A Geometric Understanding of Fermion Rest Masses Linking the Einstein and Dirac Equations via Weyl's Gauge Theory

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Abstract: We demonstrate how fermion rest masses may be understood on a strictly geometric footing, by showing how the Dirac equation is just a special case of the Einstein equation for gravitation in curved spacetime, in view of Weyl's theory of gauge (phase) invariance.

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1. Introduction

It is well-understood that the Dirac equation $(i\not{\partial} - m)\psi = 0$ may be thought of as the non-trivial square root of the relativistic energy relationship $p_\sigma p^\sigma - m^2 = 0$. For, if one writes this in flat spacetime as $\eta^{\sigma\tau} p_\sigma p_\tau = m^2$ and then applies $\eta^{\sigma\tau} = \frac{1}{2}\left(\gamma^\sigma \gamma^\tau + \gamma^\tau \gamma^\sigma\right) = \frac{1}{2}\left\{\gamma^\sigma, \gamma^\tau\right\}$ where $\eta^{\sigma\tau}$ is the contravariant Minkowski metric tensor, one first obtains $\frac{1}{2}\left(\gamma^\sigma \gamma^\tau + \gamma^\tau \gamma^\sigma\right) p_\sigma p_\tau - m^2 = 0$. Then using the Dirac-dagger notation $\not{p} \equiv \gamma^\sigma p_\sigma$ this becomes $\not{p}\not{p} = m^2$. Separating the two parts of this square root and using the resulting expression to operate from the left on a Dirac spinor *u*, yields $(\not{p} - m)u = 0$ in which the mass *m* represents eigenvalues of the daggered momentum matrix $\not{p}$. Upon promoting the spinor to a wavefunction $u \to \psi$ simultaneously with substituting $\not{p} \to i\not{\partial}$, the new wavefunction equation becomes $(i\not{\partial} - m)\psi = 0$, which is Dirac's equation. In essence, this is the path Dirac followed to derive his equation in [1], [2].

This in turn is based on the equation $d\tau^2 = g_{\sigma\tau} dx^\sigma dx^\tau$ for the spacetime metric / proper time. Specifically, if one simply converts this to $1 = g_{\sigma\tau}\left(dx^\sigma / d\tau\right)\left(dx^\tau / d\tau\right) = g_{\sigma\tau} u^\sigma \mu^\tau$ where $u^\sigma \equiv dx^\sigma / d\tau$ defines the velocity vector, and then multiplies through by a square mass m^2, then upon further defining the momentum vector $p^\sigma \equiv m u^\sigma$, one obtains $p_\sigma p^\sigma - m^2 = 0$ which is the starting point for obtaining the Dirac equation. However, as one can readily see from this well-known derivation, the mass *m* is introduced entirely by hand. It would be desirable to find a way to obtain Dirac's equation without the hand-introduction of a mass, but rather, to have the mass arise spontaneously, based strictly on a deeper understanding of the spacetime geometry.

It turns out that an exercise similar to the above using the Einstein equation $-\kappa \mathrm{T}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ enables us to do exactly that, namely, to obtain a strictly geometric interpretation of fermion rest mass in $(i\not{\partial} - m)\psi = 0$. Concurrently, we come to view Dirac's equation as just a specialized variant of Einstein's equation. This all appears to be new. We now show how this is done.

