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Fundamental Unification Theory with the Electron, the Neutrino, and Their Antiparticles

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Abstract

An SU_3 unification theory with the electron, the positron, and the neutrino is reviewed. A 10-spacetime gravidynamic unification of the internal charges and the spin is formulated, with a 16-component Majorana-Weyl fermion that consolidates the foregoing three Weyl particles with the antineutrino, and their Dirac conjugates. Vector bosons and scalar (Higgs) particles are consolidated in an antisymmetric tensor of 3rd rank, being the only tensor, apart from the graviton of 10-spacetime, that can couple to the unifying fermion. We write the Lorentz algebra of 10-spacetime in terms of the 4-spacetime Lorentz algebra and the internal O_6 factor, the latter expressed via its U_3 subalgebra, and construct the pertinent operator representations. We exhibit the complete structure of the unified gauge-Higgs couplings, indicating the source terms of particle masses. On the basis of this simple unification model, we propose the radical idea that all observed bosonic and fermionic particles, whether leptonic or hadronic, may be composed of just the underlying four fundamental fermions.

1 Introduction

This article describes the simplest and the most elegant gravidynamic unification model in a spacetime with extra dimensions. Whereas we have constructed an enormous 18-spacetime model[1], [2], [3] that can unify the spin and the internal charges of some four generations of leptons and quarks all within a 256-component fermionic Majorana-Weyl spinor of $O_{1,17}$, and whereas we have also constructed a 14-spacetime model[4], [5] that is able to consolidate two generations of quarks and leptons and their antiparticles, or in an *alternative* scenario, 8 charged leptons with their associated neutrinos, and antiparticles, all within a 64-component fermionic Weyl spinor of $O_{1,13}$, here we shall *economize*, and shall work with a 16-component fermionic Majorana-Weyl spinor of 10-spacetime, with an underlying $O_{1,9}$ symmetry. The latter spinor can accommodate only four Weyl fermions in 4-spacetime that we can identify as the electron, the neutrino, and their antiparticles. Our focus in this article will be the spectrum of bosonic particles (vectors and scalars) that would couple to the forgoing four fundamental fermions. In a 10-spacetime gravidynamic theory, these bosons would be represented by a 3rd rank antisymmetric Lorentzian tensor V_{ABC} and would get their dynamics in the effective quantum action from a coupling of the form

$$\frac{1}{3!}V_{ABC}\bar{\Psi}\gamma_{ABC}\Psi \tag{1}$$

Here, Ψ is the fermionic Majorana-Weyl spinor of $O_{1,9}$, with Dirac conjugate $\bar{\Psi}$, and the γ's are elements of the associated Dirac algebra. The main drive of the present work is to display the detailed structure of the above coupling. We shall show how this describes vector boson couplings as well as Higgs couplings for scalars that can give appropriate masses to the fermions via some non-vanishing vacuum components.

The 10-spacetime theory that will be assembled in this work can describe both the gravidynamic and the electroweak interactions of the electron and the neutrino, and their antiparticles. We shall construct the $O_{1,9}$ algebra and its pertinent tensorial and spinorial representations in a form exhibiting an internal SU_3 symmetry. Whereas, the latter SU_3 can consolidate[6], [7], [8] the electroweak interactions[9], [10], [11] of the electron and the neutrino by including the positron in the fundamental fermionic multiplet, the 10-spacetime extension will require the inclusion of the antineutrino.

Before proceeding to build our model, let us first consider and learn from the following Lagrangian density of quantum electrodynamics that describes the coupling of a Dirac field Ψ to the photon $\mathcal{A}$,

$$\overline{\Psi}(i\gamma\cdot\partial)\Psi - \mathcal{A}_\mu\overline{\Psi}\gamma_\mu\Psi - m\overline{\Psi}\Psi \tag{2}$$

Splitting Ψ into its two Weyl (chiral) parts, $\Psi = \psi + \xi$, where ψ is left-handed and ξ is right-handed, we obtain for the above,

$$\left\{\begin{array}{l} \overline{\psi}(i\gamma\cdot\partial)\psi + \overline{\xi}(i\gamma\cdot\partial)\xi \\ \\ -\mathcal{A}_\mu\left(\overline{\psi}\gamma_\mu\psi + \overline{\xi}\gamma_\mu\xi\right) \\ \\ -m(\overline{\psi}\xi + \overline{\xi}\psi) \end{array}\right\} \tag{3}$$

To the left-handed Weyl component ψ we may assign the electron e, while the right-handed component ξ, may be related to the positron e^* by writing

$$\xi \to C\tilde{\overline{e^*}} \qquad \overline{\xi} \to -\tilde{e^*}C^{-1} \tag{4}$$

Here, the tilde symbol denotes transposition, C is the antisymmetric Majorana (charge conjugation or transposition) matrix, with the property $C^{-1}\gamma_\mu C = -\tilde{\gamma}_\mu$. Hence, using these properties and replacements, the foregoing Lagrangian density takes the form

$$\left\{\begin{array}{l} \overline{e}(i\gamma\cdot\partial)e + \overline{e^*}(i\gamma\cdot\partial)e^* \\ \\ +\mathcal{A}_\mu\left(-\overline{e}\gamma_\mu e + \overline{e^*}\gamma_\mu e^*\right) \\ \\ -m\left(\overline{e}C\tilde{\overline{e^*}} - \tilde{e^*}C^{-1}e\right) \end{array}\right\} \tag{5}$$

It should be remarked that, in obtaining the kinetic terms and the coupling terms of the above expression, we used integration by parts and the property that fermionic fields anticommute. Notice how the system describes two fermionic particles, the electron e and the positron e^* that couple with opposite charge eigenvalues to the photon. Notice as well how the *Dirac mass* term corresponds to a mixing between the electron and the positron.

