $a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3$ is the only vector perpendicular to the **i**-plane. We can therefore rewrite e.g. the rotation of $\sigma_{1\|}$ as:

$$\lambda_{c1}\sigma_{1\|}=R(\theta)\sigma_{1\|}=R(\theta/2)R(\theta/2)\sigma_{1\|}=R(\theta/2)\sigma_{1\|}R(-\theta/2). \text{ (DSR)}$$

Moving a $R(\theta/2)$ factor to the right causes the angle to change sign, because of the anti-commutativity of **i** with $\sigma_{1\|}$. If we apply the same formula to the vector $a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3$ we obtain:

$$R(\theta/2)(a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3)R(-\theta/2) = R(\theta/2)R(-\theta/2)\,(a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3) = a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3,$$

because the vector $a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3$ is perpendicular to the **i**-plane, i.e. it doesn't change the sign of the angle in the factor R(–θ/2), if the factor is moved back to the left. That is to say the *double sided description of the rotation* as in (DSR) does exactly what a rotation around the axis $a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3$ is expected to do, it leaves the axis invariant and rotates only vectors in the **i**-plane. The formula (DSR) automatically takes care of the necessary projections. This corresponds well with the Cayley transformation C(–kU), which also leaves the vector $a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3$ invariant. Instead of looking at

$\lambda_{c1}\sigma_{1||}=R(\theta/2)\sigma_{1||}R(-\theta/2)$, $\lambda_{c1}\sigma_{2||}=R(\theta/2)\sigma_{2||}R(-\theta/2)$, and $\lambda_{c1}\sigma_{3||}=R(\theta/2)\sigma_{3||}R(-\theta/2)$, (LRh)

we will look straight away at:

$R(\theta/2)\sigma_1R(-\theta/2)=\{1+(1-\cos\theta)(1-a^2)\}\,\sigma_1 + \{-c\sin\theta+ab(1-\cos\theta)\}\,\sigma_2 + \{b\sin\theta+ac(1-\cos\theta)\}\,\sigma_3$

$R(\theta/2)\sigma_2R(-\theta/2)=\{c\sin\theta+ab(1-\cos\theta)\}\,\sigma_1 + \{1+(1-\cos\theta)(1-b^2)\}\,\sigma_2 + \{-a\sin\theta+bc(1-\cos\theta)\}\,\sigma_3$

$R(\theta/2)\sigma_3R(-\theta/2)=\{-b\sin\theta+ac(1-\cos\theta)\}\,\sigma_1 + \{a\sin\theta+bc(1-\cos\theta)\}\,\sigma_2 + \{1+(1-\cos\theta)(1-c^2)\}\,\sigma_3$.

Rewriting these three equations in matrix form gives:

$$\mathrm{R}\left(\frac{\vartheta}{2}\right)\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}\mathrm{R}\left(-\frac{\vartheta}{2}\right)=$$

$$=\begin{pmatrix}1+(1-\cos\vartheta)(a^2-1) & -c\sin\vartheta+ab(1-\cos\vartheta) & b\sin\vartheta+ac(1-\cos\vartheta)\\ c\sin\vartheta+ab(1-\cos\vartheta) & 1+(1-\cos\vartheta)(b^2-1) & -a\sin\vartheta+bc(1-\cos\vartheta)\\ -b\sin\vartheta+ac(1-\cos\vartheta) & a\sin\vartheta+bc(1-\cos\vartheta) & 1+(1-\cos\vartheta)(c^2-1)\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}.$$

(BR3)

The matrix obtained is exactly C(–kU), as in section III.1. Both sides of equation (BR3) leave components parallel to $a\,\sigma_1 + b\,\sigma_2 + c\,\sigma_3$ invariant. They can therefore be subtracted on both sides without changing the form of the equation:

$$\mathrm{R}\left(\frac{\vartheta}{2}\right)\begin{pmatrix}\sigma_{1||}\\ \sigma_{2||}\\ \sigma_{3||}\end{pmatrix}\mathrm{R}\left(-\frac{\vartheta}{2}\right)=\mathrm{R}(\vartheta)\begin{pmatrix}\sigma_{1||}\\ \sigma_{2||}\\ \sigma_{3||}\end{pmatrix}=C(-kU)\begin{pmatrix}\sigma_{1||}\\ \sigma_{2||}\\ \sigma_{3||}\end{pmatrix}.$$