2. Connecting the Dirac Equation to the Einstein Equation via Weyl's Gauge Theory and *Vierbein* Fields

The geometric foundation of Einstein's equation springs from the Bianchi identity $\partial_{;\sigma} R^{\alpha\beta}{}_{\mu\nu} + \partial_{;\mu} R^{\alpha\beta}{}_{\nu\sigma} + \partial_{;\nu} R^{\alpha\beta}{}_{\sigma\mu} = 0$ of Riemannian geometry, where $\partial_{;\nu}$ is the gravitationally-covariant derivative which makes well-known use of the Christoffel connections $\Gamma^{\mu}{}_{\alpha\beta}$. If one does a first index contraction of this identity while noting that $R^{\alpha\beta}{}_{\mu\nu}$ is antisymmetric in μ,ν, one obtains $\partial_{;\sigma} R^{\beta}{}_{\mu} - \partial_{;\mu} R^{\beta}{}_{\sigma} + \partial_{;\alpha} R^{\alpha\beta}{}_{\sigma\mu} = 0$ whereby two of the three terms are contracted to the Ricci tensor via $R^{\alpha\beta}{}_{\mu\alpha} = R^{\beta}{}_{\mu}$. A second contraction yields $\partial_{;\sigma} R - \partial_{;\beta} R^{\beta}{}_{\sigma} - \partial_{;\alpha} R^{\alpha}{}_{\sigma} = 0$ with the Ricci scalar $R = R^{\beta}{}_{\beta}$. With simple index gymnastics this converts over to the very well-known $\partial_{;\alpha}\left(R^{\alpha}{}_{\sigma} - \frac{1}{2}\delta^{\alpha}{}_{\sigma} R\right) = 0$. Because we also know that the local conservation of energy is represented via the mixed energy tensor $\mathrm{T}^{\alpha}{}_{\sigma}$ by the equation $\partial_{;\alpha}\mathrm{T}^{\alpha}{}_{\sigma} = 0$, one connects this to the contracted Bianchi identity in the form $-\kappa\partial_{;\alpha}\mathrm{T}^{\alpha}{}_{\sigma} = \partial_{;\alpha}\left(R^{\alpha}{}_{\sigma} - \frac{1}{2}\delta^{\alpha}{}_{\sigma} R\right) = 0$ which upon integration sans cosmological constant yields $-\kappa\mathrm{T}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. [3]

Hermann Weyl teaches [4], [5], [6] that whenever we have a field equation or a Lagrangian density for a scalar ϕ or fermion ψ field which includes a term $\partial_{;\mu}\phi$ or $\partial_{;\mu}\psi$, we should subject the field to the *local* gauge (phase) transformation $\phi \rightarrow e^{i\theta(x)}\phi$ or $\psi \rightarrow e^{i\theta(x)}\psi$ and insist the field equation or Lagrangian density remain invariant under this transformation. How does one ensure such invariance? Replace $\partial_{;\mu} \rightarrow D_{;\mu} = \partial_{;\mu} - iG_{\mu}$. So now, one promotes $\partial_{;\mu}\phi \rightarrow D_{;\mu}\phi$ and $\partial_{;\mu}\psi \rightarrow D_{;\mu}\psi$ with the consequence that ϕ or ψ acquires an interaction with a gauge field G_{μ}. If we apply Weyl's gauge recipe to $-\kappa\partial_{;\alpha}\mathrm{T}^{\alpha}{}_{\sigma} = \partial_{;\alpha}\left(R^{\alpha}{}_{\sigma} - \frac{1}{2}\delta^{\alpha}{}_{\sigma} R\right) = 0$, then we should promote $\partial_{;\alpha} \rightarrow D_{;\alpha}$ and we may write this with minor index adjustments as:

$$-\kappa g^{\sigma\tau} D_{;\sigma}\mathrm{T}_{\tau\nu} = g^{\sigma\tau} D_{;\sigma}\left(R_{\tau\nu} - \tfrac{1}{2} g_{\tau\nu} R\right) = 0 . \qquad (1)$$

Now, using the $g^{\sigma\tau}$ expressly displayed in the above, we follow the exact same recipe that Dirac used [1], [2] to convert $g^{\sigma\tau} p_{\sigma} p_{\tau} = m^2$ into $(i\partial\!\!\!/ - m)\psi = 0$ and see what happens.

Working in curved spacetime, we must make use of a local *vierbein* field $e^{\sigma}{}_{a}(x)$, [6] where Greek indexes label general spacetime coordinates and Latin indexes label local Lorentz / Minkowski coordinates. The metric tensor is then related to this in the customary manner by:

$$g^{\sigma\tau} = e^{\sigma}{}_{a} e^{\tau}{}_{b} \eta^{ab} = \tfrac{1}{2} e^{\sigma}{}_{a} e^{\tau}{}_{b}\left(\gamma^{a}\gamma^{b} + \gamma^{b}\gamma^{a}\right) = \tfrac{1}{2}\left(\Gamma^{\sigma}\Gamma^{\tau} + \Gamma^{\tau}\Gamma^{\sigma}\right) = \tfrac{1}{2}\left\{\Gamma^{\sigma}, \Gamma^{\tau}\right\}, \qquad (2)$$

