On the Classical Unification of Elementary Particle Physics with Nuclear and Hadron Physics, Electrodynamics, and Gravitation

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February 8, 2014, Revision 1

Abstract: We place the author's earlier work from [13], [14], [15], [16], [17], [18], [19], [20], [21] into the context of a classical unified field theory of elementary particle physics, nuclear and hadron physics, electrodynamics, and gravitation. This revision contains the unification of classical Maxwell and Yang-Mills electrodynamics with classical gravitation as represented by the field equation $R_{\mu\nu}=0$ of empty space, and explains how this will be further connected to the physics of nuclear matter as developed in said earlier work.

Contents

1. Introduction

Since at least the time of the ancient Greeks who believed that all of nature was constructed from the four Platonic elements of fire, water, air and earth [1] to which Aristotle added aether as a fifth element to explain the stars, a central pursuit of scientists and natural philosophers has been to understand the nature of matter in its most elemental, irreducible form. But our modern understanding of matter really emerged starting with Dmitri Mendeleev's assembly of what was to become the modern Periodic Table of the Elements [2] in which the various elements were fundamentally distinguished by their atomic weights. It would not, however, be until Rutherford and Chadwick discovered the proton and neutron in 1917 and 1933 that the foundation was laid for understanding that the atomic weights of the elements are really driven by their nuclear weights, and that the number of protons Z which for a non-ionized element is equal to the number of electrons is what establishes the basic character of an element, while the number of neutrons N together with the total number of nucleons $A=N+Z$ is what characterizes the various isotopes of any given element.

It also became clear as nuclear science developed that although nuclear weights could be very-closely characterized by the number of protons and neutrons in a nuclear isotope, these are not exactly correlated. Rather, it was found that small corrections known as the "mass defect" also exist which reflect the fact that if, for example, one were to fuse two protons together into a deuteron, the fused deuteron system of two protons would weigh slightly less than the sum of the two separate protons, and that this slight diminution of mass m could be accounted for by a commensurate release of fusion energy E in an amount equal to mc^2. This mass defect, of course, is exhibited by all of the nuclides underlying the periodic table, and is the basis upon which rests the ability to produce energy from the atomic nucleus. To this day, however, there is no commonly-accepted understanding at a precise granular level of exactly why the various mass defects are what they are. Our best understanding to date is based on a rough aggregation known as the "semi-empirical mass formula" (SEMF) [3] and various refinements of this formula that have been made over time, which works well for heavier elements but not for the lighter elements. Understanding with precision the mass defects and related binding energies of the lightest nuclides on a discrete granular level, for example ^{2}H, ^{3}H, ^{3}He, ^{4}He and various Li, Be, B, C, N and O isotopes, remains a very open question which the aggregation approach of the SEMF is ill-suited to fully explain.

Once it became clear that Mendeleev's atomic elements were themselves all constructed from protons, neutrons and electrons, and that a neutrino was also needed to "balance" the neutron in the same way that the electron balances the proton, the question naturally arose as to whether the proton, neutron, electron and neutrino are themselves "elementary," or whether these could also be further deconstructed into even-more-elementary constituents. While the redundancy of weak isospin as between both quark and lepton beta-decays has led to some interest in a variety of "preon" models with a lepton preon plus three colors of quark preon each being paired with isospin up or isospin down preons, e.g. [4], [5], the most important advance on this question came in 1964 when Gell-Mann [6] and Zweig [7], [8] proposed that the proton and the neutron and more generally the class of particles known as baryons, all comprise three yet-more-elementary fermions which Gell-Mann dubbed as "quarks." Quark theory had its origin in trying to explain the various "flavors" of baryon which Gell-Mann [9] and Ne'eman [10] had successfully characterized in 1961 by the "eightfold way." The eight baryons explained in this way included the proton and neutron which were understood to subsist in an octet of a fundamental SU(3) *flavor* group containing "up," "down" and "strange" quarks, as well as a decuplet of baryons, and an octet of mesons which include the pi and K mesons.

Quark theory advanced a year later in 1965 when several researchers proposed the necessity of a new degree of freedom to resolve certain difficulties regarding Fermion statistics / Pauli-Dirac Exclusion that arose from having three fermions in a single baryon system [11]. This came to be known as the "color" degree of freedom of "chromodynamic" theory and is now regarded as a fundamental feature of strong interaction theory based on an exact $SU(3)_C$ *color* symmetry which is separate and distinct from the approximate SU(3) *flavor* symmetry from which quark theory had originated. The interaction mediators of this color symmetry are eight bi-colored massless gluons in the adjoint $SU(3)_C$ representation, and they too, are separate and distinct from the eight bi-flavored pi and K mesons in the octet of the original SU(3) flavor theory. In the meantime, insofar as fermions are concerned, the standard view is now that there are precisely six flavors of quark and six parallel flavors of lepton, each paired into three generations of weak isospin doublets. The generations are well-characterized, but to date there is still no widely-accepted understanding as to their origins, or, as Rabi once quipped upon the discovery of the muon, "who ordered this?" has not yet been answered by any wide consensus.

Although this "inward" advancement from molecules comprising atoms, to atoms comprising nuclei and electrons, to nuclei comprising protons and neutrons, to protons and neutrons and other baryons comprising quarks would appear on its surface to be a logical, linear progression, the final progression from baryons to quarks is qualitatively different from all of the other ones. This is because molecules and atoms and nuclei and even individual protons and neutrons, as well as mesons and leptons, all exist in free, directly-observable particle states. *But quarks and gluons do not.* These are understood to be "confined" with the proton and neutron and other baryons and mesons, and so the prevailing view, which to date is confirmed by empirical observation, is that quarks and gluons may never be directly observed as free particles.

Quarks and gluons being of a qualitatively-different character from molecules and atoms and nuclei and baryons and mesons and leptons, is also what separates the modern discipline of elementary particle physics, from that of nuclear (and atomic) physics. Specifically, although in colloquial discussion it is commonplace to refer to "nuclear and particle physics" as if this is a single unified discipline, in reality it is not. This is because quarks and gluons are not free observables, but rather are confined within baryons and mesons which are the free observables, and because at the present time there is no complete, commonly-accepted understanding of how confinement works or of the dynamical interrelationships between the physics of quarks as elementary *confined* particles, and the physics of baryons as elementary *free* particles.

This fault line which separates nuclear from particle physics is concisely captured by Jaffe and Witten when they state at page 3 of the "Yang-Mills and Mass Gap" problem [12] that:

> ". . . for QCD to describe the strong force successfully . . . It must have 'quark confinement,' that is, even though the theory is described in terms of elementary fields, such as the quark fields, that transform non-trivially under SU(3), the

physical particle states—such as the proton, neutron, and pion—are SU(3)-invariant."

It is this difference between "elementary fields, such as the quark [and the gluon] fields, that transform non-trivially under SU(3)" and "the physical particle states—such as the proton, neutron, and pion—[which] are SU(3)-invariant," as well as the need to give flavor to color-neutral baryons and understand the origins of the specific baryon flavors which are protons and neutrons, which separates the elementary particle physics of colored quarks and gluons, from the nuclear physics of the colorless proton- and neutron-flavored baryons.

The purpose of this paper is to understand, and build a bridge across, this fault line between elementary confined particle fields and free physical particle states, so that that "Nuclear and Particle Physics" can indeed be developed into a fully-unified discipline. To establish that this bridge is a safe crossing between nuclear and particle physics, we also demonstrate empirical support based upon the binding and / or fusion energies of fifteen (15) distinct light nuclides as well as the proton and neutron masses themselves.

Some aspects of the development to be presented here have been previously published [13], [14], [15], [16] or preprinted [17], [18], [19], [20], [21] by the author. But this paper will refine and expand much of this earlier development and place it firmly into the context of a carefully-elaborated unification of elementary particle with nuclear and atomic physics, using nuclear mass and binding and fusion energies themselves for experimental validation.

We begin our exploration precisely where Albert Einstein ended his pursuit of electromagnetic and gravitational unification in the final scientific paper of his remarkable life.

2. Einstein's Final "Hunch"

In the final scientific paper of his life [22] which he introduced in December 1954, Albert Einstein opened at page 133 by discussing "the 'strength' of systems of field equations in general." Einstein pointed out that this discussion was "indispensable" to understanding the "problem" of non-symmetric fields. But he also stated, very importantly, that "this discussion is of intrinsic interest quite apart from the particular theory presented here." He then went on to examine three examples of field strength: First, the scalar wave equation $\partial_\sigma \partial^\sigma \phi = 0$. Second, Maxwell's equations for empty space which are the electric and magnetic charge density equations $0 = \partial_\sigma F^{\sigma\mu}$ and $0 = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu}$, respectively, with the field strength denoted as $F^{\mu\nu}$. Third, the gravitational equations for empty space $R_{\mu\nu} = 0$ for which the gravitational fields $g_{\mu\nu}$ also operate as the spacetime metric and so enjoy the metricity condition $\partial_{;\sigma} g_{\mu\nu} = 0$ for covariant differentiation.

First pointing out that such a field strength "measure can be defined which will even enable us to compare with each other the strengths of systems whose field variables differ with respect to number and kind," Einstein then found that the number of *n*th-order free coefficients for the scalar wave equation, asymptotically for large *n*, is given by

$$z \sim \binom{4}{n}\frac{z_1}{n} = \binom{4}{n}\frac{6}{n} \tag{2.1}$$

with a "coefficient of freedom" $z_1 = 6$. He then progressed to find that for the empty space Maxwell's equations *and also identically for the empty space gravitational equations*, asymptotically:

$$z \sim \binom{4}{n}\left[0 + \frac{z_1}{n}\right] = \binom{4}{n}\left[0 + \frac{12}{n}\right] \tag{2.2}$$

with $z_1 = 12$.

Einstein concluded with a remarkable understatement at page 139 that:

> "It is surprising that the gravitational equations for empty space determine their field just as strongly as do Maxwell's equations in the case of the electromagnetic field."

The above is conspicuously understated because for the last several decades of his life, Einstein worked tirelessly to try to unify classical electromagnetic field theory as represented by $0 = \partial_\sigma F^{\sigma\mu}$ and $0 = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu}$ *in vacuo*, with gravitational theory for which the vacuum equation is $R_{\mu\nu} = 0$. Viewed in this context, such an expression of "surprise" was nothing if not a masterful understatement of where Einstein's own mortality finally ended his quest for classical electromagnetic/gravitational unification. In all areas of human life, deathbed statements are accorded special weight and attention, and Einstein's statement set forth above ought to be no exception. It is a bequeathal to posterity to pick up the work of this classical unification at the exact place where Einstein was forced to set it down, and it communicates two very important points about where Einstein's intuition and his mathematical tools had delivered him to as his earthly stamina approached expiration:

First, this statement articulates Einstein's deliberately-inexplicit "hunch" that Maxwell's *system of equations* might in fact, in some way, be *one and the same* as the gravitational vacuum equation $R_{\mu\nu} = 0$. For, while Einstein did not prove the mathematical equivalence of these equations, he did prove that although Maxwell's source-free system utilizes two tensor equations $0 = \partial_\sigma F^{\sigma\mu}$ and $0 = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu}$ while the gravitational system $R_{\mu\nu} = 0$ uses a single tensor equation, and that although their "field variables differ with respect to number and kind," these two systems of equations do "determine their field just as strongly" as one another. Had Einstein been more explicit, he may well have said "I possess the strong intuitive feeling that Maxwell's equations might find some unification with gravitational theory, and might be written with identical physical content, in the simple form of $R_{\mu\nu} = 0$. But now, near the end of my life, I have only been able to prove with mathematical certainty that *in vacuo*, $z_1 = 12$ for each of these systems of equations. Posterity should carry on this pursuit to see if there something more to this than merely a 'surprising' coincidence." This is, in essence, the deathbed statement of the grandmaster of 20th century physics about the most cherished preoccupation of the final decades of his life.

Second, it is very consequential that in seeking to "compare with each other the strengths of systems whose field variables differ with respect to number and kind," and specifically by comparing the *two tensor equations* $0 = \partial_\sigma F^{\sigma\mu}$ and $0 = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu}$ with the *single tensor equation* $R_{\mu\nu} = 0$ and finding that these are equivalent equations at least insofar as the strength with which they determine their fields are concerned, Einstein was implicitly thinking about the question of what would happen if one were to combine both of Maxwell's equations together into a single tensor equation. While the "field variables [do] differ with respect to number and kind" as between Maxwell's equations and $R_{\mu\nu} = 0$, this observation motivates one to pose the highly-related question which is simply this: what would it look like, and what would the physics content be, if one were to be able combine both of Maxwell's tensor equations together into a single tensor equation, but *on their own terms, using the <u>same number and same kind</u> of field variables?*

Specifically, prior to 1905, Maxwell's equations were understood to be a system of four differential equations:

$$\begin{aligned}
&\rho = \nabla \cdot \mathbf{E} \\
&\mathbf{J} = \nabla \times \mathbf{B} - \partial \mathbf{E} / \partial t \\
&0 = \nabla \cdot \mathbf{B} \\
&0 = \nabla \times \mathbf{E} + \partial \mathbf{B} / \partial t
\end{aligned} \tag{2.3}$$

But in 1905 [23], Einstein himself showed that in tensor form the former two and latter two equations, in view of the Lorentz symmetry relating space and time, may be consolidated into the two equations:

$$\begin{aligned}
&J^\mu = \partial_\sigma F^{\sigma\mu} \\
&0 = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu}
\end{aligned} . \tag{2.4}$$

So from this view, the question Einstein was implicitly considering at the end of his life was whether there is some way to take the next step in this historical progression that he had initiated at the opening of his scientific life, by combining *both* of these tensor equations into a *single* tensor field equation.

The difficulty one confronts in trying to further combine these two equations (2.4) into a single equation in terms of the antisymmetric electromagnetic fields $F^{\mu\nu}$, rather than the symmetric gravitational fields $g_{\mu\nu}$ and the vacuum field equation $R_{\mu\nu} = 0$, stems from the fact that the magnetic monopoles in $0 = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu}$ are zero. But this in turn is a consequence of the fact that the field strength $F_{\mu\nu}$ in Maxwell's electrodynamics is specified in relation to the gauge field / vector potential G_μ according to the *abelian* equation:

$$F^{\mu\nu} = \partial^{[;\mu}G^{\nu]} = \partial^{;\mu}G^{\nu} - \partial^{;\nu}G^{\mu} \tag{2.5}$$

which via the Riemann tensor $\left[\partial_{;\mu}, \partial_{;\nu}\right]G_\alpha \equiv R^{\sigma}{}_{\alpha\mu\nu}G_\sigma$ and the first Bianchi identity $R_\tau{}^{(\nu\sigma\mu)} = R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma} = 0$ drives the monopole to zero via:

$$\begin{aligned} P^{\sigma\mu\nu} &= \partial^{;\sigma}F^{\mu\nu} + \partial^{;\mu}F^{\nu\sigma} + \partial^{;\nu}F^{\sigma\mu} \\ &= \partial^{;\sigma}\left(\partial^{;\mu}G^{\nu} - \partial^{;\nu}G^{\mu}\right) + \partial^{;\mu}\left(\partial^{;\nu}G^{\sigma} - \partial^{;\sigma}G^{;\nu}\right) + \partial^{;\nu}\left(\partial^{;\sigma}G^{\mu} - \partial^{;\mu}G^{\sigma}\right) \\ &= \left[\partial^{;\sigma}, \partial^{;\mu}\right]G^{\nu} + \left[\partial^{;\mu}, \partial^{;\nu}\right]G^{\sigma} + \left[\partial^{;\nu}, \partial^{;\sigma}\right]G^{\mu} \\ &= \left(R_\tau{}^{\nu\sigma\mu} + R_\tau{}^{\sigma\mu\nu} + R_\tau{}^{\mu\nu\sigma}\right)G^{\tau} = \mathbf{0} \end{aligned} . \tag{2.6}$$

In (2.5) and (2.6) above we have introduced the gravitationally-covariant derivative (;) by which these equations then may be applied in curved spacetime.

Yang-Mills gauge theories, on the other hand, which were developed by C. N. Yang and R. Mills in the same year of 1954 [24] when Einstein announced [22], engender no such limitation, because in these theories, the field strength is related to the gauge fields by the *non-Abelian* relationship:

$$F^{\mu\nu} = D^{[;\mu}G^{\nu]} = \partial^{;\mu}G^{\nu} - \partial^{;\nu}G^{\mu} - iG^{\mu}G^{\nu} + iG^{\nu}G^{\mu} = \partial^{;\mu}G^{\nu} - \partial^{;\nu}G^{\mu} - i\left[G^{\mu}, G^{\nu}\right], \tag{2.7}$$

where the gauge-covariant derivative $D^{;\mu} \equiv \partial^{;\mu} - iG^{\mu}$. Because Yang-Mills replaces all gravitationally-covariant derivatives $\partial^{;\mu}$ by gauge-covariant extensions $\partial^{;\mu} \rightarrow D^{;\mu} \equiv \partial^{;\mu} - iG^{\mu}$, and because $\left[G^{\mu}, G^{\nu}\right] \neq 0$, the monopoles $P_{\sigma\mu\nu} = D_{;\sigma}F_{\mu\nu} + D_{;\mu}F_{\nu\sigma} + D_{;\nu}F_{\sigma\mu}$ of Yang Mills become non-vanishing. Consequently, the question of combining the Yang-Mills extensions of (2.4), namely:

$$\begin{aligned} J^{\mu} &= D_{;\sigma}F^{\sigma\mu} = D_{;\sigma}D^{[;\sigma}G^{\mu]} \\ P_{\mu\nu\sigma} &= D_{;\sigma}F_{\mu\nu} + D_{;\mu}F_{\nu\sigma} + D_{;\nu}F_{\sigma\mu} = D_{;\sigma}D_{[;\mu}G_{\nu]} + D_{;\mu}D_{[;\nu}G_{\sigma]} + D_{;\nu}D_{[;\sigma}G_{\mu]} \end{aligned} , \tag{2.8}$$

into a single tensor equation, is a live and viable question that one may explore, precisely because the monopoles are no longer vanishing and so contain real physical content.

As the author has already shown in [13] and [21], see especially [9.1] of [21], the combining of the two tensor equations (2.8) into a single tensor equation yields a magnetic monopole $P_{\sigma\mu\nu}$ which has all of the color symmetries of a baryon, of which the proton and the neutron are the two most important flavored examples. So the short answer to the question "what happens if we combine both of Maxwell's gauge-covariantly-extended equations (2.8) into a single equation using the same field variables?" is simply this: the magnetic monopoles are found to be baryons. So, once we can find a way to add "flavor" to these baryons and develop them into protons and neutrons, we discover that what Einstein wrote about for posterity in his final paper [21] which is "of intrinsic interest quite apart from the particular [non-symmetric field] theory presented here" was no less than a springboard to a unification not only of classical gravitational theory with classical electrodynamics, but a further unification of these two theories with nuclear physics, and of elementary particle physics with nuclear physics as was discussed in the introduction.

On a personal historical note, the author makes of record that this connection of Einstein's final paper [22] to all of these unifications is not merely an afterthought to the authors' own results in [13] and [21]. In fact, Einstein's development of the concept of "the 'strength' of systems of field equations in general" was the key *ab initio* motivating factor which eventually brought the author to the results in [13] and [21]. For, after studying Einstein's paper [22] in 1984, the author emerged with two questions foremost in mind: First, *under what conditions* is Einstein's finding that Maxwell's equations and the Einstein equation *in vacuo* both have $z_1 = 12$ more than a "hunch" that these might be the same physical equations, and in fact an indication that these *are* identical equations merely represented with different "field variables [which] differ with respect to number and kind"? Second, what is the physics that results from combining Maxwell's two equations together into one equation, and can that physics be empirically validated?

