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The SU_7 Structure of 18-Dimensional Unification

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Abstract

In an 18-dimensional gravidynamic unification model the spacetime and the internal symmetries of 4 generations of leptons and quarks are consolidated in a 256-component Majorana-Weyl-Dirac fermion. In such a framework, the dynamics of vector bosons as well as Higgs scalars would be generated at the quantum level through a unified coupling of an antisymmetric tensor of rank 3 to the fermions. We exhibit the complete U_7 structure of the latter coupling. This extensive work begins by writing the Lorentz algebra of 18 dimensional spacetime in terms of its 4-dimensional Lorentz subalgebra and an internal O_{14} factor. The latter is expressed via its U_7 subalgebra. The 256-component fermion is expressed in terms of 64 Weyl fermions, and their Dirac conjugates, in 4 dimensions. Likewise the 3rd rank antisymmetric tensor is expressed in terms of vectors and scalars in 4 dimensions. The emerging picture regarding the fundamental fermions, and their interactions, would lead to aspects that are well described by a complementing O_{14} and SU_7 grand unification schemes.

1 Introduction

The consolidation of the electroweak[1], [2], [3], [4], [5], [6] $SU_2\times U_1$ and the strong SU_3 color symmetry[7], [8], [9] might proceed via the SU_5 or $SU_{N>5}$ quark-lepton[10], [11], [12], [13] grand unification schemes. Further consolidation of the quark-lepton multiplets could advance via orthogonal algebras[14], [15]. The ultimate unification of the internal and spacetime (spin) symmetries of quarks and leptons may take the road of a gravidynamic Eintein-Dirac theory in a higher-dimensional, notably an 18-dimensional[16], spacetime.

The main ingredients of a higher-dimensional Einstein-Dirac theory are a spinorial fermionic field, a vierbein field (related to the metric field), and a Lorentz connection field. The latter enters the theory as an auxilliary field that can be eliminated via its dynamical equations in favor of the vierbein and the fermion fields. The process of eliminating the higher-dimensional Lorentz connection generates a quartic coupling term in the fermion fields of the form $(\bar{\Psi}\Gamma_{ABC}\Psi)^2$, where Γ_{ABC} is a Dirac algebraic element that is totally antisymmetric in the vectorial indices (ABC). The subsequent decomposition of such a current$\times$current coupling in terms of 4-dimensional components would produce many terms that include those of low-energy weak phenomenogy. The question is, where are the vector bosons of gauge theories in such a scheme? As a matter of fact, in an Einstein-Dirac ultimate unification scheme, the vector bosons are part of an antisymmetric tensor V_{ABC}, whose dynamics are likely to arise from the effective quantum contributions resulting from a coupling of the form

$$\frac{1}{3!}V_{ABC} \times \bar{\Psi}\Gamma_{ABC}\Psi \tag{1}$$

In an 18 dimensional unification scheme[16], the fundamental fermions that can accommodate quarks and leptons constitute a 256-component Majorana-Weyl spinor. Our purpose in this article is to explore, in detail, the decomposition of the underlying $O_{1,17}$ symmetry of the above coupling through its 4-dimensional $O_{1,3}$ (Lorentz) symmetry and the O_{14} internal symmetry factor. The structure of the latter will take shape via the U_7 subalgebra[17].

In the following section, we shall begin by constructing the $O_{1,17}$ algebra in terms of $O_{1,3}$ and U_7 structural elements. This will be followed by our algebraic techniques for representing an $O_{1,17}$ vector as well as an antisymmetric tensor of rank 3 (like the one needed for the above coupling). Then we shall construct an $O_{1,17}$ algebraic representation of a Weyl spinor, and its Dirac conjugate, leading the path to the construction of the components of an antisymmetric 3rd rank tensor from fermionic bilinears. This will be used in constructing and decomposing the above boson-fermion coupling.

2 The Algebra $O_{1,17}$ in Terms of $O_{1,3}$ and U_7

The generators of the Lorentz algebra $O_{1,17}$ in 18 dimensions may be decomposed into the following set of generators:

$$\{J_{\mu\nu}, {J_a}^b, Q_{ab}, Q^{ab}, H_{\mu a}, {H_\mu}^a\} \tag{2}$$

Here, we use the Greek symbols $(\mu, \nu, \lambda, \cdots)$ to represent the vectorial indices in 4-dimensional spacetime, and the Latin symbols $(a, b, c, \cdots)$ to represent the indices of a fundamental SU_7 multiplet. In the above, the $J_{\mu\nu}$, being antisymmetric in (μ, ν), are the generators of the 4-dimensional Lorentz algebra. The ${J_a}^b$ are the generators of U_7, the trace part of which is a U_1 generator, while the traceless part is that of SU_7. The conjugate generators Q_{ab} and Q^{ab}, being antisymmetric in (a, b), would together with ${J_a}^b$ describe the O_{14} algebra. The conjugate generators $H_{\mu a}$ and ${H_\mu}^a$ would describe the coset of $O_{1,17}$ over $O_{1,3}$ and O_{14}. We begin by writing the $O_{1,3}$ commutators,

$$[J_{\mu\nu}, J_{\lambda\rho}] = (\eta_{\nu\lambda}J_{\mu\rho} - \eta_{\mu\lambda}J_{\nu\rho} + \eta_{\mu\rho}J_{\nu\lambda} - \eta_{\nu\rho}J_{\mu\lambda}) \tag{3}$$

Now ${J_a}^b, Q_{ab}$ and Q^{ab} would have vanishing commutators with $J_{\mu\nu}$. However we have

$$[J_{\mu\nu}, H_{\lambda a}] = \eta_{\nu\lambda}H_{\mu a} - \eta_{\mu\lambda}H_{\nu a} \tag{4}$$

$$[J_{\mu\nu}, {H_\lambda}^a] = \eta_{\nu\lambda}{H_\mu}^a - \eta_{\mu\lambda}{H_\nu}^a \tag{5}$$

For the generators ${J_a}^b$ of the U_7 subalgebra, we have

$$\left[{J_a}^b, {J_c}^d\right] = {\delta_c}^b{J_a}^d - {\delta_a}^d{J_c}^b \tag{6}$$

The commutators of the Q's with $J_a{}^b$ are

$$\left[J_a{}^b, Q_{cd}\right] = \delta_c{}^b Q_{ad} - \delta_d{}^b Q_{ac} \tag{7}$$

$$\left[J_a{}^b, Q^{cd}\right] = -\delta_a{}^c Q^{bd} + \delta_a{}^d Q^{bc} \tag{8}$$

and the commutators of the H's with $J_a{}^b$ are

$$\left[J_a{}^b, H_{\mu c}\right] = \delta_c{}^b H_{\mu a} \tag{9}$$

$$\left[J_a{}^b, H_\mu{}^c\right] = -\delta_a{}^c H_\mu{}^b \tag{10}$$

For the commutators of the Q's among themselves, we have

$$\left[Q_{ab}, Q_{cd}\right] = 0 \tag{11}$$

$$\left[Q^{ab}, Q^{cd}\right] = 0 \tag{12}$$

$$\left[Q_{ab}, Q^{cd}\right] = \left(\delta_b{}^c J_a{}^d - \delta_a{}^c J_b{}^d + \delta_a{}^d J_b{}^c - \delta_b{}^d J_a{}^c\right) \tag{13}$$

For the commutators of the Q's with the H's, we have

$$\left[Q_{ab}, H_{\mu c}\right] = 0 \tag{14}$$

$$\left[Q_{ab}, H_\mu{}^c\right] = \delta_b{}^c H_{\mu a} - \delta_a{}^c H_{\mu b} \tag{15}$$

$$\left[Q^{ab}, H_{\mu c}\right] = \delta_c{}^b H_\mu{}^a - \delta_c{}^a H_\mu{}^b \tag{16}$$

$$\left[Q^{ab}, H_\mu{}^c\right] = 0 \tag{17}$$

Finally, for the commutators of the H's among themselves,

$$\left[H_{\mu a}, H_{\nu b}\right] = -\eta_{\mu\nu} Q_{ab} \tag{18}$$

$$\left[H_\mu{}^a, H_\nu{}^b\right] = -\eta_{\mu\nu} Q^{ab} \tag{19}$$

$$\left[H_{\mu a}, H_\nu{}^b\right] = -\eta_{\mu\nu} J_a{}^b - \delta_a{}^b J_{\mu\nu} \tag{20}$$

We can verify that all Jacobi identities involving all generators are satisfied[1], and that all generators commute with the following quadratic (Casimir) operator:

$$\frac{1}{2} J_{\mu\nu} J_{\nu\mu} + J_a{}^b J_b{}^a + \frac{1}{2} Q_{ab} Q^{ba} + \frac{1}{2} Q^{ab} Q_{ba} - H_{\mu a} H_\mu{}^a - H_\mu{}^a H_{\mu a} \tag{21}$$

[1] **For extensive manipulations and verifications of the kind presented in this, and other articles, it would be useful to use efficient *symbolic computational software*. Interested researchers who have access to a *Mathematica* program, or similar software, and are nonetheless unable to work with such tensorial and algebraic stuctures, could obtain free consultation in that regard.**

3 The Representation of an $O_{1,17}$ Vector

In order to represent a vector in 18 dimensional spacetime, let us introduce the operators (K_μ, K_a, K^a). The commutators of these with the Lorentz generators $J_{\mu\nu}$ are

$$[J_{\mu\nu}, K_\lambda] = (\eta_{\nu\lambda} K_\mu - \eta_{\mu\lambda} K_\nu) \tag{22}$$

$$[J_{\mu\nu}, K_a] = 0 \tag{23}$$

$$[J_{\mu\nu}, K^a] = 0 \tag{24}$$

The commutators of the U_7 generators ${J_a}^b$ are

$$\left[{J_a}^b, K_\lambda\right] = 0 \tag{25}$$

$$\left[{J_a}^b, K_c\right] = {\delta_c}^b K_a \tag{26}$$

$$\left[{J_a}^b, K^c\right] = -{\delta_a}^c K^b \tag{27}$$

The commutators of Q_{ab} are

$$[Q_{ab}, K_\lambda] = 0 \tag{28}$$

$$[Q_{ab}, K_c] = 0 \tag{29}$$

$$[Q_{ab}, K^c] = ({\delta_b}^c K_a - {\delta_a}^c K_b) \tag{30}$$

The commutators of Q^{ab} are

$$\left[Q^{ab}, K_\lambda\right] = 0 \tag{31}$$

$$\left[Q^{ab}, K_c\right] = \left({\delta_c}^b K^a - {\delta_c}^a K^b\right) \tag{32}$$

$$\left[Q^{ab}, K^c\right] = 0 \tag{33}$$

The commutators of $H_{\mu a}$ are

$$[H_{\mu a}, K_\lambda] = -\eta_{\mu\lambda} K_a \tag{34}$$

$$[H_{\mu a}, K_c] = 0 \tag{35}$$

$$[H_{\mu a}, K^c] = {\delta_a}^c K_\mu \tag{36}$$

And finally, the commutators of ${H_\mu}^a$ are

$$[{H_\mu}^a, K_\lambda] = -\eta_{\mu\lambda} K^a \tag{37}$$

$$[{H_\mu}^a, K_c] = \delta_c^a K_\mu \tag{38}$$

$$[{H_\mu}^a, K^c] = 0 \tag{39}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,17}$ generators $J_{\mu\nu}$, ${J_a}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or ${H_\mu}^a$, with either of the operators K_μ, K_a, or K^a, are satisfied. Moreover any of the $O_{1,17}$ generators can be shown to commute with the following quadratic operator:

$$K_\mu K_\mu + K_a K^a + K^a K_a \tag{40}$$

We proceed now to the introduction of the multiplet that can be associated with the above operator representation, and to the construction of the infinitesimal $O_{1,17}$ transformations that act on it. The desired multiplet with components $\{B_\mu, B_a, B^a\}$ can be introduced by the vector module

$$\mathcal{B} = B_\mu K_\mu + B_a K^a + B^a K_a \tag{41}$$

Introducing the $O_{1,17}$ parameter module,

$$\mathcal{W} = \frac{1}{2}\Omega_{\mu\nu} J_{\mu\nu} + \Omega_a{}^b J_b{}^a + \frac{1}{2}\Omega_{ab} Q^{ab} + \frac{1}{2}\Omega^{ab} Q_{ab} + \Omega_{\mu a} H_\mu{}^a + \Omega_\mu{}^a H_{\mu a} \tag{42}$$

we can compute the commutator $[\mathcal{W}, \mathcal{B}]$. The latter gives a vector module whose components would define the needed infinitesimal transformations. We obtain

$$\delta B_\mu = \Omega_{\mu\nu} B_\nu + \Omega_{\mu a} B^a + \Omega_\mu{}^a B_a \tag{43}$$

$$\delta B_a = -\Omega_a{}^b B_b + \Omega_{ab} B^b - \Omega_{\mu a} B_\mu \tag{44}$$

$$\delta B^a = \Omega_b{}^a B^b + \Omega^{ab} B_b - \Omega_\mu{}^a B_\mu \tag{45}$$

