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Framework for the Effective Action of Quantum Gauge and Gravitational Fields

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Abstract

We consider the construction of a simplified framework for constructing the manifestly gauge-invariant effective action of non-Abelian quantum gauge, and gravitational, fields. The new framework modifies the bilinear terms that are associated with virtual gauge fields. This is done in a manner that rectifies the singular kernel, simplifies loop computations, and maintains manifest effective gauge invariance. Starting with the invariant Lagrangian for a general non-Abelian gauge theory, we present analysis pertaining to the derivation of the effective propagator and the effective vertices. Similar analysis is extended to the Einstein invariant gravitational Lagrangian. We discuss the possibility of seeding the elements of symmetry breaking, and structuring the underlying gauge algebra, through a mechanism of giving masses to the components of the virtual fields. This mechanism could be a substitute to the Higgs scenario in non-Abelian gauge unification models, and an alternative to compactification in extra-dimensional gravity.

1 Introduction

The conventional framework for computing loop corrections to the quantum field theory of non-Abelian gauge fields[1] is *not* manifestly gauge invariant. It is based on elaborate techniques for securing the implications of gauge invariance, in order to guarantee the unitarity and the renormalizability of the theory. These techniques[2], [3], [4], [5], [1] are based on the method of gauge-fixing and the associated introduction of ghost loops. However, in the framework of the effective action[6], [7], [8], [1] of quantum field theory, we proposed[9], many years ago, a *manifestly gauge invariant approach* to the computation of quantum loop corrections.

In our innovative approach[9], we prescribed to *constrain* the virtual counterpart of the non-Abelian gauge field with a gauge-invariant condition, and that, in order to preserve this constraint in loop computations, we must use a *gauge-covariant projection operator*. The situation is a non-Abelian generalization of the following treatment pertaining to the Abelian photon field. If, for example, we use $\mathcal{A}_\mu$ for the virtual (or quantum) part of the photon field, we must use the gauge condition $\partial_\mu \mathcal{A}_\mu = 0$. The corresponding projection operator is $\Lambda_{\mu\nu} = (\eta_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2)$. The implementation of the latter in the effective propagator would gaurantee the preservation of the constraint in all loop computations. That we can construct a gauge-covariant projection operator in non-Abelian gauge theories[9], as well as in quantum gravity, is based on a technique similar to Dirac's method[10] of modifying Poisson brackets in a manner implementing constraints.

However, our method of constructing the manifestly gauge-invariant effective action for non-Abelian quantum gauge fields, and for quantum gravity, while being well-defined in principle, leads to *very tedious computational scenarios*. Our purpose, in this article, is to point out that the manifestly gauge-invariant framework of the effective action advocates the introduction of a much simpler approach.

The new approach is based on the principle that, on the basis of an underlying functional integral definition of the effective action, *the effective bilinear terms of the virtual part of the gauge field can be modified* as convenient, *as long as manifest gauge invariance is respected*. In this way, the singular nature of the virtual dynamics can be rectified, and the framework for loop computations can be made much simpler. Notice, however, that all this would not threaten the unitarity of the theory, since the effective action would remain manifestly gauge invariant.

Another important element of our new approach concerns the possibility of *endowing the virtual part of the gauge field with mass*. This, again, is in accord with the principle of respecting the effective gauge invariance of the theory, however, with the extra benefit of securing the absence of infrared divergences in loop corrections. The possibility, in the framework of the effective action, of endowing the virtual (or quantum) part of a gauge, or non-gauge field, with mass opens the possibility of realizing *a new framework for symmetry breaking*. We shall elaborate more on this point in the concluding discussion of this article.

In the following section, we consider the construction of the effective action for a non-Abelian gauge field, and will show how to obtain the effective propagator, and the effective vertices, and the new way of handling the bilinear terms. We also consider the coupling of the non-Abelian gauge field to scalar and Dirac fields. That development will be followed by a similar treatment regarding pure quantum gravity.

2 Gauge Fields in Flat Spacetime

Let us consider the Lagrangian density $\mathcal{L}$ of a non-Abelian vector gauge field V_μ in flat spacetime. We write

$$\begin{cases} \mathcal{L} = -\frac{1}{4}\mathbf{tr}\,(F_{\mu\nu}F_{\mu\nu}) \\ \\ F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu] \end{cases} \tag{1}$$

Whereas V_μ belongs to the adjoint representation of some Lie algebra, the **tr** symbol, in the above expression, would denote the corresponding rule of writing a bilinear invariant, and the antisymmetric tensor $F_{\mu\nu}$ is the gauge-covariant curvature (or field strength) tensor.