Almost 30 years of pursuing these two questions led the author to two conclusions: As to the first question, for Maxwell's equations to be one and the same as $R_{\mu\nu} = 0$ there must be exist both electric and magnetic sources. That is, Maxwell's equations must have *both* non-vanishing electric *and* magnetic sources, such as in the form of (2.8). The middle ground (2.4) between "source-free" and "source-full" electrodynamics, in which one has non-vanishing electric sources yet vanishing magnetic sources, does *not* yield an equivalence between Maxwell's equations and $R_{\mu\nu} = 0$. As will be shown in section 6, e*lectric and magnetic source symmetry is the essential element required to establish the formal connection between Maxwell's equations and* $R_{\mu\nu} = 0$. This is a result that the author obtained in an unpublished paper in 1984 [25] that became the starting point for further development over the next three decades. This symmetry, first developed by Reinich [26] later elaborated by Wheeler [27], and which uses the Levi-Civita as laid out in [28] at pages 87-89, is commonly referred to as "duality."

Thus, what we shall show here is that Einstein's "hunch" that there is some equivalence and thus unification to be found between Maxwell's equations and the empty-space equations $R_{\mu\nu} = 0$ can only be proved if the physical universe contains magnetic charges *in addition to* electric charges. But Maxwell's electrodynamics displays a notorious absence of so-called "magnetic monopoles," which have been pursued ever since the time of Maxwell but never once validated as physically-observed entities. So to prove Einstein's "hunch," one must go beyond classical electrodynamics to study theories in which the magnetic monopoles are non-vanishing. And, for such theories to be *physically-real*, one must ultimately show that these magnetic monopoles exist in the real world in some definitively-observable form. That is, one must establish that there truly is an electric / magnetic duality with both types of source, *in the real* *physical universe*. This is what motivated the author over a period of 30 years to intensely study magnetic monopoles, and – because Yang-Mills theory had achieved demonstrable success in describing weak, electroweak and strong interactions – to thoroughly study Yang-Mills theory as perhaps the most natural way to bring about the non-vanishing magnetic monopoles needed to supply the electric / magnetic source duality symmetry required to validate Einstein's final hunch and ultimately be connected to observed physics phenomenology.

As to the second question, the author concluded that what results from combining Maxwell's two equations together into one equation using the same field variables are – via Yang-Mills theory – magnetic monopoles which are baryons, which are but a flavor-introducing step removed from the protons and neutron we observe in nuclear physics. So, by building the bridge first from particle physics to baryon physics, and then from baryon physics to nuclear physics, and finally by showing that the resulting nuclear physics can be used to explain multiple observed light nuclide binding and fusion energies as well as the proton and neutron rest masses, we validate not only the existence of magnetic monopoles in nature, but the electric and magnetic charge symmetry required to show that the hunch which Einstein articulated in the very last scientific paper of his life was a correct hunch which pointed to way to a complete classical unification among all of gravitational, electromagnetic, nuclear, and elementary particle physics.

PART I: CLASSICAL UNIFICATION OF MAXWELL'S AND YANG-MILLS' ELECTRODYNAMICS, WITH EINSTEIN'S GRAVITATIONAL THEORY IN EMPTY SPACE

3. A Brief Review of Local Energy Conservation in Maxwell's Electrodynamics

In electrodynamics, the energy tensor for a *source-free* electromagnetic field is, of course, given by the Maxwell stress energy tensor (sans the oft-employed coefficient 4π):

$$T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}} = -F^{\alpha\mu}F_{\sigma\mu} + \tfrac{1}{4}\delta^{\alpha}{}_{\sigma}F^{\mu\nu}F_{\mu\nu} \,. \tag{3.1}$$

As has already been noted in (2.5), (2.6), the identity $\partial^{;\sigma}F^{\mu\nu} + \partial^{;\mu}F^{\nu\sigma} + \partial^{;\nu}F^{\sigma\mu} = 0$, which is Maxwell's equation for vanishing magnetic charges, arises <u>if and only if</u> the field strength $F^{\mu\nu}$ is related to the gauge fields G^{ν} by the abelian relationship $F^{\mu\nu} = \partial^{[;\mu}G^{\nu]}$. But because the local conservation of energy depends upon the zero in $\partial^{;\sigma}F^{\mu\nu} + \partial^{;\mu}F^{\nu\sigma} + \partial^{;\nu}F^{\sigma\mu} = 0$, this in turn means that the local conservation of the energy density given in (3.1) is integrally-dependent upon the vanishing of the magnetic monopoles, as we shall now review. And this in turn means that for a theory in which the magnetic monopoles are *non-vanishing*, $\partial^{;\sigma}F^{\mu\nu} + \partial^{;\mu}F^{\nu\sigma} + \partial^{;\nu}F^{\sigma\mu} \neq 0$, or really, the Yang-Mills monopoles $D^{;\sigma}F^{\mu\nu} + D^{;\mu}F^{\nu\sigma} + D^{;\nu}F^{\sigma\mu} \neq 0$ which the author has shown [13], [21] are synonymous with baryons, one needs to give special attention to how the energy remains locally conserved, because the usual abelian-dependent recipe simply does not apply.

Specifically, in any general relativistic theory, the local conservation of energy is specified and enforced by requiring that $\partial_{;\mu} T^{\mu}{}_{\nu} = 0$ for whatever energy tensor one has under consideration. So as a general methodology, one looks for expressions which are identically-equal to a vector of zeroes, which expressions one then equates to $\partial_{;\mu} T^{\mu}{}_{\nu} = 0$. For example, in Riemannian geometry, one starts with $\partial_{;\alpha} R^{\tau\sigma}{}_{\mu\nu} + \partial_{;\mu} R^{\tau\sigma}{}_{\nu\alpha} + \partial_{;\nu} R^{\tau\sigma}{}_{\alpha\mu} = 0$, which is the second Bianchi identity. This is a fifth rank tensor of zeroes, but it is easily contracted along two indexes, say μ, ν. So one contracts $\partial_{;\alpha} R^{\mu\nu}{}_{\mu\nu} + \partial_{;\mu} R^{\mu\nu}{}_{\nu\alpha} + \partial_{;\nu} R^{\mu\nu}{}_{\alpha\mu} = -\partial_{;\alpha} R + 2\partial_{;\mu} R^{\mu}{}_{\alpha} = 0$ which then yields the identity $\partial_{;\mu}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R\right) = 0$, in well-known fashion. One then equates this vector of zeroes via $-\kappa\partial_{;\mu} T^{\mu}{}_{\nu} = \partial_{;\mu}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R\right) = 0$ to the local conservation of energy and momentum, and upon integration sans cosmological constant, arrives at Einstein's gravitational equation $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$.

In electrodynamics, one starts with $\partial_{;\sigma} F_{\mu\nu} + \partial_{;\mu} F_{\nu\sigma} + \partial_{;\nu} F_{\sigma\mu} = 0$. But, as noted, this is only identically equal to zero *because of the abelian field relationship* $F^{\mu\nu} = \partial^{[;\mu} G^{\nu]}$ which causes the magnetic monopoles to vanish. Multiplying through by $\frac{1}{2} F^{\mu\nu}$, one then writes Maxwell's vanishing monopole equation as $\frac{1}{2} F^{\mu\nu}\left(\partial_{;\sigma} F_{\mu\nu} + \partial_{;\mu} F_{\nu\sigma} + \partial_{;\nu} F_{\sigma\mu}\right) = 0$. This is now another identity for a vector of zeroes with a free index σ, just like $\partial_{;\mu}\left(R^{\mu}{}_{\sigma} - \frac{1}{2}\delta^{\mu}{}_{\sigma} R\right) = 0$. So this is what one now equates to the energy conservation relationship $\partial_{;\alpha} T^{\alpha}{}_{\sigma} = 0$, in the form of:

$$\partial_{;\alpha} T^{\alpha}{}_{\sigma} = \frac{1}{2} F^{\mu\nu}\left(\partial_{;\sigma} F_{\mu\nu} + \partial_{;\mu} F_{\nu\sigma} + \partial_{;\nu} F_{\sigma\mu}\right) = 0. \tag{3.2}$$

One then advances this abelian-dependent identity in the following manner:

$$\begin{aligned} 0 = \partial_{;\alpha} T^{\alpha}{}_{\sigma} &= \tfrac{1}{2} F^{\mu\nu}\left(\partial_{;\sigma} F_{\mu\nu} + \partial_{;\mu} F_{\nu\sigma} + \partial_{;\nu} F_{\sigma\mu}\right) \\ &= -F^{\alpha\mu}\partial_{;\alpha} F_{\sigma\mu} + \tfrac{1}{2} F^{\mu\nu}\partial_{;\sigma} F_{\mu\nu} \\ &= -F^{\alpha\mu}\partial_{;\alpha} F_{\sigma\mu} + \tfrac{1}{2}\delta^{\alpha}{}_{\sigma} F^{\mu\nu}\partial_{;\alpha} F_{\mu\nu} \\ &= -\partial_{;\alpha}\left(F^{\alpha\mu} F_{\sigma\mu}\right) + \tfrac{1}{4}\delta^{\alpha}{}_{\sigma}\partial_{;\alpha}\left(F^{\mu\nu} F_{\mu\nu}\right) + F_{\sigma\mu}\partial_{;\alpha} F^{\alpha\mu} \\ &= \partial_{;\alpha}\left(-F^{\alpha\mu} F_{\sigma\mu} + \tfrac{1}{4}\delta^{\alpha}{}_{\sigma} F^{\mu\nu} F_{\mu\nu}\right) + F_{\sigma\mu}\partial_{;\alpha} F^{\alpha\mu} \end{aligned} , \tag{3.3}$$

making use of $F^{\mu\nu}\partial_{;\alpha} F_{\mu\nu} = \frac{1}{2}\partial_{;\alpha}\left(F^{\mu\nu} F_{\mu\nu}\right)$ and $-F^{\alpha\mu}\partial_{;\alpha} F_{\sigma\mu} = -\partial_{;\alpha}\left(F^{\alpha\mu} F_{\sigma\mu}\right) + F_{\sigma\mu}\partial_{;\alpha} F^{\alpha\mu}$ and index renaming as required.

Of course, $J^{\mu} = \partial_{;\alpha} F^{\alpha\mu}$ is Maxell's equation for the density of an electric charge J^{μ}, which is uncovered in the term $F_{\sigma\mu}\partial_{;\alpha} F^{\alpha\mu}$. So, as Einstein first taught at [29] page 155,

$$\kappa_\sigma \equiv F_{\sigma\mu} J^\mu = F_{\sigma\mu}\partial_{;\alpha}F^{\alpha\mu} \tag{3.4}$$

"is a covariant vector the components of which are equal to the negative momentum, or, respectively, the energy which is transferred from the electric masses to the electromagnetic field per unit of time and volume. If the electric masses are free, that is, under the sole influence of the electromagnetic field, the covariant vector κ_σ will vanish." Therefore, (3.3) in view of both (3.1) and (3.4) may be written as:

$$0 = \partial_{;\alpha}T^\alpha{}_\sigma = \partial_{;\alpha}T^\alpha{}_{\sigma\ \text{Maxwell}} + \kappa_\sigma = 0\,. \tag{3.5}$$

Thus, equation (3.5), "if κ_σ vanishes, is" the equation of conservation for the energy-momentum of the Maxwell tensor. In essence, (3.2) through (3.5) are simply another way of presenting what Einstein initially developed at pages 155 and 156 of [29].

From the heart of (3.3), one may extract the very useful identity:

$$\tfrac{1}{2}F^{\mu\nu}\left(\partial_{;\sigma}F_{\mu\nu} + \partial_{;\mu}F_{\nu\sigma} + \partial_{;\nu}F_{\sigma\mu}\right) = \partial_{;\alpha}\left(-F^{\alpha\mu}F_{\sigma\mu} + \tfrac{1}{4}\delta^\alpha{}_\sigma F^{\mu\nu}F_{\mu\nu}\right) + F_{\sigma\mu}\partial_{;\alpha}F^{\alpha\mu} \tag{3.6}$$

which applies in *all* circumstances, whether we employ the abelian relationship $F^{\mu\nu} = \partial^{[;\mu}G^{\nu]}$ or the non-abelian $F^{\mu\nu} = D^{[\mu}G^{\nu]}$ of (2.7), and whether or not $\kappa_\sigma = F_{\sigma\mu}\partial_{;\alpha}F^{\alpha\mu}$ vanishes. Thus, we may somewhat invert the analysis of (3.2) through (3.5) in the manner presented by Misner, Thorne and Wheeler in §20.6 of [28], as follows:

Starting with (3.6), if the fields are abelian, $F^{\mu\nu} = \partial^{[;\mu}G^{\nu]}$, then the monopoles vanish, $\partial_{;\sigma}F_{\mu\nu} + \partial_{;\mu}F_{\nu\sigma} + \partial_{;\nu}F_{\sigma\mu} = 0$ and so (3.6) goes to zero to yield:

$$0 = \partial_{;\alpha}\left(-F^{\alpha\mu}F_{\sigma\mu} + \tfrac{1}{4}\delta^\alpha{}_\sigma F^{\mu\nu}F_{\mu\nu}\right) + F_{\sigma\mu}\partial_{;\alpha}F^{\alpha\mu}\,. \tag{3.7}$$

Then, once we identify the expression $-F^{\alpha\mu}F_{\sigma\mu} + \tfrac{1}{4}\delta^\alpha{}_\sigma F^{\mu\nu}F_{\mu\nu}$ with the Maxwell tensor $T^\alpha{}_{\sigma\ \text{Maxwell}}$ and impose the condition $\partial_{;\alpha}T^\alpha{}_{\sigma\ \text{Maxwell}} = \partial_{;\alpha}\left(-F^{\alpha\mu}F_{\sigma\mu} + \tfrac{1}{4}\delta^\alpha{}_\sigma F^{\mu\nu}F_{\mu\nu}\right) = 0$ for the local conservation of energy, (3.7) will further reduce to:

$$\kappa_\sigma \equiv F_{\sigma\mu} J^\mu = F_{\sigma\mu}\partial_{;\alpha}F^{\alpha\mu} = 0\,, \tag{3.8}$$

which is equation [20.38] of [28].

Closely related, for source-free electrodynamics, $J^\mu = 0$, and so one has the field equations:

$$
\begin{aligned}
&0=\partial_{;\sigma}F^{\sigma\mu} \\
&0=\partial_{;\sigma}F_{\mu\nu}+\partial_{;\mu}F_{\nu\sigma}+\partial_{;\nu}F_{\sigma\mu}
\end{aligned}
. \qquad (3.9)
$$

As already discussed, Einstein showed in [22] that these have a coefficient of freedom $z_1 = 12$ "surprisingly" equal to that of $R_{\mu\nu} = 0$, see (2.2) et seq. For the source-free (3.9), one will automatically also have (3.8) which then implies via (3.7) that $\partial_{;\alpha}T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}=0$, which locally conserves the Maxwell tensor. But while $J^{\mu}=0$ plus the abelian relation $F^{\mu\nu}=\partial^{[;\mu}G^{\nu]}$ implies that $\kappa_\sigma = 0$ and thus that $\partial_{;\alpha}T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}=0$, the converse is not *always* true. As noted in exercise 20.8 of [28], $\kappa_\sigma = 0$ will only conversely imply $J^{\mu}=0$ when one does *not* have $\mathbf{E}\cdot\mathbf{B}=0$ over an extended region of spacetime. But for the "generic case," ([28] page 472, ninth text line) $T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}$ together with $\partial_{;\alpha}T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}=0$ do lead to $0=\partial_{;\sigma}F^{\sigma\mu}$ as an "equation of motion." It is for these reasons that $T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}$ in (3.1) is often referred to as the stress energy tensor for *source-free* electromagnetic fields, meaning, specifically, fields which are free of both magnetic *and* electric charges as in (3.9).

Specifically, in all cases, for source-free electrodynamics (3.9), $\kappa_\sigma = 0$ in (3.8), thus $\partial_{;\alpha}T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}=0$ as in (3.5), so $T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}$ is locally conserved in the same manner that $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R$ is connected to conserved energy via $\partial_{;\mu}\left(R^{\mu}{}_{\sigma} - \frac{1}{2}\delta^{\mu}{}_{\sigma}R\right)=0$. That is, in source-free electrodynamics, the Maxwell stress energy tensor (3.1) is always locally-conserved.

But the source free electrodynamics of (3.9) is, of course, a mathematical idealization, and applies physically only to fields $F^{\mu\nu}$ passing through charge-free and baryon-free ([13], [21]) regions of spacetime. In the real world we certainly observe $J^{\mu}\neq 0$ all the time, and in today's days and age, most of the technology we use is based on the harvesting of the electrons that underlie $J^{\mu}\neq 0$. Further, because the magnetic monopoles of Yang-Mills gauge theory $P_{\sigma\mu\nu}=D_{;\sigma}F_{\mu\nu}+D_{;\mu}F_{\nu\sigma}+D_{;\nu}F_{\sigma\mu}\neq 0$ are non-vanishing, and especially if these are observed in the form of baryons as the author has shown in [13], [21], then we also observe $P_{\sigma\mu\nu}\neq 0$ all the time, even though this has only recently become known.

So, if we wish to consider the energy tensor $T^{\mu}{}_{\nu}$ for $J^{\mu}\neq 0$ generally, and especially for $\kappa_\sigma \neq 0$ and $P_{\sigma\mu\nu}\neq 0$ which comes about from the *non-abelian* gauge fields of Yang-Mills theory, then we will need to find an identity other than $P_{\sigma\mu\nu}=\partial_{;\sigma}F_{\mu\nu}+\partial_{;\mu}F_{\nu\sigma}+\partial_{;\nu}F_{\sigma\mu}=0$ as the basis for locally conserving energy via $\partial_{;\mu}T^{\mu}{}_{\nu}=0$, because this identity relies on being zeroed out via $F^{\mu\nu}=\partial^{[;\mu}G^{\nu]}$ as in (3.3) and (3.7). To fully develop "source-full" rather than "source-free" electrodynamics, we will need to use the formalism of electric / magnetic duality. And it is through this approach, that we will be able validate Einstein's final hunch in [22] about a "surprising" "strength" equivalence between the Maxwell system of equations and the empty space equation $R_{\mu\nu}=0$.

4. A Brief Review of Electric / Magnetic Duality

Edward Witten begins an examination of electric / magnetic duality at page 28 of [30]

"... with a piece of late-19th-century physics. The vacuum Maxwell equations for the electric and magnetic fields **E** and **B** ... have a symmetry under $\mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E}$ that has been known for nearly as long as the Maxwell equations themselves. This symmetry is known as duality.

The symmetry still holds in the presence of charges and currents if one adds both electric and magnetic charges and currents. In nature, such symmetry seems to be spoiled by the fact that we observe electric charges but not magnetic charges (which are usually called magnetic monopoles)."

"More fundamentally," Witten observes ...

"the symmetry seems to be violated when we derive the magnetic field from a vector potential $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$, while representing the electric field (in a static situation) as the gradient of a scalar."

"But," he points out ...

"the vector potential is not just a convenience in solving Maxwell equations. It is needed in 20th-century physics for three very good purposes:

- To write a Schrödinger equation for an electron in a magnetic field.
- To make it possible to derive Maxwell's equations from a Lagrangian.
- To write anything at all for non-Abelian gauge theory, which – in our modern understanding of elementary particle physics – is the starting point in describing the strong, weak and electromagnetic interactions."

Indeed, in the late 19th-century, the German physicist Hermann von Helmholtz believed that that the vector potential was an unnecessary element of Maxwell's theory, what was later belied for the reasons Witten lays out.

