Sedeonic theory of massive fields

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Abstract

In the present paper we develop the description of massive fields on the basis of space-time algebra of sixteen-component sedeons. The generalized sedeonic second-order equation for the potential of massive field is proposed. It is shown that this equation can be reformulated in the form of a system of Maxwell-like equations for the field strengths. We also discuss the generalized sedeonic first-order equation for massive field.

Introduction

Many attempts have been made to generalize the second-order wave equation for massive fields using different hypercomplex number systems, such as quaternions and octonions [1]-[5]. The authors discussed the possibility of constructing the field equations similar to the equations of electrodynamics but with a massive "photon". In particular they tried to represent the wave equation as the system of first-order Maxwell-like equations. The resulting Proca-Maxwell equations enclose field's strengths and potentials [2, 3, 5]. Likewise hypercomplex number generalization of the Dirac wave equation have been studied in [6]-[10]. In this approach, the wave function has a scalar-vector structure similar in nature with the potential of field and the hypercomplex Dirac-like equation can be reformulated as the wave equation for the potential of special field.

The consideration of multicomponent wave functions is an inevitable necessity in describing the spin and space-time properties of fields and quantum systems. However, quaternions consisting of a scalar and vector do not take into account the pseudoscalar and pseudovector properties of physical quantities. From this viewpoint an approach based on the use of the eight-component octonions including the scalar, vector, pseudoscalar and pseudovector components is more appropriate. However, the requirements of relativistic invariance leads to the necessity of introducing sixteen-component algebras taking into account the full symmetry with respect to the spatial and time inversion.

There are a few approaches in the development of field theory on the basis of sixteen-component structures. One of them is the application of hypernumbers sedenions, which are obtained from octonions by Cayley-Dickson extension procedure [11, 12]. However the essential imperfection of sedenions is their nonassociativity. Another approach is based on the application of hypercomplex multivectors generating associative space-time Clifford algebras. The basic idea of such multivectors is an introduction of additional noncommutative time unit vector, which is orthogonal to the space unit vectors [13]. However, the application of such multivectors in quantum mechanics and field theory is considered in general as one of abstract algebraic scheme enabling the reformulation of Klein-Gordon and Dirac equations for the multicomponent wave functions but does not touch the physical entity of these equations.

Recently we developed the space-time algebra of sixteen-component sedeons generating noncommutative associative scalar-vector Clifford algebra [14]. The sedeons take into account the

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properties of physical values with respect to the space-time inversion and realize the scalar-vector representation of Poincare group. In present paper, we use the sedeonic approach for the consideration of massive fields described by sedeonic first-order and second-order wave equations within a unified field conception.

1 Sedeonic space-time algebra

To begin with we briefly review the basic properties of sedeons [14]. The sedeonic algebra encloses four groups of values, which are differed with respect to spatial and time inversion.

1. Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
2. Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
3. Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
4. Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}}. \tag{1}$$

Let us introduce scalar-vector basis $\mathbf{a_0}$, $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$, where the value $\mathbf{a_0} \equiv 1$ is absolute scalar unit and the values $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are absolute unit vectors generating the right Cartesian basis. We introduce also four space-time scalar units $\mathbf{e_0}$, $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, where value $\mathbf{e_0} \equiv 1$ is a absolute scalar unit; $\mathbf{e_1} \equiv \mathbf{e_t}$ is a time scalar unit; $\mathbf{e_2} \equiv \mathbf{e_r}$ is a space scalar unit; $\mathbf{e_3} \equiv \mathbf{e_{tr}}$ is a space-time scalar unit. Using space-time scalar units $\mathbf{e_j}$ ($\mathbf{j} = 0,1,2,3$) and scalar-vector basis $\mathbf{a_k}$ ($\mathbf{k} = 0,1,2,3$) we can introduce unified sedeonic components V_{jk} in accordance with the following relations:

$$\begin{aligned}
&V = \mathbf{e_0} V_{00} \mathbf{a_0}, \\
&\vec{V} = \mathbf{e_0} \left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right), \\
&V_{\mathbf{t}} = \mathbf{e_1} V_{10} \mathbf{a_0}, \\
&\vec{V}_{\mathbf{t}} = \mathbf{e_1} \left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right), \\
&V_{\mathbf{r}} = \mathbf{e_2} V_{20} \mathbf{a_0}, \\
&\vec{V}_{\mathbf{r}} = \mathbf{e_2} \left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right), \\
&V_{\mathbf{tr}} = \mathbf{e_3} V_{30} \mathbf{a_0}, \\
&\vec{V}_{\mathbf{tr}} = \mathbf{e_3} \left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{2}$$

Then the sedeon (1) can be written in the following expanded form:

$$\begin{aligned}
\tilde{\mathbf{V}} = \mathbf{e_0} &\left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right) \\
+\mathbf{e_1} &\left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right) \\
+\mathbf{e_2} &\left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right) \\
+\mathbf{e_3} &\left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right).
\end{aligned} \tag{3}$$

The sedeonic components V_{jk} are numbers (complex in general). Further we will use symbol 1 instead of units $\mathbf{a_0}$ and $\mathbf{e_0}$ for simplicity.

The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and space-time units $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are presented in tables 1 and 2 respectively.

Table 1:

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2:

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_1}$	1	$i\mathbf{e_3}$	$-i\mathbf{e_2}$
$\mathbf{e_2}$	$-i\mathbf{e_3}$	1	$i\mathbf{e_1}$
$\mathbf{e_3}$	$i\mathbf{e_2}$	$-i\mathbf{e_1}$	1

In the tables and further the value i is the imaginary unit ($i^2 = -1$). Note that sedeonic units $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ and unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ generate the anticommutative algebras:

$$\begin{aligned} \mathbf{a_n a_m} &= -\mathbf{a_m a_n}, \\ \mathbf{e_n e_m} &= -\mathbf{e_m e_n}, \end{aligned} \tag{4}$$

for $\mathbf{n}$ and $\mathbf{m} = 1, 2, 3$ ($\mathbf{n} \neq \mathbf{m}$), but $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ commute with $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$:

$$\mathbf{a_n e_m} = \mathbf{e_m a_n}, \tag{5}$$

for any $\mathbf{n}$ and $\mathbf{m}$.