Proceeding to construct the elements of the effective action, we make the shift

$$V_\mu \to V_\mu + \mathcal{V}_\mu \tag{2}$$

Whereas V_μ represents the *effective* gauge field of the theory, the field $\mathcal{V}_\mu$ is the corresponding *virtual* (or quantum) field that is to be integrated over in the functional integral formulation of the effective action. It is also the field whose effective propagator, and effective vertices, are the essential building blocks for the construction of the effective action, in an associated Feynman graphic language.

Corresponding to the above shift in the vector gauge field, we obtain the following shift for the curvature tensor,

$$F_{\mu\nu} \to F_{\mu\nu} + \{\nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu - i[\mathcal{V}_\mu, \mathcal{V}_\nu]\} \tag{3}$$

Here, we have the gauge-covariant derivative of the virtual field,

$$\nabla_\mu \mathcal{V}_\nu = \partial_\mu \mathcal{V}_\nu - i[V_\mu, \mathcal{V}_\nu] \tag{4}$$

Notice that gauge covariance is meant to be *with respect to the effective gauge field* V_μ. The effective propagator and the effective vertices of the virtual field $\mathcal{V}_\mu$ must all be manifestly covariant with respect to the effective gauge field.

Now the shifted Lagrangian density takes the following form:

$$\mathcal{L} \to \mathbf{tr} \begin{pmatrix} -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} \\ \\ -\frac{1}{2} F_{\mu\nu} \{\nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu - i[\mathcal{V}_\mu, \mathcal{V}_\nu]\} \\ \\ -\frac{1}{4} \{\nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu - i[\mathcal{V}_\mu, \mathcal{V}_\nu]\} \{\nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu - i[\mathcal{V}_\mu, \mathcal{V}_\nu]\} \end{pmatrix} \tag{5}$$

Dropping the terms that are linear in the virtual field, since they cancel in the effective action formalism, using the cyclic property under the trace, and simplifying, we get

$$\mathcal{L} \to \mathbf{tr} \begin{pmatrix} -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} & (1) \\ \\ +\frac{i}{2} F_{\mu\nu} [\mathcal{V}_\mu, \mathcal{V}_\nu] - \frac{1}{2} \nabla_\mu \mathcal{V}_\nu \nabla_\mu \mathcal{V}_\nu + \frac{1}{2} \nabla_\mu \mathcal{V}_\nu \nabla_\nu \mathcal{V}_\mu & (2) \\ \\ +i \nabla_\mu \mathcal{V}_\nu [\mathcal{V}_\mu, \mathcal{V}_\nu] & (3) \\ \\ +\frac{1}{4} [\mathcal{V}_\mu, \mathcal{V}_\nu][\mathcal{V}_\mu, \mathcal{V}_\nu] & (4) \end{pmatrix} \tag{6}$$

The above terms are respectively, (1) the *classical* contribution, (2) the *bilinear*, from which the effective propagator should be obtained, (3) the *cubic*, from which the effective 3-leg vertex is obtained, and (4) the *quartic*, from which the effective 4-leg vertex should be obtained.

It is the bilinear contribution (2) that is usually problematic in a gauge theory, because the associated kernel is *singular* on a field-free background, where it takes the non-invertible form $(\partial^2 \eta_{\mu\nu} - \partial_\mu \partial_\nu)$, like in quantum electrodynamics.

Using integration by parts, and rotation under the trace, the bilinear terms would take the form

$$\mathbf{tr}\left\{\frac{1}{2}\mathcal{V}_\mu\nabla^2\mathcal{V}_\mu-\frac{3i}{2}\mathcal{V}_\mu F_{\mu\nu}\mathcal{V}_\nu+\frac{1}{2}\left(\nabla_\mu\mathcal{V}_\mu\right)^2\right\} \tag{7}$$

In our new approach, it is proposed that we drop the bilinear terms involving $\nabla_\mu\mathcal{V}_\mu$, which are gauge-invariant with respect to the effective field. Subsequently, the bilinear kernel would become nonsingular. Hence, the effective propagators are well defined by the bilinears:

$$\mathbf{tr}\left\{\frac{1}{2}\mathcal{V}_\mu\left(\nabla^2\eta_{\mu\nu}-3iF_{\mu\nu}\right)\mathcal{V}_\nu\right\} \tag{8}$$

This together with the cubic and the quartic terms,

$$\mathbf{tr}\left\{i\nabla_\mu\mathcal{V}_\nu\left[\mathcal{V}_\mu,\mathcal{V}_\nu\right]+\frac{1}{4}\left[\mathcal{V}_\mu,\mathcal{V}_\nu\right]\left[\mathcal{V}_\mu,\mathcal{V}_\nu\right]\right\} \tag{9}$$

that define the vertices, the gauge-invariant effective action is easily constructed. Of course, we must apply either our gauge-invariant and divergence-free regularization prescriptions[11], or the gauge-invariant cutoff technique[9].