The mathematical formalism used to most efficiently study electric / magnetic duality was first proposed by Reinich in [26] and later elaborated and popularized by Wheeler in [27]. It makes liberal use of the Levi-Civita tensor $\varepsilon_{\mu\nu\sigma\tau}$ which is totally antisymmetric in all spacetime indexes, and for which the covariant (lower-index) component $\varepsilon_{0123} = +1$. Because indexes in flat Minkowski spacetime are raised and lowered with $\mathrm{diag}(\eta_{\mu\nu}) = (+1,-1,-1,-1)$, the contravariant $\varepsilon^{0123} = -1$ and so $\varepsilon_{0123}\varepsilon^{0123} = -1$ is negatively-signed. This duality formalism is further elaborated by Misner, Wheeler and Throne in Exercises 3.13 and 3.14 of [28], and as developed in chapter 4 of [28], is of fundamental utility in the differential forms through which one is able to write Maxwell's equations in generally-covariant integral form via Gauss' / Stokes' theorem.

For any contravariant vector J^{τ}, second-rank antisymmetric tensor $F^{\sigma\tau}$ and third-rank antisymmetric tensor $B^{\mu\nu\sigma}$, the respective duals (*) are defined as in [3.51] of [28], by (here we use *P* rather than *B* to denote the third rank tensor because we shall wish to associate this with the magnetic monopole / baryons of Yang-Mills gauge theory):

$$*J_{\mu\nu\sigma} = \left(-g\right)^{.5} \varepsilon_{\tau\mu\nu\sigma} J^{\tau}; \quad *F_{\mu\nu} = \tfrac{1}{2!}\left(-g\right)^{.5} \varepsilon_{\sigma\tau\mu\nu} F^{\sigma\tau}; \quad *P_{\tau} = \tfrac{1}{3!}\left(-g\right)^{.5} \varepsilon_{\mu\nu\sigma\tau} P^{\mu\nu\sigma}, \tag{4.1}$$

where from [8.10a] of [28], see also equation [11] of [27], we include the $\left(-g\right)^{.5}$ factor which is needed in curved spacetime and which is equal to unity, $\left(-g\right)^{.5} = 1$, in flat spacetime. A double application of duality yields a sign reversal, $**\mathbf{F} = -\mathbf{F}$ for the second-rank duals, while it restores the sign $**\mathbf{J} = \mathbf{J}$ and $**\mathbf{P} = \mathbf{P}$. It is a good warm up exercise to show explicitly how these double-dual relationships are obtained.

For second rank duals, one writes the relationship from (4.1) twice over, as $*F_{\alpha\beta} = \tfrac{1}{2!}\left(-g\right)^{.5} \varepsilon_{\sigma\tau\alpha\beta} F^{\sigma\tau}$ and $*F^{\mu\nu} = \tfrac{1}{2!}\left(-g\right)^{-.5} \varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta}$, see equation [8.10a] of [28]. Combining, one then writes:

$$**F^{\mu\nu} = \tfrac{1}{2!}\left(-g\right)^{.5} \varepsilon^{\alpha\beta\mu\nu} *F_{\alpha\beta} = \tfrac{1}{4} \varepsilon^{\alpha\beta\mu\nu} \varepsilon_{\sigma\tau\alpha\beta} F^{\sigma\tau}. \tag{4.2}$$

This introduces the summed expression $\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\sigma\tau\alpha\beta}$, which sorts of expressions are frequently encountered in the identities of the duality formalism. In this instance, (see [3.50i] in [28]):

$$\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\sigma\tau\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\sigma\tau\alpha\beta} = -\delta^{\mu\nu\alpha}{}_{\sigma\tau\alpha} = -2!\,\delta^{\mu\nu}{}_{\sigma\tau}, \tag{4.3}$$

with the sign reversal in $\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\sigma\tau\alpha\beta} = -\delta^{\mu\nu\alpha}{}_{\sigma\tau\alpha}$ originating in the relation $\varepsilon_{0123}\varepsilon^{0123} = -1$ noted above. The fourth-rank Kronecker delta in turn is given by (see [3.50l] in [28]):

$$\delta^{\mu\nu}{}_{\sigma\tau} = \delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\tau} - \delta^{\mu}{}_{\tau}\delta^{\nu}{}_{\sigma}, \tag{4.4}$$

so that using (4.3) and (4.4) in (4.2) yields:

$$**F^{\mu\nu} = \tfrac{1}{4}\varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\sigma\tau\alpha\beta}F^{\sigma\tau} = -\tfrac{1}{2}\delta^{\mu\nu}{}_{\sigma\tau}F^{\sigma\tau} = -\tfrac{1}{2}\left(\delta^{\mu}{}_{\sigma}\delta^{\nu}{}_{\tau} - \delta^{\mu}{}_{\tau}\delta^{\nu}{}_{\sigma}\right)F^{\sigma\tau} = -F^{\mu\nu}, \tag{4.5}$$

which is the relation $**\mathbf{F} = -\mathbf{F}$. Since **=-1 and ****=+1, each * is in the nature of the imaginary number $i = \sqrt{-1}$.

For the first rank J^{τ}, we similarly combine the two expressions $*J_{\mu\nu\sigma} = \left(-g\right)^{.5} \varepsilon_{\tau\mu\nu\sigma} J^{\tau}$ and $*J^{\alpha} = \tfrac{1}{3!}\left(-g\right)^{-.5} \varepsilon^{\mu\nu\sigma\alpha} J_{\mu\nu\sigma}$ as such:

$$** J^{\alpha} = \tfrac{1}{3!}\left(-g\right)^{-.5} \varepsilon^{\mu\nu\sigma\alpha} * J_{\mu\nu\sigma} = \tfrac{1}{3!}\varepsilon^{\mu\nu\sigma\alpha}\varepsilon_{\tau\mu\nu\sigma} J^{\tau} . \tag{4.6}$$

Now, we have the summed Levi-Civita expression (see [3.50j] in [28]):

$$\varepsilon^{\mu\nu\sigma\alpha}\varepsilon_{\tau\mu\nu\sigma} = -\varepsilon^{\alpha\mu\nu\sigma}\varepsilon_{\tau\mu\nu\sigma} = \delta^{\alpha\mu\nu}{}_{\tau\mu\nu} = 2!\delta^{\alpha\mu}{}_{\tau\mu} = 3!\delta^{\alpha}{}_{\tau} . \tag{4.7}$$

Used in (4.6) one immediately discerns:

$$** J^{\alpha} = \tfrac{1}{3!}\varepsilon^{\mu\nu\sigma\alpha}\varepsilon_{\tau\mu\nu\sigma} J^{\tau} = \delta^{\alpha}{}_{\tau} J^{\tau} = J^{\alpha} , \tag{4.8}$$

which is the relation $**\mathbf{J} = \mathbf{J}$. Here, since **=1, each * is in the nature of a minus sign.

For third-rank $P^{\mu\nu\sigma}$, we combine $*P^{\delta\gamma\lambda} = \left(-g\right)^{-.5} \varepsilon^{\alpha\delta\gamma\lambda} P_{\alpha}$ and $*P_{\alpha} = \tfrac{1}{3!}\left(-g\right)^{.5} \varepsilon_{\mu\nu\sigma\alpha} P^{\mu\nu\sigma}$:

$$** P^{\delta\gamma\lambda} = \left(-g\right)^{-.5} \varepsilon^{\alpha\delta\gamma\lambda} * P_{\alpha} = \tfrac{1}{3!}\varepsilon^{\alpha\delta\gamma\lambda}\varepsilon_{\mu\nu\sigma\alpha} P^{\mu\nu\sigma} . \tag{4.9}$$

Now the Levi-Civita summation is (see [3.50h] in [28]):

$$\varepsilon^{\alpha\delta\gamma\lambda}\varepsilon_{\mu\nu\sigma\alpha} = -\varepsilon^{\delta\gamma\lambda\alpha}\varepsilon_{\mu\nu\sigma\alpha} = \delta^{\delta\gamma\lambda}{}_{\mu\nu\sigma} , \tag{4.10}$$

in which the sixth-rank Kronecker delta is given by:

$$\delta^{\delta\gamma\lambda}{}_{\mu\nu\sigma} = \delta^{\delta}{}_{\mu}\delta^{\gamma}{}_{\nu}\delta^{\lambda}{}_{\sigma} + \delta^{\delta}{}_{\nu}\delta^{\gamma}{}_{\sigma}\delta^{\lambda}{}_{\mu} + \delta^{\delta}{}_{\sigma}\delta^{\gamma}{}_{\mu}\delta^{\lambda}{}_{\nu} - \delta^{\delta}{}_{\mu}\delta^{\gamma}{}_{\sigma}\delta^{\lambda}{}_{\nu} - \delta^{\delta}{}_{\sigma}\delta^{\gamma}{}_{\nu}\delta^{\lambda}{}_{\mu} - \delta^{\delta}{}_{\nu}\delta^{\gamma}{}_{\mu}\delta^{\lambda}{}_{\sigma} . \tag{4.11}$$

Upon using (4.10) and (4.11) in (4.9) and reducing, we find that:

$$** P^{\delta\gamma\lambda} = \tfrac{1}{3!}\varepsilon^{\alpha\delta\gamma\lambda}\varepsilon_{\mu\nu\sigma\alpha} P^{\mu\nu\sigma} = \tfrac{1}{3!}\delta^{\delta\gamma\lambda}{}_{\mu\nu\sigma} P^{\mu\nu\sigma} = P^{\delta\gamma\lambda} . \tag{4.12}$$

With $\sqrt{-g} = 1$, this is the relation $**\mathbf{P} = \mathbf{P}$.

Now let us look at Maxwell's equations in view of the duality formalism, but extended via Yang-Mills gauge theory. We start with the Yang-Mills magnetic monopole equation from (2.8), which in view of the non-Abelian relationship $F^{\mu\nu} = D^{[;\mu}G^{\nu]}$ of (2.7), is non-vanishing. If we write the monopole duality relationship from (4.1) as $*P^{\tau} = \tfrac{1}{3!}\left(-g\right)^{-.5} \varepsilon^{\mu\nu\sigma\tau} P_{\mu\nu\sigma}$, and then multiply the monopole in (2.8) through by $\tfrac{1}{3!}\left(-g\right)^{-.5} \varepsilon^{\mu\nu\sigma\tau}$, one obtains:

$$\begin{aligned} *P^{\tau} &= \tfrac{1}{3!}\left(-g\right)^{-.5} \varepsilon^{\mu\nu\sigma\tau} P_{\mu\nu\sigma} = \tfrac{1}{3!}\left(-g\right)^{-.5} \varepsilon^{\mu\nu\sigma\tau} \left(D_{;\sigma}F_{\mu\nu} + D_{;\mu}F_{\nu\sigma} + D_{;\nu}F_{\sigma\mu}\right) \\ &= \tfrac{1}{3!}\left(-g\right)^{-.5} \left(D_{;\sigma}\varepsilon^{\mu\nu\sigma\tau}F_{\mu\nu} + D_{;\mu}\varepsilon^{\mu\nu\sigma\tau}F_{\nu\sigma} + D_{;\nu}\varepsilon^{\mu\nu\sigma\tau}F_{\sigma\mu}\right) \end{aligned} . \tag{4.13}$$

One may then apply the second-rank expression from (4.1) in the form $2(-g)^{.5} * F^{\mu\nu} = \varepsilon^{\sigma\tau\mu\nu} F_{\sigma\tau}$ three times over with suitable index renaming to write the above as:

$$\begin{aligned}
*P^{\tau} &= \tfrac{1}{3!}(-g)^{-.5}\left(D_{;\sigma}\varepsilon^{\mu\nu\sigma\tau}F_{\mu\nu} + D_{;\mu}\varepsilon^{\mu\nu\sigma\tau}F_{\nu\sigma} + D_{;\nu}\varepsilon^{\mu\nu\sigma\tau}F_{\sigma\mu}\right) \\
&= \tfrac{1}{3}(-g)^{-.5}\left(D_{;\sigma}\left((-g)^{.5} * F^{\sigma\tau}\right) + D_{;\sigma}\left((-g)^{.5} * F^{\sigma\tau}\right) + D_{;\sigma}\left((-g)^{.5} * F^{\sigma\tau}\right)\right) \\
&= (-g)^{-.5} D_{;\sigma}\left((-g)^{.5} * F^{\sigma\tau}\right) \\
&= D_{;\sigma} * F^{\sigma\tau}
\end{aligned} . \tag{4.14}$$

The $g = \det(g_{\mu\nu})$ term drops out in the final line, because if one expands the gauge- and gravitationally-covariant $D_{;\sigma} \to \partial_{;\sigma} - iG_{\sigma}$, the resulting $\partial_{;\sigma}(-g)^{.5} = -.5(-g)^{-.5}\partial_{;\sigma}g = 0$ by virtue of the metricity condition $\partial_{;\sigma}g_{\mu\nu} = 0$, and then the remaining $(-g)^{-.5}(-g)^{.5} = 1$ offsets to 1. Then we simply reassemble $\partial_{;\sigma} - iG_{\sigma} \to D_{;\sigma}$. So putting the above together with the Yang-Mills electric charge equation from (2.8) as well as the non-abelian relationship (2.7), we find that the Maxwell equations for Yang-Mills gauge theory, expressed via duality, are:

$$\begin{aligned}
J^{\mu} &= D_{;\sigma}F^{\sigma\mu} = D_{;\sigma}D^{[;\sigma}G^{\mu]} \\
*P^{\mu} &= D_{;\sigma} * F^{\sigma\mu} = D_{;\sigma} * D^{[;\sigma}G^{\mu]}
\end{aligned} . \tag{4.15}$$

$*P^{\tau} = \frac{1}{3!}\sqrt{-g}\,\varepsilon^{\mu\nu\sigma\tau}P_{\mu\nu\sigma}$ is the first rank dual of the magnetic monopole baryon $P_{\mu\nu\sigma}$, and in Yang-Mills gauge theory, it is clearly non-vanishing.

Based on (4.13) through (4.15), it is easy to see without repeating this calculation how the source-free electrodynamic equations (3.9), when written using duality, become:

$$\begin{aligned}
0 &= \partial_{;\sigma}F^{\sigma\mu} = \partial_{;\sigma}\partial^{[;\sigma}G^{\mu]} \\
0 &= \partial_{;\sigma} * F^{\sigma\mu} = \partial_{;\sigma} * \partial^{[;\sigma}G^{\mu]}
\end{aligned} , \tag{4.16}$$

and "have a symmetry under $\mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E}$" as excerpted from Witten's [30] at the outset of this section, cycling in the same manner as $i^4 = +1$. And, it is readily seen as well how electrodynamics with electrical sources as represented by (2.4) may be written using duality as:

$$\begin{aligned}
J^{\sigma} &= \partial_{;\sigma}F^{\sigma\mu} = \partial_{;\sigma}\partial^{[;\sigma}G^{\mu]} \\
0 &= \partial_{;\sigma} * F^{\sigma\mu} = \partial_{;\sigma} * \partial^{[;\sigma}G^{\mu]}
\end{aligned} , \tag{4.17}$$

which highlights, per Witten, how this symmetry is "spoiled by the fact that we observe electric charges but not magnetic charges." But most importantly for the present purposes, we see how the Yang-Mills extensions of Maxwell, (4.15), do restore the duality symmetry of the source-free equations (3.9), or in the words of Witten, how "the symmetry still holds in the presence of charges and currents if one adds both electric and magnetic charges and currents."

One other useful exercise before concluding this review of duality, is to develop the third-rank electric current source $*J_{\mu\nu\sigma}$. We start by *defining*:

$$*J_{\mu\nu\sigma} \equiv -\left(D_{;\sigma} *F_{\mu\nu} + D_{;\mu} *F_{\nu\sigma} + D_{;\nu} *F_{\sigma\mu}\right). \tag{4.18}$$

We keep in mind from (4.1) with a symbol renaming $P \to J$ that $*J^{\tau} = \tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\mu\nu\sigma\tau} J_{\mu\nu\sigma}$. Because **=1 for first and third rank duals, this means that $J^{\tau} = **J^{\tau} = \tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\mu\nu\sigma\tau} *J_{\mu\nu\sigma}$ is the current density J^{τ}. And also from (4.1), we know that $*F_{\mu\nu} = \tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}$. So we combine all of this with the definition (4.18) to write:

$$\begin{aligned} J^{\tau} &= \tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\mu\nu\sigma\tau} *J_{\mu\nu\sigma} = -\tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\mu\nu\sigma\tau}\left(D_{;\sigma} *F_{\mu\nu} + D_{;\mu} *F_{\nu\sigma} + D_{;\nu} *F_{\sigma\mu}\right) \\ &= -\tfrac{1}{2!}\tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\mu\nu\sigma\tau}\left(D_{;\sigma}\left((-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}\right) + D_{;\mu}\left((-g)^{.5}\varepsilon_{\alpha\beta\nu\sigma}F^{\alpha\beta}\right) + D_{;\nu}\left((-g)^{.5}\varepsilon_{\alpha\beta\sigma\mu}F^{\alpha\beta}\right)\right) \end{aligned}. \tag{4.19}$$

As in (4.14) the $(-g)^{.5}$ terms can be extracted from the derivatives because of the metricity $\partial_{;\sigma}g_{\mu\nu} = 0$ and are then cancelled via $(-g)^{-.5}(-g)^{.5} = 1$. As to the Levi-Civita tensor contractions, one has the three relationships for $\varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\alpha\beta\mu\nu}$, $\varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\alpha\beta\nu\sigma}$ and $\varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\alpha\beta\sigma\mu}$, each with slightly-varied indexes. So using (4.3) and (4.4) with simple reindexing, the balance of the calculation to reduce (4.19) proceeds as follows:

$$\begin{aligned} J^{\tau} &= -\tfrac{1}{2!}\tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\mu\nu\sigma\tau}\left(D_{;\sigma}\left((-g)^{.5}\varepsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}\right) + D_{;\mu}\left((-g)^{.5}\varepsilon_{\alpha\beta\nu\sigma}F^{\alpha\beta}\right) + D_{;\nu}\left((-g)^{.5}\varepsilon_{\alpha\beta\sigma\mu}F^{\alpha\beta}\right)\right) \\ &= -\tfrac{1}{2!}\tfrac{1}{3!}\left(\varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\alpha\beta\mu\nu}D_{;\sigma}F^{\alpha\beta} + \varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\alpha\beta\nu\sigma}D_{;\mu}F^{\alpha\beta} + \varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\alpha\beta\sigma\mu}D_{;\nu}F^{\alpha\beta}\right) \\ &= \tfrac{1}{3!}\left(\left(\delta^{\sigma}{}_{\alpha}\delta^{\tau}{}_{\beta} - \delta^{\sigma}{}_{\beta}\delta^{\tau}{}_{\alpha}\right)D_{;\sigma}F^{\alpha\beta} + \left(\delta^{\mu}{}_{\alpha}\delta^{\tau}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\tau}{}_{\alpha}\right)D_{;\mu}F^{\alpha\beta} + \left(\delta^{\mu}{}_{\alpha}\delta^{\tau}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\tau}{}_{\alpha}\right)D_{;\nu}F^{\alpha\beta}\right) \\ &= D_{;\sigma}F^{\sigma\tau} \end{aligned}. \tag{4.20}$$

This is equivalent to the Yang-Mills electric charge density $J^{\mu} = D_{;\sigma}F^{\sigma\mu}$ of (4.15), and so establishes that definition (4.18) is just another way of writing $J^{\mu} = D_{;\sigma}F^{\sigma\mu}$. This is an example of why Misner, Thorne and Wheeler state after [3.52] of [28] that these various tensors and their duals "contain precisely the same information."