We can verify that, for any two vector modules $\mathcal{A}$ and $\mathcal{B}$, the above infinitesimal transformations, acting in a like manner on the components of both, would leave invariant the following bilinear form:

$$\mathcal{A} \cdot \mathcal{B} = A_\mu B_\mu + A_a B^a + A^a B_a \tag{46}$$

4 The Antisymmetric Tensor Representation of Rank 3

An $O_{1,17}$ antisymmetric tensor representation of rank 3 would have the following $O_{1,3}$ and U_7 covariant components:

$$\{K_{\mu\nu\lambda}, K_{\mu\nu a}, K_{\mu\nu}{}^a, K_{\mu ab}, K_{\mu a}{}^b, K_\mu{}^{ab}, K_{abc}, K_{ab}{}^c, K_a{}^{bc}, K^{abc}\} \tag{47}$$

The symmetries of the above components with respect to their spacetime and SU_7 indices should be clear. Notice that the tensor $K_{\mu\nu\lambda}$ could be traded for a single-index counterpart using the 4-dimensional epsilon symbol $\epsilon_{\mu\nu\lambda\rho}$. However, it is more convenient to leave it in this form, at this stage. In order to be able to write out the infinitesimal transformations of an associated multiplet, we proceed now to the elaborate task of writing down the commutators of the above component operators with the generators of the $O_{1,17}$ algebra.

For the commutators of $J_{\mu\nu}$, we have

$$[J_{\mu\nu}, K_{\lambda\rho\sigma}] = (\eta_{\nu\lambda} K_{\mu\rho\sigma} + \eta_{\nu\rho} K_{\mu\sigma\lambda} + \eta_{\nu\sigma} K_{\mu\lambda\rho}) - (\mu \leftrightarrow \nu) \tag{48}$$

$$[J_{\mu\nu}, K_{\lambda\rho a}] = (\eta_{\nu\lambda} K_{\mu\rho a} - \eta_{\nu\rho} K_{\mu\lambda a}) - (\mu \leftrightarrow \nu) \tag{49}$$

$$[J_{\mu\nu}, K_{\lambda\rho}{}^a] = (\eta_{\nu\lambda} K_{\mu\rho}{}^a - \eta_{\nu\rho} K_{\mu\lambda}{}^a) - (\mu \leftrightarrow \nu) \tag{50}$$

$$[J_{\mu\nu}, K_{\lambda ab}] = \eta_{\nu\lambda} K_{\mu ab} - \eta_{\mu\lambda} K_{\nu ab} \tag{51}$$

$$\left[J_{\mu\nu}, K_{\lambda a}{}^{b}\right] = \eta_{\nu\lambda} K_{\mu a}{}^{b} - \eta_{\mu\lambda} K_{\nu a}{}^{b} \tag{52}$$

$$\left[J_{\mu\nu}, K_{\lambda}{}^{ab}\right] = \eta_{\nu\lambda} K_{\mu}{}^{ab} - \eta_{\mu\lambda} K_{\nu}{}^{ab} \tag{53}$$

The commutators of $J_{\mu\nu}$ with K_{abc}, $K_{ab}{}^{c}$, $K_{a}{}^{bc}$, and K^{abc} are vanishing.

Whereas $J_{a}{}^{b}$ commutes with $K_{\mu\nu\lambda}$, its commutators with the other K's are

$$\left[J_{a}{}^{b}, K_{\mu\nu c}\right] = \delta_{c}{}^{b} K_{\mu\nu a} \tag{54}$$

$$\left[J_{a}{}^{b}, K_{\mu\nu}{}^{c}\right] = -\delta_{a}{}^{c} K_{\mu\nu}{}^{b} \tag{55}$$

$$\left[J_{a}{}^{b}, K_{\mu cd}\right] = \delta_{c}{}^{b} K_{\mu ad} - \delta_{d}{}^{b} K_{\mu ac} \tag{56}$$

$$\left[J_{a}{}^{b}, K_{\mu c}{}^{d}\right] = \delta_{c}{}^{b} K_{\mu a}{}^{d} - \delta_{a}{}^{d} K_{\mu c}{}^{b} \tag{57}$$

$$\left[J_{a}{}^{b}, K_{\mu}{}^{cd}\right] = -\delta_{a}{}^{c} K_{\mu}{}^{bd} + \delta_{a}{}^{d} K_{\mu}{}^{bc} \tag{58}$$

$$\left[J_{a}{}^{b}, K_{cde}\right] = \delta_{c}{}^{b} K_{ade} + \delta_{d}{}^{b} K_{aec} + \delta_{e}{}^{b} K_{acd} \tag{59}$$

$$\left[J_{a}{}^{b}, K_{cd}{}^{e}\right] = \left(\delta_{c}{}^{b} K_{a,d}{}^{e} - \delta_{d}{}^{b} K_{ac}{}^{e}\right) - \left(\delta_{a}{}^{e} K_{cd}{}^{b}\right) \tag{60}$$

$$\left[J_{a}{}^{b}, K_{c}{}^{de}\right] = \left(\delta_{c}{}^{b} K_{a}{}^{de}\right) - \left(\delta_{a}{}^{d} K_{c}{}^{be} - \delta_{a}{}^{e} K_{c}{}^{bd}\right) \tag{61}$$

$$\left[J_{a}{}^{b}, K^{cde}\right] = -\left(\delta_{a}{}^{c} K^{bde} + \delta_{a}{}^{d} K^{bec} + \delta_{a}{}^{e} K^{bcd}\right) \tag{62}$$

The nonvanishing commutators of Q_{ab} with the K's are:

$$[Q_{ab}, K_{\mu\nu}{}^{c}] = \delta_{b}{}^{c} K_{\mu\nu a} - \delta_{a}{}^{c} K_{\mu\nu b} \tag{63}$$

$$\left[Q_{ab}, K_{\mu c}{}^{d}\right] = \delta_{a}{}^{d} K_{\mu bc} - \delta_{b}{}^{d} K_{\mu ac} \tag{64}$$

$$\left[Q_{ab}, K_{\mu}{}^{cd}\right] = -\left(\delta_{a}{}^{c} K_{\mu b}{}^{d} - \delta_{b}{}^{c} K_{\mu a}{}^{d} + \delta_{b}{}^{d} K_{\mu a}{}^{c} - \delta_{a}{}^{d} K_{\mu b}{}^{c}\right) \tag{65}$$

$$[Q_{ab}, K_{cd}{}^{e}] = (\delta_{b}{}^{e} K_{acd} - \delta_{a}{}^{e} K_{bcd}) \tag{66}$$

$$\left[Q_{ab}, K_{c}{}^{de}\right] = -\left(\delta_{b}{}^{d} K_{ac}{}^{e} - \delta_{a}{}^{d} K_{bc}{}^{e} + \delta_{a}{}^{e} K_{bc}{}^{d} - \delta_{b}{}^{e} K_{ac}{}^{d}\right) \tag{67}$$

$$\left[Q_{ab}, K^{cde}\right] = \left(\delta_{b}{}^{c} K_{a}{}^{de} + \delta_{b}{}^{d} K_{a}{}^{ec} + \delta_{b}{}^{e} K_{a}{}^{cd}\right) - (a \leftrightarrow b) \tag{68}$$

The nonvanishing commutators of Q^{ab} with the K's are:

$$\left[Q^{ab}, K_{\mu\nu c}\right] = \delta_{c}{}^{b} K_{\mu\nu}{}^{a} - \delta_{c}{}^{a} K_{\mu\nu}{}^{b} \tag{69}$$

$$\left[Q^{ab}, K_{\mu cd}\right] = \left(\delta_{c}{}^{a} K_{\mu d}{}^{b} - \delta_{c}{}^{b} K_{\mu d}{}^{a} + \delta_{d}{}^{b} K_{\mu c}{}^{a} - \delta_{d}{}^{a} K_{\mu c}{}^{b}\right) \tag{70}$$

$$\left[Q^{ab}, K_{\mu c}{}^{d}\right] = \delta_{c}{}^{b} K_{\mu}{}^{ad} - \delta_{c}{}^{a} K_{\mu}{}^{bd} \tag{71}$$

$$\left[Q^{ab}, K_{cde}\right] = \left(\delta_{c}{}^{b} K_{de}{}^{a} + \delta_{d}{}^{b} K_{ec}{}^{a} + \delta_{e}{}^{b} K_{cd}{}^{a}\right) - (a \leftrightarrow b) \tag{72}$$

$$\left[Q^{ab}, K_{cd}{}^{e}\right] = \left(-\delta_{c}{}^{b} K_{d}{}^{ae} + \delta_{d}{}^{b} K_{c}{}^{ae}\right) - (a \leftrightarrow b) \tag{73}$$

$$\left[Q^{ab}, K_{c}{}^{de}\right] = \delta_{c}{}^{b} K^{ade} - \delta_{c}{}^{a} K^{bde} \tag{74}$$

The nonvanishing commutators of $H_{\mu a}$ with the K's are:

$$[H_{\mu a}, K_{\nu\lambda\rho}] = -\left(\eta_{\mu\nu}K_{\lambda\rho a} + \eta_{\mu\rho}K_{\nu\lambda a} + \eta_{\mu\lambda}K_{\rho\nu a}\right) \tag{75}$$

$$[H_{\mu a}, K_{\nu\lambda b}] = \left(\eta_{\mu\nu}K_{\lambda ab} - \eta_{\mu\lambda}K_{\nu ab}\right) \tag{76}$$

$$\left[H_{\mu a}, K_{\nu\lambda}{}^{b}\right] = \left(\eta_{\mu\nu}K_{\lambda a}{}^{b} - \eta_{\mu\lambda}K_{\nu a}{}^{b}\right) + \left(\delta_a{}^{b}K_{\mu\nu\lambda}\right) \tag{77}$$

$$[H_{\mu a}, K_{\nu bc}] = -\eta_{\mu\nu}K_{abc} \tag{78}$$

$$[H_{\mu a}, K_{\nu b}{}^{c}] = -\eta_{\mu\nu}K_{ab}{}^{c} + \delta_a{}^{c}K_{\mu\nu b} \tag{79}$$

$$\left[H_{\mu a}, K_{\nu}{}^{bc}\right] = -\left(\eta_{\mu\nu}K_a{}^{bc}\right) - \left(\delta_a{}^{b}K_{\mu\nu}{}^{c} - \delta_a{}^{c}K_{\mu\nu}{}^{b}\right) \tag{80}$$

$$\left[H_{\mu a}, K_{bc}{}^{d}\right] = \delta_a{}^{d}K_{\mu bc} \tag{81}$$

$$\left[H_{\mu a}, K_b{}^{cd}\right] = -\delta_a{}^{c}K_{\mu b}{}^{d} + \delta_a{}^{d}K_{\mu b}{}^{c} \tag{82}$$

$$\left[H_{\mu a}, K^{bcd}\right] = \left(\delta_a{}^{b}K_{\mu}{}^{cd} + \delta_a{}^{d}K_{\mu}{}^{bc} + \delta_a{}^{c}K_{\mu}{}^{db}\right) \tag{83}$$