Thus the sedeon $\tilde{\mathbf{V}}$ is the complicated space-time object consisting of absolute scalar, time scalar, space scalar, space-time scalar, absolute vector, time vector, space vector and space-time vector.

Introducing the designations of space-time sedeon-scalars

$$\begin{aligned} \mathbf{V}_0 &= V_{00} + \mathbf{e_1} V_{10} + \mathbf{e_2} V_{20} + \mathbf{e_3} V_{30}, \\ \mathbf{V}_1 &= V_{01} + \mathbf{e_1} V_{11} + \mathbf{e_2} V_{21} + \mathbf{e_3} V_{31}, \\ \mathbf{V}_2 &= V_{02} + \mathbf{e_1} V_{12} + \mathbf{e_2} V_{22} + \mathbf{e_3} V_{32}, \\ \mathbf{V}_3 &= V_{03} + \mathbf{e_1} V_{13} + \mathbf{e_2} V_{23} + \mathbf{e_3} V_{33}, \end{aligned} \tag{6}$$

we can write the sedeon (3) in another form

$$\tilde{\mathbf{V}} = \mathbf{V}_0 + \mathbf{V}_1 \mathbf{a_1} + \mathbf{V}_2 \mathbf{a_2} + \mathbf{V}_3 \mathbf{a_3}, \tag{7}$$

or introducing the sedeon-vector

$$\vec{\mathbf{V}} = \vec{V} + \vec{V}_{\mathbf{t}} + \vec{V}_{\mathbf{r}} + \vec{V}_{\mathbf{tr}} = \mathbf{V}_1 \mathbf{a_1} + \mathbf{V}_2 \mathbf{a_2} + \mathbf{V}_3 \mathbf{a_3}, \tag{8}$$

it can be represented in following compact form:

$$\tilde{\mathbf{V}} = \mathbf{V}_0 + \vec{\mathbf{V}}. \tag{9}$$

Further we will indicate the sedeon-scalars and the sedeon-vectors with the bold capital letters.

Let us consider the sedeonic multiplication in detail. The sedeonic product of two sedeons $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ can be presented in the following form:

$$\begin{aligned} \tilde{\mathbf{A}}\tilde{\mathbf{B}} &= \left(\mathbf{A}_0 + \vec{\mathbf{A}}\right)\left(\mathbf{B}_0 + \vec{\mathbf{B}}\right) \\ &= \mathbf{A}_0 \mathbf{B}_0 + \mathbf{A}_0 \vec{\mathbf{B}} + \vec{\mathbf{A}} \mathbf{B}_0 + \left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right) + \left[\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right]. \end{aligned} \tag{10}$$

Here we denote the sedeonic scalar multiplication of two sedeon-vectors (internal product) by symbol "·" and round brackets

$$\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}\right) = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3, \tag{11}$$

and sedeonic vector multiplication (external product) by symbol "×" and square brackets

$$\begin{aligned} \left[\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right] &= i\left(\mathbf{A}_2\mathbf{B}_3 - \mathbf{A}_3\mathbf{B}_2\right)\mathbf{a_1} + i\left(\mathbf{A}_3\mathbf{B}_1 - \mathbf{A}_1\mathbf{B}_3\right)\mathbf{a_2} \\ &+ i\left(\mathbf{A}_1\mathbf{B}_2 - \mathbf{A}_2\mathbf{B}_1\right)\mathbf{a_3}. \end{aligned} \tag{12}$$

In (11) and (12) the multiplication of sedeonic components is performed in accordance with (6) and table 2. Note that in sedeonic algebra the expression for the vector product has some difference from analogous expression in Gibbs vector algebra. Let us consider three absolute vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. Then the formula for the vector triple product in sedeonic algebra has the following form:

$$\left[\vec{A} \times \left[\vec{B} \times \vec{C}\right]\right] = -\vec{B}\left(\vec{A} \cdot \vec{C}\right) + \vec{C}\left(\vec{A} \cdot \vec{B}\right). \tag{13}$$

Thus, the sedeonic product

$$\tilde{\mathbf{F}} = \tilde{\mathbf{A}}\tilde{\mathbf{B}} = \mathbf{F}_0 + \vec{\mathbf{F}} \tag{14}$$

has the following components:

$$\begin{aligned} \mathbf{F}_0 &= \mathbf{A}_0\mathbf{B}_0 + \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \mathbf{A}_3\mathbf{B}_3, \\ \mathbf{F}_1 &= \mathbf{A}_1\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_1 + i\mathbf{A}_2\mathbf{B}_3 - i\mathbf{A}_3\mathbf{B}_2, \\ \mathbf{F}_2 &= \mathbf{A}_2\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_2 + i\mathbf{A}_3\mathbf{B}_1 - i\mathbf{A}_1\mathbf{B}_3, \\ \mathbf{F}_3 &= \mathbf{A}_3\mathbf{B}_0 + \mathbf{A}_0\mathbf{B}_3 + i\mathbf{A}_1\mathbf{B}_2 - i\mathbf{A}_2\mathbf{B}_1. \end{aligned} \tag{15}$$

2 Fields described by sedeonic second-order equation

2.1 Generalized sedeonic wave equation for massive field

Let us consider the potential of massive field in the form of space-time sedeon

$$\tilde{\mathbf{W}}(\vec{r}, t) = \mathbf{W}_0(\vec{r}, t) + \vec{\mathbf{W}}(\vec{r}, t). \tag{16}$$

