Vortex theory of electromagnetism and its non-Euclidean character

Ali R. Hadjesfandiari

Department of Mechanical and Aerospace Engineering
State University of New York at Buffalo
Buffalo, NY 14260 USA
ah@buffalo.edu

April 7, 2012

Abstract

By examining the theory of relativity, as originally proposed by Lorentz and Poincaré, the fundamental relationship between space-time and matter is discovered, thus completing the theory of relativity and electrodynamics. As a result, the four-dimensional theory of general motion and the four-dimensional vortex theory of interaction are developed. It is seen that the electromagnetic four-vector potential and strength fields are the four-dimensional velocity and vorticity fields, respectively. Furthermore, the four-vector electric current density is proportional to the four-dimensional mean curvature of the four-vector potential field. This is the fundamental geometrical theory of electromagnetism, which determines the origin of electromagnetic interaction and clarifies some of the existing ambiguities. Interestingly, the governing geometry of motion and interaction is non-Euclidean.

1. Introduction

Maxwell's theory of electrodynamics is one of the greatest advances in physics. It is the most accurate physical theory known by far, which has passed many tests in a wide range of scales. This theory has also played a key role in the development of the theory of relativity, which unifies the concepts of space and time based on the work of Lorentz on space-time transformations. The Lorentz transformation originally resulted from the attempts of Lorentz and others to explain how the speed of light was observed to be independent of the reference frame, and to understand the symmetries of the laws of electrodynamics.

Based on the Lorentz ether theory, Poincaré in 1905 proposed the relativity principle as a general law of nature, including electrodynamics and gravitation. Although the Lorentz transformation is fundamental in this development, Poincaré's theory of relativity does not clearly explain its physical meaning and cannot clarify the relativistic meaning of space-time as a single entity. Despite the fact that Poincaré's theory shows a relationship between pure Lorentz transformation and hyperbolic rotation, it does not specify what is rotating. Thus, Poincaré's theory does not completely resolve fundamental aspects of space-time, including its geometry, and does not give further insight into the Maxwellian covariant electrodynamics. This is the origin of most of the troubles within the theory of relativity and electrodynamics, including the geometrization of the theory of relative accelerating motion, the explanation of the mechanisms behind the electromagnetic force, and the speculation about the existence of magnetic monopoles.

Early investigators of relativity, such as Robb, Varičak, Lewis, Wilson, and Borel [1-5] have noticed and extensively investigated the non-Euclidean geometric character of uniform relative motion, where hyperbolic geometry governs the velocity addition law. Although this non-Euclidean character is very intriguing, its fundamental meaning and its relation with space-time and particles has remained a mystery. In addition, there has not been any consistent geometric explanation of accelerating relative motion and the geometry of electromagnetism. Interestingly, Borel [5] has shown that non-Euclidean geometry is the origin of the famous Thomas-Wigner rotation. The importance of this non-Euclidean geometry and its affinity with the Minkowskian space-time in a complete theory of relativity has not been appreciated.

Electromagnetic interaction, known as the Lorentz force, is not a direct consequence of Maxwell's equations; rather this force has to be postulated in an independent manner, which is the manifest of the incompleteness of the theory. Although it has been noted that the electromagnetic field strength tensor and Lorentz force are both natural consequences of the geometric structure of Minkowskian space-time, its fundamental meaning has not been discovered.

In addition, some have argued that the existence of a magnetic monopole is compatible with fully symmetrized Maxwell's equations. It seems that only a simple modification of Maxwell's equations suffice to allow magnetic charges in electrodynamics. However, it should be noticed that no magnetic monopole has been found to this date.

These difficulties suggest that the theory of relativity of Lorentz and Poincaré needs to be modified, such that it explains:

1. The fundamental meaning of the Lorentz transformation and the geometrical structure of Minkowskian space-time;
2. The non-Euclidean geometry governing uniform relative motion and electrodynamics;
3. The fundamental theory of general accelerating motion;
4. The mechanism behind the electromagnetic interaction.

To complete the theory of relativity, we develop a fundamental geometrical theory of motion and interaction, which shows that the Lorentz force and Maxwell's equations are simple geometrical relations based on four-dimensional rotation. This development clarifies the relativity of space-time and its relationship with matter and the governing non-Euclidean geometry. This also revives the idea of the electromagnetic field as a vortex-like motion in a universal entity. Interestingly, this vortex theory also shows that the Lorentz force is a lift-like force perpendicular to the four-vector velocity.

We organize the current paper in the following manner. In Section 2, we present the theory of space-time and the geometrical theory of motion. This shows that motion of a particle is a four-dimensional rotation of its space-time body frame and clarifies the origin of the governing non-Euclidean geometry. Subsequently, in Section 3, we develop the consistent vortex theory of fundamental interaction, which shows that a Lorentz-like force is an essential character of every fundamental interaction. Therefore, every fundamental interaction is specified by a four-dimensional vortex-like field. We note that this means unification of all forces based on the geometrical theory of motion and interaction. Afterwards, in Section 4, we demonstrate all the details of this geometrical theory for electromagnetic interaction. It is seen that the electric charges are the only source of electromagnetic field and magnetic monopoles do not exist. A summary and general conclusion is presented in Section 5.

2. Theory of space-time and geometry of motion

Preliminaries and relativistic kinematics of a particle

As an inertial reference frame, a four-dimensional coordinate system $x_1x_2x_3x_4$ is considered such that $x_1x_2x_3$ is the usual space and x_4 is the axis measuring time with imaginary values, such that $x_4 = ict$. This is shown symbolically in Fig. 1 by considering a two dimensional space and one time direction. Throughout this paper, we refer to this as our specified inertial reference frame. The unit four-vector bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ are defined by

$$\begin{aligned} \mathbf{e}_1 &= (1,0,0,0) \\ \mathbf{e}_2 &= (0,1,0,0) \\ \mathbf{e}_3 &= (0,0,1,0) \\ \mathbf{e}_4 &= (0,0,0,1) \end{aligned} \tag{2.1}$$

The space-time position four-vector of the particle P can be represented by

$$\mathbf{x} = x_\mu \mathbf{e}_\mu \tag{2.2}$$

However, for simplicity, we sometimes write

$$\mathbf{x} = (\mathbf{x}, x_4) = (\mathbf{x}, ict) = (x, y, z, ict) \tag{2.3}$$

or even

$$x_\mu = (\mathbf{x}, x_4) \tag{2.4}$$

and also often use x in place of $\mathbf{x}$. The position of the massive particle in the inertial reference frame describes a path known as the world line.

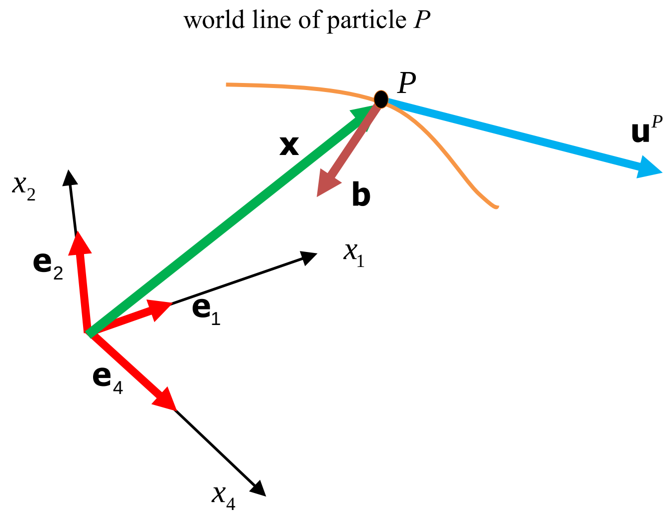

Fig. 1. World line of the particle in inertial reference frame

By considering two neighboring positions $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$ on the world line, we have

$$d\mathbf{x} = dx_\mu \mathbf{e} = (d\mathbf{x}, icdt) = (\mathbf{v}, ic)dt \tag{2.5}$$

The three-vector $\mathbf{v} = v\mathbf{e}_t$ is the velocity of particle where $\mathbf{e}_t$ is the tangential unit three-vector in the direction of $\mathbf{v}$. The square length of this infinitesimal four-vector

$$ds^2 = d\mathbf{x} \bullet d\mathbf{x} = dx_\mu dx_\mu = d\mathbf{x}^2 - c^2 dt^2 = -c^2 dt^2 \left(1 - \frac{v^2}{c^2}\right) \tag{2.6}$$

is the scalar invariant under all Lorentz transformation. The proper time between the events $d\tau$ is defined by

$$d\tau = dt\sqrt{1 - v^2/c^2} \tag{2.7}$$

Therefore,

$$ds = icd\tau \tag{2.8}$$

By using the concept of rapidity ξ, where

$$\tanh\xi = \frac{v}{c} \tag{2.9}$$

we obtain

$$dt = d\tau\cosh\xi \tag{2.10}$$

The four-vector velocity $\mathbf{u}^P = u_\mu^P \mathbf{e}_\mu$ is the rate of change of the position vector of the particle $\mathbf{x}$ with respect to its proper time

$$\mathbf{u}^P = \frac{d\mathbf{x}}{d\tau} \tag{2.11}$$

In terms of components, this relation becomes

$$u_\mu^P = \frac{dx_\mu}{d\tau} \tag{2.12}$$

For simplicity, we may drop the superscipt ^{P} and write $\mathbf{u} = u_\mu \mathbf{e}_\mu$. The space and time components of $\mathbf{u} = (\mathbf{u}, u_4)$ are

$$\mathbf{u} = \frac{\mathbf{v}}{\sqrt{1 - v^2/c^2}} = c\sinh\xi\mathbf{e}_t \tag{2.13}$$

and

$$u_4 = \frac{ic}{\sqrt{1 - v^2/c^2}} = ic\cosh\xi \tag{2.14}$$

Therefore, the four-vector velocity can be represented as

$$\mathbf{u} = c(\sinh\xi\mathbf{e}_t,\ \mathrm{i}\cosh\xi) \tag{2.15}$$

The length of the four-vector velocity is a constant, since

$$u_\mu u_\mu = \mathbf{u}^2 + u_4^{\ 2} = -c^2 \tag{2.16}$$

which shows the four-velocity is time-like.