Finally, the nonvanishing commutators of $H_{\mu}{}^{a}$ with the K's are:

$$[H_{\mu}{}^{a}, K_{\nu\lambda\rho}] = -\left(\eta_{\mu\nu}K_{\lambda\rho}{}^{a} + \eta_{\mu\rho}K_{\nu\lambda}{}^{a} + \eta_{\mu\lambda}K_{\rho\nu}{}^{a}\right) \tag{84}$$

$$[H_{\mu}{}^{a}, K_{\nu\lambda b}] = -\left(\eta_{\mu\nu}K_{\lambda b}{}^{a} - \eta_{\mu\lambda}K_{\nu b}{}^{a}\right) + \left(\delta_b{}^{a}K_{\mu\nu\lambda}\right) \tag{85}$$

$$\left[H_{\mu}{}^{a}, K_{\nu\lambda}{}^{b}\right] = \left(\eta_{\mu\nu}K_{\lambda}{}^{ab} - \eta_{\mu\lambda}K_{\nu}{}^{ab}\right) \tag{86}$$

$$[H_{\mu}{}^{a}, K_{\nu bc}] = -\left(\eta_{\mu\nu}K_{bc}{}^{a}\right) - \left(\delta_b{}^{a}K_{\mu\nu c} - \delta_c{}^{a}K_{\mu\nu b}\right) \tag{87}$$

$$[H_{\mu}{}^{a}, K_{\nu b}{}^{c}] = \eta_{\mu\nu}K_b{}^{ac} - \delta_b{}^{a}K_{\mu\nu}{}^{c} \tag{88}$$

$$\left[H_{\mu}{}^{a}, K_{\nu}{}^{bc}\right] = -\eta_{\mu\nu}K^{abc} \tag{89}$$

$$[H_{\mu}{}^{a}, K_{bcd}] = \left(\delta_b{}^{a}K_{\mu cd} + \delta_d{}^{a}K_{\mu bc} + \delta_c{}^{a}K_{\mu db}\right) \tag{90}$$

$$\left[H_{\mu}{}^{a}, K_{bc}{}^{d}\right] = \left(\delta_b{}^{a}K_{\mu c}{}^{d} - \delta_c{}^{a}K_{\mu b}{}^{d}\right) \tag{91}$$

$$\left[H_{\mu}{}^{a}, K_b{}^{cd}\right] = \delta_b{}^{a}K_{\mu}{}^{cd} \tag{92}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,17}$ generators $J_{\mu\nu}$, $J_a{}^{b}$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or $H_{\mu}{}^{a}$, with either of the operators $K_{\mu\nu\lambda}$, $K_{\mu\nu a}$, $\cdots$, are satisfied. Moreover any of the $O_{1,17}$ generators can be shown to commute with the following quadratic operator:

$$\left\{\begin{array}{l} \frac{1}{3!}K_{\mu\nu\lambda}K_{\mu\nu\lambda} \\ +\frac{1}{2}K_{\mu\nu a}K_{\mu\nu}{}^{a} + \frac{1}{2}K_{\mu\nu}{}^{a}K_{\mu\nu a} \\ +\frac{1}{2}K_{\mu ab}K_{\mu}{}^{ab} + \frac{1}{2}K_{\mu}{}^{ab}K_{\mu ab} - K_{\mu a}{}^{b}K_{\mu b}{}^{a} \\ +\frac{1}{3!}K_{abc}K^{abc} + \frac{1}{3!}K^{abc}K_{abc} + \frac{1}{2}K_{ab}{}^{c}K_c{}^{ab} + \frac{1}{2}K_a{}^{bc}K_{bc}{}^{a} \end{array}\right. \tag{93}$$

We proceed now to the introduction of the multiplet that can be associated with the above operator representation, and to the construction of the infinitesimal $O_{1,17}$ transformations that act on it. The desired multiplet with components $\{B_{\mu\nu\lambda}, B_{\mu\nu a}, \cdots\}$ can be introduced by the tensor module

$$\mathcal{B} = \left\{ \begin{array}{l} \frac{1}{3!}B_{\mu\nu\lambda}K_{\mu\nu\lambda} + \frac{1}{2}B_{\mu\nu a}K_{\mu\nu}{}^{a} + \frac{1}{2}B_{\mu\nu}{}^{a}K_{\mu\nu a} \\ +\frac{1}{2}B_{\mu ab}K_{\mu}{}^{ab} + B_{\mu a}{}^{b}K_{\mu b}{}^{a} + \frac{1}{2}B_{\mu}{}^{ab}K_{\mu ab} \\ +\frac{1}{3!}B_{abc}K^{abc} + \frac{1}{2}B_{ab}{}^{c}K_{c}{}^{ab} + \frac{1}{2}B_{a}{}^{bc}K_{bc}{}^{a} + \frac{1}{3!}B^{abc}K_{abc} \end{array} \right\} \tag{94}$$

Introducing the $O_{1,17}$ parameter module,

$$\mathcal{W} = \frac{1}{2}\Omega_{\mu\nu}J_{\mu\nu} + \Omega_{a}{}^{b}J_{b}{}^{a} + \frac{1}{2}\Omega_{ab}Q^{ab} + \frac{1}{2}\Omega^{ab}Q_{ab} + \Omega_{\mu a}H_{\mu}{}^{a} + \Omega_{\mu}{}^{a}H_{\mu a} \tag{95}$$

we can compute the commutator $[\mathcal{W}, \mathcal{B}]$. The latter gives a tensor module whose components would define the needed infinitesimal transformations.

For $B_{\mu\nu\lambda}$, we obtain

$$\delta B_{\mu\nu\lambda} = \left\{ \begin{array}{l} \Omega_{\lambda\rho}B_{\mu\nu\rho} - \Omega_{\mu\rho}B_{\lambda\nu\rho} + \Omega_{\nu\rho}B_{\lambda\mu\rho} \\ +\Omega_{\nu a}B_{\lambda\mu}{}^{a} - \Omega_{\mu a}B_{\lambda\nu}{}^{a} + \Omega_{\lambda a}B_{\mu\nu}{}^{a} \\ +\Omega_{\lambda}{}^{a}B_{\mu\nu a} - \Omega_{\mu}{}^{a}B_{\lambda\nu a} + \Omega_{\nu}{}^{a}B_{\lambda\mu a} \end{array} \right\} \tag{96}$$

For $B_{\mu\nu a}$, we obtain

$$\delta B_{\mu\nu a} = \left\{ \begin{array}{l} -\Omega_{\lambda a}B_{\lambda\mu\nu} - \Omega_{\lambda\mu}B_{\lambda\nu a} + \Omega_{\lambda\nu}B_{\lambda\mu a} \\ +\Omega_{\nu b}B_{\mu a}{}^{b} + \Omega_{ab}B_{\mu\nu}{}^{b} - \Omega_{\mu b}B_{\nu a}{}^{b} \\ -\Omega_{a}{}^{b}B_{\mu\nu b} + \Omega_{\mu}{}^{b}B_{\nu ab} - \Omega_{\nu}{}^{b}B_{\mu ab} \end{array} \right\} \tag{97}$$

For $B_{\mu\nu}{}^{a}$, we obtain

$$\delta B_{\mu\nu}{}^{a} = \left\{ \begin{array}{l} \Omega^{ab}B_{\mu\nu b} - \Omega_{\nu b}B_{\mu}{}^{ab} + \Omega_{\mu b}B_{\nu}{}^{ab} \\ +\Omega_{\lambda\nu}B_{\lambda\mu}{}^{a} - \Omega_{\lambda\mu}B_{\lambda\nu}{}^{a} + \Omega_{b}{}^{a}B_{\mu\nu}{}^{b} \\ -\Omega_{\lambda}{}^{a}B_{\lambda\mu\nu} + \Omega_{\mu}{}^{b}B_{\nu b}{}^{a} - \Omega_{\nu}{}^{b}B_{\mu b}{}^{a} \end{array} \right\} \tag{98}$$

For $B_{\mu ab}$, we obtain

$$\delta B_{\mu ab} = \left\{ \begin{array}{l} \Omega_{\mu\nu}B_{\nu ab} - \Omega_{\nu a}B_{\mu\nu b} + \Omega_{\nu b}B_{\mu\nu a} \\ +\Omega_{\mu c}B_{ab}{}^{c} - \Omega_{bc}B_{\mu a}{}^{c} + \Omega_{ac}B_{\mu b}{}^{c} \\ +\Omega_{a}{}^{c}B_{\mu bc} - \Omega_{b}{}^{c}B_{\mu ac} + \Omega_{\mu}{}^{c}B_{abc} \end{array} \right\} \tag{99}$$

For ${B_{\mu a}}^b$, we obtain

$$\delta {B_{\mu a}}^b = \left\{ \begin{array}{l} -\Omega^{bc} B_{\mu ac} - \Omega_{\mu c} {B_a}^{bc} + \Omega_{ac} {B_\mu}^{bc} \\ +\Omega_{\nu a} {B_{\mu\nu}}^b + \Omega_{\mu\nu} {B_{\nu a}}^b - {\Omega_a}^c {B_{\mu c}}^b \\ +{\Omega_c}^b {B_{\mu a}}^c + {\Omega_\mu}^c {B_{ac}}^b - {\Omega_\nu}^b B_{\mu\nu a} \end{array} \right\} \tag{100}$$

For ${B_\mu}^{ab}$, we obtain

$$\delta {B_\mu}^{ab} = \left\{ \begin{array}{l} \Omega_{\mu c} B^{abc} + \Omega_{\mu\nu} {B_\nu}^{ab} + \Omega^{bc} B^a_{\mu c} \\ -\Omega^{ac} {B_{\mu c}}^b - {\Omega_c}^a {B_\mu}^{bc} + {\Omega_c}^b {B_\mu}^{ac} \\ {\Omega_\mu}^c {B_c}^{ab} - {\Omega_\nu}^a {B_{\mu\nu}}^b + {\Omega_\nu}^b {B_{\mu\nu}}^a \end{array} \right\} \tag{101}$$

For B_{abc}, we obtain

$$\delta B_{abc} = \left\{ \begin{array}{l} -\Omega_{\mu a} B_{\mu bc} + \Omega_{\mu b} B_{\mu ac} - \Omega_{\mu c} B_{\mu ab} \\ +\Omega_{cd} {B_{ab}}^d - \Omega_{bd} {B_{ac}}^d + \Omega_{ad} {B_{bc}}^d \\ -{\Omega_a}^d B_{bcd} + {\Omega_b}^d B_{acd} - {\Omega_c}^d B_{abd} \end{array} \right\} \tag{102}$$

For ${B_{ab}}^c$, we obtain

$$\delta {B_{ab}}^c = \left\{ \begin{array}{l} \Omega^{cd} B_{abd} - \Omega_{bd} {B_a}^{cd} + \Omega_{ad} {B_b}^{cd} \\ -\Omega_{\mu b} {B_{\mu a}}^c + \Omega_{\mu a} {B_{\mu b}}^c + {\Omega_a}^d {B_{bd}}^c \\ -{\Omega_b}^d {B_{ad}}^c + {\Omega_d}^c {B_{ab}}^d - {\Omega_\mu}^c B_{\mu ab} \end{array} \right\} \tag{103}$$

For ${B_a}^{bc}$, we obtain

$$\delta {B_a}^{bc} = \left\{ \begin{array}{l} \Omega_{ad} B^{bcd} - \Omega_{\mu a} {B_\mu}^{bc} - \Omega^{cd} B^c_{ad} \\ +\Omega^{bd} {B_{ad}}^c - {\Omega_a}^d {B_d}^{bc} - {\Omega_d}^b {B_a}^{cd} \\ +{\Omega_d}^c {B_a}^{bd} - {\Omega_\mu}^b {B_{\mu a}}^c + {\Omega_\mu}^c {B_{\mu a}}^b \end{array} \right\} \tag{104}$$

Finally, for B^{abc}, we obtain

$$\delta B^{abc} = \left\{ \begin{array}{l} \Omega^{cd} {B_d}^{ab} - \Omega^{bd} {B_d}^{ac} + \Omega^{ad} {B_d}^{bc} \\ +{\Omega_d}^a B^{bcd} - {\Omega_d}^b B^{acd} + {\Omega_d}^c B^{abd} \\ -{\Omega_\mu}^a {B_\mu}^{bc} + {\Omega_\mu}^b {B_\mu}^{ac} - {\Omega_\mu}^c {B_\mu}^{ab} \end{array} \right\} \tag{105}$$