So, while (4.15) is one way to write the Yang-Mills Maxwell equations, another way to do so combines (4.18) with the latter of (2.8), and further with (2.7), as:

$$-*J_{\mu\nu\sigma} = D_{;\sigma} * F_{\mu\nu} + D_{;\mu} * F_{\nu\sigma} + D_{;\nu} * F_{\sigma\mu} = D_{;\sigma} * D_{[;\mu}G_{\nu]} + D_{;\mu} * D_{[;\nu}G_{\sigma]} + D_{;\nu} * D_{[;\sigma}G_{\mu]}$$
$$P_{\mu\nu\sigma} = D_{;\sigma}F_{\mu\nu} + D_{;\mu}F_{\nu\sigma} + D_{;\nu}F_{\sigma\mu} = D_{;\sigma}D_{[;\mu}G_{\nu]} + D_{;\mu}D_{[;\nu}G_{\sigma]} + D_{;\nu}D_{[;\sigma}G_{\mu]} . \quad (4.21)$$

While equivalent in information to (4.15), these latter expressions (4.21) lend themselves most conveniently to differential forms, in which they are written with the three-forms *J* and *P* as:

$$-*J = D*F = D*DG$$
$$P_{\mu\nu\sigma} = DF = DDG . \quad (4.22)$$

The minus sign is required for consistency with $J^{\tau} = D_{;\sigma}F^{\sigma\tau}$ as shown in (4.18) through (4.20). But in combination with the fact that **=1 for first and third rank duality, as noted after (4.8), the * in $-*J$ acts as an offsetting "quasi-minus sign" to maintain consistency with $J^{\tau} = D_{;\sigma}F^{\sigma\tau}$.

5. Several Important Duality Identities, Carried over to Yang-Mills Gauge Theories

With these preliminaries, there are two duality identities that will be of great interest here, which hold for any two antisymmetric tensors **A** and **B** in spacetime. First, from footnote 19 on page 239 of [27]:

$$A_{\mu\alpha}B^{\nu\alpha} - *A_{\mu\alpha} * B^{\nu\alpha} = \tfrac{1}{2}\delta_{\mu}{}^{\nu} A_{\sigma\tau}B^{\sigma\tau} . \quad (5.1)$$

Second, from footnote 22 on page 251 of [27], transposing indexes in $*B^{\beta\alpha} = -*B^{\alpha\beta}$ to match the index structures in (3.3) and (3.4):

$$\tfrac{1}{2}A^{\alpha\beta}\left(\partial_{;\beta}B_{\mu\alpha} + \partial_{;\mu}B_{\alpha\beta} + \partial_{;\alpha}B_{\beta\mu}\right) + *A_{\mu\alpha}\partial_{;\beta} * B^{\beta\alpha} = 0 . \quad (5.2)$$

The first identity (5.1) is best reviewed by evaluating the product $*A_{\mu\alpha} * B^{\nu\alpha}$. From (5.1) we may write $*A_{\mu\alpha} = \tfrac{1}{2!}(-g)^{.5}\,\varepsilon_{\sigma\tau\mu\alpha}A^{\sigma\tau}$ and $*B^{\nu\alpha} = \tfrac{1}{2!}(-g)^{-.5}\,\varepsilon^{\beta\rho\nu\alpha}B_{\beta\rho}$. This means that:

$$*A_{\mu\alpha} * B^{\nu\alpha} = \tfrac{1}{4}\varepsilon_{\sigma\tau\mu\alpha}\varepsilon^{\beta\rho\nu\alpha}A^{\sigma\tau}B_{\beta\rho} . \quad (5.3)$$

We have already come across a term like $\varepsilon^{\sigma\tau\mu\alpha}\varepsilon_{\beta\rho\nu\alpha}$ in (4.10) and (4.11). Applying those expressions with the indexes in (5.3) enables us to convert (5.3) into:

$$
\begin{aligned}
&*A_{\mu\alpha} * B^{\nu\alpha} = \tfrac{1}{4}\varepsilon_{\sigma\tau\mu\alpha}\varepsilon^{\beta\rho\nu\alpha}A^{\sigma\tau}B_{\beta\rho} \\
&= \tfrac{1}{4}\left(-\delta_\sigma{}^\beta\delta_\tau{}^\rho\delta_\mu{}^\nu - \delta_\tau{}^\beta\delta_\mu{}^\rho\delta_\sigma{}^\nu - \delta_\mu{}^\beta\delta_\sigma{}^\rho\delta_\tau{}^\nu + \delta_\sigma{}^\beta\delta_\mu{}^\rho\delta_\tau{}^\nu + \delta_\mu{}^\beta\delta_\tau{}^\rho\delta_\sigma{}^\nu + \delta_\tau{}^\beta\delta_\sigma{}^\rho\delta_\mu{}^\nu\right)A^{\sigma\tau}B_{\beta\rho} \\
&= \tfrac{1}{4}\left(+A^{\nu\tau}B_{\mu\tau} + A^{\nu\sigma}B_{\mu\sigma} + A^{\nu\sigma}B_{\mu\sigma} + A^{\nu\tau}B_{\mu\tau} - \delta_\mu{}^\nu A^{\sigma\tau}B_{\sigma\tau} - \delta_\mu{}^\nu A^{\sigma\tau}B_{\sigma\tau}\right) \\
&= A^{\nu\alpha}B_{\mu\alpha} - \tfrac{1}{2}\delta_\mu{}^\nu A^{\sigma\tau}B_{\sigma\tau}
\end{aligned}
\quad . \tag{5.4}
$$

Rearranging, we find a slight variant of (5.1), namely:

$$
A^{\nu\alpha}B_{\mu\alpha} - *A_{\mu\alpha} * B^{\nu\alpha} = \tfrac{1}{2}\delta_\mu{}^\nu A_{\sigma\tau}B^{\sigma\tau} . \tag{5.5}
$$

This is equivalent to (5.1) when the commutativity relationship $A^{\nu\alpha}B_{\mu\alpha} = A_{\mu\alpha}B^{\nu\alpha}$ applies, which is to say, when $A_{[\nu}{}^{\alpha}B_{\mu]\alpha} = 0$. If we set $\mathbf{A},\mathbf{B} \to \mathbf{F}$, with a bit of rearrangement this becomes:

$$
F^{\mu\alpha}F_{\nu\alpha} - *F^{\mu\alpha} * F_{\nu\alpha} = \tfrac{1}{2}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau} , \tag{5.6}
$$

which is equivalent to equation [15] in [27]. Combining $-\frac{1}{2}$ times (5.6) with the Maxwell stress-energy tensor (3.1) now enables us to now write:

$$
\begin{aligned}
T^\mu{}_{\nu\ \text{Maxwell}} &= -F^{\mu\alpha}F_{\nu\alpha} + \tfrac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau} = -\tfrac{1}{2}F^{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{2}F^{\mu\alpha}F_{\nu\alpha} + \tfrac{1}{4}\delta^\mu{}_\nu F^{\sigma\tau}F_{\sigma\tau} \\
&= -\tfrac{1}{2}\left(F^{\mu\alpha}F_{\nu\alpha} + *F^{\mu\alpha} * F_{\nu\alpha}\right)
\end{aligned}
\quad . \tag{5.7}
$$

We showed in section 3 that for source free electrodynamics (3.9), $\kappa_\sigma = 0$ in (3.8) and so via (3.5) the Maxwell tensor is locally-conserved, $\partial_{;\alpha}T^\alpha{}_{\sigma\ \text{Maxwell}} = 0$. We also observed in (4.16), echoing Witten, that source free electrodynamics has a duality "symmetry under $\mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E}$", which in terms of $F^{\mu\alpha}$ is expressed as a symmetry under $F^{\mu\nu} \to *F^{\mu\nu}$. Now, in view of (5.7), and given from (4.5) that $**F^{\mu\nu} = -F^{\mu\nu}$, we see that $T^\mu{}_{\nu\ \text{Maxwell}}$ *also* is symmetric under the duality transformation $F^{\mu\nu} \to *F^{\mu\nu}$. For, if we set $F^{\mu\nu} \to *F^{\mu\nu}$ in the final line of (5.7), we find that:

$$
T^\mu{}_{\nu\ \text{Maxwell}} = -\tfrac{1}{2}\left(F^{\mu\alpha}F_{\nu\alpha} + *F^{\mu\alpha} * F_{\nu\alpha}\right) \to -\tfrac{1}{2}\left(*F^{\mu\alpha} * F_{\nu\alpha} + F^{\mu\alpha}F_{\nu\alpha}\right) = T^\mu{}_{\nu\ \text{Maxwell}} \tag{5.8}
$$

remains completely invariant. Putting (4.16) together with (5.7), we may see the duality symmetry of source-free electrodynamics, in all respects, by assembling:

$$
\begin{aligned}
&0 = \partial_{;\sigma}F^{\sigma\mu} \\
&0 = \partial_{;\sigma} * F^{\sigma\mu} \\
&T^\mu{}_{\nu\ \text{Maxwell}} = -\tfrac{1}{2}\left(F^{\mu\alpha}F_{\nu\alpha} + *F^{\mu\alpha} * F_{\nu\alpha}\right)
\end{aligned}
\quad , \tag{5.9}
$$

which is manifestly invariant under the duality transformation $F^{\mu\nu} \to *F^{\mu\nu}$. We shall make use of this well-known observation (e.g., [27], equation [14a]) in the development to follow.

Also very pertinent to the present development, pointed out at pages 473 and 483 of [28], "Maxwell's equations of motion are fulfilled and must be fulfilled as a straight consequence of [the divergence relation $-\kappa\partial_{;\mu}T^{\mu}{}_{\nu} = \partial_{;\mu}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R\right) = 0$] plus expression [$T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}} = -F^{\alpha\mu}F_{\sigma\mu} + \frac{1}{4}\delta^{\alpha}{}_{\sigma}F^{\mu\nu}F_{\mu\nu}$] for the stress energy tensor" in the circumstance where one does not have "$\mathbf{E}\cdot\mathbf{B} = 0$ over an extended region." That is because the development in §20.6 of [28] is yet another manifestation of Einstein's "surprising" finding in [22] "that the gravitational equations for empty space determine their field just as strongly as do Maxwell's equations in the case of the electromagnetic field." It is a finding that a statement about the energy tensor such as $R_{\mu\nu} = 0$ or $T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}} = -F^{\alpha\mu}F_{\sigma\mu} + \frac{1}{4}\delta^{\alpha}{}_{\sigma}F^{\mu\nu}F_{\mu\nu}$ with $\partial_{;\alpha}T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}} = 0$ may under certain circumstance stand in as a proxy equivalent for Maxwell's equations, which all goes to the question of how one unifies classical electromagnetism with gravitational theory.

It is also of use to take the trace of (5.7), because we know that the Maxwell stress-energy tensor is traceless which means that electromagnetic fields propagate luminously. The trace equation thus yields the identity which is also duality invariant:

$$T_{\mathrm{Maxwell}} = -F^{\sigma\tau}F_{\sigma\tau} + \tfrac{1}{4}\delta^{\alpha}{}_{\alpha}F^{\sigma\tau}F_{\sigma\tau} = -F^{\sigma\tau}F_{\sigma\tau} + F^{\sigma\tau}F_{\sigma\tau} = -\tfrac{1}{2}\left(F^{\sigma\tau}F_{\sigma\tau} + *F^{\sigma\tau}*F_{\sigma\tau}\right) = 0\,. \tag{5.10}$$

Consequently, $F^{\sigma\tau}F_{\sigma\tau} = -*F^{\sigma\tau}*F_{\sigma\tau}$, which is another manifestation of how with **=-1 for second rank duality, the duality * operator behaves like $i = \sqrt{-1}$.

The second identity (5.2) is best explored by first combining (4.13) and (4.14) into the single relationship:

$$*P^{\tau} = D_{;\sigma}*F^{\sigma\tau} = \tfrac{1}{3!}\left(-g\right)^{-.5}\varepsilon^{\mu\nu\sigma\tau}\left(D_{;\sigma}F_{\mu\nu} + D_{;\mu}F_{\nu\sigma} + D_{;\nu}F_{\sigma\mu}\right), \tag{5.11}$$

and then simply renaming *F* to *B* for generality, plus some index renaming, into the form:

$$D_{;\sigma}*B^{\sigma\tau} = \tfrac{1}{3!}\left(-g\right)^{-.5}\varepsilon^{\gamma\nu\sigma\tau}\left(D_{;\sigma}B_{\gamma\nu} + D_{;\gamma}B_{\nu\sigma} + D_{;\nu}B_{\sigma\gamma}\right), \tag{5.12}$$

which applies to any second rank antisymmetric tensor $B_{\mu\nu}$. Then we front-multiply this by any antisymmetric tensor $*A_{\mu\tau}$, and apply $*A_{\mu\tau} = \frac{1}{2!}\left(-g\right)^{.5}\varepsilon_{\alpha\beta\mu\tau}A^{\alpha\beta}$ via (4.1), to obtain:

$$*A_{\mu\tau}D_{;\sigma}*B^{\sigma\tau} = \tfrac{1}{2!}\tfrac{1}{3!}\varepsilon_{\alpha\beta\mu\tau}\varepsilon^{\gamma\nu\sigma\tau}A^{\alpha\beta}\left(D_{;\sigma}B_{\gamma\nu} + D_{;\gamma}B_{\nu\sigma} + D_{;\nu}B_{\sigma\gamma}\right), \tag{5.13}$$

This once again contains a term $\varepsilon_{\alpha\beta\mu\tau}\varepsilon^{\gamma\nu\sigma\tau}$ which employs the sixth rank Kronecker delta of (4.10) and (4.11).

$$*A_{\mu\tau}D_{;\sigma}*B^{\sigma\tau}=\tfrac{1}{2!}\tfrac{1}{3!}\begin{pmatrix}-\delta_\alpha{}^\gamma\delta_\beta{}^\nu\delta_\mu{}^\sigma-\delta_\beta{}^\gamma\delta_\mu{}^\nu\delta_\alpha{}^\sigma-\delta_\mu{}^\gamma\delta_\alpha{}^\nu\delta_\beta{}^\sigma\\+\delta_\alpha{}^\gamma\delta_\mu{}^\nu\delta_\beta{}^\sigma+\delta_\mu{}^\gamma\delta_\beta{}^\nu\delta_\alpha{}^\sigma+\delta_\beta{}^\gamma\delta_\alpha{}^\nu\delta_\mu{}^\sigma\end{pmatrix}A^{\alpha\beta}\left(D_{;\sigma}B_{\gamma\nu}+D_{;\gamma}B_{\nu\sigma}+D_{;\nu}B_{\sigma\gamma}\right), \quad (5.14)$$

which is readily reduced down to the identity:

$$0=\tfrac{1}{2}A^{\sigma\tau}\left(D_{;\mu}B_{\sigma\tau}+D_{;\sigma}B_{\tau\mu}+D_{;\tau}B_{\mu\sigma}\right)+*A_{\mu\tau}D_{;\sigma}*B^{\sigma\tau}. \quad (5.15)$$

Aside from differently-named indexes, and the fact that the ordinary derivative $\partial_{;\mu}$ carries through into the gauge-covariant $D_{;\mu}$, it will be seen that (5.15) is exactly the same as (5.2) from footnote 22 on page 251 of [27].

In the event that the commutators $\left[A^{\sigma\tau},\left(D_{;\mu}B_{\sigma\tau}+D_{;\sigma}B_{\tau\mu}+D_{;\tau}B_{\mu\sigma}\right)\right]=0$ and $\left[*A_{\mu\tau},D_{;\sigma}*B^{\sigma\tau}\right]=0$, then $0=\tfrac{1}{2}\left(D_{;\mu}B_{\sigma\tau}+D_{;\sigma}B_{\tau\mu}+D_{;\tau}B_{\mu\sigma}\right)A^{\sigma\tau}+D_{;\sigma}*B^{\sigma\tau}*A_{\mu\tau}$ as well. But in non-Abelian gauge theory, where we will associate *A* and *B* with *non-commuting* field strength tensors with $F^{\mu\nu}=D^{[;\mu}G^{\nu]}=\partial^{[;\mu}G^{\nu]}-i\left[G^\mu,G^\nu\right]$ and $\left[G^\mu,G^\nu\right]\neq 0$ as in (2.7), one must treat the general case $\left[A^{\sigma\tau},\left(D_{;\mu}B_{\sigma\tau}+D_{;\sigma}B_{\tau\mu}+D_{;\tau}B_{\mu\sigma}\right)\right]\neq 0$ and $\left[*A_{\mu\tau},D_{;\sigma}*B^{\sigma\tau}\right]\neq 0$. But even in this non-commuting situation, the exact same steps (5.12) through (5.15) with a *rear* rather than front multiplication by $*A_{\mu\tau}$ yields the commuted identity, because $\varepsilon_{\alpha\beta\mu\tau}\varepsilon^{\gamma\nu\sigma\tau}=\varepsilon^{\gamma\nu\sigma\tau}\varepsilon_{\alpha\beta\mu\tau}$ may be commuted. So irrespective of commutativity, we may separately obtain:

$$0=\tfrac{1}{2}\left(D_{;\mu}B_{\sigma\tau}+D_{;\sigma}B_{\tau\mu}+D_{;\tau}B_{\mu\sigma}\right)A^{\sigma\tau}+D_{;\sigma}*B^{\sigma\tau}*A_{\mu\tau}. \quad (5.16)$$

Now let us take (5.15) and rename *A* to *F* and also *B* to *F*. Then, we take (5.15) and rename *A* to *F and also *B* to *F while applying **=-1 for second rank duality. Then, we take the commuted identity (5.16) and rename *A* to *F* and *B* to *F*. Finally, we take (5.16) and rename *A* to *F and also *B* to *F again with **=-1. The first two renamings yield the pair of identities:

$$\begin{cases}0=\tfrac{1}{2}F^{\sigma\tau}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)+*F_{\mu\tau}D_{;\sigma}*F^{\sigma\tau}\\ \quad =F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}-*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}+\tfrac{1}{2}F^{\sigma\tau}D_{;\mu}F_{\sigma\tau}\\ 0=\tfrac{1}{2}*F^{\sigma\tau}\left(D_{;\mu}*F_{\sigma\tau}+D_{;\sigma}*F_{\tau\mu}+D_{;\tau}*F_{\mu\sigma}\right)+F_{\mu\tau}D_{;\sigma}F^{\sigma\tau}\\ \quad =*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}-F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}+\tfrac{1}{2}*F^{\sigma\tau}D_{;\mu}*F_{\sigma\tau}\end{cases}. \quad (5.17)$$

The latter two renamings yield the commuted pair of identities:

$$\begin{cases} 0 = \tfrac{1}{2}\left(D_{;\mu}F_{\sigma\tau} + D_{;\sigma}F_{\tau\mu} + D_{;\tau}F_{\mu\sigma}\right)F^{\sigma\tau} + D_{;\sigma} * F^{\sigma\tau} * F_{\mu\tau} \\ \quad = D_{;\sigma}F_{\tau\mu}F^{\sigma\tau} - D_{;\sigma} * F^{\sigma\tau} * F_{\tau\mu} + \tfrac{1}{2}D_{;\mu}F_{\sigma\tau}F^{\sigma\tau} \\ 0 = \tfrac{1}{2}\left(D_{;\mu} * F_{\sigma\tau} + D_{;\sigma} * F_{\tau\mu} + D_{;\tau} * F_{\mu\sigma}\right) * F^{\sigma\tau} + D_{;\sigma}F^{\sigma\tau}F_{\mu\tau} \\ \quad = D_{;\sigma} * F_{\tau\mu} * F^{\sigma\tau} - D_{;\sigma}F^{\sigma\tau}F_{\tau\mu} + \tfrac{1}{2}D_{;\mu} * F_{\sigma\tau} * F^{\sigma\tau} \end{cases} . \tag{5.18}$$