The generalized sedeonic wave equation for massive field can be written in the following symmetric form [14]:

$$\left(i\mathbf{e_1}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}\frac{m_0 c}{\hbar}\right)\left(i\mathbf{e_1}\frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}\frac{m_0 c}{\hbar}\right)\tilde{\mathbf{W}} = 0. \tag{17}$$

Here c is the velocity of light, $\hbar$ is the Planck constant and the parameter m_0 can be interpreted as the mass of a quantum of field. The absolute gradient vector has the following form:

$$\vec{\nabla} = \frac{\partial}{\partial x}\mathbf{a_1} + \frac{\partial}{\partial y}\mathbf{a_2} + \frac{\partial}{\partial z}\mathbf{a_3}. \tag{18}$$

For convenience we introduce new operators

$$\begin{aligned} \partial &= \frac{1}{c}\frac{\partial}{\partial t}, \\ m &= \frac{m_0 c}{\hbar}. \end{aligned} \tag{19}$$

Then we can rewrite the equation (17) in compact form:

$$\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\tilde{\mathbf{W}} = 0. \tag{20}$$

Let us choose the potential in the following form:

$$\tilde{\mathbf{W}} = a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}, \tag{21}$$

where the components $a, b, c, d, \vec{A}, \vec{B}, \vec{C}$ and $\vec{D}$ are the functions of spatial coordinates and time. Introducing the scalar and vector fields strengths according to the following definitions:

$$\begin{aligned}
&e = \partial b + (\vec{\nabla} \cdot \vec{C}) + md,\\
&f = \partial a + (\vec{\nabla} \cdot \vec{D}) + mc,\\
&g = \partial d + (\vec{\nabla} \cdot \vec{A}) - mb,\\
&h = \partial c + (\vec{\nabla} \cdot \vec{B}) - ma,\\
&\vec{E} = -\partial \vec{B} - \vec{\nabla} c - i[\vec{\nabla} \times \vec{C}] - m\vec{D},\\
&\vec{F} = -\partial \vec{A} - \vec{\nabla} d + i[\vec{\nabla} \times \vec{D}] - m\vec{C},\\
&\vec{G} = -\partial \vec{D} - \vec{\nabla} a - i[\vec{\nabla} \times \vec{A}] + m\vec{B},\\
&\vec{H} = -\partial \vec{C} - \vec{\nabla} b + i[\vec{\nabla} \times \vec{B}] + m\vec{A},
\end{aligned} \tag{22}$$

we get

$$\begin{aligned}
&\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right)\\
&= -e + i\mathbf{e_1}f - i\mathbf{e_2}g + i\mathbf{e_3}h - i\vec{E} + \mathbf{e_1}\vec{F} + \mathbf{e_2}\vec{G} + \mathbf{e_3}\vec{H}
\end{aligned} \tag{23}$$

and the wave equation (20) takes the form

$$\begin{aligned}
&\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\\
&\times\left(-e + i\mathbf{e_1}f - i\mathbf{e_2}g + i\mathbf{e_3}h - i\vec{E} + \mathbf{e_1}\vec{F} + \mathbf{e_2}\vec{G} + \mathbf{e_3}\vec{H}\right) = 0.
\end{aligned} \tag{24}$$

Performing the action of operator in the left part of the equation (24), and separating the terms with different space-time properties, we obtain the system of equations for the field's strengths, similar to the system of Maxwell's equations in electrodynamics:

$$\begin{aligned}
&\partial f + (\vec{\nabla} \cdot \vec{G}) - mh = 0,\\
&\partial e + (\vec{\nabla} \cdot \vec{H}) - mg = 0,\\
&\partial h + (\vec{\nabla} \cdot \vec{E}) + mf = 0,\\
&\partial g + (\vec{\nabla} \cdot \vec{F}) + me = 0,\\
&\partial \vec{F} + \vec{\nabla} g + i[\vec{\nabla} \times \vec{G}] - m\vec{H} = 0,\\
&\partial \vec{E} + \vec{\nabla} h - i[\vec{\nabla} \times \vec{H}] - m\vec{G} = 0,\\
&\partial \vec{H} + \vec{\nabla} e + i[\vec{\nabla} \times \vec{E}] + m\vec{F} = 0,\\
&\partial \vec{G} + \vec{\nabla} f - i[\vec{\nabla} \times \vec{F}] + m\vec{E} = 0.
\end{aligned} \tag{25}$$

The proposed equations for massive field possess a specific gauge invariance. It is easy to see that fields strengths (22) and equations (25) are not changed under the following substitutions for potentials:

$$\begin{aligned}
&a \Rightarrow a + \partial\varepsilon_a - m\varepsilon_c,\\
&b \Rightarrow b + \partial\varepsilon_b - m\varepsilon_d,\\
&c \Rightarrow c + \partial\varepsilon_c + m\varepsilon_a,\\
&d \Rightarrow d + \partial\varepsilon_d + m\varepsilon_b,\\
&\vec{A} \Rightarrow \vec{A} - \vec{\nabla}\varepsilon_d,\\
&\vec{B} \Rightarrow \vec{B} - \vec{\nabla}\varepsilon_c,\\
&\vec{C} \Rightarrow \vec{C} - \vec{\nabla}\varepsilon_b,\\
&\vec{D} \Rightarrow \vec{D} - \vec{\nabla}\varepsilon_a,
\end{aligned} \tag{26}$$

where ε_a, ε_b, ε_c, ε_d, are arbitrary scalar functions, which satisfy homogeneous Klein-Gordon equation. These gauge conditions are different from those taken in electrodynamics [15].