Notice that the electrodynamic system as described above, in terms of two Weyl fermions (e, e^*) would admit an extension to an SU_2 gauge system. However, this can be seen as part of a larger SU_3 system that includes a third Weyl fermion, the neutrino. Such a system can describe electroweak interactions, and will be reviewed in the following section.

Subsequently, several sections will present the algebraic work allowing the construction of the 10-spacetime gravidynamic theory that can embed the SU_3 electroweak model, with the inclusion of the antineutrino. Important observations and remarks will be left to the concluding discussion.

2 The SU_3 Model & Boson-Fermion Couplings

The kinetic term of a system of three Weyl fermions ψ_a $(a = 1, 2, 3)$, Dirac conjugates $\overline{\psi}^a$, in 4-spacetime is invariant with respect to a global SU_3 symmetry. Promoting the latter to a local counterpart[6] would introduce an octet of gauge bosons ${V_a}^b$, the latter being traceless and corresponding to the generators of the SU_3 algebra. The resulting gauge-invariant fermionic Lagrangian takes the following form (suppressing the vectorial spacetime indices)

$$\overline{\psi}^a(i\gamma \cdot \partial)\psi_a + {V_a}^b\overline{\psi}^a\gamma\psi_b \tag{6}$$

In order to spell out the content of the above system, in terms of actual fermionic and bosonic particles, we make the following assignments. For the fermions, we write

$$\begin{cases} \psi_1 \rightarrow \nu & \overline{\psi}^1 \rightarrow \overline{\nu} \\ \psi_2 \rightarrow e & \overline{\psi}^2 \rightarrow \overline{e} \\ \psi_3 \rightarrow e^* & \overline{\psi}^3 \rightarrow \overline{e^*} \end{cases} \tag{7}$$

Notice that we have embedded the electric charge operator within SU_3 with diagonal eigenvalues $(0, -1, +1)$. In the above, we have introduced the neutrino ν, the electron e, and the positron e^*. The fermionic kinetic terms would translate like:

$$\overline{\psi}^a(i\gamma \cdot \partial)\psi_a \Rightarrow \overline{\nu}(i\gamma \cdot \partial)\nu + \overline{e}(i\gamma \cdot \partial)e + \overline{e^*}(i\gamma \cdot \partial)e^* \tag{8}$$

For the vector bosons, we have

$$\begin{cases} {V_1}^1 \to \mathcal{Z} & {V_1}^2 \to \mathcal{W}^* & {V_1}^3 \to \mathcal{Y} \\ {V_2}^1 \to \mathcal{W} & {V_2}^2 \to -\mathcal{A} - \frac{1}{2}\mathcal{Z} & {V_2}^3 \to \mathcal{X} \\ {V_3}^1 \to \mathcal{Y}^* & {V_3}^2 \to \mathcal{X}^* & {V_3}^3 \to \mathcal{A} - \frac{1}{2}\mathcal{Z} \end{cases} \tag{9}$$

Here we have introduced the photon $\mathcal{A}$, the neutral boson $\mathcal{Z}$, the charge ± 1 bosons $(\mathcal{W}, \mathcal{W}^*)$, the charge ± 2 bosons $(\mathcal{X}, \mathcal{X}^*)$, and the charge ± 1 bosons $(\mathcal{Y}, \mathcal{Y}^*)$. Notice that a bosonic bilinear invariant takes the form,

$$ {A_a}^b {A_b}^a \Rightarrow 2\mathcal{A}^2 + \frac{3}{2}\mathcal{Z}^2 + 2\mathcal{W}\mathcal{W}^* + 2\mathcal{X}\mathcal{X}^* + 2\mathcal{Y}\mathcal{Y}^* \tag{10}$$

A canonical form of the bosonic bilinears could be obtained by dividing the above by 4, and rescaling fields other than $\mathcal{A}$. However, we do not need to do that for our present purposes.

Corresponding to the foregoing assignments of the fermions and the vector bosons, the coupling term would translate as follows:

$$ {V_a}^b \overline{\psi}^a \gamma \psi_b \Rightarrow \left\{ \begin{array}{l} \mathcal{A} \times (-\bar{e}\gamma e + \overline{e^*}\gamma e^*) \\ +\frac{1}{2}\mathcal{Z} \times (-\bar{e}\gamma e + 2\overline{\nu}\gamma\nu - \overline{e^*}\gamma e^*) \\ +\mathcal{W} \times \bar{e}\gamma\nu + \mathcal{W}^* \times \overline{\nu}\gamma e \\ +\mathcal{X} \times \bar{e}\gamma e^* + \mathcal{X}^* \times \overline{e^*}\gamma e \\ +\mathcal{Y} \times \overline{\nu}\gamma e^* + \mathcal{Y}^* \times \overline{e^*}\gamma\nu \end{array} \right\} \tag{11}$$

Notice that whereas the $(\mathcal{W}, \mathcal{W}^*)$ is the charged massive particle of the electroweak theory[9], [10], [11] that would exchange the electron with the neutrino, the predicted[6] particle $(\mathcal{Y}, \mathcal{Y}^*)$ of the SU_3 theory would exchange the positron with the neutrino, and the predicted[6] doubly-charged particle $(\mathcal{X}, \mathcal{X}^*)$ would exchange the electron with the positron.