The four-acceleration $\mathbf{b} = b_\mu \mathbf{e}_\mu$ is defined as

$$\mathbf{b} = \frac{d\mathbf{u}}{d\tau} = \frac{d^2\mathbf{x}}{d\tau^2} \tag{2.17}$$

In terms of components, this relation becomes

$$b_\mu = \frac{du_\mu}{d\tau} = \frac{d^2 x_\mu}{d\tau^2} \tag{2.18}$$

The four-acceleration is always perpendicular to the four-vector velocity, as shown in Fig. 1, where

$$\mathbf{u} \bullet \mathbf{b} = u_\mu b_\mu = 0 \tag{2.19}$$

It can be easily shown that

$$\mathbf{b} = \left[\cosh^2\xi\mathbf{a} + \cosh^4\xi\left(\frac{\mathbf{v}\bullet\mathbf{a}}{c^2}\right)\mathbf{v}, i\cosh^4\xi\left(\frac{\mathbf{v}\bullet\mathbf{a}}{c}\right)\right] \tag{2.20}$$

where

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{2.21}$$

is the three-vector acceleration of the particle. The length of the four-vector acceleration can be found to be

$$|\mathbf{b}|^2 = b_\mu b_\mu = \cosh^4\xi a^2 + \cosh^6\xi\left(\frac{\mathbf{v}\bullet\mathbf{a}}{c}\right)^2 \tag{2.22}$$

Since $b_\mu b_\mu$ is positive, the four-acceleration is space-like.

It should be noticed that all four-tensors in this article are either of first or second order. For simplicity, we use the same symbols for their vector and matrix representations. Therefore, the relation

$$\mathbf{u} \bullet \mathbf{b} = 0 \tag{2.23}$$

can be written in matrix form

$$\mathbf{u}^T\mathbf{b} = 0 \tag{2.24}$$

What we have presented is the well-known relativistic kinematics of particle. However, it can be shown that the accelerating motion of the particle can be considered as the result of a four-dimensional rotation with important consequences as will be shown in the following.

Four-dimensional character of particle and vortical theory of motion

As we mentioned, Poincaré's theory of relativity can be completed by establishing the relation between matter and the Minkowskian space-time. It is postulated that the massive particle specifies its own Minkowskian space-time body frame $x_1'x_2'x_3'x_4'$, in which the particle has an attached four-velocity $\mathbf{u}^P$ with magnitude c in the time direction x_4'. Thus, the four-vector velocity of the particle in this system is represented by

$$u'^P_\mu = (0,0,0,ic) \tag{2.25}$$

whereas its representation in our inertial reference frame system $x_1x_2x_3x_4$ is

$$u^P_\mu(x) = c(\sinh\xi \mathrm{e}_t, i\cosh\xi) \tag{2.26}$$

This is shown in Fig. 2. Interestingly, we realize that the reference frame $x_1x_2x_3x_4$ is also specified with a massive particle, having the attached four-velocity $\mathbf{u}^R$ and magnitude c in the time direction x_4, where

$$u^R_\mu = (0,0,0,ic) \tag{2.27}$$

The orientation of the body frame of the particle P relative to the inertial reference frame system is specified by the orthogonal transformation four-tensor $\mathbf{\Lambda}(\mathbf{x})$ with orthogonality condition

$$\mathbf{\Lambda}(\mathbf{x})\mathbf{\Lambda}^T(\mathbf{x}) = \mathbf{\Lambda}^T(\mathbf{x})\mathbf{\Lambda}(\mathbf{x}) = \mathbf{1} \tag{2.28}$$

or

$$\Lambda_{\mu\alpha}\Lambda_{\nu\alpha} = \Lambda_{\alpha\mu}\Lambda_{\alpha\nu} = \delta_{\mu\nu} \tag{2.29}$$

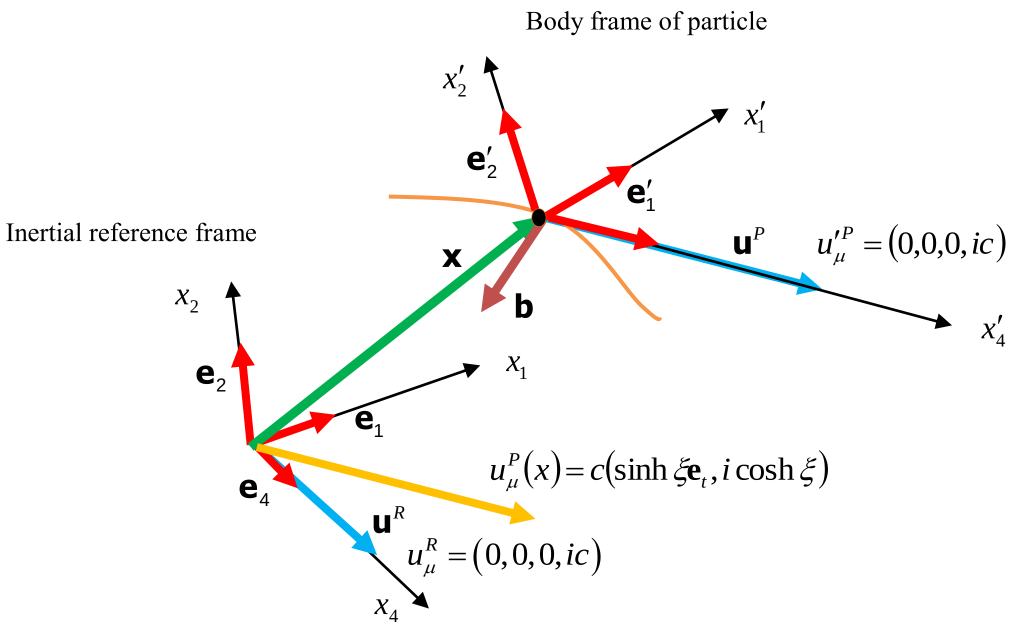

Fig. 2. Inertial reference frame and body frame.

where $\delta_{\mu\nu}$ is the Kronecker delta in four dimension. This variable orthogonal transformation $\mathbf{\Lambda}(\mathbf{x})$ is a general Lorentz transformation, which can be written as

$$\begin{aligned}\mathbf{\Lambda}(\mathbf{x}) &= \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} + (\cosh\xi - 1)\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q} + (\cosh\xi - 1)\mathbf{Q}\mathbf{e}_t\mathbf{e}_t^T & i\sinh\xi\mathbf{Q}\mathbf{e}_t \\ -i\sinh\xi\mathbf{e}_t^T & \cosh\xi \end{bmatrix}\end{aligned} \tag{2.30}$$

Here $\mathbf{Q}$ is a three-dimensional orthogonal tensor representing spatial rotation of the body frame. The boost part of this transformation depends on the rapidity vector $\boldsymbol{\xi}$ of the particle, even for accelerating motion. It is seen that the relations among the base unit four-vectors of the body frame and inertial systems are

$$\mathbf{e}'_\mu = \Lambda_{\mu\nu}\mathbf{e}_\nu \tag{2.31}$$

or

$$\mathbf{e}_\mu = \Lambda_{\nu\mu}\mathbf{e}'_\nu \tag{2.32}$$

Therefore, the angles among these directions are such that

$$\cos\left(\mathbf{e}'_{\mu}, \mathbf{e}_{\nu}\right) = \Lambda_{\mu\nu} \tag{2.33}$$

$$\cos\left(\mathbf{e}_{\mu}, \mathbf{e}'_{\nu}\right) = \Lambda_{\nu\mu} \tag{2.34}$$

The inertial reference frame and the body frame of the particle both have attached four-vector velocities $\mathbf{u}^R$ and $\mathbf{u}^P$ in their own space-time frames, respectively. The attached four-vector velocity $\mathbf{u}^P$ is rotating with the body frame of the particle, such that between its components in this frame and the inertial frame, we have

$$\mathbf{u}'^P = \mathbf{\Lambda}(\mathbf{x})\mathbf{u}^P(\mathbf{x}) \tag{2.35}$$

We may drop the superscipt ^{P} and write

$$\mathbf{u}' = \mathbf{\Lambda}(\mathbf{x})\mathbf{u}(\mathbf{x}) \tag{2.36}$$

or in terms of components as

$$u'_{\mu} = \Lambda_{\mu\nu} u_{\nu} \tag{2.37}$$

The Lorentz transformation (2.37) relates the components of four-vector velocity $\mathbf{u}'$ of particle P in its frame and its components of four-vector velocity $\mathbf{u}$ in the inertial reference frame of particle R.

This relation can also be written as

$$\mathbf{u}(\mathbf{x}) = \mathbf{\Lambda}^T(\mathbf{x})\mathbf{u}' \tag{2.38}$$

By taking the derivative of (2.28) with respect to the invariant proper time of the particle, we obtain

$$\frac{d\mathbf{\Lambda}^T}{d\tau}\mathbf{\Lambda} + \mathbf{\Lambda}^T \frac{d\mathbf{\Lambda}}{d\tau} = \mathbf{0} \tag{2.39}$$

Now by defining the four-tensor $\mathbf{\Omega}$

$$\mathbf{\Omega} = \frac{d\mathbf{\Lambda}^T}{d\tau}\mathbf{\Lambda} \tag{2.40}$$

we can see that the relation (2.39) becomes

$$\mathbf{\Omega} + \mathbf{\Omega}^T = \mathbf{0} \tag{2.41}$$

In term of components, this relation can be written as

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0 \tag{2.42}$$

which shows that $\mathbf{\Omega}$ is an anti-symmetric four-tensor. From our knowledge in non-relativistic rigid body dynamics, we realize that the four-tensor $\mathbf{\Omega}$ represents the four-tensor angular velocity of the space-time body frame of the particle measured in the inertial reference system. Interestingly, by using the relation (2.39), we rewrite the equation (2.40) as

$$\frac{d\mathbf{\Lambda}}{d\tau} = -\mathbf{\Lambda}\mathbf{\Omega} \tag{2.43}$$

which is the equation of motion of the body frame of the particle in terms of its four-tensor angular velocity $\mathbf{\Omega}(\mathbf{x})$.