We can verify that, for any two tensor modules $\mathcal{A}$ and $\mathcal{B}$, the above infinitesimal transformations, acting in a like manner on the components of both, would leave invariant the following bilinear form:

$$\mathcal{A}\cdot\mathcal{B} = \left\{ \begin{array}{l} \frac{1}{3!}A_{\mu\nu\lambda}B_{\mu\nu\lambda} + \frac{1}{2}A_{\mu\nu a}B_{\mu\nu}{}^{a} + \frac{1}{2}A_{\mu\nu}{}^{a}B_{\mu\nu a} \\ \\ +\frac{1}{2}A_{\mu ab}B_{\mu}{}^{ab} - A_{\mu a}{}^{b}B_{\mu b}{}^{a} + \frac{1}{2}A_{\mu}{}^{ab}B_{\mu ab} \\ \\ +\frac{1}{3!}A_{abc}B^{abc} + \frac{1}{2}A_{ab}{}^{c}B_{c}{}^{ab} + \frac{1}{2}A_{a}{}^{bc}B_{bc}{}^{a} + \frac{1}{3!}A^{abc}B_{abc} \end{array} \right\} \qquad (106)$$

5 The Dirac-Weyl-Majorana Spinorial Representation of $O_{1,17}$

A Dirac spinor in 18-dimensional spacetime has $2^9 = 512$ components. A Weyl (chiral) constraint would reduce this to 256 components. A Majorana constraint is a reality condition that would relate these components to the components of the Dirac conjugate. In order to construct a corresponding multiplet of Weyl spinors in 4-dimensional spacetime with components that are described by U_7 tensors, we introduce the following set of operators:

$$\{R, L_a, R_{ab}, L_{abc}, R^{abc}, L^{ab}, R^{a}, L\} \qquad (107)$$

The above entities are alternately right-handed and left-handed Weyl spinors in 4-dimensional spacetime. The indices $(a, b, \cdots)$ pertain to SU_7, with the multi-index objects totally antisymmetric. We now write the commutators of the the above operators with the generators of the $O_{1,17}$ algebra.

For the commutators with the 4-dimensional Lorentz generators, we have

$$[J_{\mu\nu}, R] = -\frac{1}{2}\gamma_{\mu\nu}R \qquad (108)$$

In the above, $\gamma_{\mu\nu}$ is a member of the Dirac algebra, being equal to $\frac{1}{2}[\gamma_\mu, \gamma_\nu]$ in terms of the Dirac matrix operators which satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. The remaining operators $\{L_a, R_{ab}, \cdots\}$ all have similar commutators with $J_{\mu\nu}$, all being Lorentzian spinors.

For the commutators with $J_a{}^b$, the generators of U_7, we have

$$\left[J_a{}^b, R\right] = -\frac{1}{2}\delta_a{}^b R \qquad (109)$$

$$\left[J_a{}^b, L_c\right] = \delta_c{}^b L_a - \frac{1}{2}\delta_a{}^b L_c \qquad (110)$$

$$\left[J_a{}^b, R_{cd}\right] = \left(\delta_c{}^b R_{ad} - \delta_d{}^b R_{ac}\right) - \frac{1}{2}\delta_a{}^b R_{cd} \qquad (111)$$

$$\left[J_a{}^b, L_{cde}\right] = \left(\delta_c{}^b L_{ade} - \delta_d{}^b L_{ace} + \delta_e{}^b L_{acd}\right) - \frac{1}{2}\delta_a{}^b L_{cde} \qquad (112)$$

$$\left[J_a{}^b, R^{cde}\right] = -\left(\delta_a{}^c R^{bde} - \delta_a{}^d R^{bce} + \delta_a{}^e R^{bcd}\right) + \frac{1}{2}\delta_a{}^b R^{cde} \qquad (113)$$

$$\left[J_a{}^b, L^{cd}\right] = -\left(\delta_a{}^c L^{bd} - \delta_a{}^d L^{bc}\right) + \frac{1}{2}\delta_a{}^b L^{cd} \tag{114}$$

$$\left[J_a{}^b, R^c\right] = -\delta_a{}^c R^b + \frac{1}{2}\delta_a{}^b R^c \tag{115}$$

$$\left[J_a{}^b, L\right] = \frac{1}{2}\delta_a{}^b L \tag{116}$$

For the commutators with Q_{ab}, we have

$$[Q_{ab}, R] = R_{ab} \tag{117}$$

$$[Q_{ab}, L_c] = L_{abc} \tag{118}$$

$$[Q_{ab}, R_{cd}] = \frac{1}{3!}\epsilon_{abcdefg} R^{efg} \tag{119}$$

$$[Q_{ab}, L_{cde}] = \frac{1}{2}\epsilon_{abcdefg} L^{fg} \tag{120}$$

$$\left[Q_{ab}, R^{cde}\right] = \left(\delta_a{}^c\delta_b{}^d R^e + \delta_a{}^e\delta_b{}^c R^d + \delta_a{}^d\delta_b{}^e R^c\right) - (a \leftrightarrow b) \tag{121}$$

$$\left[Q_{ab}, L^{cd}\right] = \left(\delta_a{}^c\delta_b^d - \delta_b{}^c\delta_a{}^d\right) L \tag{122}$$

The commutators of Q_{ab} with R_c and L are vanishing.

For the commutators with Q^{ab}, they are vanishing with R and L_c, and we have

$$\left[Q^{ab}, R_{cd}\right] = -\left(\delta_c{}^a\delta_d{}^b - \delta_c{}^b\delta_d{}^a\right) R \tag{123}$$

$$\left[Q^{ab}, L_{cde}\right] = -\left(\delta_c{}^a\delta_d{}^b L_e + \delta_e{}^a\delta_c{}^b L_d + \delta_d{}^a\delta_e{}^b L_c\right) - (a \leftrightarrow b) \tag{124}$$

$$\left[Q^{ab}, R^{cde}\right] = -\frac{1}{2}\epsilon^{abcdefg} R_{fg} \tag{125}$$

$$\left[Q^{ab}, L^{cd}\right] = -\frac{1}{3!}\epsilon^{abcdefg} L_{efg} \tag{126}$$

$$\left[Q^{ab}, R^c\right] = -R^{abc} \tag{127}$$

$$\left[Q^{ab}, L\right] = -L^{ab} \tag{128}$$

For the commutators with $H_{\mu a}$, we have

$$[H_{\mu a}, R] = -\frac{1}{\sqrt{2}}\gamma_\mu L_a \tag{129}$$

$$[H_{\mu a}, L_b] = \frac{1}{\sqrt{2}}\gamma_\mu R_{ab} \tag{130}$$

$$[H_{\mu a}, R_{bc}] = -\frac{1}{\sqrt{2}}\gamma_\mu L_{abc} \tag{131}$$

$$[H_{\mu a}, L_{bcd}] = \frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon_{abcdefg}\gamma_\mu R^{efg} \tag{132}$$

$$\left[H_{\mu a}, R^{bcd}\right] = -\frac{1}{\sqrt{2}}\left(\delta_a{}^b\gamma_\mu L^{cd} - \delta_a{}^c\gamma_\mu L^{bd} + \delta_a{}^d\gamma_\mu L^{bc}\right) \tag{133}$$

$$\left[H_{\mu a}, L^{bc}\right] = -\frac{1}{\sqrt{2}}\left(\delta_a{}^b\gamma_\mu R^c - \delta_a{}^c\gamma_\mu R^b\right) \tag{134}$$

$$\left[H_{\mu a}, R^b\right] = -\frac{1}{\sqrt{2}}\delta_a{}^c\gamma_\mu L \tag{135}$$

The commutator of $H_{\mu a}$ with L is vanishing.

For the commutators of $H_\mu{}^a$, vanishing with R, we have

$$[H_\mu{}^a, L_b] = \frac{1}{\sqrt{2}}\delta_b{}^a\gamma_\mu R \tag{136}$$

$$[H_\mu{}^a, R_{bc}] = -\frac{1}{\sqrt{2}}\left(\delta_b{}^a\gamma_\mu L_c - \delta_c{}^a\gamma_\mu L_b\right) \tag{137}$$

$$[H_\mu{}^a, L_{bcd}] = \frac{1}{\sqrt{2}}\left(\delta_b{}^a\gamma_\mu R_{cd} - \delta_c{}^a\gamma_\mu R_{bd} + \delta_d{}^a\gamma_\mu R_{bc}\right) \tag{138}$$

$$\left[H_\mu{}^a, R^{bcd}\right] = \frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon^{abcdefg}\gamma_\mu L_{efg} \tag{139}$$

$$\left[H_\mu{}^a, L^{bc}\right] = \frac{1}{\sqrt{2}}\gamma_\mu R^{abc} \tag{140}$$

$$\left[H_\mu{}^a, R^b\right] = \frac{1}{\sqrt{2}}\gamma_\mu L^{ab} \tag{141}$$

$$[H_\mu{}^a, L] = \frac{1}{\sqrt{2}}\gamma_\mu R^a \tag{142}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,17}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or $H_\mu{}^a$, with either of the operators R, L_a, R_{ab}, L_{abc}, R^{abc}, L^{ab}, R^a, or L, are satisfied.

6 The Dirac Conjugate Spinorial Representation of $O_{1,17}$

In order to be able to write Lagrangian terms for fermionic fields we must introduce the conjugate spinorial representation. This can be done with the operator set

$$\{\bar{L}, \bar{R}_a, \bar{L}_{ab}, \bar{R}_{abc}, \bar{L}^{abc}, \bar{R}^{ab}, \bar{L}^a, \bar{R}\} \tag{143}$$

All these are Dirac conjugate spinors. We now write the commutators of all the $O_{1,17}$ generators with the elements of the above set.

For the commutators with the Lorentz generators $J_{\mu\nu}$, we have

$$\left[J_{\mu\nu}, \bar{L}\right] = \frac{1}{2}\bar{L}\gamma_{\mu\nu} \tag{144}$$

and similar commutators for the remaining elements, $\bar{R}_a$, $\bar{L}_{ab}$, etc., being all Dirac conjugate spinors.

For the commutators with the U_7 generators ${J_a}^b$, we have

$$\left[{J_a}^b, \bar{L}\right] = -\frac{1}{2}{\delta_a}^b \bar{L} \tag{145}$$

$$\left[{J_a}^b, \bar{R}_c\right] = {\delta_c}^b \bar{R}_a - \frac{1}{2}{\delta_a}^b \bar{R}_c \tag{146}$$

$$\left[{J_a}^b, \bar{L}_{cd}\right] = \left({\delta_c}^b \bar{L}_{ad} - {\delta_d}^b \bar{L}_{ac}\right) - \frac{1}{2}{\delta_a}^b \bar{L}_{cd} \tag{147}$$

$$\left[{J_a}^b, \bar{R}_{cde}\right] = \left({\delta_c}^b \bar{R}_{ade} - {\delta_d}^b \bar{R}_{ace} + {\delta_e}^b \bar{R}_{acd}\right) - \frac{1}{2}{\delta_a}^b \bar{R}_{cde} \tag{148}$$

$$\left[{J_a}^b, \bar{L}^{cde}\right] = -\left({\delta_a}^c \bar{L}^{bde} - {\delta_a}^d \bar{L}^{bce} + {\delta_a}^e \bar{L}^{bcd}\right) + \frac{1}{2}{\delta_a}^b \bar{L}^{cde} \tag{149}$$

$$\left[{J_a}^b, \bar{R}^{cd}\right] = -\left({\delta_a}^c \bar{R}^{bd} - {\delta_a}^d \bar{R}^{bc}\right) + \frac{1}{2}{\delta_a}^b \bar{R}^{cd} \tag{150}$$

$$\left[{J_a}^b, \bar{L}^c\right] = -{\delta_a}^c \bar{L}^b + \frac{1}{2}{\delta_a}^b \bar{L}^c \tag{151}$$

$$\left[{J_a}^b, \bar{R}\right] = \frac{1}{2}{\delta_a}^b \bar{R} \tag{152}$$

For the commutators with Q_{ab}, we have

$$\left[Q_{ab}, \bar{L}\right] = \bar{L}_{ab} \tag{153}$$

$$\left[Q_{ab}, \bar{R}_c\right] = \bar{R}_{abc} \tag{154}$$

$$\left[Q_{ab}, \bar{L}_{cd}\right] = \frac{1}{3!}\epsilon_{abcdefg} \bar{L}^{efg} \tag{155}$$

$$\left[Q_{ab}, \bar{R}_{cde}\right] = \frac{1}{2}\epsilon_{abcdefg} \bar{R}^{fg} \tag{156}$$

$$\left[Q_{ab}, \bar{L}^{cde}\right] = \left({\delta_a}^c {\delta_b}^d \bar{L}^e + {\delta_a}^e {\delta_b}^c \bar{L}^d + {\delta_a}^d {\delta_b}^e \bar{L}^c\right) - (a \leftrightarrow b) \tag{157}$$

$$\left[Q_{ab}, \bar{R}^{cd}\right] = \left({\delta_a}^c {\delta_b}^d - {\delta_b}^c {\delta_a}^d\right) \bar{R} \tag{158}$$

The commutators of Q_{ab} with $\bar{L}^a$ and $\bar{R}$ are vanishing.