3 The $\mathbf{O}_{1,9}$ Algebra in Terms of $\mathbf{O}_{1,3}$ & $\mathbf{SU}_3$

The generators of the $\mathrm{O}_{1,9}$ algebra would consist of the following set:

$$\{J_{\mu,\nu}, {J_a}^b, Q_{ab}, Q^{ab}, H_{\mu a}, {H_\mu}^a\} \tag{12}$$

Here, the antisymmetric $J_{\mu\nu}$ are the generators of $\mathrm{O}_{1,3}$, with the indices $(\mu, \nu, \cdots)$ corresponding to 4-spacetime vectors. The generators ${J_a}^b$ are those of U_3, the traceless part corresponding to SU_3, and the trace to a U_1 factor, with the indices $(a, b, c, \ldots)$

being those of the fundamental 3-plet. The generators $Q_{a,b}$ and their conjugates Q^{ab}, being antisymmetric, would correspond to the coset of O_6 over SU_3 and U_1. The remaining generators $H_{\mu a}$ and their conjugates $H_\mu{}^a$ correspond to the coset of $O_{1,9}$ over $O_{1,3}$ and O_6. We shall write the commutators that constitute the $O_{1,9}$ algebra.

First, we have the 4-spacetime Lorentz algebra[1],

$$[J_{\mu\nu}, J_{\lambda\rho}] = (\eta_{\nu\lambda} J_{\mu\rho} - \eta_{\mu\lambda} J_{\nu\rho} + \eta_{\mu\rho} J_{\nu\lambda} - \eta_{\nu\rho} J_{\mu\lambda}) \tag{13}$$

Whereas the Lorentz generators $J_{\mu\nu}$ would commute with the generators of the internal O_6, namely $J_a{}^b$, Q_{ab}, and Q^{ab}, we have

$$[J_{\mu\nu}, H_{\lambda a}] = (\eta_{\nu\lambda} H_{\mu a} - \eta_{\mu\lambda} H_{\nu a}) \tag{14}$$

$$[J_{\mu\nu}, H_\lambda{}^a] = (\eta_{\nu\lambda} H_\mu{}^a - \eta_{\mu\lambda} H_\nu{}^a) \tag{15}$$

The generators $J_a{}^b$, Q_{ab}, and Q^{ab}, constituting the internal O_6 algebra, would satisfy the followings:

$$\left[J_a{}^b, J_c{}^d\right] = \left(\delta_c{}^b J_a{}^d - \delta_a{}^d J_c{}^b\right) \tag{16}$$

$$\left[J_a{}^b, Q_{cd}\right] = \left(\delta_c{}^b Q_{ad} - \delta_d{}^b Q_{ac}\right) \tag{17}$$

$$\left[J_a{}^b, Q^{cd}\right] = -\left(\delta_a{}^c Q^{bd} - \delta_a{}^d Q^{bc}\right) \tag{18}$$

$$[Q_{ab}, Q_{cd}] = 0 \tag{19}$$

$$\left[Q^{ab}, Q^{cd}\right] = 0 \tag{20}$$

$$\left[Q_{ab}, Q^{cd}\right] = \left(\delta_b{}^c J_a{}^d - \delta_a{}^c J_b{}^d + \delta_a{}^d J_b{}^c - \delta_b{}^d J_a{}^c\right) \tag{21}$$

The commutators of the $J_a{}^b$ generators with the H's are

$$\left[J_a{}^b, H_{\mu c}\right] = \delta_c{}^b H_{\mu a} \tag{22}$$

$$\left[J_a{}^b, H_\mu{}^c\right] = -\delta_a{}^c H_\mu{}^b \tag{23}$$

The commutators of the Q's generators with the H's are

$$[Q_{ab}, H_{\mu c}] = 0 \tag{24}$$

$$[Q_{ab}, H_\mu{}^c] = (\delta_b{}^c H_{\mu a} - \delta_a{}^c H_{\mu b}) \tag{25}$$

$$\left[Q^{ab}, H_{\mu c}\right] = \left(\delta_c{}^b H_\mu{}^a - \delta_c{}^a H_\mu{}^b\right) \tag{26}$$

$$\left[Q^{ab}, H_\mu{}^c\right] = 0 \tag{27}$$

Finally, the commutators of the H's among themselves are:

$$[H_{\mu a}, H_{\nu b}] = -\eta_{\mu\nu} Q_{ab} \tag{28}$$

$$\left[H_{\mu a}, H_\nu{}^b\right] = -\eta_{\mu\nu} J_a{}^b - \delta_a{}^b J_{\mu\nu} \tag{29}$$

[1]The alert reader should notice that in writing our Lie algebras, in this and other cited articles, we do not include the imaginary unit i on the right side of the commutator equations. We employ this tactic for the purpose of simplifying the algebraic work, and should not have adverse effects on the final results, if used properly.

$$\left[H_\mu{}^a, H_\nu{}^b\right] = -\eta_{\mu\nu} Q^{ab} \tag{30}$$

We can verify that the Jacobi identities involving any three of the $O_{1,9}$ generators $J_{\mu,\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$, or $H_\mu{}^a$, are all satisfied. Besides, any of these generators would commute with the following quadratic (Casimir) operator:

$$\frac{1}{2} J_{\mu\nu} J_{\mu\nu} - J_a{}^b J_b{}^a + \frac{1}{2} Q_{ab} Q^{ab} + \frac{1}{2} Q^{ab} Q_{ab} + H_{\mu a} H_\mu{}^a + H_\mu{}^a H_{\mu a} \tag{31}$$