The anti-symmetric tensor $\mathbf{\Omega}(\mathbf{x})$ in terms of elements in the inertial reference frame is written

$$\mathbf{\Omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & -i\eta_1/c \\ \omega_3 & 0 & -\omega_1 & -i\eta_2/c \\ -\omega_2 & \omega_1 & 0 & -i\eta_3/c \\ i\eta_1/c & i\eta_2/c & i\eta_3/c & 0 \end{bmatrix} \tag{2.44}$$

This is the general form of the four-tensor angular velocity $\mathbf{\Omega}$. By introducing three-vectors $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$, this four-tensor can be symbolically represented by

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{R}_{\boldsymbol{\omega}} & -i\frac{1}{c}\boldsymbol{\eta} \\ i\frac{1}{c}\boldsymbol{\eta}^T & 0 \end{bmatrix} \tag{2.45}$$

where the anti-symmetric matrix $\mathbf{R}_{\boldsymbol{\omega}}$, corresponding to the three-vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, is defined by

$$\mathbf{R}_{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2.46}$$

It should be noticed that, from (2.45), the elements $\Omega_{4i} = -\Omega_{i4} = i\eta_i / c$ are imaginary. The components of $\Omega_{\mu\nu}$ can be formally interpreted as follows:

- The angular velocities ω_1, ω_2 and ω_3 generate the space rotation of the body frame $x_1' x_2' x_3' x_4'$ in the planes $x_2 x_3$, $x_3 x_1$ and $x_1 x_2$, respectively.
- The imaginary angular velocities $\frac{i}{c}\eta_1$, $\frac{i}{c}\eta_2$ and $\frac{i}{c}\eta_3$ generate hyperbolic rotation of the body frame $x_1' x_2' x_3' x_4'$ in the planes $x_1 x_4$, $x_2 x_4$ and $x_3 x_4$, respectively. The quantities η_1, η_2 and η_3 are the rate of change of boost of the body frame, along the x_1, x_2 and x_3 axes, respectively.

Therefore, the space-time body frame of the particle rotates relative to the inertial system with angular velocity tensor $\mathbf{\Omega}$, which is a combination of circular and hyperbolic angular velocities $\boldsymbol{\omega}$ and $\frac{1}{c}\boldsymbol{\eta}$. Note that these interpretations of three-vector velocities are not completely compatible with our notion of rotation in non-relativistic kinematics. Although we still use the notations $\boldsymbol{\omega}$ and $\frac{i}{c}\boldsymbol{\eta}$, and call these angular velocities, these vectors cannot be taken as proper angular velocity vectors. Basically these interpretations are consistent only for an inertial co-moving observer, who measures the components of

$$\Omega'_{\mu\nu} = \Lambda_{\mu\alpha}\Lambda_{\nu\beta}\Omega_{\alpha\beta} \tag{2.47}$$

where

$$\mathbf{\Omega}' = \begin{bmatrix} \mathbf{R}_{\boldsymbol{\omega}'} & -i\frac{1}{c}\boldsymbol{\eta}' \\ i\frac{1}{c}\boldsymbol{\eta}'^T & 0 \end{bmatrix} \tag{2.48}$$

This co-moving observer can consider the vectors $\boldsymbol{\omega}'$ and $\frac{i}{c}\boldsymbol{\eta}'$ as proper angular velocity vectors.

The inverse tensor transformation of (2.47),

$$\Omega_{\mu\nu} = \Lambda_{\alpha\mu}\Lambda_{\beta\nu}\Omega'_{\alpha\beta} \tag{2.49}$$

shows that a combination of the circular and hyperbolic angular velocities $\boldsymbol{\omega}'$ and $\frac{1}{c}\boldsymbol{\eta}'$ gives the vectors $\boldsymbol{\omega}$ and $\frac{1}{c}\boldsymbol{\eta}$ under a non-Euclidean geometry transformation. One can see the appearance of apparently paradoxical effects similar to Thomas-Wigner precession [6] in this transformation. However, we now know there is no paradox at all and these effects are simply the result of the governing non-Euclidean geometry.

By noticing that the four-vector velocity of the particle is attached to its own body frame in its time direction, we realize that the four-acceleration of the particle is the result of the four-dimensional rotation of its body frame. Let us derive an expression for four-acceleration of the particle in terms of the four-tensor angular velocity $\mathbf{\Omega}(\mathbf{x})$. By taking the proper time derivative of (2.38), we obtain

$$\frac{d\mathbf{u}(\mathbf{x})}{d\tau} = \frac{d\mathbf{\Lambda}^T(\mathbf{x})}{d\tau}\mathbf{u}' \tag{2.50}$$

and by substituting from (2.36), we have

$$\frac{d\mathbf{u}(\mathbf{x})}{d\tau} = \frac{d\mathbf{\Lambda}^T(\mathbf{x})}{d\tau}\mathbf{\Lambda}(\mathbf{x})\mathbf{u}(\mathbf{x}) \tag{2.51}$$

Using the definition of four-tensor angular velocity $\mathbf{\Omega}$ in (2.40), we finally obtain

$$\frac{d\mathbf{u}(\mathbf{x})}{d\tau} = \mathbf{\Omega}(\mathbf{x})\mathbf{u}(\mathbf{x}) \tag{2.52}$$

$$\frac{du_\mu}{d\tau} = \Omega_{\mu\nu}u_\nu \tag{2.53}$$

It is seen that the four-acceleration $\frac{d\mathbf{u}(\mathbf{x})}{d\tau}$ is the result of the instantaneous four-dimensional rotation of the four-vector velocity $\mathbf{u}$ attached to the body frame and rotating with four-tensor angular velocity $\mathbf{\Omega}(\mathbf{x})$. The space and time components of this equation are

$$\frac{d\mathbf{u}}{d\tau} = \boldsymbol{\omega} \times \mathbf{u} - \frac{i}{c}\boldsymbol{\eta} u_4 \tag{2.54}$$

$$\frac{du_4}{d\tau} = i\frac{1}{c}\boldsymbol{\eta} \bullet \mathbf{u} \tag{2.55}$$

However, we notice that the more complete equation of motion for particle is the equation of instantaneous four-dimensional rotation of its space-time body frame (2.43), that is

$$\frac{d\mathbf{\Lambda}}{d\tau} + \mathbf{\Lambda}\mathbf{\Omega} = \mathbf{0} \tag{2.56}$$

which can also be written as

$$\frac{d\Lambda_{\mu\nu}}{d\tau} + \Lambda_{\mu\alpha}\Omega_{\alpha\nu} = 0 \tag{2.57}$$

General relative motion and the velocity addition law

Consider two particles A and B moving with velocities $\mathbf{v}_A = \mathbf{v}_A(t)$ and $\mathbf{v}_B = \mathbf{v}_B(t)$ relative to the inertial reference system $x_1x_2x_3x_4$. The four-vector velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ are attached four-vectors to the body frames $x_1'x_2'x_3'x_4'$ and $x_1''x_2''x_3''x_4''$ of A and B, such that

$$(\mathbf{u}_A)_A = \mathbf{\Lambda}_A\mathbf{u}_A \tag{2.58}$$

$$(\mathbf{u}_B)_B = \mathbf{\Lambda}_B\mathbf{u}_B \tag{2.59}$$

This is depicted in Fig 3. Let $(\mathbf{u}_A)_A$ and $(\mathbf{u}_B)_B$ be representations of the two four-vector velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ measured by observers attached to their corresponding body frames A and B, where

$$(\mathbf{u}_A)_A = \mathbf{u}_A' = (0,0,0,ic) \tag{2.60}$$

$$(\mathbf{u}_B)_B = \mathbf{u}_B'' = (0,0,0,ic) \tag{2.61}$$

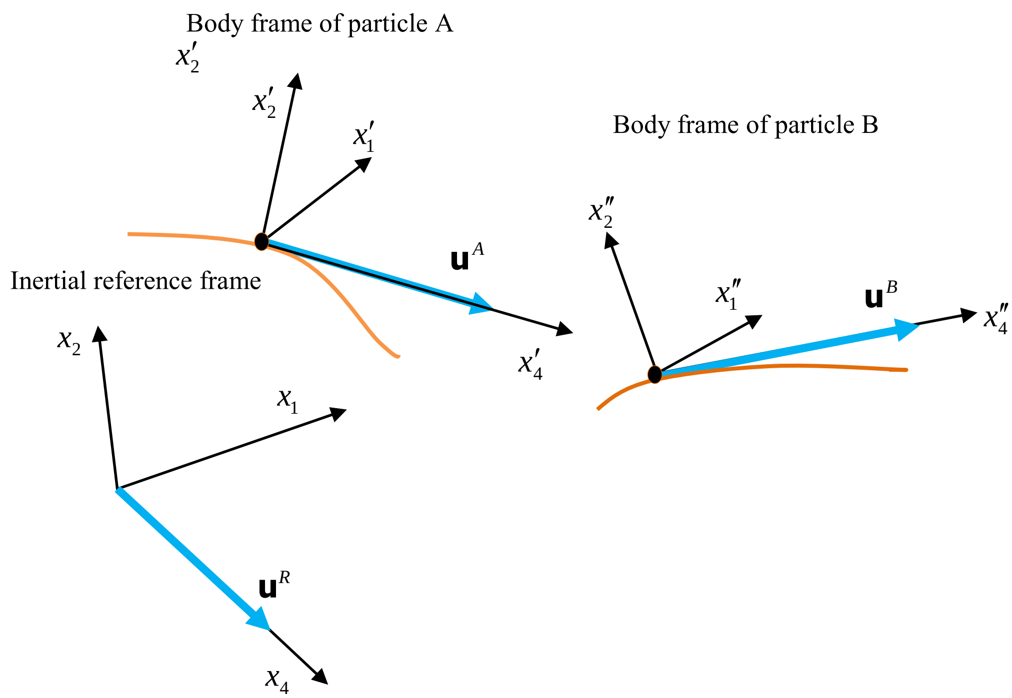

Fig. 3. Body frames in relative motion.