Multiplying each of the equations (25) to the corresponding field strength and adding these equations to each other, we obtain:

$$\begin{aligned}
&\frac{1}{2}\partial\left(f^2+e^2+h^2+g^2+\vec{F}^2+\vec{E}^2+\vec{H}^2+\vec{G}^2\right)\\
&+f\left(\vec{\nabla}\cdot\vec{G}\right)+e\left(\vec{\nabla}\cdot\vec{H}\right)+h\left(\vec{\nabla}\cdot\vec{E}\right)+g\left(\vec{\nabla}\cdot\vec{F}\right)\\
&+\left(\vec{F}\cdot\vec{\nabla}g\right)+\left(\vec{E}\cdot\vec{\nabla}h\right)+\left(\vec{H}\cdot\vec{\nabla}e\right)+\left(\vec{G}\cdot\vec{\nabla}f\right)\\
&+i\left(\vec{F}\cdot[\vec{\nabla}\times\vec{G}]\right)-i\left(\vec{E}\cdot[\vec{\nabla}\times\vec{H}]\right)\\
&+i\left(\vec{H}\cdot[\vec{\nabla}\times\vec{E}]\right)-i\left(\vec{G}\cdot[\vec{\nabla}\times\vec{F}]\right)=0.
\end{aligned} \tag{27}$$

Let us introduce the following notations:

$$w=-\frac{1}{8\pi}\left(f^2+e^2+h^2+g^2+\vec{F}^2+\vec{E}^2+\vec{H}^2+\vec{G}^2\right), \tag{28}$$

$$\vec{P}=-\frac{c}{4\pi}\left(e\vec{H}+f\vec{G}+g\vec{F}+h\vec{E}+i\left[\vec{E}\times\vec{H}\right]+i\left[\vec{G}\times\vec{F}\right]\right). \tag{29}$$

Then the equation (27) can be written as:

$$\frac{1}{c}\frac{\partial w}{\partial t}+\left(\vec{\nabla}\cdot\vec{P}\right)=0. \tag{30}$$

This expression is an analog of the Poynting theorem for massive field. The value w plays the role of the field energy density and $\vec{P}$ is a vector of energy flux density. The minus sign in expressions (28) and (29) are chosen with respect to the attractive character of charges interaction (see further Section 3.3.).

2.2 Nonhomogeneous equations for massive field

Let us consider the sedeonic equations for massive field with phenomenological source. In this case the field potential is described by sedeonic nonhomogeneous wave equation:

$$\left(i\mathbf{e_1}\partial-\mathbf{e_2}\vec{\nabla}-i\mathbf{e_3}m\right)\left(i\mathbf{e_1}\partial-\mathbf{e_2}\vec{\nabla}-i\mathbf{e_3}m\right)\tilde{\mathbf{W}}=\tilde{\mathbf{J}}. \tag{31}$$

By analogy with electrodynamics we consider the source of field in the form of incomplete sedeon [16, 17]:

$$\tilde{\mathbf{J}}=-i\mathbf{e_1}4\pi\rho_s-\mathbf{e_2}\frac{4\pi}{c}\vec{j}_s, \tag{32}$$

where ρ_s is a volume density of charge and $\vec{j}_s$ is volume density of current. In this case the sedeonic potential $\tilde{\mathbf{W}}$ can be written in the following form

$$\tilde{\mathbf{W}}=i\mathbf{e_1}b+\mathbf{e_2}\vec{C}, \tag{33}$$

where $b(\vec{r},t)$ is a scalar part (time component) and $\vec{C}(\vec{r},t)$ is a vector part (space component) of potential. In this case we have only the following nonzero field's strengths

$$\begin{aligned}
&e=\partial b+\left(\vec{\nabla}\cdot\vec{C}\right),\\
&g=-mb,\\
&\vec{E}=-i\left[\vec{\nabla}\times\vec{C}\right],\\
&\vec{F}=-m\vec{C},\\
&\vec{H}=-\partial\vec{C}-\vec{\nabla}b,
\end{aligned} \tag{34}$$

and the equation (31) can be rewritten as

$$\begin{aligned}&\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\left(-e - i\mathbf{e_2}g - i\vec{E} + \mathbf{e_1}\vec{F} + \mathbf{e_3}\vec{H}\right)\\ &= -i\mathbf{e_1}4\pi\rho_s - \mathbf{e_2}\frac{4\pi}{c}\vec{j}_s.\end{aligned} \tag{35}$$

Then we obtain the following equations for the field strengths:

$$\begin{aligned}&\partial e + (\vec{\nabla}\cdot\vec{H}) - mg = 4\pi\rho_s,\\ &(\vec{\nabla}\cdot\vec{E}) = 0,\\ &\partial g + (\vec{\nabla}\cdot\vec{F}) + me = 0,\\ &\partial\vec{F} + \vec{\nabla}g - m\vec{H} = 0,\\ &\partial\vec{E} - i[\vec{\nabla}\times\vec{H}] = 0,\\ &\partial\vec{H} + \vec{\nabla}e + i[\vec{\nabla}\times\vec{E}] + m\vec{F} = -\frac{4\pi}{c}\vec{j}_s,\\ &i[\vec{\nabla}\times\vec{F}] - m\vec{E} = 0.\end{aligned} \tag{36}$$

On the other hand, applying the operator $(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m)$ to the equation (35) we obtain the following wave equations for the field strengths:

$$\begin{aligned}&(\partial^2 - \triangle + m^2)e = 4\pi(\partial\rho_s + \frac{1}{c}(\vec{\nabla}\cdot\vec{j}_s)),\\ &(\partial^2 - \triangle + m^2)g = -4\pi m\rho_s,\\ &(\partial^2 - \triangle + m^2)\vec{F} = -\frac{4\pi}{c}m\vec{j}_s,,\\ &(\partial^2 - \triangle + m^2)\vec{E} = -i\frac{4\pi}{c}[\vec{\nabla}\times\vec{j}_s],\\ &(\partial^2 - \triangle + m^2)\vec{H} = -4\pi(\frac{1}{c}\partial\vec{j}_s + \vec{\nabla}\rho_s).\end{aligned} \tag{37}$$

Assuming the charge conservation

$$\partial\rho_s + \frac{1}{c}(\vec{\nabla}\cdot\vec{j}_s) = 0 \tag{38}$$

we can choose the field strength e equal to zero. This is equivalent to the following gauge condition (see (34)):

$$\partial b + (\vec{\nabla}\cdot\vec{C}) = 0 \tag{39}$$

similar to the Lorentz gauge.