4 The Representation of an $O_{1,9}$ Vector

Let us introduce the operators (K_μ, K_a, K^a). The commutators of these with the Lorentz generators $J_{\mu\nu}$ are

$$[J_{\mu\nu}, K_\lambda] = (\eta_{\nu\lambda} K_\mu - \eta_{\mu\lambda} K_\nu) \tag{32}$$

$$[J_{\mu\nu}, K_a] = 0 \tag{33}$$

$$[J_{\mu\nu}, K^a] = 0 \tag{34}$$

The commutators with the U_3 generators $J_a{}^b$,

$$\left[J_a{}^b, K_\mu\right] = 0 \tag{35}$$

$$\left[J_a{}^b, K_c\right] = \delta_c{}^b K_a \tag{36}$$

$$\left[J_a{}^b, K^c\right] = -\delta_a^c K^b \tag{37}$$

The commutators with Q_{ab},

$$[Q_{ab}, K_\mu] = 0 \tag{38}$$

$$[Q_{ab}, K_c] = 0 \tag{39}$$

$$[Q_{ab}, K^c] = \delta_b{}^c K_a - \delta_a{}^c K_b \tag{40}$$

The commutators with Q^{ab},

$$\left[Q^{ab}, K_\mu\right] = 0 \tag{41}$$

$$\left[Q^{ab}, K_c\right] = \delta_c{}^b K^a - \delta_c{}^a K^b \tag{42}$$

$$\left[Q^{ab}, K^c\right] = 0 \tag{43}$$

The commutators with $H_{\mu a}$,

$$[H_{\mu a}, K_\nu] = -\eta_{\mu\nu} K_a \tag{44}$$

$$[H_{\mu a}, K_b] = 0 \tag{45}$$

$$\left[H_{\mu a}, K^b\right] = \delta_a{}^b K_\mu \tag{46}$$

The commutators with $H_\mu{}^a$,

$$[H_\mu{}^a, K_\nu] = -\eta_{\mu\nu} K^a \tag{47}$$

$$[H_\mu{}^a, K_b] = \delta_b{}^a K_\mu \tag{48}$$

$$\left[H_\mu{}^a, K^b\right] = 0 \tag{49}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,9}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or $H_\mu{}^a$, with either of the operators K_μ, K_a, or K^a, are satisfied. Moreover any of the $O_{1,9}$ generators can be shown to commute with the following quadratic operator:

$$K_\mu K_\mu + K_a K^a + K^a K_a \tag{50}$$

We proceed now to the introduction of the multiplet that can be associated with the above operator representation, and to the construction of the infinitesimal $O_{1,9}$ transformations that act on it. The desired multiplet with components $\{B_\mu, B_a, B^a\}$ can be introduced by the vector module

$$\mathcal{B} = B_\mu K_\mu + B_a K^a + B^a K_a \tag{51}$$

Introducing the $O_{1,9}$ parameter module,

$$\mathcal{W} = \frac{1}{2}\Omega_{\mu\nu} J_{\mu\nu} + \Omega_a{}^b J_b{}^a + \frac{1}{2}\Omega_{ab} Q^{ab} + \frac{1}{2}\Omega^{ab} Q_{ab} + \Omega_{\mu a} H_\mu{}^a + \Omega_\mu{}^a H_{\mu a} \tag{52}$$

we can compute the commutator $[\mathcal{W}, \mathcal{B}]$. The latter gives a vector module whose components would define the needed infinitesimal transformations. We obtain

$$\delta B_\mu = \Omega_{\mu\nu} B_\nu + \Omega_{\mu a} B^a + \Omega_\mu{}^a B_a \tag{53}$$

$$\delta B_a = -\Omega_a{}^b B_b + \Omega_{ab} B^b - \Omega_{\mu a} B_\mu \tag{54}$$

$$\delta B^a = \Omega_b{}^a B^b + \Omega^{a,b} B_b - \Omega_\mu{}^a B_\mu \tag{55}$$

We can verify that, for any two vector modules $\mathcal{A}$ and $\mathcal{B}$, the above infinitesimal transformations, acting in a like manner on the components of both, would leave invariant the following bilinear form:

$$\mathcal{A} \cdot \mathcal{B} = A_\mu B_\mu + A_a B^a + A^a B_a \tag{56}$$

5 The Antisymmetric Tensor Representation of Rank 3

An $O_{1,9}$ antisymmetric tensor representation of rank 3 would have the following $O_{1,3}$ and U_3 covariant components:

$$\{K_{\mu\nu\lambda}, K_{\mu\nu a}, K_{\mu\nu}{}^a, K_{\mu ab}, K_{\mu a}{}^b, K_\mu{}^{ab}, K_{abc}, K_{ab}{}^c, K_a{}^{bc}, K^{abc}\} \tag{57}$$

The symmetries of the above components with respect to their spacetime and SU_3 indices should be clear. Notice that the tensor $K_{\mu\nu\lambda}$ could be traded for a single-index counterpart using the 4-dimensional epsilon symbol $\epsilon_{\mu\nu\lambda\rho}$. However, it is convenient to leave it in this form, at this stage. Likewise, the 2-index SU_3 tensors can be traded for single-index counterparts, and the 3-index SU_3 tensors can be traded for scalars, all using the pertinent epsilon symbol. Again, it is more convenient to leave them as such. The replacements can be made later after the couplings are constructed.