For the commutators of Q^{ab}, vanishing with $\bar{L}$ and $\bar{R}_a$, we have

$$\left[Q^{ab}, \bar{L}_{cd}\right] = -\left({\delta_c}^a {\delta_d}^b - {\delta_c}^b {\delta_d}^a\right) \bar{L} \tag{159}$$

$$\left[Q^{ab}, \bar{R}_{cde}\right] = -\left({\delta_c}^a {\delta_d}^b \bar{R}_e + {\delta_e}^a {\delta_c}^b \bar{R}_d + {\delta_d}^a {\delta_e}^b \bar{R}_c\right) - (a \leftrightarrow b) \tag{160}$$

$$\left[Q^{ab}, \bar{L}^{cde}\right] = -\frac{1}{2}\epsilon^{abcdefg} \bar{L}_{fg} \tag{161}$$

$$\left[Q^{ab}, \bar{R}^{cd}\right] = -\frac{1}{3!}\epsilon^{abcdefg} \bar{R}_{efg} \tag{162}$$

$$\left[Q^{ab}, \bar{L}^{c}\right] = -\bar{L}^{abc} \tag{163}$$

$$\left[Q^{ab}, \bar{R}\right] = -\bar{R}^{ab} \tag{164}$$

For the commutators of $H_{\mu a}$, we have

$$\left[H_{\mu a}, \bar{L}\right] = -\frac{1}{\sqrt{2}}\bar{R}_{a}\gamma_{\mu} \tag{165}$$

$$\left[H_{\mu a}, \bar{R}_{b}\right] = \frac{1}{\sqrt{2}}\bar{L}_{ab}\gamma_{\mu} \tag{166}$$

$$\left[H_{\mu a}, \bar{L}_{bc}\right] = -\frac{1}{\sqrt{2}}\bar{R}_{abc}\gamma_{\mu} \tag{167}$$

$$\left[H_{\mu a}, \bar{R}_{bcd}\right] = \frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon_{abcdefg}\bar{L}^{efg}\gamma_{\mu} \tag{168}$$

$$\left[H_{\mu a}, \bar{L}^{bcd}\right] = -\frac{1}{\sqrt{2}}\left(\delta_{a}{}^{b}\bar{R}^{cd}\gamma_{\mu} - \delta_{a}{}^{c}\bar{R}^{bd}\gamma_{\mu} + \delta_{a}{}^{d}\bar{R}^{bc}\gamma_{\mu}\right) \tag{169}$$

$$\left[H_{\mu a}, \bar{R}^{bc}\right] = -\frac{1}{\sqrt{2}}\left(\delta_{a}{}^{b}\bar{L}^{c}\gamma_{\mu} - \delta_{a}{}^{c}\bar{L}^{b}\gamma_{\mu}\right) \tag{170}$$

$$\left[H_{\mu a}, \bar{L}^{b}\right] = -\frac{1}{\sqrt{2}}\delta_{a}{}^{b}\bar{R}\gamma_{\mu} \tag{171}$$

The commutator of $H_{\mu a}$ with $\bar{R}$ is vanishing.
For the commutators of $H_{\mu}{}^{a}$, vanishing with $\bar{L}$, we have

$$\left[H_{\mu}{}^{a}, \bar{R}_{b}\right] = \frac{1}{\sqrt{2}}\delta_{b}{}^{a}\bar{L}\gamma_{\mu} \tag{172}$$

$$\left[H_{\mu}{}^{a}, \bar{L}_{bc}\right] = -\frac{1}{\sqrt{2}}\left(\delta_{b}{}^{a}\bar{R}_{c}\gamma_{\mu} - \delta_{c}{}^{a}\bar{R}_{b}\gamma_{\mu}\right) \tag{173}$$

$$\left[H_{\mu}{}^{a}, \bar{R}_{bcd}\right] = \frac{1}{\sqrt{2}}\left(\delta_{b}{}^{a}\bar{L}_{cd}\gamma_{\mu} - \delta_{c}{}^{a}\bar{L}_{bd}\gamma_{\mu} + \delta_{d}{}^{a}\bar{L}_{bc}\gamma_{\mu}\right) \tag{174}$$

$$\left[H_{\mu}{}^{a}, \bar{L}^{bcd}\right] = \frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon^{abcdefg}\bar{R}_{efg}\gamma_{\mu} \tag{175}$$

$$\left[H_{\mu}{}^{a}, \bar{R}^{bc}\right] = \frac{1}{\sqrt{2}}\bar{L}^{abc}\gamma_{\mu} \tag{176}$$

$$\left[H_{\mu}{}^{a}, \bar{L}^{b}\right] = \frac{1}{\sqrt{2}}\bar{R}^{ab}\gamma_{\mu} \tag{177}$$

$$\left[H_{\mu}{}^{a}, \bar{R}\right] = \frac{1}{\sqrt{2}}\bar{L}^{a}\gamma_{\mu} \tag{178}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,17}$ generators $J_{\mu\nu}$, $J_{a}{}^{b}$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or $H_{\mu}{}^{a}$, with either of the operators $\bar{L}$, $\bar{R}_{a}$, $\bar{L}_{ab}$, $\bar{R}_{abc}$, $\bar{L}^{abc}$, $\bar{R}^{ab}$, $\bar{L}^{a}$, or $\bar{R}$, are satisfied.

Having constructed the Lorentz covariant as well as the U_7 covariant algebraic representations for a fundamental $O_{1,17}$ spinor and its Dirac conjugate, we can verify that all generators $J_{\mu\nu}$, ${J_a}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or ${H_\mu}^a$, of $O_{1,17}$ do commute with the following quadratic operator:

$$\bar{L}L - \bar{R}_a\bar{R}^a - \frac{1}{2}\bar{L}_{ab}L^{ab} + \frac{1}{3!}\bar{R}_{abc}R^{abc} + \frac{1}{3!}\bar{L}^{abc}L_{abc} - \frac{1}{2}\bar{R}^{ab}R_{ab} - \bar{L}^aL_a + \bar{R}R \tag{179}$$

We now proceed to construct the spinorial multiplet modules and the infinitesimal transformations of their components.

7 Fundamental Spinorial Multiplet

We introduce an SU_7 covariant multiplet of Weyl spinors of 4 dimensional spacetime, via the following module:

$$\Psi = \left\{ \begin{array}{l} \bar{R}\xi + \bar{L}^a\chi_a + \frac{1}{2}\bar{R}^{ab}\xi_{ab} + \frac{1}{3!}\bar{L}^{abc}\chi_{abc} \\ \\ +\frac{1}{3!}\bar{R}_{abc}\xi^{abc} + \frac{1}{2}\bar{L}_{ab}\chi^{ab} + \bar{R}_a\xi^a + \bar{L}\chi \end{array} \right\} \tag{180}$$

Notice that the ξ's are left-handed Weyl spinors, while the χ's are righthanded. We also introduce the Dirac conjugate module:

$$\bar{\Psi} = \left\{ \begin{array}{l} \bar{\xi}R + \bar{\chi}^aL_a + \frac{1}{2}\bar{\xi}^{ab}R_{ab} + \frac{1}{3!}\bar{\chi}^{abc}L_{abc} \\ \\ +\frac{1}{3!}\bar{\xi}_{abc}R^{abc} + \frac{1}{2}\bar{\chi}_{ab}L^{ab} + \bar{\xi}_aR^a + \bar{\chi}L \end{array} \right\} \tag{181}$$

Now, with the $O_{1,17}$ parameter module,

$$\mathcal{W} = \frac{1}{2}\Omega_{\mu\nu}J_{\mu\nu} + {\Omega_a}^b{J_b}^a + \frac{1}{2}\Omega_{ab}Q^{ab} + \frac{1}{2}\Omega^{ab}Q_{ab} + \Omega_{\mu a}{H_\mu}^a + {\Omega_\mu}^aH_{\mu a} \tag{182}$$

we can compute the commutators $[\mathcal{W},\Psi]$ and $[\mathcal{W},\bar{\Psi}]$. These give corresponding spinorial modules whose components would define the $O_{1,17}$ infinitesimal transformations.

For the infinitesimal transformations of the Ψ components, we have

$$\delta\xi = \frac{1}{2}\Omega^{ab}\xi_{ab} + \frac{1}{2}{\Omega_a}^a\xi - \frac{1}{\sqrt{2}}{\Omega_\mu}^a\gamma_\mu\chi_a + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi \tag{183}$$

$$\delta\chi_a = \frac{1}{2}\Omega^{bc}\chi_{abc} - {\Omega_a}^b\chi_b + \frac{1}{2}{\Omega_b}^b\chi_a + \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\xi + \frac{1}{\sqrt{2}}{\Omega_\mu}^b\gamma_\mu\xi_{ab} + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\chi_a \tag{184}$$

$$\delta\xi_{ab} = \left\{ \begin{array}{l} -\Omega_{ab}\xi + \frac{1}{12}\epsilon_{abcdefg}\Omega^{cd}\xi^{efg} + {\Omega_a}^c\xi_{bc} - {\Omega_b}^c\xi_{ac} - \frac{1}{2}{\Omega_c}^c\xi_{ab} \\ \\ -\frac{1}{\sqrt{2}}\Omega_{\mu b}\gamma_\mu\chi_a + \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\chi_b - \frac{1}{\sqrt{2}}{\Omega_\mu}^c\gamma_\mu\chi_{abc} + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi_{ab} \end{array} \right\} \tag{185}$$

$$\delta\chi_{abc} = \left\{ \begin{array}{l} -\Omega_{ab}\chi_c + \Omega_{ac}\chi_b - \Omega_{bc}\chi_a + \frac{1}{4}\epsilon_{abcdefg}\Omega^{de}\chi^{fg} \\ -\Omega_a{}^d\chi_{bcd} + \Omega_b{}^d\chi_{acd} - \Omega_c{}^d\chi_{abd} + \frac{1}{2}\Omega_d{}^d\chi_{abc} \\ +\frac{1}{\sqrt{2}}\Omega_{\mu c}\gamma_\mu\xi_{ab} - \frac{1}{\sqrt{2}}\Omega_{\mu b}\gamma_\mu\xi_{ac} + \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\xi_{bc} \\ \frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon_{abcdefg}\Omega_\mu{}^d\gamma_\mu\xi^{efg} + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\chi_{abc} \end{array} \right\} \tag{186}$$

$$\delta\xi^{abc} = \left\{ \begin{array}{l} -\frac{1}{4}\epsilon^{abcdefg}\Omega_{de}\xi_{fg} + \Omega^{ab}\xi^c - \Omega^{ac}\xi^b + \Omega^{bc}\xi^a \\ +\Omega_d{}^a\xi^{bcd} - \Omega_d{}^b\xi^{acd} + \Omega_d{}^c\xi^{abd} - \frac{1}{2}\Omega_d{}^d\xi^{abc} \\ -\frac{1}{\sqrt{2}}\Omega_\mu{}^c\gamma_\mu\chi^{ab} + \frac{1}{\sqrt{2}}\Omega_\mu{}^b\gamma_\mu\chi^{ac} - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\chi^{bc} \\ +\frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon^{abcdefg}\Omega_{\mu d}\gamma_\mu\chi_{efg} + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi^{abc} \end{array} \right\} \tag{187}$$

$$\delta\chi^{ab} = \left\{ \begin{array}{l} -\frac{1}{12}\epsilon^{abcdefg}\Omega_{cd}\chi_{efg} + \Omega^{ab}\chi - \Omega_c{}^a\chi^{bc} + \Omega_c{}^b\chi^{ac} - \frac{1}{2}\Omega_c{}^c\chi^{ab} \\ -\frac{1}{\sqrt{2}}\Omega_\mu{}^b\gamma_\mu\xi^a + \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\xi^b + \frac{1}{\sqrt{2}}\Omega_{\mu c}\gamma_\mu\xi^{abc} + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\chi^{ab} \end{array} \right\} \tag{188}$$

$$\delta\xi^a = -\frac{1}{2}\Omega_{bc}\xi^{abc} + \Omega_b{}^a\xi^b - \frac{1}{2}\Omega_b{}^b\xi^a - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\chi + \frac{1}{\sqrt{2}}\Omega_{\mu b}\gamma_\mu\chi^{ab} + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi^a \tag{189}$$

$$\delta\chi = -\frac{1}{2}\Omega_{ab}\chi^{ab} - \frac{1}{2}\Omega_a{}^a\chi + \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\xi^a + \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\chi \tag{190}$$