Therefore, we have the interesting relation

$$(\mathbf{u}_A)_A = (\mathbf{u}_B)_B = (0,0,0,ic) \tag{2.62}$$

However, we notice that an inertial observer in the inertial reference frame measures different components for four-vectors $\mathbf{u}_A$ and $\mathbf{u}_B$. The equality (2.62) for components of $(\mathbf{u}_A)_A$ and $(\mathbf{u}_B)_B$ is the origin of the non-Euclidean character of relative motion, which we next explore.

The transformations $\mathbf{\Lambda}_A = \mathbf{\Lambda}_A(t)$ and $\mathbf{\Lambda}_B = \mathbf{\Lambda}_B(t)$ represent the orientation of the body frames of particles A and B relative to the inertial reference frame. For these transformations, we explicitly have

$$\mathbf{\Lambda}_A(t) = \begin{bmatrix} \mathbf{Q}_A + (\cosh\xi_A - 1)\mathbf{Q}_A\mathbf{e}_{tA}\mathbf{e}_{tA}^T & i\sinh\xi_A\mathbf{Q}_A\mathbf{e}_{tA} \\ -i\sinh\xi_A\mathbf{e}_{tA}^T & \cosh\xi_A \end{bmatrix} \tag{2.63}$$

and

$$\mathbf{\Lambda}_B(t) = \begin{bmatrix} \mathbf{Q}_B + (\cosh\xi_B - 1)\mathbf{Q}_B\mathbf{e}_{tB}\mathbf{e}_{tB}^T & i\sinh\xi_B\mathbf{Q}_B\mathbf{e}_{tB} \\ -i\sinh\xi_B\mathbf{e}_{tB}^T & \cosh\xi_B \end{bmatrix} \tag{2.64}$$

By using (2.62) to combine (2.58) and (2.59), we obtain

$$\mathbf{u}_A = \mathbf{\Lambda}_A^T\mathbf{\Lambda}_B\mathbf{u}_B \tag{2.65}$$

The orientation of the body frame B relative to A at time t is denoted by $\mathbf{\Lambda}_{B/A}$ where

$$\mathbf{\Lambda}_B = \mathbf{\Lambda}_A\mathbf{\Lambda}_{B/A} \tag{2.66}$$

From this relation, we have

$$\mathbf{\Lambda}_{B/A} = \mathbf{\Lambda}_A^T\mathbf{\Lambda}_B \tag{2.67}$$

Therefore, (2.65) becomes

$$\mathbf{u}_A = \mathbf{\Lambda}_{B/A}\mathbf{u}_B \tag{2.68}$$

which can also be written as

$$\mathbf{u}_B = \mathbf{\Lambda}_{B/A}^T\mathbf{u}_A \tag{2.69}$$

We notice that $\mathbf{\Lambda}_{B/A}$ is the relative Lorentz transformation from body frame A to body frame B measured by the inertial reference frame at time t. Therefore, all the relations are relative to this observer at time t.

Now we derive the relations relative to the observer attached to the body frame A. For this, we notice that the velocity of particle B relative to particle A measured by the observer in the body frame of A is

$$(\mathbf{u}_B)_A = (\mathbf{u}_{B/A})_A = \mathbf{\Lambda}_A\mathbf{u}_B \tag{2.70}$$

By substituting for $\mathbf{u}_B$ from (2.59), we obtain

$$(\mathbf{u}_B)_A = (\mathbf{u}_{B/A})_A = \mathbf{\Lambda}_A\mathbf{\Lambda}_B^T(\mathbf{u}_B)_B \tag{2.71}$$

We also have the obvious relation

$$\left(\mathbf{u}_A\right)_A = \left(\mathbf{\Lambda}_{B/A}\right)_A \left(\mathbf{u}_{B/A}\right)_A \tag{2.72}$$

which can be written as

$$\left(\mathbf{u}_{B/A}\right)_A = \left(\mathbf{\Lambda}_{B/A}\right)_A^T \left(\mathbf{u}_A\right)_A \tag{2.73}$$

By comparing (2.71) and (2.72) and using (2.70), we obtain the relation

$$\left(\mathbf{\Lambda}_{B/A}\right)_A^T = \mathbf{\Lambda}_A \mathbf{\Lambda}_B^T \tag{2.74}$$

which can be written as

$$\left(\mathbf{\Lambda}_{B/A}\right)_A = \mathbf{\Lambda}_B \mathbf{\Lambda}_A^T \tag{2.75}$$

By using the relation (2.67), we obtain

$$\left(\mathbf{\Lambda}_{B/A}\right)_A = \mathbf{\Lambda}_A \mathbf{\Lambda}_{B/A} \mathbf{\Lambda}_A^T \tag{2.76}$$

Interestingly, this is the transformation for tensor $\mathbf{\Lambda}_{B/A}$ from the inertial reference frame to the body frame A. What we have is the development of the general theory of relative motion. This transformation clearly shows that the geometry governing the attached three-vector and three-tensors is non-Euclidean.

Explicitly from (2.70), we have

$$\left(\mathbf{u}_{B/A}\right)_A = \begin{bmatrix} \mathbf{Q}_A + (\cosh\xi_A - 1)\mathbf{Q}_A \mathbf{e}_{tA}\mathbf{e}_{tA}^T & i\sinh\xi_A \mathbf{Q}_A \mathbf{e}_{tA} \\ -i\sinh\xi_A \mathbf{e}_{tA}^T & \cosh\xi_A \end{bmatrix} \begin{bmatrix} c\sinh\xi_B \mathbf{e}_{tB} \\ ic\cosh\xi_B \end{bmatrix} \tag{2.77}$$

From this, we obtain the relations

$$\left(\sinh\xi_{B/A}\mathbf{e}_{tB/A}\right)_A = -\sinh\xi_A \cosh\xi_B \mathbf{Q}_A \mathbf{e}_{tA} + \mathbf{Q}_A[\mathbf{1} + (\cosh\xi_A - 1)\mathbf{e}_{tA}\mathbf{e}_{tA}^T]\sinh\xi_B \mathbf{e}_{tB} \tag{2.78}$$

$$\left(\cosh\xi_{B/A}\right)_A = \cosh\xi_A \cosh\xi_B - \sinh\xi_A \sinh\xi_B \mathbf{e}_{tA} \bullet \mathbf{e}_{tB} \tag{2.79}$$

These relations are the manifest of hyperbolic geometry governing the velocity addition law, which applies even for accelerating particles. This property holds for all attached four-vectors and four-tensors. Inertial observers relate components of attached four-vectors and four-tensors by Lorentz transformations. This is the origin of non-Euclidean geometry governing the three-vector and three-tensors. As we saw, the addition of three-vector velocities follow hyperbolic geometry. Thus,

$$\begin{aligned}&\left(\sinh\xi_{B/A}\mathbf{e}_{tB/A}\right)_A\\&=\mathbf{Q}_A\left\{\left[(\cosh\xi_A-1)\sinh\xi_B\left(\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}\right)-\sinh\xi_A\cosh\xi_B\right]\mathbf{e}_{tA}+\sinh\xi_B\mathbf{e}_{tB}\right\}\end{aligned}\tag{2.80}$$

$$\left(\cosh\xi_{B/A}\right)_A=\cosh\xi_A\cosh\xi_B-\sinh\xi_A\sinh\xi_B\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}\tag{2.81}$$

If the relative rapidity $\left(\xi_{B/A}\right)_A$ is nonzero, we can divide (2.80) by (2.81) to obtain

$$\begin{aligned}&\left(\tanh\xi_{B/A}\mathbf{e}_{tB/A}\right)_A\\&=\mathbf{Q}_A\frac{\left[(\cosh\xi_A-1)\tanh\xi_B\left(\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}\right)-\sinh\xi_A\right]\mathbf{e}_{tA}+\tanh\xi_B\mathbf{e}_{tB}}{\cosh\xi_A-\sinh\xi_A\tanh\xi_B\mathbf{e}_{tA}\bullet\mathbf{e}_{tB}}\end{aligned}\tag{2.82}$$

Finally, it should be noticed that these relations hold despite the fact that the transformation

$$x'_\mu=\Lambda_{\mu\nu}x_\nu\tag{2.83}$$

is not valid among accelerating systems. The general transformation must be written for four-vector velocities, not positions. What we have developed here is the completion of Poincare's relativity for accelerating systems.

3. Vortex theory of fundamental interaction

After completing the theory of accelerating motion, we are ready to develop the theory of fundamental interaction. Symmetry of nature suggests a common mechanism for all fundamental interactions. Therefore, based on the established four-dimensional theory of motion in Section 2, in this section, we develop the general theory of fundamental interaction.