Now, let us consider the coupling to a scalar field, which belongs to some multiplet representation of the underlying gauge algebra. The gauge-invariant kinetic term of a multiplet of scalars ϕ takes the form:

$$\begin{cases}\mathcal{L}_\phi=\mathbf{tr}\left\{\frac{1}{2}\nabla_\mu\phi\nabla_\mu\phi\right\}+\cdots\\[2ex] \nabla_\mu\phi=\partial_\mu\phi-i\left[V_\mu,\phi\right]\end{cases} \tag{10}$$

Here, the dots $(\cdots)$ represent the terms of a possible scalar field potential, that we suppress in the present discussion.

We make the simultaneous shifting of the vector gauge field, and of the scalar field,

$$\begin{cases}V_\mu\rightarrow V_\mu+\mathcal{V}_\mu\\[2ex] \phi\rightarrow\phi+\varphi\end{cases} \tag{11}$$

Here, φ is the virtual (or quantum) part of the scalar multiplet, the analog of $\mathcal{V}_\mu$. Accordingly, the covariant derivative $\nabla_\mu\phi$ of the effective scalar field gets the following shift:

$$\nabla_\mu\phi\rightarrow\nabla_\mu\phi+\nabla_\mu\varphi-i\left[\mathcal{V}_\mu,\phi\right]-i\left[\mathcal{V}_\mu,\varphi\right] \tag{12}$$

Subsequently, the Lagrangian acquires the following shift:

$$\mathcal{L}_\phi\rightarrow\mathbf{tr}\begin{pmatrix}\frac{1}{2}\nabla_\mu\phi\nabla_\mu\phi\\[2ex] +\nabla_\mu\phi\left\{\nabla_\mu\varphi-i\left[\mathcal{V}_\mu,\phi\right]-i\left[\mathcal{V}_\mu,\varphi\right]\right\}\\[2ex] \frac{1}{2}\left\{\nabla_\mu\varphi-i\left[\mathcal{V}_\mu,\phi\right]-i\left[\mathcal{V}_\mu,\varphi\right]\right\}\left\{\nabla_\mu\varphi-i\left[\mathcal{V}_\mu,\phi\right]-i\left[\mathcal{V}_\mu,\varphi\right]\right\}\end{pmatrix} \tag{13}$$

Again, dropping the linear terms, manipulating and simplifying, using integration by parts and the rotation property under the trace, and dropping the terms involving $\nabla_\mu \mathcal{V}_\mu$, we get

$$\mathcal{L}_\phi \to \mathbf{tr} \begin{pmatrix} \frac{1}{2}\nabla_\mu \phi \nabla_\mu \phi & (1) \\ -\frac{1}{2}\varphi \nabla^2 \varphi + 2i\mathcal{V}_\mu [\nabla_\mu, \varphi] - \frac{1}{2}[\mathcal{V}_\mu, \phi][\mathcal{V}_\mu, \phi] & (2) \\ -i\nabla_\mu \varphi [\mathcal{V}_\mu, \varphi] - [\mathcal{V}_\mu, \phi][\mathcal{V}_\mu, \varphi] & (3) \\ -\frac{1}{2}[\mathcal{V}_\mu, \varphi][\mathcal{V}_\mu, \varphi] & (4) \end{pmatrix} \quad (14)$$

In the above, the first line (1) gives the classical contribution of the scalar field to the effective action, the second line (2) gives the *bilinear* terms. These contain besides the gauge-invariant kinetic term of the virtual scalar, a mixing term between the virtual scalar and the virtual vector, and a contribution to the virtual vector bilinear. The third line (3) gives the effective gauge-invariant 3-leg scalar-scalar-vector and scalar-vector-vector vertices. The fourth line (4) gives the effective 4-leg vertices, with two vector legs and two scalar legs.