For the infinitesimal transformations of the $\bar{\Psi}$ components, we have

$$\delta\bar{\xi} = -\frac{1}{2}\Omega_{ab}\bar{\xi}^{ab} - \frac{1}{2}\Omega_a{}^a\bar{\xi} - \frac{1}{4}\Omega_{\mu\nu}\bar{\xi}\gamma_{\mu\nu} + \frac{1}{\sqrt{2}}\Omega_{\mu a}\bar{\chi}^a\gamma_\mu \tag{191}$$

$$\delta\bar{\chi}^a = -\frac{1}{2}\Omega_{bc}\bar{\chi}^{abc} + \Omega_b{}^a\bar{\chi}^b - \frac{1}{2}\Omega_b{}^b\bar{\chi}^a - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\bar{\xi}\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_{\mu b}\bar{\xi}^{ab}\gamma_\mu - \frac{1}{4}\Omega_{\mu\nu}\bar{\chi}^a\gamma_{\mu\nu} \tag{192}$$

$$\delta\bar{\xi}^{ab} = \left\{ \begin{array}{l} -\frac{1}{12}\epsilon^{abcdefg}\Omega_{cd}\bar{\xi}_{efg} + \Omega^{ab}\bar{\xi} - \Omega_c{}^a\bar{\xi}^{bc} + \Omega_c{}^b\bar{\xi}^{ac} - \frac{1}{2}\Omega_c{}^c\bar{\xi}^{ab} \\ -\frac{1}{4}\Omega_{\mu\nu}\bar{\xi}^{ab}\gamma_{\mu\nu} - \frac{1}{\sqrt{2}}\Omega_\mu{}^b\bar{\chi}^a\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_\mu{}^a\bar{\chi}^b\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_{\mu c}\bar{\chi}^{abc}\gamma_\mu \end{array} \right\} \tag{193}$$

$$\delta\bar{\chi}^{abc} = \left\{ \begin{array}{l} -\frac{1}{4}\epsilon^{abcdefg}\Omega_{de}\bar{\chi}_{fg} + \Omega^{bc}\bar{\chi}^a - \Omega^{ac}\bar{\chi}^b + \Omega^{ab}\bar{\chi}^c \\ +\Omega_d{}^a\bar{\chi}^{bcd} - \Omega_d{}^b\bar{\chi}^{acd} + \Omega_d{}^c\bar{\chi}^{abd} - \frac{1}{2}\Omega_d{}^d\bar{\chi}^{abc} + \frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon^{abcdefg}\Omega_{\mu d}\bar{\xi}_{efg}\gamma_\mu \\ -\frac{1}{\sqrt{2}}\Omega_\mu{}^c\bar{\xi}^{ab}\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_\mu{}^b\bar{\xi}^{ac}\gamma_\mu - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\bar{\xi}^{bc}\gamma_\mu - \frac{1}{4}\Omega_{\mu\nu}\bar{\chi}^{abc}\gamma_{\mu\nu} \end{array} \right\} \tag{194}$$

$$\delta\bar{\xi}_{abc} = \left\{ \begin{array}{l} -\Omega_{bc}\bar{\xi}_a + \Omega_{ac}\bar{\xi}_b - \Omega_{ab}\bar{\xi}_c + \frac{1}{4}\epsilon_{abcdefg}\Omega^{de}\bar{\xi}^{fg} \\ -\Omega_a{}^d\bar{\xi}_{bcd} + \Omega_b{}^d\bar{\xi}_{acd} - \Omega_c{}^d\bar{\xi}_{abd} + \frac{1}{2}\Omega_d{}^d\bar{\xi}_{abc} + \frac{1}{\sqrt{2}}\frac{1}{3!}\epsilon_{abcdefg}\Omega_\mu{}^d\bar{\chi}^{efg}\gamma_\mu \\ +\frac{1}{\sqrt{2}}\Omega_{\mu c}\bar{\chi}_{ab}\gamma_\mu - \frac{1}{\sqrt{2}}\Omega_{\mu b}\bar{\chi}_{ac}\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_{\mu a}\bar{\chi}_{bc}\gamma_\mu - \frac{1}{4}\Omega_{\mu\nu}\bar{\xi}_{abc}\gamma_{\mu\nu} \end{array} \right\} \tag{195}$$

$$\delta\bar{\chi}_{ab} = \left\{ \begin{array}{l} -\Omega_{ab}\bar{\chi} + \frac{1}{12}\epsilon_{abcdefg}\Omega^{cd}\bar{\chi}^{efg} + \Omega_a{}^c\bar{\chi}_{bc} - \Omega_b{}^c\bar{\chi}_{ac} + \frac{1}{2}\Omega_c{}^c\bar{\chi}_{ab} \\ \\ -\frac{1}{\sqrt{2}}\Omega_{\mu b}\bar{\xi}_a\gamma_\mu + \frac{1}{\sqrt{2}}\Omega_{\mu a}\bar{\xi}_b\gamma_\mu - \frac{1}{\sqrt{2}}\Omega_\mu{}^c\bar{\xi}_{abc}\gamma_\mu - \frac{1}{4}\Omega_{\mu\nu}\bar{\chi}_{ab}\gamma_{\mu\nu} \end{array} \right\} \tag{196}$$

$$\delta\bar{\xi}_a = \frac{1}{2}\Omega^{bc}\bar{\xi}_{abc} - \Omega_a{}^b\bar{\xi}_b + \frac{1}{2}\Omega_b{}^b\bar{\xi}_a + \frac{1}{\sqrt{2}}\Omega_{\mu a}\bar{\chi}\gamma_\mu - \frac{1}{4}\Omega_{\mu\nu}\bar{\xi}_a\gamma_{\mu\nu} + \frac{1}{\sqrt{2}}\Omega_\mu{}^b\bar{\chi}_{ab}\gamma_\mu \tag{197}$$

$$\delta\bar{\chi} = \frac{1}{2}\Omega^{ab}\bar{\chi}_{ab} + \frac{1}{2}\Omega_a{}^a\bar{\chi} - \frac{1}{4}\Omega_{\mu\nu}\bar{\chi}\gamma_{\mu\nu} - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\bar{\xi}_a\gamma_\mu \tag{198}$$

Having written the infinitesimal transformations we can use them to show that the following kinetic spinorial bilinear is invariant:

$$\bar{\Psi}(\gamma\cdot\partial)\Psi = \left\{ \begin{array}{l} \bar{\chi}(\gamma\cdot\partial)\chi - \bar{\xi}_a(\gamma\cdot\partial)\xi^a - \frac{1}{2}\bar{\chi}_{ab}(\gamma\cdot\partial)\chi^{ab} + \frac{1}{3!}\bar{\xi}_{abc}(\gamma\cdot\partial)\xi^{abc} \\ \\ +\frac{1}{3!}\bar{\chi}^{abc}(\gamma\cdot\partial)\chi_{abc} - \frac{1}{2}\bar{\xi}^{ab}(\gamma\cdot\partial)\xi_{ab} - \bar{\chi}^a(\gamma\cdot\partial)\chi_a + \bar{\xi}(\gamma\cdot\partial)\xi \end{array} \right\} \tag{199}$$

The above expression applies to an $O_{1,17}$ Weyl module. However, we know that we must also apply the Majorana condition which relates components of Ψ to components of $\tilde{\bar{\Psi}}$, where the tilde denotes transposition. The following 4-dimensional Lorentz covariant, as well as the SU_7 covariant, rules would represent the implementation of the Majorana condition:

$$\left\{ \begin{array}{ll} \chi \to C\tilde{\bar{\xi}} & \bar{\chi} \to -\tilde{\xi}C^{-1} \\ \\ \chi_a \to C\tilde{\bar{\xi}}_a & \bar{\chi}^a \to -\tilde{\xi}^a C^{-1} \\ \\ \chi^{ab} \to C\tilde{\bar{\xi}}^{ab} & \bar{\chi}_{ab} \to -\tilde{\xi}_{ab}C^{-1} \\ \\ \chi_{abc} \to C\tilde{\bar{\xi}}_{abc} & \bar{\chi}^{abc} \to -\tilde{\xi}^{abc}C^{-1} \end{array} \right. \tag{200}$$

In the above, C is the Majorana (or charge-conjugation matrix) which acts on the 4-dimensional spinorial indices. Noting that $C^{-1}\gamma_\mu C = -\tilde{\gamma}_\mu$, the above rules would lead to eliminate the χ components in the kinetic bilinears in favor of the ξ components, and our system of 128 Weyl fermions would reduce to a 64-component counterpart.

8 The Composition of a Vector Multiplet from Fermionic Bilinears

Here we give the composition of an $O_{1,17}$ vector multiplet with components $\{V_\mu, V_a, V^a\}$ from the ξ and χ components of a fundamental spinorial multiplet:

$$V_\mu = \frac{1}{\sqrt{2}} \left(\begin{array}{l} \bar{\chi}\gamma_\mu\chi + \bar{\xi}_a\gamma_\mu\xi^a - \frac{1}{2}\bar{\chi}_{ab}\gamma_\mu\chi^{ab} - \frac{1}{3!}\bar{\xi}_{abc}\gamma_\mu\xi^{abc} \\ \\ +\frac{1}{3!}\bar{\chi}^{abc}\gamma_\mu\chi_{abc} + \frac{1}{2}\bar{\xi}^{ab}\gamma_\mu\xi_{ab} - \bar{\chi}^a\gamma_\mu\chi_a - \bar{\xi}\gamma_\mu\xi \end{array} \right) \tag{201}$$

$$V_a = \begin{pmatrix} -\bar{\xi}_a \chi - \bar{\chi}_{ab}\xi^b + \frac{1}{2}\bar{\xi}_{abc}\chi^{bc} \\ -\frac{1}{36}\epsilon_{abcdefg}\bar{\chi}^{bcd}\xi^{efg} \\ -\frac{1}{2}\bar{\xi}^{bc}\chi_{abc} + \bar{\chi}^b\xi_{ab} + \bar{\xi}\chi_a \end{pmatrix} \tag{202}$$

$$V^a = \begin{pmatrix} \bar{\chi}\xi^a - \bar{\xi}_b\chi^{ab} - \frac{1}{2}\bar{\chi}_{bc}\xi^{abc} \\ +\frac{1}{36}\epsilon^{abcdefg}\bar{\xi}_{bcd}\chi_{efg} \\ +\frac{1}{2}\bar{\chi}^{abc}\xi_{bc} + \bar{\xi}^{ab}\chi_b - \bar{\chi}^a\xi \end{pmatrix} \tag{203}$$

Using the $O_{1,17}$ infinitesimal transformations of components on both sides, we can verify that the above expressions are identities.

9 The $O_{1,17}$ Couplings of a Vector to a Weyl Fermion

Using the foregoing composition of an $O_{1,17}$ vector in terms of the components of a Weyl fermion, we can now construct the couplings. Starting with a bilinear invariant of two vector modules $\mathcal{V}$ and $\mathcal{W}$,

$$\mathcal{V} \cdot \mathcal{W} = V_\mu W_\mu + V_a W^a + V^a W_a \tag{204}$$

we then replace the components W_μ, W_a, and W^a, by their compositions in terms of the ξ and χ fermionic fields. We obtain the following Lorentz invariant, and U_7 invariant, coupling terms. These will be given according to the associated vector field.