In order to be able to write out the infinitesimal transformations of an associated multiplet, we proceed to the elaborate task of writing down the commutators of the above component operators with the generators of the $O_{1,9}$ algebra.

For the commutators of $J_{\mu\nu}$, we have

$$[J_{\mu\nu}, K_{\lambda\rho\sigma}] = (\eta_{\nu\lambda}K_{\mu\rho\sigma} + \eta_{\nu\rho}K_{\mu\sigma\lambda} + \eta_{\nu\sigma}K_{\mu\lambda\rho}) - (\mu \leftrightarrow \nu) \tag{58}$$

$$[J_{\mu\nu}, K_{\lambda\rho a}] = (\eta_{\nu\lambda}K_{\mu\rho a} - \eta_{\nu\rho}K_{\mu\lambda a}) - (\mu \leftrightarrow \nu) \tag{59}$$

$$[J_{\mu\nu}, K_{\lambda\rho}{}^{a}] = (\eta_{\nu\lambda}K_{\mu\rho}{}^{a} - \eta_{\nu\rho}K_{\mu\lambda}{}^{a}) - (\mu \leftrightarrow \nu) \tag{60}$$

$$[J_{\mu\nu}, K_{\lambda ab}] = \eta_{\nu\lambda}K_{\mu ab} - \eta_{\mu\lambda}K_{\nu ab} \tag{61}$$

$$\left[J_{\mu\nu}, K_{\lambda a}{}^{b}\right] = \eta_{\nu\lambda}K_{\mu a}{}^{b} - \eta_{\mu\lambda}K_{\nu a}{}^{b} \tag{62}$$

$$\left[J_{\mu\nu}, K_{\lambda}{}^{ab}\right] = \eta_{\nu\lambda}K_{\mu}{}^{ab} - \eta_{\mu\lambda}K_{\nu}{}^{ab} \tag{63}$$

The commutators of $J_{\mu\nu}$ with K_{abc}, $K_{ab}{}^{c}$, $K_{a}{}^{bc}$, and K^{abc} are vanishing.

Whereas $J_{a}{}^{b}$ commutes with $K_{\mu\nu\lambda}$, its commutators with the other K's are

$$\left[J_{a}{}^{b}, K_{\mu\nu c}\right] = \delta_{c}{}^{b}K_{\mu\nu a} \tag{64}$$

$$\left[J_{a}{}^{b}, K_{\mu\nu}{}^{c}\right] = -\delta_{a}{}^{c}K_{\mu\nu}{}^{b} \tag{65}$$

$$\left[J_{a}{}^{b}, K_{\mu cd}\right] = \delta_{c}{}^{b}K_{\mu ad} - \delta_{d}{}^{b}K_{\mu ac} \tag{66}$$

$$\left[J_{a}{}^{b}, K_{\mu c}{}^{d}\right] = \delta_{c}{}^{b}K_{\mu a}{}^{d} - \delta_{a}{}^{d}K_{\mu c}{}^{b} \tag{67}$$

$$\left[J_{a}{}^{b}, K_{\mu}{}^{cd}\right] = -\delta_{a}{}^{c}K_{\mu}{}^{bd} + \delta_{a}{}^{d}K_{\mu}{}^{bc} \tag{68}$$

$$\left[J_{a}{}^{b}, K_{cde}\right] = \delta_{c}{}^{b}K_{ade} + \delta_{d}{}^{b}K_{aec} + \delta_{e}{}^{b}K_{acd} \tag{69}$$

$$\left[J_{a}{}^{b}, K_{cd}{}^{e}\right] = \left(\delta_{c}{}^{b}K_{a,d}{}^{e} - \delta_{d}{}^{b}K_{ac}{}^{e}\right) - \left(\delta_{a}{}^{e}K_{cd}{}^{b}\right) \tag{70}$$

$$\left[J_{a}{}^{b}, K_{c}{}^{de}\right] = \left(\delta_{c}{}^{b}K_{a}{}^{de}\right) - \left(\delta_{a}{}^{d}K_{c}{}^{be} - \delta_{a}{}^{e}K_{c}{}^{bd}\right) \tag{71}$$

$$\left[J_{a}{}^{b}, K^{cde}\right] = -\left(\delta_{a}{}^{c}K^{bde} + \delta_{a}{}^{d}K^{bec} + \delta_{a}{}^{e}K^{bcd}\right) \tag{72}$$