We conclude this section by considering the coupling of a non-Abelian gauge field to a Dirac fermion Ψ, having a Dirac conjugate $\bar{\Psi}$, and belonging to some representation of the underlying gauge algebra. The associated Lagrangian takes the form:

$$\begin{cases} \mathcal{L}_\psi = \bar{\Psi}(i\gamma \cdot \nabla - m)\Psi \\ \nabla_\mu \Psi = (\partial_\mu - iV_\mu)\Psi \end{cases} \quad (15)$$

Again, we make the simultaneous shifts,

$$\begin{cases} V_\mu \to V_\mu + \mathcal{V}_\mu \\ \Psi \to \Psi + \psi \end{cases} \quad (16)$$

whereas V_μ and Ψ are the effective vector gauge boson, and the effective spinor fermion, respectively, the fields $\mathcal{V}_\mu$ and ψ are the respective virtual (or quantum) counterparts.

After dropping the terms that are linear in the virtual fields, the shifted Lagrangian takes the form

$$\mathcal{L}_\psi = \begin{pmatrix} \bar{\Psi}(i\gamma \cdot \nabla - m)\Psi & (1) \\ \bar{\psi}(\gamma \cdot \mathcal{V})\Psi + \bar{\Psi}(\gamma \cdot \mathcal{V})\psi + \bar{\psi}(i\gamma \cdot \nabla - m)\psi & (2) \\ \bar{\psi}(\gamma \cdot \mathcal{V})\psi & (3) \end{pmatrix} \quad (17)$$

Here, we have the classical contribution (1) of the gauge-invariant Ψ Lagrangian to the effective action, the bilinears (2) which show a mixing between the virtual vector $\mathcal{V}_\mu$

and the virtual fermion ψ together with a ψ bilinear, and a term (3) that gives 3-leg fermion-fermion-vector vertex.

In the following, we show that the mixing term in the above can be eliminated by redefining the virtual fermion ψ. Such a technique is also applicable for the earlier mixing that occurs in the vector-scalar bilinears. Now, the terms

$$\bar{\psi}(\gamma\cdot\mathcal{V})\Psi+\bar{\Psi}(\gamma\cdot\mathcal{V})\psi+\bar{\psi}\,(i\gamma\cdot\nabla-m)\,\psi \tag{18}$$

can be written in the following form,

$$(\bar{\psi}+\bar{\chi})\,(i\gamma\cdot\nabla-m)\,(\psi+\chi)-\bar{\chi}\,(i\gamma\cdot\nabla-m)\,\chi \tag{19}$$

with χ and $\bar{\chi}$ given by

$$\begin{cases} \chi=(i\gamma\cdot\nabla-m)^{-1}\,(\gamma\cdot\mathcal{V})\Psi \\ \\ \bar{\chi}=\bar{\Psi}(\gamma\cdot\mathcal{V})\,(i\gamma\cdot\nabla-m)^{-1} \end{cases} \tag{20}$$

Now, since the virtual fields ψ and $\bar{\psi}$ are integrated over, in the associated functional integral formulation of the effective action, we can make the shifts

$$\psi\to\psi-\chi \qquad \bar{\psi}\to\bar{\psi}-\bar{\chi} \tag{21}$$

and we are left with the *effective bilinears*,

$$\bar{\psi}\,(i\gamma\cdot\nabla-m)\,\psi-\bar{\Psi}(\gamma\cdot\mathcal{V})\,(i\gamma\cdot\nabla-m)^{-1}\,(\gamma\cdot\mathcal{V})\Psi \tag{22}$$

The first term of the above would give the effective gauge-covariant propagator

$$(i\gamma\cdot\nabla-m)^{-1} \tag{23}$$

of the virtual fermion, while the second term would contribute the following *gauge covariant insertion* in the effective propagator of the virtual vector field,

$$\bar{\Psi}\gamma_\mu\,(i\gamma\cdot\nabla-m)^{-1}\,\gamma_\nu\Psi \tag{24}$$

The above insertion describes an effective fermion propagator, and two external lines corresponding to the effective fields Ψ and $\bar{\Psi}$.

3 Covariant Quantum Gravity

Let us consider the generally covariant classical action density of pure Einstein gravidynamics,

$$\mathcal{L}=\sqrt{g}g^{\mu\nu}R_{\mu\nu} \tag{25}$$

where we have the Ricci tensor,

$$R_{\mu\nu}=\partial_\mu\Gamma^\lambda_{\nu\lambda}-\partial_\lambda\Gamma^\lambda_{\mu\nu}-\Gamma^\lambda_{\mu\nu}\Gamma^\rho_{\lambda\rho}+\Gamma^\lambda_{\mu\rho}\Gamma^\rho_{\nu\lambda} \tag{26}$$