First for the couplings to the vector V_μ, we have

$$V_\mu \times \frac{1}{\sqrt{2}} \begin{pmatrix} -\bar{\xi}\gamma_\mu\xi + \bar{\chi}\gamma_\mu\chi + \bar{\xi}_a\gamma_\mu\xi^a - \bar{\chi}^a\gamma_\mu\chi_a \\ +\frac{1}{2}\bar{\xi}^{ab}\gamma_\mu\xi_{ab} - \frac{1}{2}\bar{\chi}_{ab}\gamma_\mu\chi^{ab} - \frac{1}{3!}\bar{\xi}_{abc}\gamma_\mu\xi^{abc} + \frac{1}{3!}\bar{\chi}^{abc}\gamma_\mu\chi_{abc} \end{pmatrix} \tag{205}$$

For the couplings to the scalar 7-plet V_a, we have

$$V_a \times \begin{pmatrix} \bar{\chi}\xi^a - \bar{\chi}^a\xi - \bar{\xi}_b\chi^{ab} + \bar{\xi}^{ab}\chi_b \\ -\frac{1}{2}\bar{\chi}_{bc}\xi^{abc} + \frac{1}{2}\bar{\chi}^{abc}\xi_{bc} + \frac{1}{36}\epsilon^{abcdefg}\bar{\xi}_{bcd}\chi_{efg} \end{pmatrix} \tag{206}$$

For the couplings to the conjugate scalar 7-plet V^a, we have

$$V^a \times \begin{pmatrix} \bar{\xi}\chi_a - \bar{\xi}_a\chi - \bar{\chi}_{ab}\xi^b + \bar{\chi}^b\xi_{ab} \\ +\frac{1}{2}\bar{\xi}_{abc}\chi^{bc} - \frac{1}{2}\bar{\xi}^{bc}\chi_{abc} - \frac{1}{36}\epsilon_{abcdefg}\bar{\chi}^{bcd}\xi^{efg} \end{pmatrix} \tag{207}$$

10 The Composition of a Tensor Multiplet from Fermionic Bilinears

Here we give the composition of an $O_{1,17}$ tensor multiplet with components $B_{\mu\nu\lambda}$, $B_{\mu\nu a}$, ${B_{\mu\nu}}^{a}$, etc., from the ξ and χ components of a fundamental spinorial multiplet:

$$B_{\mu\nu\lambda} = \frac{1}{2}\begin{pmatrix} \bar{\chi}\gamma_{\mu\nu\lambda}\chi + \bar{\xi}_a\gamma_{\mu\nu\lambda}\xi^a - \frac{1}{2}\bar{\chi}_{ab}\gamma_{\mu\nu\lambda}\chi^{ab} - \frac{1}{3!}\bar{\xi}_{abc}\gamma_{\mu\nu\lambda}\xi^{abc} \\ +\frac{1}{3!}\bar{\chi}^{abc}\gamma_{\mu\nu\lambda}\chi_{abc} + \frac{1}{2}\bar{\xi}^{ab}\gamma_{\mu\nu\lambda}\xi_{ab} - \bar{\chi}^a\gamma_{\mu\nu\lambda}\chi_a - \bar{\xi}\gamma_{\mu\nu\lambda}\xi \end{pmatrix} \tag{208}$$

$$B_{\mu\nu a} = -\frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\xi}_a\gamma_{\mu\nu}\chi + \bar{\chi}_{ab}\gamma_{\mu\nu}\xi^b - \frac{1}{2}\bar{\xi}_{abc}\gamma_{\mu\nu}\chi^{bc} \\ +\frac{1}{36}\epsilon_{abcdefg}\bar{\chi}^{bcd}\gamma_{\mu\nu}\xi^{efg} \\ +\frac{1}{2}\bar{\xi}^{bc}\gamma_{\mu\nu}\chi_{abc} - \bar{\chi}^b\gamma_{\mu\nu}\xi_{ab} - \bar{\xi}\gamma_{\mu\nu}\chi_a \end{pmatrix} \tag{209}$$

$${B_{\mu\nu}}^{a} = \frac{1}{\sqrt{2}}\begin{pmatrix} \bar{\chi}\gamma_{\mu\nu}\xi^a - \bar{\xi}_b\gamma_{\mu\nu}\chi^{ab} - \frac{1}{2}\bar{\chi}_{bc}\gamma_{\mu\nu}\xi^{abc} \\ \frac{1}{36}\epsilon^{abcdefg}\bar{\xi}_{bcd}\gamma_{\mu\nu}\chi_{efg} \\ +\frac{1}{2}\bar{\chi}^{abc}\gamma_{\mu\nu}\xi_{bc} + \bar{\xi}^{ab}\gamma_{\mu\nu}\chi_b - \bar{\chi}^a\gamma_{\mu\nu}\xi \end{pmatrix} \tag{210}$$

$$B_{\mu ab} = -\begin{pmatrix} \bar{\xi}\gamma_\mu\xi_{ab} + \bar{\chi}^c\gamma_\mu\chi_{abc} - \frac{1}{12}\epsilon_{abcdefg}\bar{\xi}^{cd}\gamma_\mu\xi^{efg} \\ -\frac{1}{12}\epsilon_{abcdefg}\bar{\chi}^{cde}\gamma_\mu\xi^{fg} + \bar{\xi}_{abc}\gamma_\mu\xi^c + \bar{\chi}_{ab}\gamma_\mu\chi \end{pmatrix} \tag{211}$$

$${B_{\mu a}}^{b} = \left\{ \begin{array}{l} \frac{1}{2}\delta_a^b \begin{pmatrix} \bar{\chi}\gamma_\mu\chi + \bar{\xi}_c\gamma_\mu\xi^c - \frac{1}{2}\bar{\chi}_{cd}\gamma_\mu\chi^{cd} \\ -\frac{1}{3!}\bar{\xi}_{cde}\gamma_\mu\xi^{cde} - \frac{1}{3!}\bar{\chi}^{cde}\gamma_\mu\chi_{cde} \\ -\frac{1}{2}\bar{\xi}^{cd}\gamma_\mu\xi_{cd} + \bar{\chi}^c\gamma_\mu\chi_c + \bar{\xi}\gamma_\mu\xi \end{pmatrix} \\ + \begin{pmatrix} -\bar{\xi}_a\gamma_\mu\xi^b + \bar{\chi}_{ac}\gamma_\mu\chi^{bc} + \frac{1}{2}\bar{\xi}_{acd}\gamma_\mu\xi^{bcd} \\ +\frac{1}{2}\bar{\chi}^{bcd}\gamma_\mu\chi_{acd} + \bar{\xi}^{bc}\gamma_\mu\xi_{ac} - \bar{\chi}^b\gamma_\mu\chi_a \end{pmatrix} \end{array} \right\} \tag{212}$$

$${B_\mu}^{ab} = \begin{pmatrix} -\bar{\xi}^{ab}\gamma_\mu\xi - \bar{\chi}^{abc}\gamma_\mu\chi_c + \frac{1}{12}\epsilon^{abcdefg}\bar{\xi}_{cde}\gamma_\mu\xi_{fg} \\ +\frac{1}{12}\epsilon^{abcdefg}\bar{\chi}_{cd}\gamma_\mu\chi_{efg} - \bar{\xi}_c\gamma_\mu\xi^{abc} - \bar{\chi}\gamma_\mu\chi^{ab} \end{pmatrix} \tag{213}$$

$$B_{abc} = \sqrt{2}\begin{pmatrix} \bar{\xi}\chi_{abc} + \frac{1}{3!}\epsilon_{abcdefg}\bar{\chi}^d\xi^{efg} - \frac{1}{4}\epsilon_{abcdefg}\bar{\xi}^{de}\chi^{fg} \\ -\frac{1}{3!}\epsilon_{abcdefg}\bar{\chi}^{def}\xi^g + \bar{\xi}_{abc}\chi \end{pmatrix} \tag{214}$$

$$B_{ab}{}^{c} = \left\{ \begin{array}{l} \frac{1}{\sqrt{2}} \left(\begin{array}{l} \bar{\xi}_a \chi + \bar{\chi}_{ad} \xi^d - \frac{1}{2} \bar{\xi}_{ade} \chi^{de} \\ -\frac{1}{2} \bar{\xi}^{de} \chi_{ade} + \bar{\chi}^d \xi_{ad} + \bar{\xi} \chi_a \end{array} \right) \delta_b{}^c - (a \leftrightarrow b) \\ +\sqrt{2} \left(\begin{array}{l} \bar{\chi}_{ab} \xi^c - \bar{\xi}_{abd} \chi^{cd} - \frac{1}{24} \epsilon_{abdefgh} \bar{\chi}^{cde} \xi^{fgh} \\ -\frac{1}{24} \epsilon_{abdefgh} \bar{\chi}^{def} \xi^{cgh} - \bar{\xi}^{cd} \chi_{abd} + \bar{\chi}^c \xi_{ab} \end{array} \right) \end{array} \right\} \tag{215}$$

$$B_a{}^{bc} = \left\{ \begin{array}{l} -\frac{1}{\sqrt{2}} \delta_a{}^b \left(\begin{array}{l} \bar{\chi} \xi^c - \bar{\xi}_d \chi^{cd} - \frac{1}{2} \bar{\chi}_{de} \xi^{cde} \\ -\frac{1}{2} \bar{\chi}^{cde} \xi_{de} - \bar{\xi}^{cd} \chi_d + \bar{\chi}^c \xi \end{array} \right) - (b \leftrightarrow c) \\ +\sqrt{2} \left(\begin{array}{l} \bar{\xi}_a \chi^{bc} + \bar{\chi}_{ad} \xi^{bcd} - \frac{1}{24} \epsilon^{bcdefgh} \bar{\xi}_{ade} \chi_{fgh} \\ -\frac{1}{24} \epsilon^{bcdefgh} \bar{\xi}_{def} \chi_{agh} + \bar{\chi}^{bcd} \xi_{ad} + \bar{\xi}^{bc} \chi_a \end{array} \right) \end{array} \right\} \tag{216}$$

$$B^{abc} = -\sqrt{2} \left(\begin{array}{l} \bar{\chi}^{abc} \xi + \frac{1}{3!} \epsilon^{abcdefg} \bar{\xi}_{def} \chi_g - \frac{1}{4} \epsilon^{abcdefg} \bar{\chi}_{de} \xi_{fg} \\ -\frac{1}{3!} \epsilon^{abcdefg} \bar{\xi}_d \chi_{efg} + \bar{\chi} \xi^{abc} \end{array} \right) \tag{217}$$

Using the $O_{1,17}$ infinitesimal transformations of components on both sides, we can verify that the above expressions are identities.

11 The $O_{1,17}$ Couplings of a Tensor to a Weyl Fermion

Using the foregoing composition of an $O_{1,17}$ tensor in terms of the components of a Weyl fermion, we can now construct the couplings. Starting with a bilinear invariant of two tensor modules $\mathcal{V}$ and $\mathcal{W}$,

$$\mathcal{V} \cdot \mathcal{W} = \left\{ \begin{array}{l} \frac{1}{3!} V_{\mu\nu\lambda} + \frac{1}{2} V_{\mu\nu a} W_{\mu\nu}{}^a + \frac{1}{2} V_{\mu\nu}{}^a W_{\mu\nu a} \\ +\frac{1}{2} V_{\mu ab} W_{\mu}{}^{ab} - V_{\mu a}{}^b W_{\mu b}{}^a + \frac{1}{2} V_{\mu}{}^{ab} W_{\mu ab} \\ +\frac{1}{3!} V_{abc} W^{abc} + \frac{1}{2} V_{ab}{}^c W_c{}^{ab} + \frac{1}{2} V_a{}^{bc} W_{bc}{}^a + \frac{1}{3!} V^{abc} W_{abc} \end{array} \right\} \tag{218}$$

we then replace the components $W_{\mu\nu\lambda}$, $W_{\mu\nu a}$, $W_{\mu\nu}{}^a$, etc., by their compositions in terms of the ξ and χ fermionic fields. We obtain the following Lorentz invariant, and U_7 invariant, coupling terms. These will be given according to the associated tensor field components. Notice that these couplings are given, for greater generality, without implementing the Majorana constraint that would eliminate the χ components in favor of the $\bar{\xi}$ components, as discussed before.