where we define $\Gamma^{\sigma}(x) \equiv \gamma^{a} e^{\sigma}{}_{a}$. This is simply a generalization of $\eta^{\sigma\tau} = \frac{1}{2}\left\{\gamma^{\sigma}, \gamma^{\tau}\right\}$ into curved spacetime. We shall continue to employ the usual Dirac-dagger notation, but now define this in curved spacetime such that for any arbitrary vector B_σ, we have $\not{B} \equiv \Gamma^\sigma B_\sigma = \gamma^a e^\sigma{}_a B_\sigma$. In the special case where $B^\sigma = \partial^{;\sigma}$, i.e., where $\not{B} = \not{\partial} = \Gamma^\sigma \partial_{;\sigma} = \gamma^a e^\sigma{}_a \partial_{;\sigma}$ is a covariant derivative, beyond the Christoffel connection $\Gamma^\mu{}_{\alpha\beta}$ this derivative takes on additional terms involving a *vierbein* connection $\omega_\mu{}^{ab}$ with one spacetime and two Lorentz indexes, in known fashion.

Let us now make use of (2) in the $g^{\sigma\tau} D_{;\sigma} \mathrm{T}_{\tau\nu} = 0$ portion of (1). This enables us to write the anticommutator equation:

$$g^{\sigma\tau} D_{;\sigma} \mathrm{T}_{\tau\nu} = \tfrac{1}{2}\left(\Gamma^\sigma \Gamma^\tau + \Gamma^\tau \Gamma^\sigma\right) D_{;\sigma} \mathrm{T}_{\tau\nu} = \tfrac{1}{2}\left(\not{D}\not{\mathrm{T}}_\nu + \not{\mathrm{T}}_\nu \not{D}\right) = \tfrac{1}{2}\left\{\not{D}, \not{\mathrm{T}}_\nu\right\} = 0 \quad , \tag{3}$$

where $\not{T}_\nu = \Gamma^\tau \mathrm{T}_{\tau\nu}$ is a “half-daggered” energy tensor daggered in one index while retaining one free index. We then use this to operate from the left on a Dirac wavefunction as such:

$$\tfrac{1}{2}\left(\not{\mathrm{T}}_\nu \not{D} + \not{D}\not{\mathrm{T}}_\nu\right)\psi = \tfrac{1}{2}\left\{\not{\mathrm{T}}_\nu, \not{D}\right\}\psi = 0 \quad . \tag{4}$$

This is the Einstein equation used as an operator equation for fermions, no more and no less. The bears exactly the same relationship to the energy conservation relation $-\kappa g^{\sigma\tau} D_{;\sigma} \mathrm{T}_{\tau\nu} = 0$ linked to geometry via $g^{\sigma\tau} D_{;\sigma}\left(R_{\tau\nu} - \tfrac{1}{2} g_{\tau\nu} R\right) = 0$ in (1), as Dirac’s equation $\left(i\not{\partial} - m\right)\psi = 0$ bears to the metric equation $d\tau^2 = g_{\sigma\tau} dx^\sigma dx^\tau$. Note that the operation $\not{D}\not{\mathrm{T}}_\nu$ will contain both Christoffel and *vierbein* connections.

Equation (4) raises the prospect of introducing a fermion rest mass *without* having to do so by hand, and in the process, of obtaining a geometric understanding of this mass linking Einstein’s equation to Dirac’s. This is in contrast to when we multiply $1 = g_{\sigma\tau} u^\sigma \mu^\tau$ by a hand-added m^2 without ever explaining anything about the mass *per se*. Specifically, we take Dirac’s equation $\left(i\not{\partial} - m\right)\psi = 0$, regard this in curved spacetime such that $\not{\partial} = \Gamma^\sigma \partial_{;\sigma} = \gamma^a e^\sigma{}_a \partial_{;\sigma}$, and use Weyl’s gauge prescription to introduce a gauge interaction by promoting $\not{\partial} \to \not{D}$, thus writing:

$$\left(\not{D} + im\right)\psi = 0 \, . \tag{5}$$

We have also multiplied through by $-i$, which allows us to contrast (5) directly with (4), as one should now do.