The equation of motion for a particle in the inertial reference frame system is given by

$$m\frac{d\mathbf{u}}{d\tau}=\boldsymbol{F}\tag{3.1}$$

where $\boldsymbol{F}$ is the four-vector Minkowski force. This force is the result of interaction of the particle with a field, such as an electromagnetic field. By substituting from (2.52) in (3.1), we obtain

$$\boldsymbol{F} = m\boldsymbol{\Omega} \bullet \mathbf{u} \tag{3.2}$$

which can be written in terms of components as

$$F_{\mu} = m\Omega_{\mu\nu}u_{\nu} \tag{3.3}$$

Since $\boldsymbol{\Omega}$ is anti-symmetric, we have the relation

$$\boldsymbol{F} \bullet \mathbf{u} = F_{\mu}u_{\mu} = m\Omega_{\mu\nu}u_{\mu}u_{\nu} = 0 \tag{3.4}$$

which means the four-vector Minkowski force $\boldsymbol{F}$ is perpendicular to the four-vector velocity $\mathbf{u}$. The relation (3.2) shows that this force depends on the four-vector velocity $\mathbf{u}$ and the four-tensor angular velocity $\boldsymbol{\Omega}$ of the body frame. Therefore, the four-tensor angular velocity $\boldsymbol{\Omega}$ must depend on the field strength. The simplest admissible field strength is characterized by a four-tensor $\boldsymbol{\Theta}(\mathbf{x})$ such that

$$m\boldsymbol{\Omega}(\tilde{\mathbf{x}}) = \alpha\boldsymbol{\Theta}(\tilde{\mathbf{x}}) \tag{3.5}$$

where $\tilde{\mathbf{x}}$ denotes the position of the particle and the scalar α is a property of the particle, called the charge of the particle, which depends on the type of interaction. Consequently, we can postulate a fundamental interaction to be an interaction characterized by an anti-symmetric strength tensor field $\boldsymbol{\Theta}(\mathbf{x})$. At the position of the particle $\tilde{\mathbf{x}}$, the space-time body frame of particle rotates with four-tensor angular velocity $\boldsymbol{\Omega}(\tilde{\mathbf{x}})$, such that

$$\boldsymbol{\Omega}(\tilde{\mathbf{x}}) = \frac{\alpha}{m}\boldsymbol{\Theta}(\tilde{\mathbf{x}}) \tag{3.6}$$

$$\Omega_{\mu\nu} = \frac{\alpha}{m}\Theta_{\mu\nu}(\tilde{x}) \tag{3.7}$$

This shows that the angular velocity $\boldsymbol{\Omega}$ depends on the particle through the ratio $\frac{\alpha}{m}$. As a result, the equation of motion for the four-dimensional rotation of the space-time body frame of the particle becomes

$$\frac{d\boldsymbol{\Lambda}}{d\tau} = -\frac{\alpha}{m}\boldsymbol{\Lambda} \bullet \boldsymbol{\Theta}(\tilde{\mathbf{x}}) \tag{3.8}$$

This is the complete form of the equation of motion for the particle. However, we are more familiar with the equation of motion in the form

$$m\frac{d\mathbf{u}}{d\tau}=\boldsymbol{F} \tag{3.9}$$

where $\boldsymbol{F}$ is the Minkowski force defined as

$$\boldsymbol{F}=\alpha\boldsymbol{\Theta}(\tilde{\mathbf{x}})\mathbf{u} \tag{3.10}$$

It is seen that this force looks like the Lorentz force in electrodynamics, which depends on the charge α of the particle and its four-vector velocity $\mathbf{u}$. Therefore, the equation of motion becomes

$$m\frac{d\mathbf{u}}{d\tau}=\alpha\boldsymbol{\Theta}(\tilde{\mathbf{x}})\mathbf{u} \tag{3.11}$$

In terms of components, this equation can be written as

$$m\frac{du_\mu}{d\tau}=\alpha\Theta_{\mu\nu}(\tilde{x})u_\nu \tag{3.12}$$

One can see that the anti-symmetric strength tensor $\boldsymbol{\Theta}$ looks like a four-dimensional vorticity field analogous to the three-dimensional vorticity in rotational fluid flow. Thus, we consider a four-vector velocity-like field $\mathbf{V}=(\mathbf{V},V_4)=V_\mu\mathbf{e}_\mu$ induced to the space-time of the inertial reference frame. It is seen that the vorticity-like strength tensor $\boldsymbol{\Theta}=\Theta_{\mu\nu}\mathbf{e}_\mu\mathbf{e}_\nu$ is the four-dimensional curl of this four-vector velocity, where

$$\Theta_{\mu\nu}(x)=\partial_\nu V_\mu-\partial_\mu V_\nu \tag{3.13}$$

The general form of this anti-symmetric vorticity-like tensor in terms of elements is

$$\boldsymbol{\Theta}=\begin{bmatrix} 0 & -w_3 & w_2 & -ih_1/c \\ w_3 & 0 & -w_1 & -ih_2/c \\ -w_2 & w_1 & 0 & -ih_3/c \\ ih_1/c & ih_2/c & ih_3/c & 0 \end{bmatrix} \tag{3.14}$$

with three-vectors $\mathbf{w}$ and $\mathbf{h}$. Thus, this four-tensor can be represented symbolically by

$$\mathbf{\Theta} = \begin{bmatrix} \mathbf{R}_{\mathrm{w}} & -i\frac{1}{c}\mathbf{h} \\ i\frac{1}{c}\mathbf{h}^T & 0 \end{bmatrix} \tag{3.15}$$

where the anti-symmetric matrix $\mathbf{R}_{\mathrm{w}}$ corresponds to the three-vector $\mathbf{w} = (w_1, w_2, w_3)$ and is defined by

$$\mathbf{R}_{\mathrm{w}} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \tag{3.16}$$

It should be noticed that the elements $\Theta_{4i} = -\Theta_{i4} = ih_i/c$ are imaginary. By decomposition of the four-tensor vorticity defined by (3.13), we obtain

$$\frac{1}{c}\mathbf{h} = \frac{1}{c}\frac{\partial \mathbf{V}}{\partial t} - i\nabla V_4 \tag{3.17}$$

$$\mathbf{w} = \nabla \times \mathbf{V} \tag{3.18}$$

We recognize that the vector $\mathbf{w}$ is analogous to the vorticity of the three-vector velocity in fluid mechanics. The components of $\mathbf{\Theta}$ can be interpreted as follows:

- The components w_1, w_2 and w_3 are the vorticity-like components of the four-vector velocity-like $\mathbf{V}$ in planes x_2x_3, x_3x_1 and x_1x_2, respectively.
- The imaginary components $\frac{i}{c}h_1$, $\frac{i}{c}h_2$ and $\frac{i}{c}h_3$ are the imaginary vorticity-like components of the four-vector velocity-like $\mathbf{V}$ in planes x_1x_4, x_2x_4 and x_3x_4, respectively.

Therefore, the vectors $\mathbf{w}$ and $\mathbf{h}/c$ are the circular and hyperbolic vorticity of the four-vector velocity field $\mathbf{V}$, respectively.

By taking the curl of (3.17) and divergence of (3.18), we obtain the compatibility equations

$$\nabla \times \mathbf{h} = \frac{\partial \mathbf{w}}{\partial t} \tag{3.19}$$

$$\nabla \bullet \mathbf{w} = 0 \tag{3.20}$$

These equations are the necessary condition for consistency of given three-vector vorticity fields $\mathbf{w}$ and $\mathbf{h}$. The covariant form of these equations is the consistency condition

$$\partial_\sigma \Theta_{\mu\nu} + \partial_\mu \Theta_{\nu\sigma} + \partial_\nu \Theta_{\sigma\mu} = 0 \tag{3.21}$$

which is the necessary condition to obtain the four-vector velocity V_μ from the four-dimensional vorticity field $\Theta_{\mu\nu}$. However, the four-vector V_μ is not uniquely determined from compatible four-tensor vorticity $\Theta_{\mu\nu}$. Indeed, the new field

$$V_\mu \to V_\mu + \partial_\mu \lambda \tag{3.22}$$

does not change the vorticity field $\Theta_{\mu\nu}$. Such transformation is called a gauge transformation, in which the function λ is a function of position coordinate x_μ. This gauge freedom allows us to choose the gauge constraint

$$\partial_\mu V_\mu = 0 \tag{3.23}$$

This constraint can be interpreted as an incompressibility condition on the four-vector velocity field $\mathbf{V}$.

It seems that defining a quantity, which represents a measure of curvature of the four-velocity field $\mathbf{V}$, is also important in our development of the fundamental interaction. An analogy with continuum mechanics suggests that the four-vector $\partial_\nu \Theta_{\mu\nu}$ can be taken as a measure of curvature of the four-vector velocity field V_μ [7, 8]. Actually, it turns out that the *four-vector mean curvature* of this four-velocity field may be defined as

$$K_\mu = -\frac{1}{6} \partial_\nu \Theta_{\mu\nu} \tag{3.24}$$

where $\mathbf{K} = K_\mu \mathbf{e}_\mu$. The coefficient $\frac{1}{6}$ is chosen such that the definition of the mean curvature four-vector is compatible with the definition of the mean curvature in higher dimensions [9]. The space and time components of the mean curvature four-vector can be written in terms of circular and hyperbolic vorticities as $\mathbf{K} = (\mathbf{K}, K_4)$ with

$$\mathbf{K} = -\frac{1}{6}\left(\nabla \times \mathbf{w} - \frac{1}{c^2}\frac{\partial \mathbf{h}}{\partial t}\right) \tag{3.25}$$

$$K_4 = -i\frac{1}{6c}\nabla \bullet \mathbf{h} \tag{3.26}$$

However, in terms of the four-vector velocity V_μ, the mean curvature four-vector becomes

$$K_\mu = -\frac{1}{6}\left(\partial_\nu \partial_\nu V_\mu - \partial_\mu \partial_\nu V_\nu\right) \tag{3.27}$$

By using the gauge constraint (3.23), we obtain the form

$$K_\mu = -\frac{1}{6}\partial_\nu \partial_\nu V_\mu \tag{3.28}$$

From this, the space and time components of the mean curvature four-vector are seen to be

$$\mathbf{K} = -\frac{1}{6}\left(\nabla^2 \mathbf{V} - \frac{1}{c^2}\frac{\partial^2 \mathbf{V}}{\partial t^2}\right) \tag{3.29}$$

$$K_4 = -\frac{1}{6}\left(\nabla^2 V_4 - \frac{1}{c^2}\frac{\partial^2 V_4}{\partial t^2}\right) \tag{3.30}$$

It should be noted that the components K_1, K_2, K_3 and K_4 at each point in the inertial system $x_1x_2x_3x_4$ are the mean curvature of the hyper-planes parallel to the coordinate hyper-planes $x_2x_3x_4$, $x_3x_4x_1$, $x_4x_1x_2$, and $x_1x_2x_3$ caused by the four-vector velocity components V_1, V_2, V_3 and V_4, respectively.