11.1 Couplings to the Axial Vector $V_{\mu\nu\lambda}$

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the tensor $V_{\mu\nu\lambda}$. Notice that the latter tensor can be traded for an axial vector using the

4-dimensional epsilon, while the Dirac operator $\gamma_{\mu\nu\lambda}$ can likewise be traded for $\gamma_\mu\gamma_5$.

$$V_{\lambda\mu\nu}\times\frac{1}{12}\left(\begin{array}{l}-\bar{\xi}\gamma_{\lambda\mu\nu}\xi+\bar{\chi}\gamma_{\lambda\mu\nu}\chi+\bar{\xi}_a\gamma_{\lambda\mu\nu}\xi^a-\bar{\chi}^a\gamma_{\lambda\mu\nu}\chi_a\\ +\frac{1}{2}\bar{\xi}^{ab}\gamma_{\lambda\mu\nu}\xi_{ab}-\frac{1}{2}\bar{\chi}_{ab}\gamma_{\lambda\mu\nu}\chi^{ab}-\frac{1}{3!}\bar{\xi}_{abc}\gamma_{\lambda\mu\nu}\xi^{abc}+\frac{1}{3!}\bar{\chi}^{abc}\gamma_{\lambda\mu\nu}\chi_{abc}\end{array}\right) \tag{219}$$

11.2 Couplings to the Tensor 7-Plets $V_{\mu\nu a}$

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the tensor $V_{\mu\nu a}$. Notice that the latter is a 7-plet of field-strength-like tensors.

$$V_{\mu\nu a}\times\frac{1}{2\sqrt{2}}\left(\begin{array}{l}\bar{\chi}\gamma_{\mu\nu}\xi^a-\bar{\chi}^a\gamma_{\mu\nu}\xi-\bar{\xi}_b\gamma_{\mu\nu}\chi^{ab}+\bar{\xi}^{ab}\gamma_{\mu\nu}\chi_b\\ -\frac{1}{2}\bar{\chi}_{bc}\gamma_{\mu\nu}\xi^{abc}+\frac{1}{2}\bar{\chi}^{abc}\gamma_{\mu\nu}\xi_{bc}+\frac{1}{36}\epsilon^{abcdefg}\bar{\xi}_{bcd}\gamma_{\mu\nu}\chi_{efg}\end{array}\right) \tag{220}$$

11.3 Couplings to the Conjugate Tensor 7-Plet ${V_{\mu\nu}}^a$

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the conjugate tensor ${V_{\mu\nu}}^a$.

$${V_{\mu\nu}}^a\times\frac{1}{2\sqrt{2}}\left(\begin{array}{l}\bar{\xi}\gamma_{\mu\nu}\chi_a-\bar{\xi}_a\gamma_{\mu\nu}\chi-\bar{\chi}_{ab}\gamma_{\mu\nu}\xi^b+\bar{\chi}^b\gamma_{\mu\nu}\xi_{ab}\\ +\frac{1}{2}\bar{\xi}_{abc}\gamma_{\mu\nu}\chi^{bc}-\frac{1}{2}\bar{\xi}^{bc}\gamma_{\mu\nu}\chi_{abc}-\frac{1}{36}\epsilon_{abcdefg}\bar{\chi}^{bcd}\gamma_{\mu\nu}\xi^{efg}\end{array}\right) \tag{221}$$

11.4 Couplings to the Vector 21-Plet $V_{\mu ab}$

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the vector 21-plet $V_{\mu ab}$. Notice that the latter is part of the O_{14} gauge multiplet.

$$V_{\mu ab}\times\frac{1}{2}\left(\begin{array}{l}-\bar{\chi}\gamma_\mu\chi^{ab}-\bar{\xi}^{ab}\gamma_\mu\xi-\bar{\xi}_c\gamma_\mu\xi^{abc}-\bar{\chi}^{abc}\gamma_\mu\chi_c\\ +\frac{1}{12}\epsilon^{abcdefg}\bar{\xi}_{cde}\gamma_\mu\xi_{fg}+\frac{1}{12}\epsilon^{abcdefg}\bar{\chi}_{cd}\gamma_\mu\chi_{efg}\end{array}\right) \tag{222}$$

11.5 Couplings to the U_7 Vector Bosons

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the vectors ${V_{\mu a}}^b$. Notice that the latter correspond to the vector gauge bosons of the U_7 algebra.

$$\left\{\begin{array}{l}{V_{\mu a}}^b\times\left(\begin{array}{l}\bar{\xi}_b\gamma_\mu\xi^a+\bar{\chi}^a\gamma_\mu\chi_b-\bar{\chi}_{bc}\gamma_\mu\chi^{ac}-\bar{\xi}^{ac}\gamma_\mu\xi_{bc}\\ -\frac{1}{2}\bar{\xi}_{bcd}\gamma_\mu\xi^{acd}-\frac{1}{2}\bar{\chi}^{acd}\gamma_\mu\chi_{bcd}\end{array}\right)\\ -\frac{1}{2}{V_{\mu a}}^a\times\left(\begin{array}{l}\bar{\xi}\gamma_\mu\xi+\bar{\chi}\gamma_\mu\chi+\bar{\xi}_b\gamma_\mu\xi^b+\bar{\chi}^b\gamma_\mu\chi_b\\ -\frac{1}{2}\bar{\chi}_{bc}\gamma_\mu\chi^{bc}-\frac{1}{2}\bar{\xi}^{bc}\gamma_\mu\xi_{bc}-\frac{1}{3!}\bar{\xi}_{bcd}\gamma_\mu\xi^{bcd}-\frac{1}{3!}\bar{\chi}^{bcd}\gamma_\mu\chi_{bcd}\end{array}\right)\end{array}\right. \tag{223}$$

Notice that we can simplify the above, if convenient, by decomposing the adjoint U_7 multiplet ${V_{\mu a}}^b$ into a traceless part, and a trace part.

11.6 Couplings to the Conjugate Vector 21-Plet ${V_\mu}^{ab}$

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the conjugate vector 21-plet ${V_\mu}^{ab}$. Notice that the latter is part of the O_{14} gauge multiplet.

$$ {V_\mu}^{ab} \times \frac{1}{2} \left(\begin{array}{l} -\bar{\xi}\gamma_\mu \xi_{ab} - \bar{\chi}_{ab}\gamma_\mu \chi - \bar{\chi}^c \gamma_\mu \chi_{abc} - \bar{\xi}_{abc}\gamma_\mu \xi^c \\ +\frac{1}{12}\epsilon_{abcdefg}\bar{\chi}^{cde}\gamma_\mu \chi^{fg} + \frac{1}{12}\epsilon_{abcdefg}\bar{\xi}^{cd}\gamma_\mu \xi^{efg} \end{array} \right) \tag{224} $$

11.7 Couplings to the Scalar Multiplet V_{abc}

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the scalar multiplet V_{abc}.

$$ V_{abc} \times \frac{1}{3\sqrt{2}} \left(\begin{array}{l} -\bar{\chi}\xi^{abc} - \bar{\chi}^{abc}\xi + \frac{1}{4}\epsilon^{abcdefg}\bar{\chi}_{de}\xi_{fg} \\ +\frac{1}{3!}\epsilon^{abcdefg}\bar{\xi}_d \chi_{efg} - \frac{1}{3!}\epsilon^{abcdefg}\bar{\xi}_{def}\chi_g \end{array} \right) \tag{225} $$

11.8 Couplings to the Scalar Multiplet ${V_{ab}}^c$

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the scalar multiplet ${V_{ab}}^c$.

$$ \frac{1}{\sqrt{2}} \left\{ \begin{array}{l} {V_{ab}}^c \times \left(\bar{\xi}_c \chi^{ab} + \bar{\xi}^{ab}\chi_c + \bar{\chi}_{cd}\xi^{abd} + \bar{\chi}^{abd}\xi_{cd}\right) \\ -\frac{1}{24}\epsilon^{abcdefg}{V_{ab}}^h \times \left(\bar{\xi}_{cde}\chi_{hfg} + \bar{\xi}_{hcd}\chi_{efg}\right) \\ +{V_{ab}}^b \left(\bar{\chi}\xi^a + \bar{\chi}^a \xi - \bar{\xi}_c \chi^{ac} - \bar{\xi}^{ac}\chi_c - \frac{1}{2}\bar{\chi}_{cd}\xi^{acd} - \frac{1}{2}\bar{\chi}^{acd}\xi_{cd}\right) \end{array} \right\} \tag{226} $$

11.9 Couplings to the Conjugate Scalar Multiplet ${V_a}^{bc}$

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the Conjugatc scalar multiplet ${V_a}^{bc}$.

$$ \frac{1}{\sqrt{2}} \left\{ \begin{array}{l} {V_a}^{bc} \times \left(-\bar{\chi}_{bc}\xi^a - \bar{\chi}^a \xi_{bc} + \bar{\xi}_{bcd}\chi^{ad} + \bar{\xi}^{ad}\chi_{bcd}\right) \\ +\frac{1}{24}\epsilon_{abcdefg}{V_h}^{ab} \times \left(\bar{\chi}^{cde}\xi^{hfg} + \bar{\chi}^{hcd}\xi^{efg}\right) \\ -{V_a}^{ab} \left(\bar{\xi}\chi_b + \bar{\xi}_b \chi + \bar{\chi}_{bc}\xi^c + \bar{\chi}^c \xi_{bc} - \frac{1}{2}\bar{\xi}_{bcd}\chi^{cd} - \frac{1}{2}\bar{\xi}^{cd}\chi_{bcd}\right) \end{array} \right\} \tag{227} $$

11.10 Couplings to the Conjugate Scalar Multiplet V^{abc}

Here we give the invariant $O_{1,17}$ couplings of the U_7 multiplets of Weyl fermions to the conjugate scalar multiplet V^{abc}.

$$V^{abc} \times \frac{1}{3\sqrt{2}} \left(\begin{array}{l} \bar{\xi}\chi_{abc} + \bar{\xi}_{abc}\chi - \frac{1}{4}\epsilon_{abcdefg}\bar{\xi}^{de}\chi^{fg} \\ +\frac{1}{3!}\epsilon_{abcdefg}\bar{\chi}^{d}\xi^{efg} - \frac{1}{3!}\epsilon_{abcdefg}\bar{\chi}^{def}\xi^{g} \end{array} \right) \tag{228}$$

12 Discussion

In this article, we have displayed the structure of the fundamental spin-$\frac{1}{2}$ fermionic SU_7 multiplets that constitute a Weyl (or a Majorana-Weyl) spinor in an 18-dimensional gravidynamic unification[16] model. We have also displayed the structure of the bosonic multiplets. These are spin-1 (gauge) and spin-0 (Higgs) particles that could arise from the quantum contributions to the effective action. We have decomposed the pertinent fundamental coupling in the unified scheme.

At this point, the scheme could be decomposed further to display the *color* SU_3 and the *family* SU_3 structure of the particles and the couplings. The subsequent steps are straightforward and could follow the work done[17] in connection with grand-unified O_{14} and SU_7. The features and phenomenology discussed there, pertaining to family structure of quarks and leptons, as well as the family structure of the W-like vector bosons, and the issue of evading the dilemma of mirror quark-lepton generations, would all be applicable to the present work.

We shall return, in other articles, to the important problem of symmetry breaking in an effective action framework. We would direct our concern towards a possible role to be played by the scalar particles that take part in the foregoing unified fermion-boson couplings. These scalar particles would be important in selecting the structure of symmetry breaking via their mass-generating vacuum components.

It is clear that our ability to handle the foregoing extensive algebraic work should give greater attention to the underlying theory, and its relevance to the description of some ultimate unified reality, in spite of its immensity. If the *replication* of hadronic constituents (quarks) and leptons cannot be explained otherwise, the 18-dimensional Einstein-Dirac unification scheme, with a fundamental Majorana-Weyl fermionic structure, would seem to be the correct choice to take, and a confrontation with high-energy collider phenomenology must take place. A confrontation with the other scheme of supersymmetric E_8 gauge unification[18] should also come to light.

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