and the metric connection

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) \tag{27}$$

Here too, the covariant scheme for computing the effective action follows from the general methods that pertain to gauge field theory. We introduce a virtual part $\phi_{\mu\nu}$ to the metric field $g_{\mu\nu}$, and make the shift

$$g_{\mu\nu} \to g_{\mu\nu} + \phi_{\mu\nu} \tag{28}$$

Correspondingly, we obtain the following shift for the metric connection,

$$\Gamma^{\lambda}_{\mu\nu} \to \frac{1}{2} \left(\frac{1}{1+\phi} \right)^{\lambda\rho} \left\{ \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) + \left(\partial_\mu \phi_{\nu\rho} + \partial_\nu \phi_{\mu\rho} - \partial_\rho \phi_{\mu\nu} \right) \right\} \tag{29}$$

Now using

$$\nabla_\mu \phi_{\lambda\rho} = \partial_\mu \phi_{\lambda\rho} - \Gamma^{\alpha}_{\mu\lambda} \phi_{\alpha\rho} - \Gamma^{\alpha}_{\mu\rho} \phi_{\lambda\alpha} \tag{30}$$

we obtain

$$\Gamma^{\lambda}_{\mu\nu} \to \frac{1}{2} \left(\frac{1}{1+\phi} \right)^{\lambda}_{\ \sigma} g^{\sigma\rho} \left(\begin{array}{c} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu} \right) + \left(\nabla_\mu \phi_{\nu\rho} + \nabla_\nu \phi_{\mu\rho} - \nabla_\rho \phi_{\mu\nu} \right) \\ + \left(\Gamma^{\alpha}_{\mu\nu} \phi_{\alpha\rho} + \Gamma^{\alpha}_{\mu\rho} \phi_{\nu\alpha} + \Gamma^{\alpha}_{\nu\mu} \phi_{\alpha\rho} + \Gamma^{\alpha}_{\nu\rho} \phi_{\mu\alpha} - \Gamma^{\alpha}_{\rho\mu} \phi_{\alpha\nu} - \Gamma^{\alpha}_{\rho\nu} \phi_{\mu\alpha} \right) \end{array} \right) \tag{31}$$

Now since ϕ and Γ are symmetric in their lower indices, some terms cancel, and using the fact that the metric and its inverse can pass through the covariant derivative, we obtain

$$\Gamma^{\lambda}_{\mu\nu} \to \left(\frac{1}{1+\phi} \right)^{\lambda}_{\ \sigma} \left\{ \Gamma^{\sigma}_{\mu\nu} + \frac{1}{2} \left(\nabla_\mu \phi^{\sigma}_{\nu} + \nabla_\nu \phi^{\sigma}_{\mu} - \nabla^{\sigma} \phi_{\mu\nu} \right) + \Gamma^{\alpha}_{\mu\nu} \phi^{\sigma}_{\alpha} \right\} \tag{32}$$

Notice that the coefficient of $\Gamma^{\lambda}_{\mu\nu}$ is the above is just

$$\left(\frac{1}{1+\phi} \right)^{\lambda}_{\ \sigma} \left(\delta^{\sigma}_{\alpha} + \phi^{\sigma}_{\alpha} \right) = \delta^{\lambda}_{\alpha} \tag{33}$$

Hence the metric connection is shifted in the following form:

$$\begin{cases} \Gamma^{\lambda}_{\mu\nu} \to \Gamma^{\lambda}_{\mu\nu} + \gamma^{\lambda}_{\mu\nu} \\ \gamma^{\lambda}_{\mu\nu} = \frac{1}{2} \left(\frac{1}{1+\phi} \right)^{\lambda}_{\ \rho} \left(\nabla_\mu \phi^{\rho}_{\nu} + \nabla_\nu \phi^{\rho}_{\mu} - \nabla^{\rho} \phi_{\mu\nu} \right) \end{cases} \tag{34}$$

Since ϕ^{ρ}_{λ} is symmetric, we obtain the contracted expression:

$$\gamma^{\lambda}_{\mu\lambda} = \frac{1}{2} \left(\frac{1}{1+\phi} \right)^{\lambda}_{\ \rho} \nabla_\mu \phi^{\rho}_{\lambda} \tag{35}$$

We should emphasize that ∇_μ is the Einstein covariant derivative with respect to the effective metric connection $\Gamma(g)$, and that indices are lowered and raised by the effective metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, respectively.