2.3 Stationary field of point scalar source

In the stationary case $\vec{j}_s = 0$ and potential of the field can be chosen as

$$\tilde{\mathbf{W}} = i\mathbf{e_1}b\,(\vec{r})\,. \tag{40}$$

Then we have only two nonzero field components

$$\begin{aligned}g &= -mb,\\ \vec{H} &= -\vec{\nabla}b\end{aligned} \tag{41}$$

and the following field equations:

$$\begin{aligned}&(\vec{\nabla}\cdot\vec{H}) - mg = 4\pi\rho_s,\\ &\vec{\nabla}g - m\vec{H} = 0,\\ &[\vec{\nabla}\times\vec{H}] = 0.\end{aligned} \tag{42}$$

Let us calculate the field produced by a scalar stationary point source

$$\tilde{\mathbf{J}} = -i\mathbf{e_1} 4\pi q_s \delta(\vec{r}), \tag{43}$$

where q_s is the point charge and $\delta(\vec{r})$ is delta function. Then stationary wave equation (31) can be written in spherical coordinates as

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{m_0^2c^2}{\hbar^2}\right) b(r) = -4\pi q_s \delta(\vec{r}). \tag{44}$$

The partial solution of the equation (44), which decays at $r \to \infty$, is

$$b = \frac{q_s}{r}\exp\left(-\frac{m_0 c}{\hbar} r\right). \tag{45}$$

This is the Yukava potential [18]. The stationary field has scalar and vector components

$$g = \frac{m_0 c}{\hbar}\frac{q_s}{r}\exp\left(-\frac{m_0 c}{\hbar} r\right), \tag{46}$$

$$\vec{H} = \left(\frac{1}{r} + \frac{m_0 c}{\hbar}\right)\frac{q_s}{r}\exp\left(-\frac{m_0 c}{\hbar} r\right)\vec{r}_0, \tag{47}$$

where $\vec{r}_0$ is a unit radial vector.

Let us consider the interaction of two point charges due to the overlap of their fields. Taking into account that the field in this case is the sum of the two fields $g = g_1 + g_2$ and $\vec{H} = \vec{H}_1 + \vec{H}_2$ the energy of interaction is equal (see expression (28))

$$W_{ss} = -\frac{1}{4\pi}\int \{g_1 g_2 + (H_1 \cdot H_2)\} dV, \tag{48}$$

where the integral is over all space. This expression can be derived analytically:

$$W_{ss} = -\frac{q_{s1}q_{s2}}{R}\exp\left(-\frac{m_0 c}{\hbar} R\right), \tag{49}$$

where R is the distance between the point charges. By definition we assume the attractive interaction between charges.

3 Fields described by sedeonic first-order equation

3.1 Homogeneous sedeonic first-order equation

Let us consider the special class of massive field, which is described by homogeneous sedeonic first-order equation:

$$\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3} m\right)\tilde{\mathbf{W}} = 0. \tag{50}$$

In equation (50) the basis elements $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ and $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ play the role of the space-time operators, which transform the wave function by means of component permutation. Choosing potential $\tilde{\mathbf{W}}$ in the form (21) we find that sedeonic equation (50) is equivalent to the following system

$$\begin{aligned}
&\partial a + (\vec{\nabla}\cdot\vec{D}) + mc = 0,\\
&\partial b + (\vec{\nabla}\cdot\vec{C}) + md = 0,\\
&\partial c + (\vec{\nabla}\cdot\vec{B}) - ma = 0,\\
&\partial d + (\vec{\nabla}\cdot\vec{A}) - mb = 0,\\
&\partial\vec{A} + \vec{\nabla} d - i[\vec{\nabla}\times\vec{D}] + m\vec{C} = 0,\\
&\partial\vec{B} + \vec{\nabla} c + i[\vec{\nabla}\times\vec{C}] + m\vec{D} = 0,\\
&\partial\vec{C} + \vec{\nabla} b - i[\vec{\nabla}\times\vec{B}] - m\vec{A} = 0,\\
&\partial\vec{D} + \vec{\nabla} a + i[\vec{\nabla}\times\vec{A}] - m\vec{B} = 0.
\end{aligned} \tag{51}$$

In fact, these equations describe the special field [10, 14] with zero field strengths (see for comparison the expressions (22)).

3.2 Plane wave solution

Let us consider the plane wave solution of equation (50) in detail. In this case the potential can be written as

$$\tilde{\mathbf{W}} = \tilde{\mathbf{U}}\ \exp\left\{-i\omega t + i\left(\vec{k}\cdot\vec{r}\right)\right\}, \tag{52}$$

where ω is a frequency and $\vec{k}$ is an absolute wave vector; the amplitude of the wave $\mathbf{U}$ does not depend on the coordinates and time. In this case, the dependence of frequency on the wave vector has two branches:

$$\omega_{\pm} = \pm\sqrt{c^2k^2 + \frac{m_0^2c^4}{\hbar^2}}\ . \tag{53}$$

Let us consider the amplitude of the wave function in the form of (21):

$$\tilde{\mathbf{U}} = a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}, \tag{54}$$

where a, b, c, d, $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ are arbitrary constants. Then the solution can be written as

$$\begin{aligned}\tilde{\mathbf{W}} = &\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right)\\ &\times\exp\left\{-i\omega_{\pm}t + i\left(\vec{k}\cdot\vec{r}\right)\right\}.\end{aligned} \tag{55}$$