The nonvanishing commutators of Q_{ab} with the K's are:

$$[Q_{ab}, K_{\mu\nu}{}^{c}] = \delta_{b}{}^{c}K_{\mu\nu a} - \delta_{a}{}^{c}K_{\mu\nu b} \tag{73}$$

$$\left[Q_{ab}, K_{\mu c}{}^{d}\right] = \delta_{a}{}^{d}K_{\mu bc} - \delta_{b}{}^{d}K_{\mu ac} \tag{74}$$

$$\left[Q_{ab}, K_{\mu}{}^{cd}\right] = -\left(\delta_{a}{}^{c}K_{\mu b}{}^{d} - \delta_{b}{}^{c}K_{\mu a}{}^{d} + \delta_{b}{}^{d}K_{\mu a}{}^{c} - \delta_{a}{}^{d}K_{\mu b}{}^{c}\right) \tag{75}$$

$$[Q_{ab}, K_{cd}{}^{e}] = (\delta_{b}{}^{e}K_{acd} - \delta_{a}{}^{e}K_{bcd}) \tag{76}$$

$$\left[Q_{ab}, K_{c}{}^{de}\right] = -\left(\delta_{b}{}^{d}K_{ac}{}^{e} - \delta_{a}{}^{d}K_{bc}{}^{e} + \delta_{a}{}^{e}K_{bc}{}^{d} - \delta_{b}{}^{e}K_{ac}{}^{d}\right) \tag{77}$$

$$\left[Q_{ab}, K^{cde}\right] = \left(\delta_{b}{}^{c}K_{a}{}^{de} + \delta_{b}{}^{d}K_{a}{}^{ec} + \delta_{b}{}^{e}K_{a}{}^{cd}\right) - (a \leftrightarrow b) \tag{78}$$

The nonvanishing commutators of Q^{ab} with the K's are:

$$\left[Q^{ab}, K_{\mu\nu c}\right] = \delta_{c}{}^{b}K_{\mu\nu}{}^{a} - \delta_{c}{}^{a}K_{\mu\nu}{}^{b} \tag{79}$$

$$\left[Q^{ab}, K_{\mu cd}\right] = \left(\delta_c{}^a K_{\mu d}{}^b - \delta_c{}^b K_{\mu d}{}^a + \delta_d{}^b K_{\mu c}{}^a - \delta_d{}^a K_{\mu c}{}^b\right) \tag{80}$$

$$\left[Q^{ab}, K_{\mu c}{}^d\right] = \delta_c{}^b K_\mu{}^{ad} - \delta_c{}^a K_\mu{}^{bd} \tag{81}$$

$$\left[Q^{ab}, K_{cde}\right] = \left(\delta_c{}^b K_{de}{}^a + \delta_d{}^b K_{ec}{}^a + \delta_e{}^b K_{cd}{}^a\right) - (a \leftrightarrow b) \tag{82}$$

$$\left[Q^{ab}, K_{cd}{}^e\right] = \left(-\delta_c{}^b K_d{}^{ae} + \delta_d{}^b K_c{}^{ae}\right) - (a \leftrightarrow b) \tag{83}$$

$$\left[Q^{ab}, K_c{}^{de}\right] = \delta_c{}^b K^{ade} - \delta_c{}^a K^{bde} \tag{84}$$

The nonvanishing commutators of $H_{\mu a}$ with the K's are:

$$\left[H_{\mu a}, K_{\nu\lambda\rho}\right] = -\left(\eta_{\mu\nu} K_{\lambda\rho a} + \eta_{\mu\rho} K_{\nu\lambda a} + \eta_{\mu\lambda} K_{\rho\nu a}\right) \tag{85}$$

$$\left[H_{\mu a}, K_{\nu\lambda b}\right] = \left(\eta_{\mu\nu} K_{\lambda ab} - \eta_{\mu\lambda} K_{\nu ab}\right) \tag{86}$$

$$\left[H_{\mu a}, K_{\nu\lambda}{}^b\right] = \left(\eta_{\mu\nu} K_{\lambda a}{}^b - \eta_{\mu\lambda} K_{\nu a}{}^b\right) + \left(\delta_a{}^b K_{\mu\nu\lambda}\right) \tag{87}$$

$$\left[H_{\mu a}, K_{\nu bc}\right] = -\eta_{\mu\nu} K_{abc} \tag{88}$$

$$\left[H_{\mu a}, K_{\nu b}{}^c\right] = -\eta_{\mu\nu} K_{ab}{}^c + \delta_a{}^c K_{\mu\nu b} \tag{89}$$

$$\left[H_{\mu a}, K_\nu{}^{bc}\right] = -\left(\eta_{\mu\nu} K_a{}^{bc}\right) - \left(\delta_a{}^b K_{\mu\nu}{}^c - \delta_a{}^c K_{\mu\nu}{}^b\right) \tag{90}$$

$$\left[H_{\mu a}, K_{bc}{}^d\right] = \delta_a{}^d K_{\mu bc} \tag{91}$$

$$\left[H_{\mu a}, K_b{}^{cd}\right] = -\delta_a{}^c K_{\mu b}{}^d + \delta_a{}^d K_{\mu b}{}^c \tag{92}$$

$$\left[H_{\mu a}, K^{bcd}\right] = \left(\delta_a{}^b K_\mu{}^{cd} + \delta_a{}^d K_\mu{}^{bc} + \delta_a{}^c K_\mu{}^{db}\right) \tag{93}$$