Now corresponding to the above shift of the metric connection, $\Gamma \to \Gamma + \gamma$, we obtain the following shift of the Ricci tensor,

$$\left\{ \begin{array}{l} R_{\mu\nu} \to R_{\mu\nu} + \mathcal{R}_{\mu\nu} \\ \\ \mathcal{R}_{\mu\nu} = \nabla_\mu \gamma^\lambda_{\nu\lambda} - \nabla_\lambda \gamma^\lambda_{\mu\nu} - \gamma^\lambda_{\mu\nu}\gamma^\rho_{\lambda\rho} + \gamma^\lambda_{\mu\rho}\gamma^\rho_{\nu\lambda} \end{array} \right. \tag{36}$$

The Lagrangian density would then transform such as

$$\sqrt{g} g^{\mu\nu} R_{\mu\nu} \to \sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu \left\{R^\mu_\nu + \mathcal{R}^\mu_\nu\right\} \tag{37}$$

Substituting for $\mathcal{R}^\mu_\nu$, we obtain

$$\left\{ \begin{array}{l} \sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu R^\mu_\nu \\ \\ +\sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu \left\{\nabla_\alpha \gamma^\lambda_{\nu\lambda} - \nabla_\lambda \gamma^\lambda_{\alpha\nu} - \gamma^\lambda_{\alpha\nu}\gamma^\rho_{\lambda\rho} + \gamma^\lambda_{\alpha\rho}\gamma^\rho_{\nu\lambda}\right\} g^{\mu\alpha} \end{array} \right. \tag{38}$$

Before substituting for γ, we must use the relation

$$\left. \begin{array}{r} \sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu \left\{\nabla_\alpha \gamma^\lambda_{\nu\lambda} - \nabla_\lambda \gamma^\lambda_{\alpha\nu}\right\} g^{\mu\alpha} \\ \\ = 2\sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu \left\{\gamma^\lambda_{\alpha\nu}\gamma^\rho_{\lambda\rho} - \gamma^\lambda_{\alpha\rho}\gamma^\rho_{\nu\lambda}\right\} g^{\mu\alpha} \end{array} \right\} \tag{39}$$

This can be comprehended by comparing with the derivation of the action in the Palatini formalism with derivatives replaced by covariant one. Hence we have for the transformed Lagrangian density

$$\left\{ \begin{array}{l} \sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu R^\mu_\nu \\ \\ +\sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu \left\{\gamma^\lambda_{\alpha\nu}\gamma^\rho_{\lambda\rho} - \gamma^\lambda_{\alpha\rho}\gamma^\rho_{\nu\lambda}\right\} g^{\mu\alpha} \end{array} \right. \tag{40}$$

Substituting for the γ's in terms of ϕ, simplifying and rearranging, we obtain for the shifted Lagrangian density,

$$\left\{ \begin{array}{l} \sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu R^\mu_\nu \\ \\ +\sqrt{g}\sqrt{\det(1+\phi)} \left(\frac{1}{1+\phi}\right)^\nu_\mu \left(\frac{1}{1+\phi}\right)^\rho_\lambda \left(\frac{1}{1+\phi}\right)^\beta_\alpha \times \\ \\ \left\{\frac{1}{4}\nabla^\mu\phi^\lambda_\beta\nabla_\nu\phi^\alpha_\rho + \frac{1}{2}\nabla^\mu\phi^\lambda_\nu\nabla_\rho\phi^\alpha_\beta - \frac{1}{2}\nabla^\mu\phi^\lambda_\beta\nabla_\rho\phi^\alpha_\nu - \frac{1}{4}\nabla^\mu\phi^\lambda_\rho\nabla_\nu\phi^\alpha_\beta\right\} \end{array} \right. \tag{41}$$

The expansions of $\det(1+\phi)$ and $(1+\phi)^{-1}$ in the above expression, in tensorial powers of ϕ^ν_μ, would yield the *infinite set of gravitational couplings associated with the virtual graviton.*

The computation of the effective quantum action would proceed in the usual perturbative loop expansion, using effective propagators and vertices. For that purpose, one must specify the nature of the background about which the quantum expansion is developed. We might wish to take as part of $g_{\mu\nu}$ any of the nontrivial classical configurations. In that case, we must expand $\phi_{\mu\nu}$ in terms of the eigenstates associated with the covariant bilinear kernel. However, the simplest approach is to take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is *a perturbation to flat spacetime in the effective background*. The effective action is obtained as a perturbative series in $h_{\mu\nu}$. However, principles of covariance would require the computation of only the lowest order terms. From this, the full covariant contribution may be deduced.