Substituting this expression in the original equation (50) we get:

$$\begin{aligned}&\left(\mathbf{e_1}\frac{\omega_{\pm}}{c} - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}\frac{m_0c}{\hbar}\right)\\ &\times\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right) = 0.\end{aligned} \tag{56}$$

For convenience we introduce the following notation:

$$\begin{aligned}\omega' &= \frac{\omega_{\pm}}{c},\\ m &= \frac{m_0c}{\hbar},\end{aligned} \tag{57}$$

then equation (56) can be rewritten as

$$\begin{aligned}&\left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right)\\ &\times\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d + i\vec{A} + \mathbf{e_1}\vec{B} + \mathbf{e_2}\vec{C} - \mathbf{e_3}\vec{D}\right) = 0.\end{aligned} \tag{58}$$

For fixed $\vec{k}$ let us represent the vector constants in (54) in the form

$$\begin{aligned}\vec{A} &= \vec{A}_{\|} + \vec{A}_{\perp},\\ \vec{B} &= \vec{B}_{\|} + \vec{B}_{\perp},\\ \vec{C} &= \vec{C}_{\|} + \vec{C}_{\perp},\\ \vec{D} &= \vec{D}_{\|} + \vec{D}_{\perp},\end{aligned} \tag{59}$$

where the vectors $\vec{A}_{\|}$, $\vec{B}_{\|}$, $\vec{C}_{\|}$ and $\vec{D}_{\|}$ are parallel to the vector $\vec{k}$ while the vectors $\vec{A}_{\perp}$, $\vec{B}_{\perp}$, $\vec{C}_{\perp}$ and $\vec{D}_{\perp}$ are perpendicular to $\vec{k}$. Then performing the multiplication in (58), we obtain the following system of algebraic equations:

$$i\omega' b - ikC_{\|} - md = 0, \tag{60}$$

$$\omega' a - kD_{\|} + imc = 0, \tag{61}$$

$$-\omega' d + kA_{\parallel} + imb = 0, \tag{62}$$
$$\omega' c - kB_{\parallel} - ima = 0, \tag{63}$$
$$\omega' B_{\parallel} - kc + imD_{\parallel} = 0, \tag{64}$$
$$i\omega' A_{\parallel} - ikd - mC_{\parallel} = 0, \tag{65}$$
$$i\omega' D_{\parallel} - ika + mB_{\parallel} = 0, \tag{66}$$
$$i\omega' C_{\parallel} - ikb + mA_{\parallel} = 0, \tag{67}$$
$$\omega' \vec{B}_{\perp} - i\left[\vec{k} \times \vec{C}_{\perp}\right] + im\vec{D}_{\perp} = 0, \tag{68}$$
$$i\omega' \vec{A}_{\perp} - \left[\vec{k} \times \vec{D}_{\perp}\right] - m\vec{C}_{\perp} = 0, \tag{69}$$
$$i\omega' \vec{D}_{\perp} + \left[\vec{k} \times \vec{A}_{\perp}\right] + m\vec{B}_{\perp} = 0, \tag{70}$$
$$i\omega' \vec{C}_{\perp} - \left[\vec{k} \times \vec{B}_{\perp}\right] + m\vec{A}_{\perp} = 0, \tag{71}$$

where the values $A_{\parallel}$, $B_{\parallel}$, $C_{\parallel}$ and $D_{\parallel}$ are the projections of the vectors $\vec{A}_{\parallel}$, $\vec{B}_{\parallel}$, $\vec{C}_{\parallel}$ and $\vec{D}_{\parallel}$ on the vector $\vec{k}$.

Let us solve this system of equations. From (70) and (71) we find

$$\vec{D}_{\perp} = \frac{im}{\omega'}\vec{B}_{\perp} + \frac{i}{\omega'}\left[\vec{k} \times \vec{A}_{\perp}\right], \tag{72}$$
$$\vec{C}_{\perp} = \frac{im}{\omega'}\vec{A}_{\perp} - \frac{i}{\omega'}\left[\vec{k} \times \vec{B}_{\perp}\right]. \tag{73}$$

Using (53) one can easily check that for arbitrary $A_{\perp}$ and $B_{\perp}$ the equations (68) and (69) are fulfilled.

As a next step from equations (60)-(63) we obtain:

$$C_{\parallel} = \frac{\omega'}{k}b + i\frac{m}{k}d, \tag{74}$$
$$D_{\parallel} = \frac{\omega'}{k}a + i\frac{m}{k}c, \tag{75}$$
$$A_{\parallel} = \frac{\omega'}{k}d - i\frac{m}{k}b, \tag{76}$$
$$B_{\parallel} = \frac{\omega'}{k}c - i\frac{m}{k}a. \tag{77}$$

One can check that these solution fulfill the equations (64)-(67).