Although we have investigated the kinematics of the four-vector velocity $\mathbf{V}$ by defining the vorticity field $\mathbf{\Theta}$ and mean curvature $\mathbf{K}$, we have not specified how this four-vector velocity field $\mathbf{V}$ is created by charges. We develop this theory by defining the current density as follows.

Let ρ_A be the volume density of charges moving with the three-vector velocity field $\mathbf{v}^A$. The charge current density $\mathbf{J}_A$ is defined as

$$\mathbf{J}_A = (\mathbf{J}_A, J_{A4}) = (\mathbf{J}_A, i\rho_A c) = \rho_A(\mathbf{v}^A, ic) \tag{3.31}$$

which satisfies the continuity equation

$$\partial_\mu J_{A\mu} = \nabla \bullet \mathbf{J}_A + \frac{\partial \rho_A}{\partial t} = 0 \tag{3.32}$$

It is postulated that the four-vector charge current density $\mathbf{J}_A$ induces the four-vector fundamental interacting velocity field $\mathbf{V} = V_\mu \mathbf{e}_\mu = (\mathbf{V}, V_4)$ to the reference inertial space-time, such that $\mathbf{J_A}$ is proportional to the mean curvature four-vector $\mathbf{K}$. Therefore, we have

$$\mathbf{K} = -\frac{1}{6}\beta \mathbf{J}_A \tag{3.33}$$

$$K_\mu = -\frac{1}{6}\beta J_{A\mu} \tag{3.34}$$

where β is a constant depending on the interaction type and can be either positive or negative. By using the defintion K_μ in (3.24) we can rewrite the relation (3.34) in the form

$$\partial_\nu \Theta_{\mu\nu} = \beta J_{A\mu} \tag{3.35}$$

This equation can be considered as the equation of motion for vorticities. In terms of four-vector velocity, it becomes

$$\partial_\nu \partial_\nu V_\mu - \partial_\mu \partial_\nu V_\nu = \beta J_{A\mu} \tag{3.36}$$

After using the gauge constraint (3.23), this reduces to

$$\partial_\nu \partial_\nu V_\mu = \beta J_{A\mu} \tag{3.37}$$

4. Geometry of electrodynamics

Vortex Theory of Electromagnetic Interaction

One can see that the electromagnetic interaction is completely compatible with this geometrical vortex theory of interaction, where the coupling quantity α in (3.5) is recognized as the electric charge q. This shows that the Maxwellian theory is actually a model for any other fundamental interaction.

The four-vector current density J_E is defined as

$$\mathsf{J}_E = \left(\mathbf{J}_E, J_{E4}\right) = \left(\mathbf{J}_E, i\rho_E c\right) = \rho_E(\mathbf{v}_E, ic) \tag{4.1}$$

where ρ_E is the electric charge density moving with three-velocity field $\mathbf{v}_E$ in space. This four-vector density satisfies the continuity equation

$$\partial_\mu J_{E\mu} = \nabla \bullet \mathbf{J}_E + \frac{\partial \rho_E}{\partial t} = 0 \tag{4.2}$$

In the theory of electrodynamics, electric charges, through the four-vector electric current density J_E, induce the electromagnetic four-vector potential $\mathsf{A} = A_\mu \mathsf{e}_\mu = (\mathbf{A}, A_4)$ in the space-time corresponding to the inertial reference frame. The space component $\mathbf{A}$ is the magnetic vector potential and the time component A_4 is related to the electric scalar potential ϕ as

$$A_4 = i\frac{1}{c}\phi \tag{4.3}$$

Based on the vortex theory of interaction developed here, $-\mathsf{A}$ is the four-vector electromagnetic velocity created in the space-time. Obviously, the four-dimensional vorticity of this four-velocity field is the electromagnetic strength four-tensor $\mathsf{F} = F_{\mu\nu}\mathsf{e}_\mu\mathsf{e}_\nu$, where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.4}$$

The gauge freedom allows us to have

$$\partial_\mu A_\mu = \nabla \bullet \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0 \tag{4.5}$$

which is the Lorentz gauge constraint.

By considering the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \tag{4.6}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.7}$$

we obtain

$$\mathbf{F} = \begin{bmatrix} 0 & B_3 & -B_2 & -iE_1/c \\ -B_3 & 0 & B_1 & -iE_2/c \\ B_2 & -B_1 & 0 & -iE_3/c \\ iE_1/c & iE_2/c & iE_3/c & 0 \end{bmatrix} = \begin{bmatrix} -\mathbf{R}_B & -i\frac{1}{c}\mathbf{E}^T \\ i\frac{1}{c}\mathbf{E} & 0 \end{bmatrix} \tag{4.8}$$

We can see that the electromagnetic vorticity four-tensor field $\mathbf{F}$ is a combination of hyperbolic electromagnetic vorticity $\frac{1}{c}\mathbf{E}$ and circular electromagnetic vorticity $-\mathbf{B}$. This is an amazing result explaining the geometrical character of the six components of the electromagnetic field $\mathbf{F}$, as follows:

- The components $-B_1$, $-B_2$ and $-B_3$ are the circular vorticity-like components of the electromagnetic field in planes x_2x_3, x_3x_1 and x_1x_2, respectively.
- The components $\frac{1}{c}E_1$, $\frac{1}{c}E_2$ and $\frac{1}{c}E_3$ are the hyperbolic vorticity-like components of the electromagnetic field in planes x_1x_4, x_2x_4 and x_3x_4, respectively.

For a particle with mass m and electric charge q at position $\tilde{\mathbf{x}}$, the four-tensor vorticity $\mathbf{F}$ is transformed to the four-tensor angular velocity $\mathbf{\Omega}$ of the space-time body frame of particle, such that

$$\mathbf{\Omega}(\tilde{\mathbf{x}}) = \frac{q}{m}\mathbf{F}(\tilde{\mathbf{x}}) \tag{4.9}$$

One can see that this is a linear transformation with mapping constant $\frac{q}{m}$. Therefore, the effect of electromagnetic interaction on the charged particle is nothing but the instantaneous four-dimensional rotation of its body frame. The hyperbolic and circular angular velocities of the body frame are

$$\frac{\boldsymbol{\eta}}{c} = \frac{q}{mc}\mathbf{E}(\tilde{x}) \tag{4.10}$$

and

$$\boldsymbol{\omega} = -\frac{q}{m}\mathbf{B}(\tilde{x}) \tag{4.11}$$

respectively.

The equation of motion for rotation of the space-time body frame of the particle is

$$\frac{d\mathbf{\Lambda}}{d\tau} = -\frac{q}{m}\mathbf{\Lambda} \bullet \mathbf{F}(\tilde{\mathbf{x}}) \tag{4.12}$$

However, the classical equation of motion for the particle is

$$m\frac{du_{\mu}}{d\tau} = qF_{\mu\nu}(\tilde{x})u_{\nu} \tag{4.13}$$

where the right hand side is the electromagnetic four-vector Lorentz force. The space and time components of this equation are

$$m\frac{d\mathbf{u}}{dt} = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right) = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{4.14}$$

$$\frac{d}{dt}\frac{mc^2}{\sqrt{1-v^2/c^2}} = q\mathbf{E} \bullet \mathbf{v} \tag{4.15}$$

respectively. The first equation is the equation of motion, where its right hand side is the familiar Lorentz force. The second equation is the rate at which the electromagnetic field does work on the particle and changes its energy. Now we realize that the electromagnetic strength field tensor and Lorentz force vector are both natural consequences of the geometric structure of relative space time.

The compatibility equation for $F_{\mu\nu}$ is

$$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \tag{4.16}$$

This equation is the covariant form of Maxwell's homogeneous equations

$$\nabla \bullet \mathbf{B} = 0 \tag{4.17}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{4.18}$$

Equation (4.17) is known as Gauss' law for magnetism and equation (4.18) is Faraday's law of induction. Therefore, Maxwell's homogeneous equations are the necessary kinematical compatibility equations for the circular and hyperbolic vorticities.

Now we consider the relation between electric charges and the electromagnetic field that these charges create. The covariant form of the governing equation for strength or vorticity tensor $F_{\mu\nu}$ due to the electric current density is

$$\partial_\nu F_{\mu\nu} = \mu_0 J_{E\mu} \tag{4.19}$$

which is the compact form of Maxwell's inhomogeneous equations

$$\nabla \bullet \mathbf{E} = \rho_E / \varepsilon_0 \tag{4.20}$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}_E \tag{4.21}$$

Equation (4.20) is Gauss' law and (4.21) represents Ampere's law with Maxwell's correction. The constants ε_0 and μ_0 are the permittivity and permeability of free space, respectively. As we know, the relation $\frac{1}{\mu_0 \varepsilon_0} = c^2$ holds.