In the expansion of the shifted Lagrangian density, with respect to ϕ, we use the series

$$\begin{cases} \sqrt{\det(1+\phi)} = e^{\frac{1}{2}\operatorname{tr}\ln(1+\phi)} \\ = 1 + \frac{1}{2}(\phi) + \left\{\frac{1}{8}(\phi)^2 - \frac{1}{4}\phi_\mu^\nu\phi_\nu^\mu\right\} \\ + \left\{\frac{1}{48}(\phi)^3 - \frac{1}{8}(\phi)\phi_\mu^\nu\phi_\nu^\mu + \frac{1}{6}\phi_\mu^\nu\phi_\nu^\lambda\phi_\lambda^\mu\right\} \\ + \left\{\frac{1}{384}(\phi)^4 - \frac{1}{32}(\phi)^2\phi_\mu^\nu\phi_\nu^\mu + \frac{1}{12}(\phi)\phi_\mu^\nu\phi_\nu^\lambda\phi_\lambda^\mu + \frac{1}{32}(\phi_\mu^\nu\phi_\nu^\mu)^2 - \frac{1}{8}\phi_\mu^\nu\phi_\nu^\lambda\phi_\lambda^\rho\phi_\rho^\mu\right\} \\ + \cdots \end{cases} \tag{42}$$

Here (ϕ) without indices stands for the trace part ϕ_μ^μ. We also use the series,

$$\left(\frac{1}{1+\phi}\right)_\mu^\nu = \delta_\mu^\nu - \phi_\mu^\nu + \phi_\mu^\lambda\phi_\lambda^\nu - \phi_\mu^\lambda\phi_\lambda^\rho\phi_\rho^\nu + \phi_\mu^\lambda\phi_\lambda^\rho\phi_\rho^\sigma\phi_\sigma^\nu - \cdots \tag{43}$$

The computation of the one-loop contribution requires the expansion only to second order in ϕ (the bilinears). However, two-loop contributions require the expansion to order ϕ^4, etc.

In the remainder of this section, we shall examine only the bilinears that involve the covariant derivatives of the virtual tensor field ϕ_μ^ν. These are

$$\sqrt{g}\left\{\frac{1}{4}\nabla^\mu\phi_\alpha^\lambda\nabla_\mu\phi_\lambda^\alpha + \frac{1}{2}\nabla^\mu\phi_\mu^\lambda\nabla_\lambda\phi_\alpha^\alpha - \frac{1}{2}\nabla^\mu\phi_\alpha^\lambda\nabla_\lambda\phi_\mu^\alpha - \frac{1}{4}\nabla^\mu\phi_\lambda^\lambda\nabla_\mu\phi_\alpha^\alpha\right\} \tag{44}$$

In flat spacetime where $g_{\mu\nu} = \eta_{\mu\nu}$, the bilinear kernel corresponding to the above is singular. In order to obtain a non-singular counterpart, we shall manipulate terms and drop those that contain a factor like $\nabla^\mu\phi_\mu^\lambda$. So we drop the second term, while the third term becomes, using integration by parts:

$$-\frac{1}{2}\nabla^\mu\phi_\alpha^\lambda\nabla_\lambda\phi_\mu^\alpha = \frac{1}{2}\phi_\alpha^\lambda\nabla_\mu\nabla_\lambda\phi^{\mu\alpha} = \frac{1}{2}\phi_\alpha^\lambda\left[\nabla_\mu, \nabla_\lambda\right]\phi^{\mu\alpha} \tag{45}$$

Computing the commutator, we can express the above in terms of curvature tensors. Hence, up to curvature corrections, which may all be important when computing the effective action (however, see the concluding discussion section), the remaining bilinears become

$$\sqrt{g}\left\{\frac{1}{4}\nabla^{\mu}\phi^{\lambda}_{\alpha}\nabla_{\mu}\phi^{\alpha}_{\lambda}-\frac{1}{4}\nabla^{\mu}\phi^{\lambda}_{\lambda}\nabla_{\mu}\phi^{\alpha}_{\alpha}\right\} \tag{46}$$

These would give us a *non-singular* kernel, and a corresponding gauge-covariant effective propagator for the virtual graviton.