According to (DSR) I simplified the first expression, to have only $R(\theta)=\lambda_{c1}$ to the left. Multiplying by $\sigma_{1||}$ from the right we obtain:

$$\mathrm{R}(\vartheta)\begin{pmatrix}\sigma_{1||}\sigma_{1||}\\ \sigma_{2||}\sigma_{1||}\\ \sigma_{3||}\sigma_{1||}\end{pmatrix}=\lambda_{c1}\begin{pmatrix}\sigma_{1||}\sigma_{1||}\\ \sigma_{2||}\sigma_{1||}\\ \sigma_{3||}\sigma_{1||}\end{pmatrix}=\lambda_{c1}\begin{pmatrix}1-a^2\\ -ab-c\mathbf{i}\\ -ac+b\mathbf{i}\end{pmatrix}=C(-kU)\begin{pmatrix}\sigma_{1||}\sigma_{1||}\\ \sigma_{2||}\sigma_{1||}\\ \sigma_{3||}\sigma_{1||}\end{pmatrix}=C(-kU)\begin{pmatrix}1-a^2\\ -ab-c\mathbf{i}\\ -ac+b\mathbf{i}\end{pmatrix}.$$

(BR3s)

This is seen to exactly be the Cayley transformation eigenvector equation $C(-kU)=\lambda_{c1}\mathbf{x}_1=(\cos\theta+\mathbf{i}\sin\theta)\,\mathbf{x}_1$, i.e. equ. (Lc3.1) in section III.3. Using $\sigma_{2||}$ or $\sigma_{3||}$ instead of $\sigma_{1||}$ from the right, we obtain the other forms of $\mathbf{x}_1$ mentioned in section III.3. Reversing the order of all geometric products of vectors involved in (BR3s) yields equation (Lc3.2) of section III.3.

Using the laws governing the real geometric algebra R_3, and properly rewriting the action of the eigenvalues λ_{c1} and λ_{c2} as in equ. (DSR), we have therefore obtained all relationships for the three-dimensional anti-symmetric matrix U and its Cayley transform C(–kU) in a purely constructive manner.

V. Conclusion

In this paper the eigenvalues and eigenvectors of anti-symmetric matrices in two and three dimensions and of their Cayley transformations were briefly reviewed as described by standard linear algebra. Traditionally the imaginary and complex eigenvalues and the complex eigenvectors have no direct real geometric interpretation. In an effort to unravel the real geometric interpretation I used the geometric algebras of the plane and Eulidean three-space respectively. All relationships involving imaginary and complex values or components found an explanation. The imaginary and complex eigenvalues were found to act like rotation operators on duplets and triplets of vectors. In three-space the triplets of vectors were found to be the projections of the coordinate vectors onto the plane of rotation, unique up to an arbitrary constant rotation in the plane. Geometric multiplication of this triplet of vectors with any other vector (termed reference vector) in the plane of rotation gives the three components equivalent to a complex three-component eigenvector in the standard treatment.

The imaginary eigenvalues corresponded to real space +90 degree or –90 degree rotation operators and the complex eigenvalues of the Cayley transformations to arbitrary real space rotations in the plane of rotation, depending on the scalar parameter k of the Cayley transformation.

Finally, for both two and three dimensions a bottom up explanation was found in a *constructive* way, only assuming the two- and three-dimensional geometric algebras. This explanation yielded expressions and equations that completely resemble the eigenvalue and eigenvector relationships for anti-symmetric matrices and their Cayley transformations.

The extension to Euclidean and Minkowskian four-space might be a natural next step. This way the interpretation could be further refined and new insights in areas of physics whose theories are based on such spaces might be gained. I especially think of the special theory of relativity, electrodynamics and relativistic quantum mechanics. Some preliminary calculations seem to indicate that the extension to four-spaces will not be trivial.

Independent of the question for higher dimensions, the two and three dimensional treatment has already yielded a definitive and very satisfactory picture of how imaginary and complex eigenvalues and complex eigenvectors should truly be understood in terms of real geometry. Something undergraduate students, exposed to linear algebra for the first time, as well as scientists who teach and apply it will certainly appreciate.

VI. Acknowledgements

I first of all thank God for the joy of studying his creation: "…since the creation of the world God's invisible qualities – his eternal power and divine nature – have been clearly seen, being understood from what has been made…"[8]. I thank my wife for encouraging my research in many ways. I further thank my very helpful and teaching assistant T. Ido for the good cooperation in teaching linear algebra. Dr. H. Ishi helped me through discussion and comments on this work. I thank Prof. O. Giering and Prof. J.S.R. Chisholm for attracting my interest to geometry as such. Fukui University provided a good research environment for conducting this research. I thank K. Shinoda (Kyoto) for his prayerful support and encouragement for me.

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