The non-homogeneous equation (4.19) expresses the relation between the derivative of the four-dimensional vorticity and electric density current four-vector. As a result, it can be taken as the equation of motion for electromagnetic vorticities. This relation may also be written as

$$K_{E\mu} = -\frac{1}{6}\mu_0 J_{E\mu} \tag{4.22}$$

where $K_{E\mu}$ is the mean curvature four-vector of the electromagnetic four-vector velocity $-A_\mu$. Therefore, Maxwell's inhomogeneous equations turn out to express the mean curvature four-vector of the four-vector electromagnetic velocity. This is an amazing result in the theory of electrodynamics, which shows that the four-vector density current $J_{E\mu}$ is a measure of the four-vector mean curvature of electromagnetic four-vector velocity $-A_\mu$. This is the geometrical explanation of Maxwell's inhomogeneous equations. The space and time components of (4.22) are

$$\mathbf{K}_E = -\frac{1}{6}\mu_0 \mathbf{J}_E \tag{4.23}$$

$$K_{E4} = -i\frac{1}{6c}\frac{1}{\varepsilon_0}\rho_E \tag{4.24}$$

Interestingly, we notice that the electric charge density is proportional to the imaginary mean curvature of the scalar potential ϕ in the time direction.

The mean curvature four-vector $K_{E\mu}$ in terms of the vorticity field and four-vector velocity becomes

$$K_{E\mu} = -\frac{1}{6}\partial_\nu F_{\mu\nu} \tag{4.25}$$

$$K_{E\mu} = -\frac{1}{6}\left(\partial_\mu \partial_\nu A_\nu - \partial_\nu \partial_\nu A_\mu\right) \tag{4.26}$$

respectively. If we use the Lorentz gauge in (4.5), this relation reduces to

$$K_{E\mu} = \frac{1}{6}\partial_\nu \partial_\nu A_\mu \tag{4.27}$$

Therefore, for the space and time components of the mean curvature four-vector, we obtain

$$\mathbf{K}_E = \frac{1}{6}\left(\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \tag{4.28}$$

$$K_{E4} = i\frac{1}{6c}\left(\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}\right) \tag{4.29}$$

These components can also be written in terms of circular and hyperbolic electromagnetic vorticities as

$$\mathbf{K}_E = -\frac{1}{6}\left(\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \tag{4.30}$$

$$K_{E4} = -i\frac{1}{6c}\nabla \bullet \mathbf{E} \tag{4.31}$$

Therefore, point charges, such as electrons, create concentrated mean curvature of the electromagnetic four-vector velocity field.

It is also interesting to note that in regions of space, where there is no charge or current, the electromagnetic field has no mean curvature

$$K_{E\mu} = 0 \tag{4.32}$$

Accordingly, plane electromagnetic waves, which are the solution to homogeneous Maxwell's equations, have no mean curvature anywhere in space.

What we have shown is that the electrodynamics is compatible with the four-dimensional vortex theory of interaction, where

$$\alpha \to q \tag{4.33}$$

$$\mathbf{J}_A \to \mathbf{J}_E \tag{4.34}$$

$$\mathbf{V} \to -\mathbf{A} \tag{4.35}$$

$$\mathbf{\Theta} \to \mathbf{F} \tag{4.36}$$

$$\mathbf{w} \to -\mathbf{B} \tag{4.37}$$

$$\mathbf{h} \to \mathbf{E} \tag{4.38}$$

It is amazing to note that that the theory of electrodynamics is based on geometrical concepts, such as the four-tensor vorticity and the four-vector mean curvature. Consequently, one can realize that the theory of electrodynamics is a treasure, which contains all clues for understanding space, time, motion and interaction.

Now it is time to explore more about the universal fundamental entity in which particles create their space-time and interact through vorticity fields. It turns out that the study of the electromagnetic energy-momentum tensor and Maxwell stress tensor is significant in this investigation.

Electromagnetic Energy-Momentum Tensor

Relative to the space-time inertial reference frame, the Lorentz force per unit volume on a medium with a charge density ρ_E and current density $\mathbf{J}_E$ is given by

$$\mathbf{f} = \rho_E \mathbf{E} + \mathbf{J}_E \times \mathbf{B} \tag{4.39}$$

The generalization of this force in covariant electrodynamics is

$$f_\mu = F_{\mu\nu} J_{E\nu} \tag{4.40}$$

where $f_\mu = (\mathbf{f}, f_4)$ is the force-density four-vector with

$$f_4 = \frac{i}{c} \mathbf{J}_E \bullet \mathbf{E} \tag{4.41}$$

We note that

$$\frac{\partial w}{\partial t} = \mathbf{J}_E \bullet \mathbf{E} \tag{4.42}$$

is the work done per unit time per unit volume by the electric field on moving charges. Therefore

$$f_4 = \frac{i}{c} \frac{\partial w}{\partial t} \tag{4.43}$$

By substituting $J_{E\mu}$ from the equations of motion of the electromagnetic field

$$\partial_\nu F_{\mu\nu} = \mu_0 J_{E\mu} \tag{4.44}$$

and some tensor algebra, we obtain

$$f_\mu = \partial_\nu T_{\mu\nu} \tag{4.45}$$

where $T_{\mu\nu}$ is the electromagnetic energy-momentum tensor, defined by

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\sigma}F_{\sigma\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta} \right) \tag{4.46}$$

The explicit form of the components of this four-tensor in terms of $\mathbf{E}$ and $\mathbf{B}$ are the Maxwell stress tensor

$$T_{ij} = \varepsilon_0 (E_i E_j - \frac{1}{2}E_k E_k \delta_{ij}) + \frac{1}{\mu_0}(B_i B_j - \frac{1}{2}B_k B_k \delta_{ij}) \tag{4.47}$$

the electromagnetic energy density

$$T_{44} = u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2 \right) \tag{4.48}$$

and

$$T_{4i} = T_{i4} = -\frac{i}{c}\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})_i = -\frac{i}{c}S_i \tag{4.49}$$

where the Poynting vector $\mathbf{S}$ is defined by

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \tag{4.50}$$

Therefore, the symmetric four-tensor $T_{\mu\nu}$ can be written in schematic matrix form as

$$\mathbf{T} = \begin{bmatrix} T_{ij} & -\frac{i}{c}\mathrm{S} \\ -\frac{i}{c}\mathrm{S} & u \end{bmatrix} \tag{4.51}$$

The traction or force exerted by this field on a unit area of a surface in space with unit normal vector n_i is

$$T_{ij}n_j = T_i^{(n)} \tag{4.52}$$

Through this analogy with continuum mechanics, we can take $T_{\mu\nu}$ as a four-stress tensor. The time-space components of (4.45) are

$$f_i = \frac{\partial T_{ij}}{\partial x_j} - \frac{1}{c^2}\frac{\partial S_i}{\partial t} \tag{4.53}$$

$$-icf_4 = -\frac{\partial S_j}{\partial x_j} - \frac{\partial u}{\partial t} \tag{4.54}$$

Integrating these relations over a volume V bounded by surface A, and using the divergence theorem, we obtain

$$\int_V f_i dV + \frac{1}{c^2}\frac{\partial}{\partial t}\int_V S_i dV = \int_A T_{ij} n_j dA \tag{4.55}$$

$$-ic\int_V f_4 dV + \frac{\partial}{\partial t}\int_V u dV + \int_A S_i n_i dA = 0 \tag{4.56}$$

In addition, we note that the relation

$$T_{\mu\nu} = \frac{1}{\mu_0}\left(F_{\mu\sigma}F_{\sigma\nu} + \frac{1}{4}\delta_{\mu\nu}F_{\alpha\beta}F_{\alpha\beta} \right) \tag{4.57}$$

looks like a constitutive relation for four-stress tensor $T_{\mu\nu}$ in term of the four-tensor electromagnetic vorticity $F_{\mu\nu}$ in the universal entity. In linear continuum mechanics, the constitutive equations relate the stress tensor linearly to strain or strain rate, but the energy density is a quadratic function of strain or strain rate tensor. However, what we are dealing with here is the four-dimensional analogous case in which the stress four-tensor $\mathbf{T}$ is a quadratic function of the vorticity four-tensor $\mathbf{\Omega}$ in the universal entity. Thus, the universal entity behaves like a continuum in which charged particles create stresses and electromagnetic vorticities. Interestingly, the point charged particles, which create concentrated curvature of the four-vector electromagnetic velocity, are singularities of these vorticities and four-stress tensors. Therefore, we realize that the Minkowski-Lorentz forces exerted on these point particles can be considered as four-dimensional lift-like forces. Although the idea of using vorticity looks very interesting, historical accounts show that it is not completely new. This development is similar to the efforts of investigators of ether theory. Ether was the term used to describe a medium for the propagation of electromagnetic waves. Whittaker [10] gives a detailed account of these investigations through which we learn that Maxwell considered a rotational character for the magnetic field and a translational character for the electric field. We also learn that Larmor [11] considered that the ether was separate from matter and that particles, such as electrons, serve as source of vortices in this ether. What we have developed here can be considered the completion of Larmor's ether theory.

Interestingly, we have used similar ideas about stress and vorticity as in continuum mechanics, but here in a four-dimensional context. In our development, the magnetic field has the same character as circular rotation, but the electric field has the character of hyperbolic rotation. It is well justified to call our fundamental universal entity the historical ether out of respect, which now is represented by four-dimensional space-time systems. Therefore, in the new view, particles specify their space-time body frames in the ether and interact with each other through four-vorticity and four-stress that the particles create in the ether. As we mentioned, the Lorentz force

$$F_{\mu} = qF_{\mu\nu}(\tilde{x})u_{\nu} \tag{4.58}$$

is analogous to the lift force in fluid dynamics. The lift on an airfoil is perpendicular to the velocity of flow past the surface. This is the mechanical analogy of the four-vector electromagnetic Lorentz force.