If we further use (4.15) and (4.21) in the above two identities as applicable, from the top lines of (5.17) and (5.18) we find the further equivalent identities:

$$\begin{cases} 0 = \tfrac{1}{2}F^{\sigma\tau}P_{\mu\sigma\tau} + *F_{\mu\tau} * P^{\tau} \\ 0 = -\tfrac{1}{2} * F^{\sigma\tau} * J_{\mu\sigma\tau} + F_{\mu\tau}J^{\tau} \end{cases} \tag{5.19}$$

and

$$\begin{cases} 0 = \tfrac{1}{2}P_{\mu\sigma\tau}F^{\sigma\tau} + *P^{\tau} * F_{\mu\tau} \\ 0 = -\tfrac{1}{2} * J_{\mu\sigma\tau} * F^{\sigma\tau} + J^{\tau}F_{\mu\tau} \end{cases} . \tag{5.20}$$

The top identity (5.19) may be reconfirmed using $*F_{\mu\tau} = \tfrac{1}{2!}(-g)^{.5}\varepsilon_{\sigma\gamma\mu\tau}F^{\sigma\gamma}$ and $*P^{\tau} = \tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\alpha\beta\delta\tau}P_{\alpha\beta\delta}$ from (4.1), with the aid of (4.10) and (4.11), to obtain:

$$*F_{\mu\tau} * P^{\tau} = \tfrac{1}{2!}(-g)^{.5}\varepsilon_{\sigma\gamma\mu\tau}F^{\sigma\gamma}\tfrac{1}{3!}(-g)^{-.5}\varepsilon^{\alpha\beta\delta\tau}P_{\alpha\beta\delta} = \tfrac{1}{2!}\tfrac{1}{3!}\varepsilon_{\sigma\gamma\mu\tau}\varepsilon^{\alpha\beta\delta\tau}F^{\sigma\gamma}P_{\alpha\beta\delta} = -F^{\sigma\tau}\tfrac{1}{2}P_{\mu\sigma\tau} . \tag{5.21}$$

The bottom identity (5.19) may then be easily reconfirmed by keeping in mind from (4.15) that in a theory with both electric and magnetic charge densities, whenever the field strength $F^{\sigma\tau} \to *F^{\sigma\tau}$, at the very same time the electric charge densities will become magnetic charge densities $J^{\tau} \to *P^{\tau}$. So starting with (5.19), top, we need to go in the reverse direction from magnetic to electric charges. Thus, in (5.19), top, we set $*F^{\sigma\tau} \to F^{\sigma\tau}$ which means that $**F^{\sigma\tau} = -F^{\sigma\tau} \to *F^{\sigma\tau}$ via **=-1 for second rank duality, and at the same time we must set $*P^{\tau} \to J^{\tau}$ which means that $**P_{\mu\sigma\tau} = P_{\mu\sigma\tau} \to *J_{\mu\sigma\tau}$ via **=1 for first and third rank duality. This is simply the application, using Reinich [26] and Wheeler [27] duality, of Witten's statement in [30] that the "symmetry under $\mathbf{E} \to \mathbf{B}, \mathbf{B} \to -\mathbf{E}$ . . . still holds in the presence of charges and currents if one adds both electric and magnetic charges and currents." Performing these precise substitutions into (5.19), top, leads to (5.19), bottom. Then (5.20) are simply the commuted identities of (5.19).

It is also important to keep in mind that the *mathematical identities* (5.19), (5.20) for fields and sources are *independently-derived* identities from the identities (5.17) and (5.18) that purely involve field configurations. So, if one knew (5.17) and (5.18) as *a priori* identities, and also knew (5.19) and (5.20) *a priori*, it would be possible to hold them side by side, and immediately deduce the four relationships between sources and fields given by:

$$
\begin{aligned}
&J^{\mu} = D_{;\sigma} F^{\sigma\mu} \\
&P_{\mu\sigma\tau} = D_{;\mu} F_{\sigma\tau} + D_{;\sigma} F_{\tau\mu} + D_{;\tau} F_{\mu\sigma} \\
&-{}^{*}J_{\mu\sigma\tau} = D_{;\mu}{}^{*}F_{\sigma\tau} + D_{;\sigma}{}^{*}F_{\tau\mu} + D_{;\tau}{}^{*}F_{\mu\sigma} \\
&{}^{*}P^{\mu} = D_{;\sigma}{}^{*}F^{\sigma\mu}
\end{aligned}
\quad . \qquad (5.22)
$$

The first two are of course Maxwell's equations in customary form extended to Yang-Mills theory, and the latter two are alternate expressions for the same physical content which, because of the duality relationships (4.1), possess the "same-information-content" as highlighted by [28] at page 88. The use of one or the other form will often depend on circumstance and mathematical convenience.

With all of the foregoing, we are now finally prepared to examine the conservation of energy, and the energy tensor, for a Yang-Mills gauge theory in which there are both electric and magnetic charges. We shall show how the energy tensor is that of the vacuum, $T_{\mu\nu} = 0$, which of course means that $R_{\mu\nu} = 0$, which will validate Einstein's final hunch discussed in section 2 that the relationship between Maxwell's system of equations and "the gravitational equations for empty space" is more than just "surprising," and is in fact a signal of gravitational and electromagnetic unification. This in turn will take us on a path to discover baryons out of the Yang-Mills magnetic monopoles, and nuclear physics out of electrodynamics and elementary particle physics.

6. The Classical Unification of Maxwell's "Source-Full" Electrodynamics with Einstein's Gravitational Theory

We reviewed at the outset of section 3 how the local conservation of energy / momentum is established through a zero divergence of the energy tensor, $\partial_{;\mu} T^{\mu}{}_{\nu} = 0$. With four free indexes, this is a set of four equations, and $\partial_{;\mu} T^{\mu}{}_{\nu}$ is set equal to a four-vector of zeroes. The time component equation establishes energy conservation, while the three space components establish momentum conservation along each of the three spatial components of motion. We reviewed how the Einstein equation may be uncovered by connecting this vector of zeroes with the four-vector of zeroes in the contracted Bianchi identity $\partial_{;\mu}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R\right) = 0$ to obtain $-\kappa\partial_{;\mu} T^{\mu}{}_{\nu} = \partial_{;\mu}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R\right) = 0$ which upon integration without cosmological constant yields $-\kappa T^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R$. And we saw that in electrodynamics, one follows a similar procedure, but instead supplies this vector of zeroes using $\frac{1}{2} F^{\mu\nu}\left(\partial_{;\sigma} F_{\mu\nu} + \partial_{;\mu} F_{\nu\sigma} + \partial_{;\nu} F_{\sigma\mu}\right) = 0$, which is also identically equal to zero, *but only because of the abelian field relationship* $F^{\mu\nu} = \partial^{[;\mu} G^{\nu]}$. In Yang-Mills gauge theory, the non-Abelian field relationship is $F^{\mu\nu} = D^{[;\mu} G^{\nu]}$ of (2.7), and as a result the magnetic monopoles of (2.8) are $P_{\mu\nu\sigma} = D_{;\sigma} F_{\mu\nu} + D_{;\mu} F_{\nu\sigma} + D_{;\nu} F_{\sigma\mu} \neq 0$, which is no longer equal to zero. As a consequence, the identity used to enforce energy conservation in electrodynamics migrates to $\frac{1}{2}F^{\mu\nu}\left(D_{;\sigma}F_{\mu\nu}+D_{;\mu}F_{\nu\sigma}+D_{;\nu}F_{\sigma\mu}\right)\neq 0$, and because this is no longer equal to zero, it can no longer be connected to $\partial_{;\mu}T^{\mu}{}_{\nu}=0$. So, to make a long story short, we now must find a new "vector of zeroes" to connect to $\partial_{;\mu}T^{\mu}{}_{\nu}=0$, if we wish to properly describe the energy tensor and the conservation of energy momentum in a non-abelian, Yang-Mills gauge theory for which the identity $\frac{1}{2}F^{\mu\nu}\left(D_{;\sigma}F_{\mu\nu}+D_{;\mu}F_{\nu\sigma}+D_{;\nu}F_{\sigma\mu}\right)\neq 0$ because the magnetic charges are no longer vanishing. We need a new identity to replace $\frac{1}{2}F^{\mu\nu}\left(\partial_{;\sigma}F_{\mu\nu}+\partial_{;\mu}F_{\nu\sigma}+\partial_{;\nu}F_{\sigma\mu}\right)=0$ with a new vector of zeroes.

The identities (5.17) and (5.18) all fit the bill as "vector of zeroes" which may be set to $\partial_{;\sigma}T^{\sigma}{}_{\mu}=0$ to establish the local conservation of energy. Indeed, the precise vector of zeroes $\frac{1}{2}F^{\sigma\tau}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)$ used for this very same purpose in source-free electrodynamics, see (3.2), is already part of $0=\frac{1}{2}F^{\sigma\tau}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)+{}^{*}F_{\mu\tau}D_{;\sigma}{}^{*}F^{\sigma\tau}$ which is the top identity in (5.17). So the questions we must now consider is: which of the four identities in (5.17) and (5.18), or *what combination of these identities*, do we use?

First, we take some guidance from (5.9), where we see that in source-free electrodynamics, which is symmetric under duality $F^{\mu\nu}\rightarrow{}^{*}F^{\mu\nu}$, the Maxwell tensor $T^{\mu}{}_{\nu\ \mathrm{Maxwell}}$ is itself also symmetric under duality. We surmise therefore, for "source-full" electrodynamics which contains both electric and magnetic charges, that the energy tensor should likewise remain invariant under a $F^{\mu\nu}\rightarrow{}^{*}F^{\mu\nu}$ transformation. We can construct such an identity by adding together the two separate identities in (5.17) to form the single identity:

$$\begin{aligned}0&=\tfrac{1}{2}F^{\sigma\tau}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)+{}^{*}F_{\mu\tau}D_{;\sigma}{}^{*}F^{\sigma\tau}+\tfrac{1}{2}{}^{*}F^{\sigma\tau}\left(D_{;\mu}{}^{*}F_{\sigma\tau}+D_{;\sigma}{}^{*}F_{\tau\mu}+D_{;\tau}{}^{*}F_{\mu\sigma}\right)+F_{\mu\tau}D_{;\sigma}F^{\sigma\tau}\\&=F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+{}^{*}F^{\sigma\tau}D_{;\sigma}{}^{*}F_{\tau\mu}-F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}-{}^{*}F_{\tau\mu}D_{;\sigma}{}^{*}F^{\sigma\tau}+\tfrac{1}{2}F^{\sigma\tau}D_{;\mu}F_{\sigma\tau}+\tfrac{1}{2}{}^{*}F^{\sigma\tau}D_{;\mu}{}^{*}F_{\sigma\tau}\end{aligned}\quad .(6.1)$$

Just like $T^{\mu}{}_{\nu\ \mathrm{Maxwell}}=-\frac{1}{2}\left(F^{\mu\alpha}F_{\nu\alpha}+{}^{*}F^{\mu\alpha}{}^{*}F_{\nu\alpha}\right)$, this is manifestly invariant under duality. We may form a separate identity by combining both of (5.18) to form:

$$\begin{aligned}0&=\tfrac{1}{2}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)F^{\sigma\tau}+D_{;\sigma}{}^{*}F^{\sigma\tau}{}^{*}F_{\mu\tau}+\tfrac{1}{2}\left(D_{;\mu}{}^{*}F_{\sigma\tau}+D_{;\sigma}{}^{*}F_{\tau\mu}+D_{;\tau}{}^{*}F_{\mu\sigma}\right){}^{*}F^{\sigma\tau}+D_{;\sigma}F^{\sigma\tau}F_{\mu\tau}\\&=D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}+D_{;\sigma}{}^{*}F_{\tau\mu}{}^{*}F^{\sigma\tau}-D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}-D_{;\sigma}{}^{*}F^{\sigma\tau}{}^{*}F_{\tau\mu}+\tfrac{1}{2}D_{;\mu}F_{\sigma\tau}F^{\sigma\tau}+\tfrac{1}{2}D_{;\mu}{}^{*}F_{\sigma\tau}{}^{*}F^{\sigma\tau}\end{aligned}\quad .(6.2)$$

This is just the commuted version of (6.2) and it too is $F^{\mu\nu}\rightarrow{}^{*}F^{\mu\nu}$ symmetric. Now the question becomes, mat we use (6.1) alone, or (6.2) alone, in connection with $T^{\mu}{}_{\nu\ \mathrm{Maxwell}}$? Or, do we have to also combine these in some way?

Now, as noted after (5.15), for an abelian gauge theory with $F^{\mu\nu}=\partial^{[;\mu}G^{\nu]}$ and $\left[G^{\mu},G^{\nu}\right]=0$, (6.1) and (6.2) will be redundant identities. So we can get by with using only one of these. As a strictly mathematical warm-up for the complete non-Abelian Yang-Mills solution to this question, let us rewrite (6.1) as an abelian relationship, by setting $D_{;\mu} \to \partial_{;\mu}$, and regarding the field strength tensors to be fully commuting. Thus, (6.1) simplifies to:

$$\begin{aligned} 0 &= \tfrac{1}{2} F^{\sigma\tau} \left(\partial_{;\mu} F_{\sigma\tau} + \partial_{;\sigma} F_{\tau\mu} + \partial_{;\tau} F_{\mu\sigma} \right) + {}^*F_{\mu\tau} \partial_{;\sigma} {}^*F^{\sigma\tau} + \tfrac{1}{2} {}^*F^{\sigma\tau} \left(\partial_{;\mu} {}^*F_{\sigma\tau} + \partial_{;\sigma} {}^*F_{\tau\mu} + \partial_{;\tau} {}^*F_{\mu\sigma} \right) + F_{\mu\tau} \partial_{;\sigma} F^{\sigma\tau} \\ &= F^{\sigma\tau} \partial_{;\sigma} F_{\tau\mu} + {}^*F^{\sigma\tau} \partial_{;\sigma} {}^*F_{\tau\mu} - F_{\tau\mu} \partial_{;\sigma} F^{\sigma\tau} - {}^*F_{\tau\mu} \partial_{;\sigma} {}^*F^{\sigma\tau} + \tfrac{1}{2} F^{\sigma\tau} \partial_{;\mu} F_{\sigma\tau} + \tfrac{1}{2} {}^*F^{\sigma\tau} \partial_{;\mu} {}^*F_{\sigma\tau} \end{aligned} . \quad (6.3)$$

Of course, when $F^{\mu\nu} = \partial^{[;\mu} G^{\nu]}$, the monopoles are zeroed out, but let us for the moment adopt the late 19th century stance of Helmholtz and simply regard the $F^{\mu\nu}$ as ordinary antisymmetric tensors with no relation to a gauge field, simply to flesh out the mathematics, recognizing that this is an "unphysical" view.

Regarding (6.3) as no more than a mathematical identity involving antisymmetric tensors, we first observe by way of (5.10) for the luminosity of electromagnetic energy that:

$$\tfrac{1}{2} F^{\sigma\tau} \partial_{;\mu} F_{\sigma\tau} + \tfrac{1}{2} {}^*F^{\sigma\tau} \partial_{;\mu} {}^*F_{\sigma\tau} = \tfrac{1}{4} \partial_{;\mu} \left(F^{\sigma\tau} F_{\sigma\tau} + {}^*F^{\sigma\tau} {}^*F_{\sigma\tau} \right) = \tfrac{1}{4} \partial_{;\mu} (0) = 0 . \quad (6.4)$$

Therefore (6.3) may be partially reduced to:

$$\begin{aligned} 0 &= \tfrac{1}{2} F^{\sigma\tau} \left(\partial_{;\mu} F_{\sigma\tau} + \partial_{;\sigma} F_{\tau\mu} + \partial_{;\tau} F_{\mu\sigma} \right) + {}^*F_{\mu\tau} \partial_{;\sigma} {}^*F^{\sigma\tau} + \tfrac{1}{2} {}^*F^{\sigma\tau} \left(\partial_{;\mu} {}^*F_{\sigma\tau} + \partial_{;\sigma} {}^*F_{\tau\mu} + \partial_{;\tau} {}^*F_{\mu\sigma} \right) + F_{\mu\tau} \partial_{;\sigma} F^{\sigma\tau} \\ &= F^{\sigma\tau} \partial_{;\sigma} F_{\tau\mu} + {}^*F^{\sigma\tau} \partial_{;\sigma} {}^*F_{\tau\mu} - F_{\tau\mu} \partial_{;\sigma} F^{\sigma\tau} - {}^*F_{\tau\mu} \partial_{;\sigma} {}^*F^{\sigma\tau} \end{aligned} . \quad (6.5)$$

Now let's work with the identity on the final line above, which we may write as:

$$F^{\sigma\tau} \partial_{;\sigma} F_{\tau\mu} + {}^*F^{\sigma\tau} \partial_{;\sigma} {}^*F_{\tau\mu} = F_{\tau\mu} \partial_{;\sigma} F^{\sigma\tau} + {}^*F_{\tau\mu} \partial_{;\sigma} {}^*F^{\sigma\tau} . \quad (6.6)$$

But if both sides of this equation are equal, than each of these sides is also equal to ½ times the sum of both sides, that is (if $A = B$, then $A = B = \tfrac{1}{2}(A + B)$):

$$\begin{aligned} & F^{\sigma\tau} \partial_{;\sigma} F_{\tau\mu} + {}^*F^{\sigma\tau} \partial_{;\sigma} {}^*F_{\tau\mu} \\ &= F_{\tau\mu} \partial_{;\sigma} F^{\sigma\tau} + {}^*F_{\tau\mu} \partial_{;\sigma} {}^*F^{\sigma\tau} \\ &= \tfrac{1}{2} \left(F^{\sigma\tau} \partial_{;\sigma} F_{\tau\mu} + F_{\tau\mu} \partial_{;\sigma} F^{\sigma\tau} + {}^*F^{\sigma\tau} \partial_{;\sigma} {}^*F_{\tau\mu} + {}^*F_{\tau\mu} \partial_{;\sigma} {}^*F^{\sigma\tau} \right) \end{aligned} . \quad (6.7)$$

At this point we reach an important juncture which highlights the commutativity issues that need to be accounted for as between (6.1) and (6.2). Because we are momentarily assuming abelian, commuting gauge theory, we may commute $\left[F_{\tau\mu}, \partial_{;\sigma} F^{\sigma\tau} \right] = 0$ as well as $\left[{}^*F^{\sigma\tau}, \partial_{;\sigma} {}^*F_{\tau\mu} \right]$. With these commutations, plus one final consolidation, the identity (6.7) finally becomes:

$$
\begin{aligned}
&F^{\sigma\tau}\partial_{;\sigma}F_{\tau\mu}+*F^{\sigma\tau}\partial_{;\sigma}*F_{\tau\mu}\\
&=F_{\tau\mu}\partial_{;\sigma}F^{\sigma\tau}+*F_{\tau\mu}\partial_{;\sigma}*F^{\sigma\tau}\\
&=\tfrac{1}{2}\left(F^{\sigma\tau}\partial_{;\sigma}F_{\tau\mu}+F_{\tau\mu}\partial_{;\sigma}F^{\sigma\tau}+*F^{\sigma\tau}\partial_{;\sigma}*F_{\tau\mu}+*F_{\tau\mu}\partial_{;\sigma}*F^{\sigma\tau}\right).\\
&=\tfrac{1}{2}\left(\partial_{;\sigma}F_{\tau\mu}F^{\sigma\tau}+F_{\tau\mu}\partial_{;\sigma}F^{\sigma\tau}+\partial_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}+*F_{\tau\mu}\partial_{;\sigma}*F^{\sigma\tau}\right)\\
&=\tfrac{1}{2}\partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau}+*F_{\tau\mu}*F^{\sigma\tau}\right)
\end{aligned}
\tag{6.8}
$$