Finally, the nonvanishing commutators of $H_\mu{}^a$ with the K's are:

$$\left[H_\mu{}^a, K_{\nu\lambda\rho}\right] = -\left(\eta_{\mu\nu} K_{\lambda\rho}{}^a + \eta_{\mu\rho} K_{\nu\lambda}{}^a + \eta_{\mu\lambda} K_{\rho\nu}{}^a\right) \tag{94}$$

$$\left[H_\mu{}^a, K_{\nu\lambda b}\right] = -\left(\eta_{\mu\nu} K_{\lambda b}{}^a - \eta_{\mu\lambda} K_{\nu b}{}^a\right) + \left(\delta_b{}^a K_{\mu\nu\lambda}\right) \tag{95}$$

$$\left[H_\mu{}^a, K_{\nu\lambda}{}^b\right] = \left(\eta_{\mu\nu} K_\lambda{}^{ab} - \eta_{\mu\lambda} K_\nu{}^{ab}\right) \tag{96}$$

$$\left[H_\mu{}^a, K_{\nu bc}\right] = -\left(\eta_{\mu\nu} K_{bc}{}^a\right) - \left(\delta_b{}^a K_{\mu\nu c} - \delta_c{}^a K_{\mu\nu b}\right) \tag{97}$$

$$\left[H_\mu{}^a, K_{\nu b}{}^c\right] = \eta_{\mu\nu} K_b{}^{ac} - \delta_b{}^a K_{\mu\nu}{}^c \tag{98}$$

$$\left[H_\mu{}^a, K_\nu{}^{bc}\right] = -\eta_{\mu\nu} K^{abc} \tag{99}$$

$$\left[H_\mu{}^a, K_{bcd}\right] = \left(\delta_b{}^a K_{\mu cd} + \delta_d{}^a K_{\mu bc} + \delta_c{}^a K_{\mu db}\right) \tag{100}$$

$$\left[H_\mu{}^a, K_{bc}{}^d\right] = \left(\delta_b{}^a K_{\mu c}{}^d - \delta_c{}^a K_{\mu b}{}^d\right) \tag{101}$$

$$\left[H_\mu{}^a, K_b{}^{cd}\right] = \delta_b{}^a K_\mu{}^{cd} \tag{102}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,9}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$ or $H_\mu{}^a$, with either of the operators $K_{\mu\nu\lambda}$, $K_{\mu\nu a}$, $\cdots$, are satisfied.[2]

[2] **For extensive manipulations and verifications of the kind presented in this, and other articles, it would be useful to use efficient symbolic computational software. Readers who have access to a *Mathematica* program can inspect the illustrative notebooks that are embedded in our article: "Symbolic Manipulations in High-Energy Theory", *N.S.B. Computing*, NSBC-NB-007, http://www.vixra.org/abs/1402.0033**

Moreover any of the $O_{1,9}$ generators can be shown to commute with the following quadratic operator:

$$\left\{\begin{array}{l} \frac{1}{3!}K_{\mu\nu\lambda}K_{\mu\nu\lambda} \\ +\frac{1}{2}K_{\mu\nu a}K_{\mu\nu}{}^{a}+\frac{1}{2}K_{\mu\nu}{}^{a}K_{\mu\nu a} \\ +\frac{1}{2}K_{\mu ab}K_{\mu}{}^{ab}+\frac{1}{2}K_{\mu}{}^{ab}K_{\mu ab}-K_{\mu a}{}^{b}K_{\mu b}{}^{a} \\ +\frac{1}{3!}K_{abc}K^{abc}+\frac{1}{3!}K^{abc}K_{abc}+\frac{1}{2}K_{ab}{}^{c}K_{c}{}^{ab}+\frac{1}{2}K_{a}{}^{bc}K_{bc}{}^{a} \end{array}\right. \tag{103}$$

We proceed now to the introduction of the multiplet that can be associated with the above operator representation, and to the construction of the infinitesimal $O_{1,9}$ transformations that act on it. The desired multiplet with components $\{B_{\mu\nu\lambda}, B_{\mu\nu a}, \cdots\}$ can be introduced by the tensor module

$$\mathcal{B}=\left\{\begin{array}{l} \frac{1}{3!}B_{\mu\nu\lambda}K_{\mu\nu\lambda}+\frac{1}{2}B_{\mu\nu a}K_{\mu\nu}{}^{a}+\frac{1}{2}B_{\mu\nu}{}^{a}K_{\mu\nu a} \\ +\frac{1}{2}B_{\mu ab}K_{\mu}{}^{ab}+B_{\mu a}{}^{b}K_{\mu b}{}^{a}+\frac{1}{2}B_{\mu}{}^{ab}K_{\mu ab} \\ +\frac{1}{3!}B_{abc}K^{abc}+\frac{1}{2}B_{ab}{}^{c}K_{c}{}^{ab}+\frac{1}{2}B_{a}{}^{bc}K_{bc}{}^{a}+\frac{1}{3!}B^{abc}K_{abc} \end{array}\right\} \tag{104}$$

Introducing the $O_{1,9}$ parameter module,

$$\mathcal{W}=\frac{1}{2}\Omega_{\mu\nu}J_{\mu\nu}+\Omega_{a}{}^{b}J_{b}{}^{a}+\frac{1}{2}\Omega_{ab}Q^{ab}+\frac{1}{2}\Omega^{ab}Q_{ab}+\Omega_{\mu a}H_{\mu}{}^{a}+\Omega_{\mu}{}^{a}H_{\mu a} \tag{105}$$

we can compute the commutator $[\mathcal{W},\mathcal{B}]$. The latter gives a tensor module whose components would define the needed infinitesimal transformations.