Thus the sedeon $\tilde{\mathbf{U}}$ has the form

$$\begin{aligned} \tilde{\mathbf{U}} &= a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d \\ &+ \{i\omega' d + mb + \mathbf{e_1}\omega' c - i\mathbf{e_1}ma + \mathbf{e_2}\omega' b + i\mathbf{e_2}md - \mathbf{e_3}\omega' a - i\mathbf{e_3}mc\}\frac{\vec{k}}{k^2} \\ &+ i\vec{A}_{\perp} + \mathbf{e_1}\vec{B}_{\perp} + i\mathbf{e_2}\frac{m}{\omega'}\vec{A}_{\perp} - i\mathbf{e_3}\frac{m}{\omega'}\vec{B}_{\perp} \\ &- i\mathbf{e_3}\frac{1}{\omega'}\left[\vec{k} \times \vec{A}_{\perp}\right] - i\mathbf{e_2}\frac{1}{\omega'}\left[\vec{k} \times \vec{B}_{\perp}\right]. \end{aligned} \tag{78}$$

Note that this expression can be rewritten in the following form:

$$\begin{aligned} \tilde{\mathbf{U}} &= \left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right) \\ &\times \left\{i\mathbf{e_2}\frac{\vec{k}}{k^2}\left(a + i\mathbf{e_1}b - i\mathbf{e_2}c - i\mathbf{e_3}d\right) + i\mathbf{e_1}\frac{1}{\omega'}\vec{A}_{\perp} + \frac{1}{\omega'}\vec{B}_{\perp}\right\}. \end{aligned} \tag{79}$$

Substituted this amplitude into (58) one can see that this equation is satisfied for any parameters $a,\ b,\ c,\ d,\ \vec{A}_\perp,\ \vec{B}_\perp$ because the expression in round brackets is sedeonic zero divisor. Indeed it is simple to check that

$$\left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right)\left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right) = 0. \tag{80}$$

In general, the plane wave solution for the equation (50) can be written in the following sedeonic form:

$$\tilde{\mathbf{W}} = \left(\mathbf{e_1}\omega' - i\mathbf{e_2}\vec{k} - i\mathbf{e_3}m\right)\tilde{\mathbf{M}}\exp\left\{-i\omega t + i\left(\vec{k}\cdot\vec{r}\right)\right\}, \tag{81}$$

where $\tilde{\mathbf{M}}$ is an arbitrary sedeon with constant components. In this case after performing multiplication in (81) we obtain that the components of the resulting sedeon are defined only by 8 independent combinations of the sedeon $\tilde{\mathbf{M}}$ components. Note that the internal structure of this wave is changed under space and time inversion.

In massless case the dispersion relation is

$$\omega_\pm = \pm\ ck \tag{82}$$

and plane wave solution can be written as

$$\tilde{\mathbf{W}} = \left(\mathbf{e_t}\frac{\omega_\pm}{c} - i\mathbf{e_r}\vec{k}\right)\tilde{\mathbf{M}}\exp\left\{-i\omega_\pm t + i\left(\vec{k}\cdot\vec{r}\right)\right\}. \tag{83}$$

Let us analyze the structure of the plane wave (83) in detail. We suppose that wave vector is directed along z axis. Then the first-order equation (50) can be rewritten in the following equivalent form:

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{e_{tr}a_3}\frac{\partial}{\partial z}\right)\tilde{\mathbf{W}}' = 0, \tag{84}$$

where $\tilde{\mathbf{W}}' = i\mathbf{e_t}\tilde{\mathbf{W}}$. Using (82) and (83) we can write solution of (84) in the following form:

$$\tilde{\mathbf{W}}'_+ = -\left(1 + \mathbf{e_{tr}a_3}\right)k\tilde{\mathbf{M}}\exp\left\{-i\omega_+ t + ikz\right\}, \tag{85}$$

and

$$\tilde{\mathbf{W}}'_- = \left(1 - \mathbf{e_{tr}a_3}\right)k\tilde{\mathbf{M}}\exp\left\{-i\omega_- t + ikz\right\}. \tag{86}$$

Note that the wave function $\tilde{\mathbf{W}}'_+$ describes the positive branch of dispersion law (82) that corresponds, for example, to the "antiparticle", while $\tilde{\mathbf{W}}'_-$ describes the negative branch that corresponds to the "particle" state. Besides, as it is seen the wave functions (85) and (86) are the eigenfunctions of spin operator [21]:

$$\hat{S}_z = \frac{1}{2}\ \mathbf{e_{tr}a_3}. \tag{87}$$

Indeed it is simple to check that

$$\hat{S}_z\tilde{\mathbf{W}}' = S_z\tilde{\mathbf{W}}', \tag{88}$$

where eigenvalue $S_z = \pm 1/2$. It is seen that plane waves (85) and (86) correspond to the different eigenvalues S_z. Thus $\tilde{\mathbf{W}}'_+$ describes "antiparticle" state with spirality $S_z = +1/2$, while $\tilde{\mathbf{W}}'_-$ describes "particle" state with spirality $S_z = -1/2$. However in the case of massive field the plane wave (81) has more complicated space-time structure.

3.3 Nonhomogeneous first-order equation

Let us consider the nonhomogeneous equation corresponding to the equation (50)

$$\left(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m\right)\tilde{\mathbf{W}} = \tilde{\mathbf{I}}. \tag{89}$$

Here $\tilde{\mathbf{I}}$ is the field source. Choosing the potential $\tilde{\mathbf{W}}$ in the form (21), we obtain the following equation for the field strengths:

$$-e + i\mathbf{e_1}f - i\mathbf{e_2}g + i\mathbf{e_3}h - i\vec{E} + \mathbf{e_1}\vec{F} + \mathbf{e_2}\vec{G} + \mathbf{e_3}\vec{H} = \mathbf{I}_0 + \vec{\mathbf{I}}. \tag{90}$$

This equation means that the strengths of this field are nonzero only in the region of field source.