So, now we return to (6.3), and because it is a vector identity of zeroes, we connect this to the local energy conservation relationship $\partial_{;\sigma}T^{\sigma}{}_{\nu}=0$. We then apply $F^{\sigma\tau}F_{\sigma\tau}+*F^{\sigma\tau}*F_{\sigma\tau}$ as well as the identity (6.8). Showing all of the steps, what we find is that:

$$
\begin{aligned}
&\partial_{;\sigma}T^{\sigma}{}_{\nu}=0\\
&=\tfrac{1}{2}F^{\sigma\tau}\left(\partial_{;\mu}F_{\sigma\tau}+\partial_{;\sigma}F_{\tau\mu}+\partial_{;\tau}F_{\mu\sigma}\right)+*F_{\mu\tau}\partial_{;\sigma}*F^{\sigma\tau}+\tfrac{1}{2}*F^{\sigma\tau}\left(\partial_{;\mu}*F_{\sigma\tau}+\partial_{;\sigma}*F_{\tau\mu}+\partial_{;\tau}*F_{\mu\sigma}\right)+F_{\mu\tau}\partial_{;\sigma}F^{\sigma\tau}\\
&=F^{\sigma\tau}\partial_{;\sigma}F_{\tau\mu}+*F^{\sigma\tau}\partial_{;\sigma}*F_{\tau\mu}-F_{\tau\mu}\partial_{;\sigma}F^{\sigma\tau}-*F_{\tau\mu}\partial_{;\sigma}*F^{\sigma\tau}+\tfrac{1}{2}F^{\sigma\tau}\partial_{;\mu}F_{\sigma\tau}+\tfrac{1}{2}*F^{\sigma\tau}\partial_{;\mu}*F_{\sigma\tau}\\
&=\tfrac{1}{2}\partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau}+*F_{\tau\mu}*F^{\sigma\tau}\right)-\tfrac{1}{2}\partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau}+*F_{\tau\mu}*F^{\sigma\tau}\right)\\
&=\tfrac{1}{2}\partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau}+*F_{\tau\mu}*F^{\sigma\tau}-F_{\tau\mu}F^{\sigma\tau}-*F_{\tau\mu}*F^{\sigma\tau}\right)\\
&=\tfrac{1}{2}\partial_{;\sigma}\left(0\right)
\end{aligned}
\quad .(6.9)
$$

This may also be written in terms of the sources *J* and *P* via abelian versions of (2.8), (4.15) and (4.18), as:

$$
\partial_{;\sigma}T^{\sigma}{}_{\nu}=0=\tfrac{1}{2}F^{\sigma\tau}P_{\mu\sigma\tau}+*F_{\mu\tau}P^{\tau}-\tfrac{1}{2}*F^{\sigma\tau}*J_{\mu\sigma\tau}+F_{\mu\tau}J^{\tau}=\tfrac{1}{2}\partial_{;\sigma}\left(0\right). \tag{6.10}
$$

Of course, we started out with a vector of zeroes being equal to $\partial_{;\sigma}T^{\sigma}{}_{\nu}$. But the goal, as it was in (3.3) for source-free electrodynamics, is to identify an integrable expression that can then be identified directly with the energy tensor itself rather than with its divergence. What we come across in (6.9) is now a *tensor of zeroes* in the form of $\partial_{;\sigma}T^{\sigma}{}_{\nu}=\tfrac{1}{2}\partial_{;\sigma}\left(0\right)=0$. Integrating each side without a cosmological constant, we find rather simply that $T^{\sigma}{}_{\nu}=0$. And of course, as is well-known, because the Einstein equation $-\kappa T^{\mu}{}_{\nu}=R^{\mu}{}_{\nu}-\tfrac{1}{2}\delta^{\mu}{}_{\nu}R$ has a trace $\kappa T=R$, it is readily invertible to $R^{\mu}{}_{\nu}=-\kappa\left(T^{\mu}{}_{\nu}-\tfrac{1}{2}\delta^{\mu}{}_{\nu}T\right)$. So in strictly geometric terms, (6.9) says that:

$$
R_{\mu\nu}=0 \tag{6.11}
$$

for *abelian*, *source-full*, *duality-invariant* electrodynamics. This is the direct proof of Einstein's final hunch discussed in section 2 in which he found it "surprising that the gravitational equations for empty space determine their field just as strongly as do Maxwell's equations in the case of the electromagnetic field." The reason for this is that at least for abelian gauge theory, when there are both electric and magnetic sources with duality symmetry, $R_{\mu\nu}=0$ is an equivalent statement of Maxwell's equations, *and so unifies classical electrodynamics with gravitation*, reaching the objective that Einstein pursued during the last several decades of his life and which he handed off to posterity by noting this "surprising" finding. This is likewise the next step in what is developed by Misner, Thorne and Wheeler in §20.6 of [28], as discussed following (3.6). In that analysis, it is shown how the choice of the Maxwell energy tensor and the conservation of that energy tensor leads "in the generic case" to Maxwell's *source-free* electrodynamics. In (6.11), we find that the choice of the vacuum $R_{\mu\nu}=0$ together with energy conservation also leads to Maxwell's electrodynamics, but now, what we have is abelian, *source-full* electrodynamics. It is the duality symmetry of the sources that creates the balances needed to zero out the energy tensor.

As stated at the end of section 2, (6.11) is the particular result which the author discovered in 1984 [25], which became the starting point for much of the author's subsequent research. For, while (6.11) was developed for an abelian gauge theory with both electric and magnetic sources, and duality symmetry, this is not a *physical* theory but rather is a formal, mathematical unification, because in Abelian gauge theory magnetic sources vanish as soon as one introduces a gauge field. And, as Witten points out in [30] as reviewed at the start of section 4, "the vector potential is not just a convenience in solving Maxwell equations. It is needed in 20th-century physics for three very good purposes" which were not understood back when Helmholtz suggested that there might be no real need for a vector potential and before Hermann Weyl had developed gauge theory [31], [32], [33] in which the gauge field/ vector potential assumed an indispensable and central role. So while Einstein's final hunch about there being some equivalence between Maxwell's system of equations and the empty space equations $R_{\mu\nu}=0$ is confirmed *mathematically* by (6.11), taking the next, vital step toward *physical* confirmation requires a) the existence of magnetic monopoles *as part of a theory which includes a vector potential*, which is realized by the use of Yang-Mills gauge theory; b) identification of the magnetic monopoles with something that may be observed, which the author has shown in [13] and [21] are baryons; and c) validation of all of the foregoing using empirical, numeric data from at least some baryons, and specifically, by closely explaining a number of observed binding and fusion energies for the proton and neutron [14], [17], [18], [19] as well as the observed proton and neutron rest masses [20].

So in the next section, we will use the result (6.11) for mathematical guidance, and shall see whether this same result $R_{\mu\nu}=0$ can be reproduced in Yang-Mills gauge theory in which the gauge fields are non-commuting, i.e., non-abelian, and in which the magnetic monopoles really do become non-vanishing as a result of the field strength defined in relation to the vector/gauge potential in (2.7) by $F^{\mu\nu}=D^{[;\mu}G^{\nu]}=\partial^{[;\mu}G^{\nu]}-i\left[G^{\mu},G^{\nu}\right]$.

7. The Classical Unification of Yang-Mills' Electrodynamics with Einstein's Gravitational Theory

Let us now return to the non-Abelian identity (6.1) and see how far we can progress toward $R_{\mu\nu}=0$ before being required to also employ (6.2). As we shall shortly see, it is the step we took at (6.8) of commuting $\left[F_{\tau\mu},\partial_{;\sigma}F^{\sigma\tau}\right]=0$ and $\left[*F^{\sigma\tau},\partial_{;\sigma}*F_{\tau\mu}\right]$,which we cannot do at will for Yang-Mills theory, which will require combining (6.1) and (6.2) in a particular manner.

Corresponding with (6.4), let us first develop the term $\frac{1}{2}F^{\sigma\tau}D_{;\mu}F_{\sigma\tau}+\frac{1}{2}*F^{\sigma\tau}D_{;\mu}*F_{\sigma\tau}$ from (6.1). Using variants of (4.1), and keeping in mind the cancellation of the $(-g)^{\pm.5}$ terms as we have previously seen, e.g., after (4.14), we may write:

$$\begin{aligned}&\tfrac{1}{2}*F^{\sigma\tau}D_{;\mu}*F_{\sigma\tau}=\tfrac{1}{2}\left(\tfrac{1}{2!}(-g)^{-.5}\varepsilon^{\mu\nu\sigma\tau}F_{\mu\nu}\right)D_{;\mu}\left(\tfrac{1}{2!}(-g)^{.5}\varepsilon_{\alpha\beta\sigma\tau}F^{\alpha\beta}\right)\\ &=\tfrac{1}{2}\tfrac{1}{2!}\tfrac{1}{2!}\varepsilon^{\mu\nu\sigma\tau}\varepsilon_{\alpha\beta\sigma\tau}F_{\mu\nu}D_{;\mu}F^{\alpha\beta}=-\tfrac{1}{2}\tfrac{1}{2!}\tfrac{1}{2!}\delta^{\mu\nu\sigma}{}_{\alpha\beta\sigma}F_{\mu\nu}D_{;\mu}F^{\alpha\beta}=-\tfrac{1}{2}\tfrac{1}{2!}\delta^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu}D_{;\mu}F^{\alpha\beta}\ .\\ &=-\tfrac{1}{2}\tfrac{1}{2!}\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}-\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right)F_{\mu\nu}D_{;\mu}F^{\alpha\beta}=-\tfrac{1}{2}F^{\sigma\tau}D_{;\mu}F_{\sigma\tau}\end{aligned}\qquad(7.1)$$

So based what is effectively the relation **=-1 for second-rank duality rooted in the underlying calculation (4.2) to (4.5), we find that (6.4) does generalize to:

$$\tfrac{1}{2}F^{\sigma\tau}D_{;\mu}F_{\sigma\tau}+\tfrac{1}{2}*F^{\sigma\tau}D_{;\mu}*F_{\sigma\tau}=0\,.\qquad(7.2)$$

This means that (6.1) reduces to:

$$\begin{aligned}0&=\tfrac{1}{2}F^{\sigma\tau}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)+*F_{\mu\tau}D_{;\sigma}*F^{\sigma\tau}+\tfrac{1}{2}*F^{\sigma\tau}\left(D_{;\mu}*F_{\sigma\tau}+D_{;\sigma}*F_{\tau\mu}+D_{;\tau}*F_{\mu\sigma}\right)+F_{\mu\tau}D_{;\sigma}F^{\sigma\tau}\\ &=F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}-F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}-*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}\end{aligned}\qquad(7.3)$$

and that via an identical **=-1 type calculation, (6.2)

$$\tfrac{1}{2}D_{;\mu}F_{\sigma\tau}F^{\sigma\tau}+\tfrac{1}{2}D_{;\mu}*F_{\sigma\tau}*F^{\sigma\tau}=0\,,\qquad(7.4)$$

so that (6.2) reduces to:

$$\begin{aligned}0&=\tfrac{1}{2}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)F^{\sigma\tau}+D_{;\sigma}*F^{\sigma\tau}*F_{\mu\tau}+\tfrac{1}{2}\left(D_{;\mu}*F_{\sigma\tau}+D_{;\sigma}*F_{\tau\mu}+D_{;\tau}*F_{\mu\sigma}\right)*F^{\sigma\tau}+D_{;\sigma}F^{\sigma\tau}F_{\mu\tau}\\ &=D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}+D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}-D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}-D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}\end{aligned}\ .\qquad(7.5)$$

As a result, we now may extract from (7.3) and (7.5), with the further help of $A=B=\frac{1}{2}(A+B)$ the respective identities which generalize (6.7):

$$\begin{aligned}&F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}\\ &=F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}+*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}\\ &=\tfrac{1}{2}\left(F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}+*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}\right)\end{aligned}\ ,\qquad(7.6)$$

$$
\begin{aligned}
&D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}+D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}\\
&=D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}+D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}\\
&=\tfrac{1}{2}\left(D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}+D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}+D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}+D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}\right)
\end{aligned}
\quad . \tag{7.7}
$$

Now we hit a roadblock that is uniquely a consequence of the non-commuting nature of non-Abelian, Yang-Mills gauge theory. In (7.6) we would like, for example, as in (6.8), to be able to use $\left[F^{\sigma\tau},D_{;\sigma}F_{\tau\mu}\right]=0$ to write $F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}=D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}+F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}$ toward the goal of then collapsing via the product rule $\partial A\cdot B+A\cdot\partial B=\partial\left(A\cdot B\right)$. Similarly we would like to use $\left[*F^{\sigma\tau},D_{;\sigma}*F_{\tau\mu}\right]=0$ to write $*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}\rightarrow D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}$, but cannot. This is because these field expressions do not commute in Yang-Mills gauge theory. So what do we do?

This is where we now start to interplay (7.6) and (7.7), because (7.7) does contain the exact field terms that we cannot get to in (7.6) because the non-Abelian fields do not allow us to commute the way we can in abelian gauge theory. So, for non-abelian gauge theory, the “vector of zeroes” that we need to connect to $\partial_{;\sigma}T^{\sigma}{}_{\mu}=0$ will need to involve a combination of (6.1) and (6.2), which thus far have advanced to (7.3) and (7.5).

Therefore, let us now combine (6.1) and (6.2) in view of (7.3) and (7.5) (see (7.2)) by *subtracting* (6.2) from (6.1) to fashion the combined identity:

$$
\begin{aligned}
0=&\tfrac{1}{2}F^{\sigma\tau}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)+*F_{\mu\tau}D_{;\sigma}*F^{\sigma\tau}+\tfrac{1}{2}*F^{\sigma\tau}\left(D_{;\mu}*F_{\sigma\tau}+D_{;\sigma}*F_{\tau\mu}+D_{;\tau}*F_{\mu\sigma}\right)+F_{\mu\tau}D_{;\sigma}F^{\sigma\tau}\\
&-\tfrac{1}{2}\left(D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}\right)F^{\sigma\tau}-D_{;\sigma}*F^{\sigma\tau}*F_{\mu\tau}-\tfrac{1}{2}\left(D_{;\mu}*F_{\sigma\tau}+D_{;\sigma}*F_{\tau\mu}+D_{;\tau}*F_{\mu\sigma}\right)*F^{\sigma\tau}-D_{;\sigma}F^{\sigma\tau}F_{\mu\tau}\\
&=F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}-F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}-*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}+\tfrac{1}{2}F^{\sigma\tau}D_{;\mu}F_{\sigma\tau}+\tfrac{1}{2}*F^{\sigma\tau}D_{;\mu}*F_{\sigma\tau}\\
&-D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}-D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}+D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}+D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}-\tfrac{1}{2}D_{;\mu}F_{\sigma\tau}F^{\sigma\tau}-\tfrac{1}{2}D_{;\mu}*F_{\sigma\tau}*F^{\sigma\tau}\\
&=F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}-F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}-*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}\\
&-D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}-D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}+D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}+D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}\\
&=\left(D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}+F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}\right)+\left(D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}\right)\\
&-\left(D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}+F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}\right)-\left(D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}+*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}\right)
\end{aligned}
\quad .\tag{7.8}
$$

The reason we have subtracted rather than added is rather simple: this now gives us four pairs of terms, $D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}+F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}$, $D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}$, $-D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}-F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}$ and $-D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}-*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}$ which are structured just as we would like them to be to now pursue a consolidation $\partial A\cdot B+A\cdot\partial B\rightarrow\partial\left(A\cdot B\right)$ using the product rule. That is, it is the desire to combine terms using the product rule which blocked us from proceeding at (7.6) and (7.7) that motivates the combining of (6.1) and (6.2) through subtraction rather than addition. Now let us work with these four pairs of terms.

One cannot immediately use $\partial A \cdot B + A \cdot \partial B \to \partial\left(A \cdot B\right)$ because the terms in (7.8) contain gauge covariant derivatives. So, for example, the terms:

$$
\begin{aligned}
&D_{;\sigma}F^{\sigma\tau}F_{\tau\mu} + F^{\sigma\tau}D_{;\sigma}F_{\tau\mu} = \left(\partial_{;\sigma} - iG_\sigma\right)F^{\sigma\tau}F_{\tau\mu} + F^{\sigma\tau}\left(\partial_{;\sigma} - iG_\sigma\right)F_{\tau\mu} \\
&= \partial_{;\sigma}F^{\sigma\tau}F_{\tau\mu} + F^{\sigma\tau}\partial_{;\sigma}F_{\tau\mu} - iG_\sigma F^{\sigma\tau}F_{\tau\mu} - iF^{\sigma\tau}G_\sigma F_{\tau\mu} \\
&= \partial_{;\sigma}\left(F^{\sigma\tau}F_{\tau\mu}\right) - iG_\sigma F^{\sigma\tau}F_{\tau\mu} - iF^{\sigma\tau}G_\sigma F_{\tau\mu}
\end{aligned}
\tag{7.9}
$$

contain extra the extra terms $G_\sigma F^{\sigma\tau}F_{\tau\mu} + F^{\sigma\tau}G_\sigma F_{\tau\mu}$ arising from the gauge-covariant derivatives. Similarly, for the other three term pairs:

$$
\begin{aligned}
&D_{;\sigma} * F^{\sigma\tau} * F_{\tau\mu} + * F^{\sigma\tau}D_{;\sigma} * F_{\tau\mu} = \left(\partial_{;\sigma} - iG_\sigma\right) * F^{\sigma\tau} * F_{\tau\mu} + * F^{\sigma\tau}\left(\partial_{;\sigma} - iG_\sigma\right) * F_{\tau\mu} \\
&= \partial_{;\sigma}\left(*F^{\sigma\tau} * F_{\tau\mu}\right) - iG_\sigma * F^{\sigma\tau} * F_{\tau\mu} - i * F^{\sigma\tau}G_\sigma * F_{\tau\mu}
\end{aligned}
,
\tag{7.10}
$$

$$
\begin{aligned}
&-D_{;\sigma}F_{\tau\mu}F^{\sigma\tau} - F_{\tau\mu}D_{;\sigma}F^{\sigma\tau} = -\left(\partial_{;\sigma} - iG_\sigma\right)F_{\tau\mu}F^{\sigma\tau} - F_{\tau\mu}\left(\partial_{;\sigma} - iG_\sigma\right)F^{\sigma\tau} \\
&= -\partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau}\right) + iG_\sigma F_{\tau\mu}F^{\sigma\tau} + iF_{\tau\mu}G_\sigma F^{\sigma\tau}
\end{aligned}
,
\tag{7.11}
$$

$$
\begin{aligned}
&-D_{;\sigma} * F_{\tau\mu} * F^{\sigma\tau} - *F_{\tau\mu}D_{;\sigma} * F^{\sigma\tau} = -\left(\partial_{;\sigma} - iG_\sigma\right) * F_{\tau\mu} * F^{\sigma\tau} - *F_{\tau\mu}\left(\partial_{;\sigma} - iG_\sigma\right) * F^{\sigma\tau} \\
&= -\partial_{;\sigma}\left(*F_{\tau\mu} * F^{\sigma\tau}\right) + iG_\sigma * F_{\tau\mu} * F^{\sigma\tau} + i * F_{\tau\mu}G_\sigma * F^{\sigma\tau}
\end{aligned}
.
\tag{7.12}
$$