Contrasting, we see that both (4) and (5) contain two additive terms. The left hand terms have the respective forms $\tfrac{1}{2}\not{\mathrm{T}}_\nu \not{D}\psi$ and $\not{D}\psi$. The right hand terms are $\tfrac{1}{2}\not{D}\not{\mathrm{T}}_\nu\psi$ and $im\psi$. This suggests that perhaps the fermion mass can be interpreted via a commutator $\left[\not{\mathrm{T}}_\nu, \not{D}\right]$. Specifically: If we take both (4) and (5) to be true equations, with (4) being the Einstein equation represented in Dirac form with a Weyl supplement $\not{\partial} \to \not{D}$, and with (5) being Dirac’s equation with a gauge field in curved spacetime, then we see that Dirac’s equation (5) may be embedded as a special case of Einstein’s equation (4) if we set:

$$[\mathbb{D},\mathrm{T}_\nu]\psi = (\mathbb{D}\mathrm{T}_\nu - \mathrm{T}_\nu\mathbb{D})\psi = 2i\mathrm{T}_\nu m\psi \ . \tag{6}$$

Specifically, if we now substitute (6) into (4), we obtain:

$$\tfrac{1}{2}(\mathrm{T}_\nu\mathbb{D} + \mathbb{D}\mathrm{T}_\nu)\psi = \tfrac{1}{2}(\mathrm{T}_\nu\mathbb{D} + \mathrm{T}_\nu\mathbb{D} + 2i\mathrm{T}_\nu m)\psi = \mathrm{T}_\nu(\mathbb{D} + im)\psi = 0 \ . \tag{7}$$

In the special case where $(\mathbb{D}+im)\psi = 0$ separately from the further left-multiplication with the matrix $\mathrm{T}_\nu = \Gamma^\sigma \mathrm{T}_{\sigma\nu} = \gamma^a e^\sigma{}_a \mathrm{T}_{\sigma\nu}$, this becomes identically equivalent to Dirac's equation represented in the form of (5). So, consolidating (6), we see that:

$$\mathrm{T}_\nu m\psi = \tfrac{1}{2}[\mathrm{T}_\nu, i\mathbb{D}]\psi \text{ ; or alternatively, } \left(\tfrac{1}{2}[\mathrm{T}_\nu, i\mathbb{D}] - \mathrm{T}_\nu m\right)\psi = 0 \ . \tag{8}$$

If (8) is true, then Dirac's equation $(\mathbb{D}+im)\psi = 0$ with a gauge field coupling $\partial \to \mathbb{D}$ as in (5) is just a special case of the Einstein equation (1) with Weyl's gauge supplement $\partial \to \mathbb{D}$ as represented in the Dirac form (4). In general, Dirac's equation is that of Einstein, in the form $\tfrac{1}{2}\{\mathrm{T}_\nu, \mathbb{D}\}\psi = 0$ of (4), or with mass revealed, $\mathrm{T}_\nu(\mathbb{D}+im)\psi = 0$ of (7). These interrelationships do not appear to have previously been found.

3. Conclusion

From the foregoing, fermion rest mass is given a strictly geometric interpretation in terms of eigenvalues of the commutation $[\mathrm{T}_\nu, i\mathbb{D}]$ of a half-daggered energy tensor T_ν with the daggered gauge-covariant derivative $\mathbb{D}$ of Weyl using *vierbein* fields in accordance with (2). The fermion mass no longer needs to be regarded as something added "by hand," and this should help us understand how to "reveal" a fermion rest mass via spontaneous symmetry breaking. The correspondence $p \leftrightarrow i\partial$ which is such a familiar part of the quantum mechanical landscape is now shown to have an analogous correspondence $m \leftrightarrow i\mathbb{D}$ with fermion rest mass when used in the form of (8). Finally, with the fermion rest mass now understood as in (8), Dirac's equation written in the form of (4) as $\tfrac{1}{2}\{\mathrm{T}_\nu, \mathbb{D}\}\psi = 0$, is simply a variant of Einstein's equation $-\kappa\mathrm{T}_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R$ of which $(\mathbb{D}+im)\psi = 0$ is a special case. This is all analogous to how the usual Dirac equation $(i\partial - m)\psi = 0$ is just a variant of the metric relationship $d\tau^2 = g_{\sigma\tau}dx^\sigma dx^\tau$.

References

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