Let us consider the sedeonic source in the following form:

$$\tilde{\mathbf{I}} = -i\mathbf{e_2}4\pi\rho_v + \mathbf{e_1}\frac{4\pi}{c}\vec{j}_v. \qquad (91)$$

where ρ_v is a volume density of charge and $\vec{j}_v$ is volume density of current. In this case the equation (90) is rewritten as

$$-i\mathbf{e_2}g + \mathbf{e_1}\vec{F} = -i\mathbf{e_2}4\pi\rho_v + \mathbf{e_1}\frac{4\pi}{c}\vec{j}_v, \qquad (92)$$

Applying the operator $(i\mathbf{e_1}\partial - \mathbf{e_2}\vec{\nabla} - i\mathbf{e_3}m)$ to the equation (92) and separating the values with different space-time properties we obtain the following equations for the field strengths:

$$\begin{aligned}
&g = 4\pi\rho_v, \\
&\vec{F} = \frac{4\pi}{c}\vec{j}_v, \\
&\partial g + (\vec{\nabla}\cdot\vec{F}) = 4\pi[\partial\rho_v + \frac{1}{c}(\vec{\nabla}\cdot\vec{j}_v)], \\
&[\vec{\nabla}\times\vec{F}] = \frac{4\pi}{c}[\vec{\nabla}\times\vec{j}_v], \\
&\partial\vec{F} + \vec{\nabla}g = 4\pi[\frac{1}{c}\partial\vec{j}_v + \vec{\nabla}\rho_v].
\end{aligned} \qquad (93)$$

Assuming charge conservation

$$\partial\rho_v + \frac{1}{c}(\vec{\nabla}\cdot\vec{j}_v) = 0 \qquad (94)$$

we have the following gauge condition:

$$\partial g + (\vec{\nabla}\cdot\vec{F}) = 0 \qquad (95)$$

which is similar to the Lorentz gauge, but for the field strengths.

Let us consider the a stationary field generated by a scalar point source

$$\mathbf{I}_0 = -i\mathbf{e_2}4\pi q_v\delta(\vec{r}), \qquad (96)$$

where q_v is the point charge. Then the intensity of the scalar field is

$$g_v(\vec{r}) = 4\pi q_v\delta(\vec{r}). \qquad (97)$$

This field is non-zero only in the region of source. It indicates that two point charges interact only if they are at the same point of space. The interaction energy for two point charges q_{v1} and q_{v2} is equal

$$W_{vv} = -\frac{1}{4\pi}\int g_{v1}g_{v2}dV = -4\pi q_{v1}q_{v2}\delta(\vec{R}), \qquad (98)$$

where $\vec{R}$ is the vector of distance between point charges.

Moreover one can suppose the interaction between q_s and q_v charges due to the overlap of scalar fields g_s and g_v. In the case of point q_s and q_v the fields are determined by the expressions (46) and (97), so that the interaction energy is equal to:

$$W_{sv} = -\frac{1}{4\pi}\int g_s g_v dV. \qquad (99)$$

As a result, we get:

$$W_{sv} = -\frac{m_0 c}{\hbar}\frac{q_s q_v}{R}\exp\left(-\frac{m_0 c}{\hbar}R\right), \qquad (100)$$

where R is the distance between charges.

4 Discussion

The algebra of space-time sedeons can be considered as the scalar-vector variant of Clifford algebra with specific commutation and multiplication rules. The sedeonic basis elements $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ are responsible for the spatial rotation, while the elements $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are responsible for the space-time inversions. In contrast to the Heaviside-Gibbs vector algebra the multiplication rules for vector basis in sedeonic algebra contain the imaginary unit (see Table 1). It allows realizing the scalar-vector algebra with Clifford product [16]. Apparently, such possibility of vector basis multiplication was pointed first by A.Macfarlane [19]. The non-commutative $\mathbf{e_n}$ basis plays an important role. First, it allows to take into account the space-time properties of the field potentials (33) and sources (32) explicitly, because the scalar and vector potentials as well as charge density and current density are the temporal and spatial components of corresponding four-vectors. This ensures the correct Lorentz transformation of these values from one inertial frame to another [14]. On the other hand, the non-commutative $\mathbf{e_n}$ basis allows us to represent the second-order differential operator in the Klein-Gordon equation as the product of two identical first-order operators and thereby write the sedeonic wave equation in a symmetric form.

The sedeonic approach enables constructing a description of massive fields in complete analogy with classical electrodynamics. In particular, we have shown the correct way to introduce the fields described by the system of equations (25) similar to the Maxwell's equations. In contrast to the Proca-Maxwell equations for the fields and field potentials [2, 3, 5] the equations (25) contain only the field strengths. This allowed us to generalize the concepts of density of energy and density of energy flux for the massive field and derive the relations for the field energy and momentum similar to the Poynting theorem in electrodynamics.

In the particular case of a stationary point source the solution of the sedeonic wave equation is the potential of the Yukawa type. It allows to assume that the charge q_s can be interpreted as the baryon charge and apparently the equations (25) can be used for the description of the baryon field. On the other hand, the nonhomogeneous sedeonic first-order equation (89) describes the specific short-acting fields. In this case the charge q_v can be interpreted as the lepton charge and apparently the equations (89) can be used for the description of the lepton field. The fact that the homogeneous first-order and second-order octonic and sedeonic wave equations for the quantum particles interacting with electromagnetic field describe the particles with spin 1/2 was previously discussed in Refs. [10, 20, 21].

5 Summary

Thus, in this paper we have considered the sedeonic generalization of equations describing the massive fields. It was shown that this approach allows one to construct the theory of massive fields analogous to the theory of massless electromagnetic field in classical electrodynamics.

We have considered the sedeonic second-order wave equation for the sedeonic potential of the massive field. It was demonstrated that this wave equation can be represented as a system of first-order equations for the field intensities similar to the system of Maxwell's equations. We generalized the concepts of density of energy and density of energy flux for the massive field. The relation for the energy and momentum of massive field similar to the Poynting theorem in electrodynamics have been derived. Besides, we considered the stationary massive fields generated by point sources (charges). The interaction energy of two point charges as a function of the distance between them have been calculated.

Additionally, we have discussed the massive fields described by first-order sedeonic wave equation. It was shown that the intensity of these fields are nonzero only in the region of field sources that leads to the specific short-range interaction between charges field sources.

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