As a consequence of (7.9) through (7.12), the final set of terms in (7.8) reduces as such:

$$
\begin{aligned}
&D_{;\sigma}F^{\sigma\tau}F_{\tau\mu} + F^{\sigma\tau}D_{;\sigma}F_{\tau\mu} + D_{;\sigma} * F^{\sigma\tau} * F_{\tau\mu} + * F^{\sigma\tau}D_{;\sigma} * F_{\tau\mu} \\
&-D_{;\sigma}F_{\tau\mu}F^{\sigma\tau} - F_{\tau\mu}D_{;\sigma}F^{\sigma\tau} - D_{;\sigma} * F_{\tau\mu} * F^{\sigma\tau} - *F_{\tau\mu}D_{;\sigma} * F^{\sigma\tau} \\
&= \partial_{;\sigma}\left(F^{\sigma\tau}F_{\tau\mu} + *F^{\sigma\tau} * F_{\tau\mu}\right) - \partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau} + *F_{\tau\mu} * F^{\sigma\tau}\right) \\
&-i\left(G_\sigma F^{\sigma\tau}F_{\tau\mu} + G_\sigma * F^{\sigma\tau} * F_{\tau\mu}\right) + i\left(G_\sigma F_{\tau\mu}F^{\sigma\tau} + G_\sigma * F_{\tau\mu} * F^{\sigma\tau}\right) \\
&-i\left(F^{\sigma\tau}G_\sigma F_{\tau\mu} + *F^{\sigma\tau}G_\sigma * F_{\tau\mu}\right) + i\left(F_{\tau\mu}G_\sigma F^{\sigma\tau} + *F_{\tau\mu}G_\sigma * F^{\sigma\tau}\right)
\end{aligned}
.
\tag{7.13}
$$

Once again, duality helps us reduce. For example, using (4.1) and variants:

$$
\begin{aligned}
&* F^{\sigma\tau} * F_{\tau\mu} = \tfrac{1}{2!}\tfrac{1}{2!}\varepsilon^{\delta\gamma\sigma\tau}\varepsilon_{\alpha\beta\tau\mu}F_{\delta\gamma}F^{\alpha\beta} = -\tfrac{1}{2!}\tfrac{1}{2!}\varepsilon^{\delta\gamma\sigma\tau}\varepsilon_{\alpha\beta\mu\tau}F_{\delta\gamma}F^{\alpha\beta} = \tfrac{1}{2!}\tfrac{1}{2!}\delta^{\delta\gamma\sigma}{}_{\alpha\beta\mu}F_{\delta\gamma}F^{\alpha\beta} \\
&= \tfrac{1}{2!}\tfrac{1}{2!}\left(\delta^\delta{}_\alpha\delta^\gamma{}_\beta\delta^\sigma{}_\mu + \delta^\delta{}_\beta\delta^\gamma{}_\mu\delta^\sigma{}_\alpha + \delta^\delta{}_\mu\delta^\gamma{}_\alpha\delta^\sigma{}_\beta - \delta^\delta{}_\alpha\delta^\gamma{}_\mu\delta^\sigma{}_\beta - \delta^\delta{}_\mu\delta^\gamma{}_\beta\delta^\sigma{}_\alpha - \delta^\delta{}_\beta\delta^\gamma{}_\alpha\delta^\sigma{}_\mu\right)F_{\delta\gamma}F^{\alpha\beta} \\
&= F_{\tau\mu}F^{\sigma\tau} + \tfrac{1}{2}\delta^\sigma{}_\mu F_{\sigma\tau}F^{\sigma\tau}
\end{aligned}
.
\tag{7.14}
$$

Referring to terms in (7.13), this means that:

$$F^{\sigma\tau}F_{\tau\mu} + {}^*F^{\sigma\tau}\,{}^*F_{\tau\mu} = F^{\sigma\tau}F_{\tau\mu} + F_{\tau\mu}F^{\sigma\tau} + \tfrac{1}{2}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau} \tag{7.15}$$

which is a non-abelian variant on the Maxwell tensor as we can see by rewriting this as:

$$-\tfrac{1}{2}\left(F^{\sigma\tau}F_{\mu\tau} + {}^*F^{\sigma\tau}\,{}^*F_{\mu\tau}\right) = -\tfrac{1}{2}\left(F^{\sigma\tau}F_{\mu\tau} + F_{\mu\tau}F^{\sigma\tau}\right) + \tfrac{1}{4}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau} \tag{7.16}$$

and thinking about the abelian commutation $F^{\sigma\tau}F_{\mu\tau} = F_{\mu\tau}F^{\sigma\tau}$ which we are *not* allowed to do in a non-abelian theory, but which would yield $T^{\sigma}{}_{\mu\mathrm{Maxwell}} = -F^{\sigma\tau}F_{\mu\tau} + \tfrac{1}{4}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau}$ if we were.

Next, we consider ${}^*F_{\tau\mu}\,{}^*F^{\sigma\tau}$ in (7.13), which is the commuted form of ${}^*F^{\sigma\tau}\,{}^*F_{\tau\mu}$ calculated in (7.14). Here, a like-calculation gives:

$$\begin{aligned} &{}^*F_{\tau\mu}\,{}^*F^{\sigma\tau} = \tfrac{1}{2!}\tfrac{1}{2!}\varepsilon_{\alpha\beta\tau\mu}\varepsilon^{\delta\gamma\sigma\tau}F^{\alpha\beta}F_{\delta\gamma} = -\tfrac{1}{2!}\tfrac{1}{2!}\varepsilon_{\alpha\beta\mu\tau}\varepsilon^{\delta\gamma\sigma\tau}F^{\alpha\beta}F_{\delta\gamma} \\ &= \tfrac{1}{2!}\tfrac{1}{2!}\delta_{\alpha\beta\mu}{}^{\delta\gamma\sigma}F^{\alpha\beta}F_{\delta\gamma} = F^{\sigma\tau}F_{\tau\mu} + \tfrac{1}{2}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau} \end{aligned} \tag{7.17}$$

which is simply (7.14) with $F_{\tau\mu}$ and $F^{\sigma\tau}$ commuted. Consequently, in (7.13), this means that:

$$F_{\tau\mu}F^{\sigma\tau} + {}^*F_{\tau\mu}\,{}^*F^{\sigma\tau} = F_{\tau\mu}F^{\sigma\tau} + F^{\sigma\tau}F_{\tau\mu} + \tfrac{1}{2}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau}, \tag{7.18}$$

which similarly is a non-abelian variant on the Maxwell tensor. Most importantly to pinpointing an integrable divergence for the energy tensor, in (7.13), using (7.15) and (7.18), the term

$$\begin{aligned} &\partial_{;\sigma}\left(F^{\sigma\tau}F_{\tau\mu} + {}^*F^{\sigma\tau}\,{}^*F_{\tau\mu}\right) - \partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau} + {}^*F_{\tau\mu}\,{}^*F^{\sigma\tau}\right) \\ &= \partial_{;\sigma}\left(F^{\sigma\tau}F_{\tau\mu} + F_{\tau\mu}F^{\sigma\tau} + \tfrac{1}{2}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau}\right) - \partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau} + F^{\sigma\tau}F_{\tau\mu} + \tfrac{1}{2}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau}\right) \\ &= \partial_{;\sigma}\left(F^{\sigma\tau}F_{\tau\mu} + F_{\tau\mu}F^{\sigma\tau} + \tfrac{1}{2}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau} - F_{\tau\mu}F^{\sigma\tau} - F^{\sigma\tau}F_{\tau\mu} - \tfrac{1}{2}\delta^{\sigma}{}_{\mu}F_{\sigma\tau}F^{\sigma\tau}\right) \\ &= \partial_{;\sigma}\left(0^{\sigma}{}_{\mu}\right) \end{aligned}, \tag{7.19}$$

So we have found the term to which we will want to connect $\partial_{;\sigma}T^{\sigma}{}_{\mu} = 0$ in order to identify $T^{\sigma}{}_{\mu} = 0$ and thus extract the empty-space gravitational equation $R_{\mu\nu} = 0$ of (6.11) as the gravitational equation *even for non-abelian electrodynamics*. But now we need to take care of the remaining terms in (7.13) which contain the gauge field G_{σ} and which arise only because of the very fact that Yang-Mills theory employs the gauge-covariant derivative $D_{;\sigma}$ rather than the ordinary covariant derivative $\partial_{;\sigma}$.

As to four of the eight G_{σ}-containing terms in (7.13), we may use (7.15) and (7.18) to eliminate:

$$
\begin{aligned}
&-i\left(G_\sigma F^{\sigma\tau}F_{\tau\mu}+G_\sigma *F^{\sigma\tau}*F_{\tau\mu}\right)+i\left(G_\sigma F_{\tau\mu}F^{\sigma\tau}+G_\sigma *F_{\tau\mu}*F^{\sigma\tau}\right)\\
&=iG_\sigma\left(F_{\tau\mu}F^{\sigma\tau}+*F_{\tau\mu}*F^{\sigma\tau}-F^{\sigma\tau}F_{\tau\mu}-*F^{\sigma\tau}*F_{\tau\mu}\right)\\
&=iG_\sigma\left(F_{\tau\mu}F^{\sigma\tau}+F^{\sigma\tau}F_{\tau\mu}+\tfrac{1}{2}\delta^\sigma{}_\mu F_{\sigma\tau}F^{\sigma\tau}-F^{\sigma\tau}F_{\tau\mu}-F_{\tau\mu}F^{\sigma\tau}-\tfrac{1}{2}\delta^\sigma{}_\mu F_{\sigma\tau}F^{\sigma\tau}\right)=0
\end{aligned}
\quad . \tag{7.20}
$$

As to the remaining four G_σ-containing terms in (7.13), we again use duality (4.1) as in (7.14) and (7.17) to obtain:

$$
\begin{aligned}
&*F^{\sigma\tau}G_\sigma *F_{\tau\mu}=\tfrac{1}{2!}\tfrac{1}{2!}\varepsilon^{\delta\gamma\sigma\tau}\varepsilon_{\alpha\beta\tau\mu}F_{\delta\gamma}G_\sigma F^{\alpha\beta}=-\tfrac{1}{2!}\tfrac{1}{2!}\varepsilon^{\delta\gamma\sigma\tau}\varepsilon_{\alpha\beta\mu\tau}F_{\delta\gamma}G_\sigma F^{\alpha\beta}=\tfrac{1}{2!}\tfrac{1}{2!}\delta^{\delta\gamma\sigma}{}_{\alpha\beta\mu}F_{\delta\gamma}G_\sigma F^{\alpha\beta}\\
&=\tfrac{1}{2!}\tfrac{1}{2!}\left(\delta^\delta{}_\alpha\delta^\gamma{}_\beta\delta^\sigma{}_\mu+\delta^\delta{}_\beta\delta^\gamma{}_\mu\delta^\sigma{}_\alpha+\delta^\delta{}_\mu\delta^\gamma{}_\alpha\delta^\sigma{}_\beta-\delta^\delta{}_\alpha\delta^\gamma{}_\mu\delta^\sigma{}_\beta-\delta^\delta{}_\mu\delta^\gamma{}_\beta\delta^\sigma{}_\alpha-\delta^\delta{}_\beta\delta^\gamma{}_\alpha\delta^\sigma{}_\mu\right)F_{\delta\gamma}G_\sigma F^{\alpha\beta}\\
&=F_{\tau\mu}G_\sigma F^{\sigma\tau}+\tfrac{1}{2}F_{\sigma\tau}G_\mu F^{\sigma\tau}
\end{aligned}
\quad . \tag{7.21}
$$

Consequently, the term:

$$
F^{\sigma\tau}G_\sigma F_{\tau\mu}+*F^{\sigma\tau}G_\sigma *F_{\tau\mu}=F^{\sigma\tau}G_\sigma F_{\tau\mu}+F_{\tau\mu}G_\sigma F^{\sigma\tau}+\tfrac{1}{2}F_{\sigma\tau}G_\mu F^{\sigma\tau}\,. \tag{7.22}
$$

Likewise to (7.21), we may use duality to find the commuted relationship:

$$
*F_{\tau\mu}G_\sigma *F^{\sigma\tau}=F^{\sigma\tau}G_\sigma F_{\tau\mu}+\tfrac{1}{2}F_{\sigma\tau}G_\mu F^{\sigma\tau} \tag{7.23}
$$

so that:

$$
F_{\tau\mu}G_\sigma F^{\sigma\tau}+*F_{\tau\mu}G_\sigma *F^{\sigma\tau}=F_{\tau\mu}G_\sigma F^{\sigma\tau}+F^{\sigma\tau}G_\sigma F_{\tau\mu}+\tfrac{1}{2}F_{\sigma\tau}G_\mu F^{\sigma\tau}\,. \tag{7.24}
$$

Therefore, on the very bottom line of (7.13), we may use (7.22) and (7.24) to eliminate:

$$
\begin{aligned}
&-i\left(F^{\sigma\tau}G_\sigma F_{\tau\mu}+*F^{\sigma\tau}G_\sigma *F_{\tau\mu}\right)+i\left(F_{\tau\mu}G_\sigma F^{\sigma\tau}+*F_{\tau\mu}G_\sigma *F^{\sigma\tau}\right)\\
&=i\left(F_{\tau\mu}G_\sigma F^{\sigma\tau}+*F_{\tau\mu}G_\sigma *F^{\sigma\tau}-F^{\sigma\tau}G_\sigma F_{\tau\mu}*F^{\sigma\tau}G_\sigma *F_{\tau\mu}\right)\\
&=i\left(F_{\tau\mu}G_\sigma F^{\sigma\tau}+F^{\sigma\tau}G_\sigma F_{\tau\mu}+\tfrac{1}{2}F_{\sigma\tau}G_\mu F^{\sigma\tau}-F^{\sigma\tau}G_\sigma F_{\tau\mu}-F_{\tau\mu}G_\sigma F^{\sigma\tau}-\tfrac{1}{2}F_{\sigma\tau}G_\mu F^{\sigma\tau}\right)=0
\end{aligned}
\quad . \tag{7.25}
$$

Now consolidating all of our calculations, we may use (7.19) and (7.20) and (7.25) to reduce (7.13) to:

$$
\begin{aligned}
&D_{;\sigma}F^{\sigma\tau}F_{\tau\mu}+F^{\sigma\tau}D_{;\sigma}F_{\tau\mu}+D_{;\sigma}*F^{\sigma\tau}*F_{\tau\mu}+*F^{\sigma\tau}D_{;\sigma}*F_{\tau\mu}\\
&-D_{;\sigma}F_{\tau\mu}F^{\sigma\tau}-F_{\tau\mu}D_{;\sigma}F^{\sigma\tau}-D_{;\sigma}*F_{\tau\mu}*F^{\sigma\tau}-*F_{\tau\mu}D_{;\sigma}*F^{\sigma\tau}\\
&=\partial_{;\sigma}\left(F^{\sigma\tau}F_{\tau\mu}+*F^{\sigma\tau}*F_{\tau\mu}\right)-\partial_{;\sigma}\left(F_{\tau\mu}F^{\sigma\tau}+*F_{\tau\mu}*F^{\sigma\tau}\right)\\
&=\partial_{;\sigma}\left(0^\sigma{}_\mu\right)
\end{aligned}
\quad . \tag{7.26}
$$

We then use (7.26) in (7.8), and also write the original subtraction of (6.2) from (6.1) using commutators, so at to arrive at our final identity:

$$
\begin{aligned}
0 &= \tfrac{1}{2} F^{\sigma\tau}\left(D_{;\mu}F_{\sigma\tau} + D_{;\sigma}F_{\tau\mu} + D_{;\tau}F_{\mu\sigma}\right) + {*F_{\mu\tau}} D_{;\sigma} {*F^{\sigma\tau}} + \tfrac{1}{2} {*F^{\sigma\tau}}\left(D_{;\mu} {*F_{\sigma\tau}} + D_{;\sigma} {*F_{\tau\mu}} + D_{;\tau} {*F_{\mu\sigma}}\right) + F_{\mu\tau} D_{;\sigma} F^{\sigma\tau} \\
&\quad - \tfrac{1}{2}\left(D_{;\mu}F_{\sigma\tau} + D_{;\sigma}F_{\tau\mu} + D_{;\tau}F_{\mu\sigma}\right) F^{\sigma\tau} - D_{;\sigma} {*F^{\sigma\tau}} {*F_{\mu\tau}} - \tfrac{1}{2}\left(D_{;\mu} {*F_{\sigma\tau}} + D_{;\sigma} {*F_{\tau\mu}} + D_{;\tau} {*F_{\mu\sigma}}\right) {*F^{\sigma\tau}} - D_{;\sigma}F^{\sigma\tau}F_{\mu\tau} \\
&= \tfrac{1}{2}\left[F^{\sigma\tau}, \left(D_{;\mu}F_{\sigma\tau} + D_{;\sigma}F_{\tau\mu} + D_{;\tau}F_{\mu\sigma}\right)\right] + \left[{*F_{\mu\tau}}, D_{;\sigma} {*F^{\sigma\tau}}\right] \\
&\quad + \tfrac{1}{2}\left[{*F^{\sigma\tau}}, \left(D_{;\mu} {*F_{\sigma\tau}} + D_{;\sigma} {*F_{\tau\mu}} + D_{;\tau} {*F_{\mu\sigma}}\right)\right] + \left[F_{\mu\tau}, D_{;\sigma}F^{\sigma\tau}\right] \\
&= \partial_{;\sigma}\left(0^{\sigma}{}_{\mu}\right)
\end{aligned}
\quad .(7.27)
$$

This identity is an identity of non-abelian gauge theory in which there are both electric and magnetic sources, it is invariant under the duality transformation $F^{\mu\nu} \to {*F^{\mu\nu}}$, and it comprises a vector of zeroes which is the spacetime divergence of a tensor of zeroes. Thus, as a final step, we connect this to the divergence of the conserved energy tensor, and also introduce the electric and magnetic sources directly via (2.8), (4.15) and (4.18) to arrive at (contrast the abelian result (6.10)):

$$
\begin{aligned}
\partial_{;\sigma}T^{\sigma}{}_{\mu} = 0 &= \tfrac{1}{2}\left[F^{\sigma\tau}, \left(D_{;\mu}F_{\sigma\tau} + D_{;\sigma}F_{\tau\mu} + D_{;\tau}F_{\mu\sigma}\right)\right] + \left[{*F_{\mu\tau}}, D_{;\sigma} {*F^{\sigma\tau}}\right] \\
&\quad + \tfrac{1}{2}\left[{*F^{\sigma\tau}}, \left(D_{;\mu} {*F_{\sigma\tau}} + D_{;\sigma} {*F_{\tau\mu}} + D_{;\tau} {*F_{\mu\sigma}}\right)\right] + \left[F_{\mu\tau}, D_{;\sigma}F^{\sigma\tau}\right] \quad . \\
&= \tfrac{1}{2}\left[F^{\sigma\tau}, P_{\mu\sigma\tau}\right] + \left[{*F_{\mu\tau}}, {*P^{\tau}}\right] - \tfrac{1}{2}\left[{*F^{\sigma\tau}}, {*J_{\mu\sigma\tau}}\right] + \left[F_{\mu\tau}, J^{\tau}\right] = \partial_{;\sigma}0
\end{aligned}
\qquad (7.28)
$$

So the subtraction of identity (6.2) from (6.1) turns out to introduce the four commutators $\left[F^{\sigma\tau}, P_{\mu\sigma\tau}\right]$, $\left[{*F_{\mu\tau}}, {*P^{\tau}}\right]$, $\left[{*F^{\sigma\tau}}, {*J_{\mu\sigma\tau}}\right]$ and $\left[F_{\mu\tau}, J^{\tau}\right]$ of sources with field strength densities, and is effectively a classical equation of motion relating all sources and fields. The overarching result in (7.28), however, is $0 = \partial_{;\sigma}T^{\sigma}{}_{\mu} = \partial_{;\sigma}0$ which integrates sans cosmological constant to $T^{\sigma}{}_{\mu} = 0$ and which, upon inverting the Einstein equation as we did after (6.10< once again yields the result (6.11), that:

$$
R_{\mu\nu} = 0 \qquad (7.29)
$$

In (7.28), once we connect up $\partial_{;\sigma}T^{\sigma}{}_{\mu} = 0$ to the Yang Mills identity, it is important to keep in mind that because $F^{\sigma\tau} = \lambda^{i}F^{i\,\sigma\tau} = F^{\sigma\tau}_{AB} = \lambda^{i}{}_{AB}F^{i\,\sigma\tau}$ is really an NxN Yang-Mills matrix of $N^2 - 1$ bivectors $F^{i\,\sigma\tau}$ for any specific gauge group SU(N) for which the generators λ^{i} maintain the commutator relationship $\left[\lambda_i, \lambda_j\right] = if_{ijk}\lambda_k$ using the group structure constants f_{ijk}, that the energy tensor $T^{\sigma}{}_{\mu}$ will now contain an additional NxN Yang-Mills matrix character, which is to say that $\partial_{;\sigma}T^{\sigma}{}_{\mu} = 0_{\mu}$, once all indexes are explicit, really has the structure $\partial_{;\sigma}T^{\sigma}{}_{\mu AB} = 0_{\mu AB}$. In general however, we shall suppress showing these Yang-Mills indexes explicitly unless specifically warranted to make a particular point or carry out a specific calculation, as it will eventually be when it comes time to calculate the binding energies of the monopole baryons to demonstrate how they do match up closely with empirical nuclear data. This means that in Yang-Mills, $R_{\mu\nu} = R_{\mu\nu AB} = 0$ has an implicit NxN Yang-Mills matrix structure as well, on top of its 4x4 symmetric spacetime structure. So gravitational theory, in this way, inherits the non-commuting attributes of Yang-Mills gauge theory.