4 Discussion

In the preceding two sections we have scrutinized the manner of deriving the effective propagators, and the effective vertices, for constructing the effective action of quantum field theory, with particular reference to non-Abelian gauge, and gravitational, fields. We have proposed that the *bilinear terms* associated with the virtual quanta corresponding to the gauge fields, which normally give singular kernels, *can be modified by removing gauge-covariant terms* (typically involving covariant divergences of the vector or tensor fields). The real justification for such modification is based on the fact that, in the underlying functional integral derivation of the effective action[9], the perturbative loop expansion is developed around a Gaussian integral of the form

$$\int(\,\mathrm{d}\varphi)\,e^{\frac{i}{2}W_{ij}(\phi)\varphi_i\varphi_j} \tag{47}$$

Here, in compact notation, ϕ_i represents the effective fields of the theory, φ_i the virtual fields that are integrated over, and $W_{ij}(\phi)$ the bilinear kernel. For gauge theories, the latter is usually singular and needs modification. Such modification is admissible since the functional integration measure, in the above, can be defined conveniently. However, we *must do the modification in a manifestly gauge-invariant manner* (with respect to the effective gauge fields), and that indeed corresponds to our suggestions of the foregoing sections.

We have shown that the modified bilinear kernel takes the general form of a covariant Laplacian ∇^2, plus *terms that contain the curvature tensors*. The latter terms are harmless, and would represent *gauge covariant insertions* in the effective propagators, leading to various contributions in the loop expansion of the effective action. However, great simplifications would take place, especially in quantum gravity computations, if these insertions were absent. That we can drop these insertions from the outset can also be justified as an admissible and convenient *definition* of the Gaussian functional integral, and the associated integration measure.

However, assuming that the fundamental kernel of an effective gauge, or gravitational, quantum field theory is taken to be of the form ∇^2, it is also justifiable to modify the latter to (∇^2+m^2), where m^2 corresponds to a *gauge-invariant effective mass term* associated with the virtual quantum part of the gauge field. This arbitrary parameter is a welcome addition that would regularize any possible infrared divergences, and might be related to the scale of the underlying physics.

We come now to an important point pertaining to symmetry breaking in quantum field theory. Whereas the modern approach to this problem is connected with Higgs scalar fields[1], here we recognize a new approach that depends on a *mechanism of implementing masses to virtual field components*. Consider, for example, a gauge theory with an underlying Lie algebra G that needs to be broken into two maximal subalgebraic factors H_1 and H_2. Let the associated virtual gauge field $\mathcal{V}$ be decomposed into components $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}\}$, that correspond to the respective subalegbras and the coset space of G over $H_1 \times H_2$. Giving these split virtual field components effective propagators with different mass parameters would clearly introduce a symmetry breaking mechanism for the effective gauge theory. Giving different masses to split components is not only limited to the virtual parts of gauge fields, but can be extended to all fundamental fields of the theory, bosonic and fermionic, that belong to decomposable multiplets of the underlying Lie algebra. In this way, symmetry breaking in the effective unified gauge theory can be implemented in any desirable way via the masses of virtual (quantum) field components, rather than by a classical Higgs potential.

At this point, the important question is this: Introducing a host of mass parameters into the very stucture of the effective action framework, *is there a possibility of relating these masses to each other*? The answer is: Computing the *vacuum contributions* of the effective action would furnish us with a function containing all these mass parameters, as well as the coupling constants. This function, or rather this *effective potential*, must be minimized with respect to all mass paremeters. This would yield the desirable relations that one needs.

Moving to the case of quantum gravity, our approach of endowing virtual fields with masses could have important impact on the extra dimensional extensions that are intended for gravidynamic unification schemes. Whereas prominent approaches to higher dimensional gravity theories would employ the mechanism of compactification for handling the extra dimensions[12], [13], [14], here we have a possible technique for *suppressing the dynamical effects of extra dimensions*, again through our mass implementation mechanism. Let the virtual part of a higher-dimensional graviton be denoted by ϕ_{MN}. This would decompose, for instance, into $\{\phi_{\mu\nu}, \phi_{\mu i}, \phi_{ij}\}$ where $\phi_{\mu\nu}$ is the 4-dimensional virtual graviton, $\phi_{\mu i}$ are vectors, and ϕ_{ij} are scalars. Giving these split components effective propagators with different mass parameters would implement a symmetry breaking mechanism into the effective higher-dimensional action. This would be reflected in producing different kinetic behavior for particles in the visible 4 dimensions, from those in the rather invisible extra dimensions (this is the effect of *asymmetrical renormalization* speculated about long time ago[15]). Here, giving different masses to split components is not only limited to the virtual graviton field, but can be extended to all fundamental fields of the higher-dimensional theory, bosonic and fermionic, whether tensorial or spinorial.

We shall return to practical applications of the foregoing ideas in other articles. The road is clear for the computation of higher-loop contributions in unified gauge theories, and in higher-dimensional gravidynamic schemes.

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