Clarifying the concepts of ether and space-time and the development of the vortex theory of electromagnetism are important steps in the completion of Poincaré's theory of relativity and covariant electrodynamics. These achievements enable us to understand more about modern physics and resolve some difficulties, even in a classical view. Interestingly, the geometrical theory of electromagnetic interaction resolves the speculation about magnetic monopole, which is addressed in the following section.

Magnetic Monopole Does Not Exist

As mentioned in Section 2, the existence of magnetic monopole is apparently compatible with the fully symmetrized Maxwell's equations. It seems only modification of Maxwell's equations suffices to permit magnetic charges to exist in electrodynamics. However, the geometric vortex theory of electromagnetic resolves this quest by showing the impossibility of the existence of magnetic monopole in the universe as follows.

The magnetic field $\mathbf{B}$ is the space electromagnetic vorticity induced to the ether relative to the inertial reference frame. This is analogous to the vorticity field in a rotational fluid flow. From non-relativistic fluid mechanics, we know that the vorticity is the curl of the velocity field of the fluid, which equals twice the angular velocity of the fluid element. By definition, the vorticity field is source-less, which is the necessary compatibility condition for a realistic fluid flow. We notice the same character for the electromagnetic vorticity $\mathbf{B}$. The magnetic field $\mathbf{B}$ is the curl of the electromagnetic velocity vector field $\mathbf{A}$. Thus,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.59}$$

This definition requires

$$\nabla \bullet \mathbf{B} = 0 \tag{4.60}$$

which is the kinematical compatibility equation stating that the magnetic vorticity field has no source. This equation is the necessary condition for the existence of vector potential $\mathbf{A}$ for a given magnetic field $\mathbf{B}$. Existence of a magnetic monopole would violate this trivial kinematical compatibility equation with bizarre consequences as we illustrate further.

Let us assume that there is a point magnetic monopole of strength q_m at the origin. Therefore, in SI units

$$\nabla \bullet \mathbf{B} = \mu_0 q_m \delta^{(3)}(\mathbf{x}) \tag{4.61}$$

and the static magnetic field is then given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{\mathbf{r}} \tag{4.62}$$

However, the relation (4.61) contradicts the kinematical compatibility (4.60). The magnetic field of a magnetic monopole cannot be represented by a vector potential $\mathbf{A}$. Interestingly, based on the Helmholtz decomposition theorem, this field can only be represented by a scalar potential [12, 13]

$$\phi_m(\mathbf{x}) = \frac{\mu_0}{4\pi}\frac{q_m}{r} \tag{4.63}$$

where the magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = -\nabla\phi_m \tag{4.64}$$

But this is absurd, because the electromagnetic vorticity vector field $\mathbf{B}$ has to be always represented by curl of the electromagnetic velocity vector $\mathbf{A}$. Therefore, magnetic monopoles cannot exist. We realize that the magnetic field $\mathbf{B}$ is only generated by moving electric charges.

It has been long speculated that magnetic monopoles might not exist, because there is no complete symmetry between $\mathbf{B}$ and $\mathbf{E}$. This is due to the fact that $\mathbf{B}$ is a pseudo-vector, but $\mathbf{E}$ is a polar vector. What we have here is the confirmation of this correct speculation that there is no duality between $\mathbf{E}$ and $\mathbf{B}$ in electrodynamics. We have shown that the magnetic field $\mathbf{B}$ has the character of a circular vorticity field and is divergence free.

However, we realize that the electric field $\mathbf{E}$ has the character of a hyperbolic vorticity with electric charges as its sources, where

$$\nabla \bullet \mathbf{E} = \rho_E / \varepsilon_0 \tag{4.65}$$

This explanation is actually the clarification of Larmor's ether theory.

As mentioned previously, the electric charge q of a particle has the property of a kinematical coupling, which maps the four-dimensional electromagnetic vorticity at the position of the particle to the angular velocity of its body frame. We have shown that electric charge is the only coupling present. Furthermore, there is no need for any other coupling. Based on the developed geometry of electromagnetism, it is naïve to assume that a simplistic modification of Maxwell's equations suffice to allow the existence of magnetic charges in electrodynamics.

5. Conclusions

The theory of relativity has been completed by establishing the fundamental relation between space-time and particles. This theory shows that every particle specifies a Minkowskian space-time body frame in a universal entity, here referred to as ether, and moves in the time direction with speed *c* in that frame. The relative motion of particles is actually the result of relative four-dimensional rotation of their corresponding space-time body frames. This aspect of space-time shows that the pure Lorentz transformations represent the relative four-dimensional orientation among the space-time body frames of uniformly translating particles. Inertial observers in these frames relate the components of four-vectors and four-tensors by Lorentz transformation. This is the origin of non-Euclidean geometry governing the three-vector and three-tensor components. The hyperbolic geometry of the velocity addition law for uniform motions is the manifest for this fact.

We also realize that the orthogonal transformations are not restricted to relative uniform motion. The relative motion of accelerating particles is also represented by varying orthogonal transformations. This establishes the general theory of motion and fundamental interaction. The acceleration of a particle is the result of the instantaneous rotation of its space-time body frame in the ether. This instantaneous rotation is specified by a four-dimensional angular velocity tensor in the inertial reference frame. The hyperbolic part of this rotation is in fact the accelerating motion. However, there is also a circular space rotation, which is observed in some phenomena, such as the spin precession of a stationary charged particle in a magnetic field.

Based on the theory of motion, the geometrical character of fundamental interaction has also been discovered. This development shows that a Lorentz-like Minkowski force is an essential feature of the simplest model for every fundamental interaction, which is represented by an anti-symmetric strength four-tensor field with characteristics of a vorticity field. This four-vorticity tensor is a combination of three-vector circular and hyperbolic vorticities. Particles interact with each other via the four-vorticity, which the particles induce in the ether.

This vortex theory gives a clear geometrical explanation of electrodynamics, which is a model for any other interaction. We realize that the electromagnetic strength field is a four-dimensional vorticity field. The magnetic and the electric fields are the circular and hyperbolic vorticity-like fields, respectively. Through this theory, we realize that the homogeneous Maxwell's equations are the necessary compatibility equations for the electromagnetic vorticity vectors, whereas the inhomogeneous Maxwell's equations govern the motion of these vorticities. Geometrically, the inhomogeneous Maxwell's equations are the relation for mean curvature four-vector of electromagnetic velocity field. The charge current density four-vector is proportional to the mean curvature four-vector of electromagnetic four-vector velocity.

Moreover, the energy-momentum four-tensor has the character of a four-stress tensor and its expression in terms of electromagnetic vorticities is a constitutive relation. This reveals the mechanical character of the Lorentz force as a four-dimensional lift-like force perpendicular to the four-vector velocity. The circular vortical character of magnetic field clearly shows why magnetic monopole cannot exist. Therefore, electric charges are the only source of the electromagnetic field.

It should be emphasized that in this paper nothing has changed in the original Maxwell's equations. We have just shown that the theory of electromagnetism is much more important than previously thought. Our theory of space, time, motion, interaction and the governing geometry is the result of understanding the apparently subtle characteristics of this theory. We have discovered the fascinating fundamental geometry of electromagnetism by understanding the relation between particle and space-time, four-dimensional rotational motion, and vortex characteristics of the interaction field. Continuum mechanics has played an essential role in this achievement. Vortex theory in fluid mechanics and rigid body dynamics enable us to reveal the true character of space-time, motion, and interaction. Amazingly, everything in electrodynamics and the theory of relativity is about rotation, which has not been recognized completely before.

References

[1] S. Walter, *The non-Euclidean style of Minkowskian relativity: Vladimir Varicak's non-Euclidean program*, in J. Gray, *The Symbolic Universe: Geometry and Physics* (Oxford University Press): 91–127 (1999).

[2] V. Varičak, Anwendung der Lobatschefskijschen Geometrie in der Relativtheorie, Physikalische Zeitschrift 11, 93–96 (1910).

[3] V. Varičak, Über die nichteuklidisch Interpretation der Relativtheorie, Jahresbericht der deutschen Mathematiker-Vereinigung 21, 103–127 (1912).

[4] V. Varičak. *Darstellung der Relativitätstheorie im dreidimensionalen Lobatchefskijschen Raume* (Zagreb: Zaklada, 1924).

In English: *Relativity in three dimensional Lobachevsky space* (LuLu.com, 2006).

[5] É. Borel, La théorie de la relativité et la cinématique, Comptes Rendus des séances de l'Académie des Sciences 156 (1913): 215–217.

[6] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1999).

[7] A. R. Hadjesfandiari, and G. F. Dargush, 'Couple stress theory for solids', *International Journal of Solids and Structures*, 48, 2496-2510 (2011).

[8] A. R. Hadjesfandiari, and G. F. Dargush, 'Couple stress theory for polar solids', http://imechanica.org/files/Hadj_PolarSolids_2010_02.pdf.

[9] M. Spivak, *A comprehensive introduction to differential geometry (*Publish or Perish Press, 1999).

[10] E. T. Whittaker, *A history of the theories of aether and electricity* (Philosophical Library, New York; Thomas Nelson, Edinburgh—London, 1954).

[11] J. Larmor, *Aether and Matter* (Cambridge, At the University Press, 1900).

[12] A. R. Hadjesfandiari, Character of the magnetic monopole, International Journal of Materials and Product Technology, 34, 37-53 (2009).

[13] A. R. Hadjesfandiari, Field of the magnetic monopole, http://arxiv.org/abs /physics/0701232.