In (7.29), once again, Einstein's final hunch in [22] as to a connection between Maxwell's equations and the gravitational equation $R_{\mu\nu} = 0$ for empty space, is validated by the fact that $R_{\mu\nu} = 0$, coupled with a locally-conserved energy tensor $\partial_{;\sigma} T^{\sigma}{}_{\mu} = 0$, can, as shown in (7.28)**,** be identically matched up with Maxwell's equations as extended to non-abelian Yang-Mills gauge theory with a duality symmetry between electric sources and the non-vanishing magnetic sources that arise from Yang-Mills gauge theory. Stated differently, (7.28) and (7.29) are the concrete representation, for non-Abelian gauge theory, of Einstein's "surprising" finding in [22] that " the gravitational equations for empty space determine their field just as strongly as do Maxwell's equations in the case of the electromagnetic field." To extend and encapsulate the lessons of Misner, Thorne and Wheeler in §20.6 of [28], $R_{\mu\nu} = 0$ for pure spacetime geometry, together with the condition $\partial_{;\sigma} T^{\sigma}{}_{\mu} = 0$ that energy must be locally conserved, is all that is needed to reproduce Maxwell's equations as represented in (7.28) for a non-abelian gauge theory (which therefore encompasses strong and weak interactions), and as represented in (6.9) and (6.10) for abelian, source-full electrodynamics. Because the magnetic monopoles in (7.28) are non-vanishing, and subsist in a theory that does contain gage fields which are related to the field strength according to the non-abelian relationship $F^{\mu\nu} = D^{[;\mu} G^{\nu]} = \partial^{[;\mu} G^{\nu]} - i\left[G^{\mu}, G^{\nu}\right]$ of (2.7), this theory is a viable theory of nature insofar as it does utilize the vector potential which "is not just a convenience in solving Maxwell equations [but] is needed in 20$^{\text{th}}$-century physics for [the] three very good purposes" identified by Witten in [30], not to mention that the vector potential is the central defining element of the gauge theory of Hermann Weyl [31], [32], [33].

This completes Part I of the development here. Now, we need to establish that the non-vanishing magnetic monopole of Yang-Mills theory to exist in the physical universe, and are in fact the baryons which in their proton and neutron flavors, are at the heart of the material universe.

PART II: CLASSICAL UNIFICATION OF NUCLEAR AND ELEMENTARY PARTICLE PHYSICS

8. In the Beginning: How Nature Springs Forth Luminous Energy and Matter from the Geometrodynamic Vacuum

When Einstein made the "surprising" finding in his final paper [22] that "that the gravitational equations for empty space determine their field just as strongly as do Maxwell's equations in the case of the electromagnetic field," as discussed in section 2, he really begged two questions. The first of these is whether this was more than just a "surprise," and rather an indication that there is in fact some formal equivalence between electrodynamic theory and gravitational theory under some defined set of circumstances. As we have now demonstrated in section 6 for "source-full" electrodynamics and section 7 for Yang-Mills electrodynamics, and throughout the development of Part I, there is indeed a formal equivalence between the gravitational equation $R_{\mu\nu}=0$ for empty space, and duality-symmetric, source-full electrodynamics, with **=-1 from the duality symmetry of second rank fields and (for non-abelian Yang-Mills theory) the commutation of field densities with sources in (7.28) resulting in the necessary balancing to cancel all constituent contributions to the energy tensor down to zero.

Second, by defining a "measure . . . which will even enable us to compare with each other the strengths of systems whose field variables *differ with respect to number and kind*," and by showing the two separate Maxwell equations $0=\partial_\sigma F^{\sigma\mu}$ and $0=\partial_\sigma F_{\mu\nu}+\partial_\mu F_{\nu\sigma}+\partial_\nu F_{\sigma\mu}$ in covariant form to be equivalent in strength to the one covariant equation $R_{\mu\nu}=0$ of empty gravitational spacetime, Einstein begged the question as to whether the logical consolidation which he himself started in 1905 in [23] of Maxwell's four equations (2.3) down to the two spacetime-covariant equations (2.4) was itself just a way station on path to a fuller consolidation in which one would speak simply of "the Maxwell equation," singular, as a single covariant equation that contains and unites all the physical content of Maxwell's theory. Simply put: what would happen if one were to combine both of Maxwell's equations into a single equation that contained the combined physical content of both? Would the "combination of the parts" yield more physics than the separate parts, and if so, what would that physics be teaching us about? This is a different question from whether there is a crossover connection to gravitational theory in the form of $R_{\mu\nu}=0$, and is rather a question about whether electrodynamics, as a self-contained theory, *on its own terms*, has in fact yet been fully advanced to its logical and physical and historical conclusion? That is the question we shall now begin to explore, and it will deepen the electrodynamic and gravitational unification of Part I by revealing unification among elementary particle physics, nuclear and hadron physics, and electrodynamics, in which the magnetic monopoles of Yang-Mills theory are understood to be baryons which confine their quarks and gluons and interact via meson exchange.

A fair portion of what will now be reviewed has already been developed at length by the author in previous papers [13], [14], [15], [16] and preprints [17], [18], [19], [20], [21] and so that development will not be repeated here. Instead, the review to follow will now focus on two main purposes: First, we shall simplify and streamline and clarify and consolidate the development of these earlier papers wherever the opportunity presents itself. Secondly, the opportunity in fact does best present itself, when exploring the fashion in which these earlier results are implicitly a unification among elementary particle physics, nuclear and hadron physics, and electrodynamics, by making that unification explicit and clear. Consequently, the presentation to follow will be that of a unified field theory, in which the author's previous work pertaining to baryons including protons and neutrons being the magnetic monopoles of Yang-Mills gauge theory, the solution in [21] to the Mass Gap Problem [12] including the development of quantum Yang-Mills theory and proving the existence of a non-trivial quantum Yang–Mills theory on $\mathbb{R}^4$ for any simple gauge group G, and the connections to empirical light nuclide binding data and the proton and neutron masse themselves, will all be assembled together into the context of a unified field theory springing forth from the vacuum $R_{\mu\nu}=0$ of empty space. And so, from that seemingly-empty stage of the spacetime vacuum, we shall show how nature springs forth luminous and material energy, and all of the natural phenomenology we observe in the physical universe.

It has often been envisioned that a unified field theory might start at its summit with a simple, single equation and very small number of principles, and that from that one equation and few principles, one would be able to systematically unfold by deduction other equations and principles which, when fully elaborated and mined for their physical content, would reveal and explain everything that is observed in the physical world. In that spirit, we begin in the vacuum, with $R_{\mu\nu}=0$, which we uncovered in (6.11) for source-full Maxwell electrodynamics, and in (7.29) for Yang-Mills electrodynamics where $R_{\mu\nu AB}=0$. *This is the single, simple, master equation of nature*. The way forward from there is revealed in spirit by §20.6 of [28], where at page 473, Misner, Thorne and Wheeler show how Maxwell's equations of motion "are fulfilled and must be fulfilled as a straight consequence of Einstein's field equation . . . plus . . . the stress energy tensor." The key point of Einstein's field equation is that it emanates from connecting the contacted Bianchi identity $\partial_{;\mu}\left(R^{\mu}{}_{\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}R\right)=0$ to the local conservation of energy $\partial_{;\mu}T^{\mu}{}_{\nu}=0$ as the vector of zeroes $-\kappa\partial_{;\mu}T^{\mu}{}_{\nu}=\partial_{;\mu}\left(R^{\mu}{}_{\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}R\right)=0$, and then integrating sans cosmological constant to obtain $-\kappa T^{\mu}{}_{\nu}=R^{\mu}{}_{\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}R$. That is, the divergence of Einstein's equation is a set of equations for local conservation of energy and three components of momentum, connected to and enforced by pure Riemannian geometry. While the stress energy tensor used in §20.6 of [28] is $T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}=-F^{\alpha\mu}F_{\sigma\mu}+\frac{1}{4}\delta^{\alpha}{}_{\sigma}F^{\mu\nu}F_{\mu\nu}$ of (3.1), the point of the stress energy tensor is not this particular energy tensor, but rather, *the very fact that one <u>has</u> an energy tensor to begin with*. So what Misner, Thorne and Wheeler are pointing out more generally is that one starts out with an energy tensor – some energy tensor – (which in abelian gauge theory has ten independent components), one starts out with the principle that that energy tensor is conserved (which is an equation with four independent components), and that by having an energy tensor coupled with the principle that that energy tensor is conserved, one may deduce Maxwell's electrodynamics in some variation. So, let us now take those steps starting with the energy tensor $R_{\mu\nu}=0$ of (7.29).

In the beginning:

$$R_{\mu\nu}=0. \tag{8.2}$$

This is the equation for the pure, empty geometry of spacetime. Via $-\kappa T^{\mu}{}_{\nu}=R^{\mu}{}_{\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}R$, we obtain the stress energy tensor $T^{\mu}{}_{\nu}=0$. Via $-\kappa\partial_{;\mu}T^{\mu}{}_{\nu}=\partial_{;\mu}\left(R^{\mu}{}_{\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}R\right)=0$ we require energies in the vacuum to be locally conserved. But to create light and electronic and nuclear matter we will need a different identity from the vector of zeroes $\partial_{;\mu}\left(R^{\mu}{}_{\nu}-\frac{1}{2}\delta^{\mu}{}_{\nu}R\right)=0$. We will need another identity – also a vector of zeroes – that contains electrodynamic fields $F^{\mu\nu}$ and sources J^{μ}, $P_{\sigma\mu\nu}$. As to the fields $F^{\mu\nu}$, we have the mathematical identity pairs (5.17) and (5.18) which contain the gauge-covariant derivatives $D_{;\sigma}=\partial_{;\sigma}-iG_{\sigma}$ and so apply even to non-abelian gauge theory in which the fields $F^{\mu\nu}$ do not commute. For abelian gauge theory, these identities continue to apply, but with the specialization $D_{;\sigma} \to \partial_{;\sigma}$ and with the fields now commuting, which commutativity collapses (5.17) and (5.18) together into a single identity pair. So in place of the Bianchi divergence $\partial_{;\mu}\left(R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R\right) = 0$, we may fashion (5.17) and (5.18) into the field identity of (7.28), connect this to the conservation requirement $\partial_{;\sigma}T^{\sigma}{}_{\mu} = 0$, and write the local energy-momentum conservation equation:

$$\begin{aligned}\partial_{;\sigma}T^{\sigma}{}_{\mu} &= \partial_{;\sigma}\left(R^{\sigma}{}_{\mu} - \tfrac{1}{2}\delta^{\sigma}{}_{\mu}R\right) = 0 \\ &= \tfrac{1}{2}\left[F^{\sigma\tau}, \left(D_{;\mu}F_{\sigma\tau} + D_{;\sigma}F_{\tau\mu} + D_{;\tau}F_{\mu\sigma}\right)\right] + \left[*F_{\mu\tau}, D_{;\sigma} * F^{\sigma\tau}\right] \\ &\quad + \tfrac{1}{2}\left[*F^{\sigma\tau}, \left(D_{;\mu} * F_{\sigma\tau} + D_{;\sigma} * F_{\tau\mu} + D_{;\tau} * F_{\mu\sigma}\right)\right] + \left[F_{\mu\tau}, D_{;\sigma}F^{\sigma\tau}\right] = \partial_{;\sigma}0\end{aligned} \quad . \tag{8.2}$$

Both $\partial_{;\sigma}\left(R^{\sigma}{}_{\mu} - \frac{1}{2}\delta^{\sigma}{}_{\mu}R\right)$ and the expression containing $F^{\sigma\tau}$ are a vector of zeroes by *mathematical* identity, while $\partial_{;\sigma}T^{\sigma}{}_{\mu} = 0$ is a *physical* principle of local conservation that we impose upon the energy tensor. However, the particular combination of fields $F^{\sigma\tau}$ shown in the above, because of duality and the commutators, conspires to also contain $\partial_{;\sigma}T^{\sigma}{}_{\mu} = \partial_{;\sigma}0$, which means up to a cosmological constant, that $R_{\mu\nu} = 0$. So (8.2) is another way of saying that $R_{\mu\nu} = 0$, and that the associated energy momentum is locally conserved. This is §20.6 of [28], extended to Yang-Mills electrodynamics with both electric and magnetic sources.

But there are some other mathematical identities that duality permits us to derive, and those involve first rank J^{μ}, $*P^{\mu}$ and third rank antisymmetric $P_{\mu\sigma\tau}$, $*J_{\mu\sigma\tau}$ source tensors. Those are the mathematical identities (5.19) and (5.20), which may also be combined together into the mathematical identity:

$$0 = \tfrac{1}{2}\left[F^{\sigma\tau}, P_{\mu\sigma\tau}\right] + \left[*F_{\mu\tau}, *P^{\tau}\right] - \tfrac{1}{2}\left[*F^{\sigma\tau}, *J_{\mu\sigma\tau}\right] + \left[F_{\mu\tau}, J^{\tau}\right] \tag{8.3}$$

which is yet another vector of zeroes that is shown embedded into (7.28). However, when we separate out this embedding as we have done above, then the *comparison* of the mathematical *field identity* in (8.2) with the mathematical *source and field* identity (8.3) immediately allows us to deduce the four interrelationships previously embedded in (7.28):

$$\begin{aligned}J^{\mu} &= D_{;\sigma}F^{\sigma\mu} \\ P_{\mu\sigma\tau} &= D_{;\mu}F_{\sigma\tau} + D_{;\sigma}F_{\tau\mu} + D_{;\tau}F_{\mu\sigma} \\ -*J_{\mu\sigma\tau} &= D_{;\mu} * F_{\sigma\tau} + D_{;\sigma} * F_{\tau\mu} + D_{;\tau} * F_{\mu\sigma} \\ *P^{\mu} &= D_{;\sigma} * F^{\sigma\mu}\end{aligned} \quad . \tag{8.4}$$

This is a verbatim reproduction of (5.22). So now, out of the vacuum $R_{\mu\nu}=0$ and the conservation $\partial_{;\sigma}T^{\sigma}{}_{\mu}=0$ of its associated energy-momentum, we have, by a careful construction of mathematical identities, derived the Maxwell equations of Yang-Mills electrodynamics, which by being Yang-Mills equations, are capable of accommodating the weak and strong interactions. As previously reviewed, although (8.4) contains four tensor equations, the "same-information-content" ([28] at page 88) character of duality (4.1) allows us to capture all of the physics information contained in (8.4) by using only one of the electric source and one of the magnetic source equations. Often, it is the first two equations that one uses.

One can then proceed to the further specialization $D_{;\sigma}\to\partial_{;\sigma}$ and impose an abelian condition $F^{\mu\nu}=\partial^{[;\mu}G^{\nu]}$ in lieu of the non-abelian condition $F^{\mu\nu}=D^{[;\mu}G^{\nu]}=\partial^{[;\mu}G^{\nu]}-i\left[G^{\mu},G^{\nu}\right]$ of (2.7), so that all of the fields now commute. In this instance the top two equations (8.5) reduce to the equation pair (2.4):

$$\begin{aligned}J^{\mu}&=\partial_{\sigma}F^{\sigma\mu}\\0&=\partial_{\sigma}F_{\mu\nu}+\partial_{\mu}F_{\nu\sigma}+\partial_{\nu}F_{\sigma\mu}\end{aligned}. \tag{8.5}$$

of ordinary Maxwell electrodynamics with electric sources but without magnetic sources. In the further specialization of source-free electrodynamics, we set $J^{\mu}=\partial_{\sigma}F^{\sigma\mu}=0$. This implies as in (3.8) that $\kappa_{\sigma}\equiv F_{\sigma\mu}J^{\mu}=F_{\sigma\mu}\partial_{;\alpha}F^{\alpha\mu}=0$, which in turn means that (3.5) reduces to the conservation relationship $0=\partial_{;\alpha}T^{\alpha}{}_{\sigma}=\partial_{;\alpha}T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}$ in which the Maxwell stress energy tensor:

$$T^{\alpha}{}_{\sigma\ \mathrm{Maxwell}}=-F^{\alpha\mu}F_{\sigma\mu}+\tfrac{1}{4}\delta^{\alpha}{}_{\sigma}F^{\mu\nu}F_{\mu\nu} \tag{8.6}$$

of (3.1) is the conserved energy tensor. This, of course, is a traceless energy tensor, $T^{\sigma}{}_{\sigma\ \mathrm{Maxwell}}=0$, and so it represents the luminous energy of light propagation.

So in Part I we worked forward from source-free Abelian electrodynamics to source-full Yang-Mills electrodynamics and connected all of this to the gravitational equation $R_{\mu\nu}=0$ of empty space. Now, in this section, we have reversed all of that development, by starting with $R_{\mu\nu}=0$ and local energy-momentum conservation to deductively work backwards to Yang-Mills' and Maxwell's electrodynamics. This is the way in which classical gravitation becomes unified with both the Maxwell and Yang-Mills variants of classical electrodynamics: We start with a geometrodynamic vacuum with locally-conserved energy, and from that vacuum, nature springs forth both light and matter. That (8.6) represents light and (8.5) represents electrodynamics is well-established. The balance of the development here – as the author has previously established in [13] and [21], and supported with empirical data in [14], [16], [17], [18], [19], [20] – will be to review how and why equations (8.4) are the equations of baryonic and nuclear matter, wherein the Yang-Mills magnetic monopoles $P_{\mu\sigma\tau}=D_{;\mu}F_{\sigma\tau}+D_{;\sigma}F_{\tau\mu}+D_{;\tau}F_{\mu\sigma}$

are one and the same as baryons, which monopoles $P_{\mu\sigma\tau}$, in their most important and interesting flavors, include the observed protons and neutrons at "the heart of matter." [34]

9. To Be Added

Subsequent sections to be added will review the work of [13], [14], [15], [16], [17], [18], [19], [20], [21], placing them all into the above-established context of a unified field theory.

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