For $B_{\mu\nu\lambda}$, we obtain

$$\delta B_{\mu\nu\lambda}=\left\{\begin{array}{l} \Omega_{\lambda\rho}B_{\mu\nu\rho}-\Omega_{\mu\rho}B_{\lambda\nu\rho}+\Omega_{\nu\rho}B_{\lambda\mu\rho} \\ +\Omega_{\nu a}B_{\lambda\mu}{}^{a}-\Omega_{\mu a}B_{\lambda\nu}{}^{a}+\Omega_{\lambda a}B_{\mu\nu}{}^{a} \\ +\Omega_{\lambda}{}^{a}B_{\mu\nu a}-\Omega_{\mu}{}^{a}B_{\lambda\nu a}+\Omega_{\nu}{}^{a}B_{\lambda\mu a} \end{array}\right\} \tag{106}$$

For $B_{\mu\nu a}$, we obtain

$$\delta B_{\mu\nu a}=\left\{\begin{array}{l} -\Omega_{\lambda a}B_{\lambda\mu\nu}-\Omega_{\lambda\mu}B_{\lambda\nu a}+\Omega_{\lambda\nu}B_{\lambda\mu a} \\ +\Omega_{\nu b}B_{\mu a}{}^{b}+\Omega_{ab}B_{\mu\nu}{}^{b}-\Omega_{\mu b}B_{\nu a}{}^{b} \\ -\Omega_{a}{}^{b}B_{\mu\nu b}+\Omega_{\mu}{}^{b}B_{\nu ab}-\Omega_{\nu}{}^{b}B_{\mu ab} \end{array}\right\} \tag{107}$$

For $B_{\mu\nu}{}^{a}$, we obtain

$$\delta B_{\mu\nu}{}^{a} = \left\{ \begin{array}{l} \Omega^{ab} B_{\mu\nu b} - \Omega_{\nu b} B_{\mu}{}^{ab} + \Omega_{\mu b} B_{\nu}{}^{ab} \\ \\ +\Omega_{\lambda\nu} B_{\lambda\mu}{}^{a} - \Omega_{\lambda\mu} B_{\lambda\nu}{}^{a} + \Omega_{b}{}^{a} B_{\mu\nu}{}^{b} \\ \\ -\Omega_{\lambda}{}^{a} B_{\lambda\mu\nu} + \Omega_{\mu}{}^{b} B_{\nu b}{}^{a} - \Omega_{\nu}{}^{b} B_{\mu b}{}^{a} \end{array} \right\} \tag{108}$$

For $B_{\mu ab}$, we obtain

$$\delta B_{\mu ab} = \left\{ \begin{array}{l} \Omega_{\mu\nu} B_{\nu ab} - \Omega_{\nu a} B_{\mu\nu b} + \Omega_{\nu b} B_{\mu\nu a} \\ \\ +\Omega_{\mu c} B_{ab}{}^{c} - \Omega_{bc} B_{\mu a}{}^{c} + \Omega_{ac} B_{\mu b}{}^{c} \\ \\ +\Omega_{a}{}^{c} B_{\mu bc} - \Omega_{b}{}^{c} B_{\mu ac} + \Omega_{\mu}{}^{c} B_{abc} \end{array} \right\} \tag{109}$$

For $B_{\mu a}{}^{b}$, we obtain

$$\delta B_{\mu a}{}^{b} = \left\{ \begin{array}{l} -\Omega^{bc} B_{\mu ac} - \Omega_{\mu c} B_{a}{}^{bc} + \Omega_{ac} B_{\mu}{}^{bc} \\ \\ +\Omega_{\nu a} B_{\mu\nu}{}^{b} + \Omega_{\mu\nu} B_{\nu a}{}^{b} - \Omega_{a}{}^{c} B_{\mu c}{}^{b} \\ \\ +\Omega_{c}{}^{b} B_{\mu a}{}^{c} + \Omega_{\mu}{}^{c} B_{ac}{}^{b} - \Omega_{\nu}{}^{b} B_{\mu\nu a} \end{array} \right\} \tag{110}$$

For $B_{\mu}{}^{ab}$, we obtain

$$\delta B_{\mu}{}^{ab} = \left\{ \begin{array}{l} \Omega_{\mu c} B^{abc} + \Omega_{\mu\nu} B_{\nu}{}^{ab} + \Omega^{bc} B^{a}_{\mu c} \\ \\ -\Omega^{ac} B_{\mu c}{}^{b} - \Omega_{c}{}^{a} B_{\mu}{}^{bc} + \Omega_{c}{}^{b} B_{\mu}{}^{ac} \\ \\ \Omega_{\mu}{}^{c} B_{c}{}^{ab} - \Omega_{\nu}{}^{a} B_{\mu\nu}{}^{b} + \Omega_{\nu}{}^{b} B_{\mu\nu}{}^{a} \end{array} \right\} \tag{111}$$

For B_{abc}, we obtain

$$\delta B_{abc} = \left\{ \begin{array}{l} -\Omega_{\mu a} B_{\mu bc} + \Omega_{\mu b} B_{\mu ac} - \Omega_{\mu c} B_{\mu ab} \\ \\ +\Omega_{cd} B_{ab}{}^{d} - \Omega_{bd} B_{ac}{}^{d} + \Omega_{ad} B_{bc}{}^{d} \\ \\ -\Omega_{a}{}^{d} B_{bcd} + \Omega_{b}{}^{d} B_{acd} - \Omega_{c}{}^{d} B_{abd} \end{array} \right\} \tag{112}$$

For $B_{ab}{}^{c}$, we obtain

$$\delta B_{ab}{}^{c} = \left\{ \begin{array}{l} \Omega^{cd} B_{abd} - \Omega_{bd} B_{a}{}^{cd} + \Omega_{ad} B_{b}{}^{cd} \\ \\ -\Omega_{\mu b} B_{\mu a}{}^{c} + \Omega_{\mu a} B_{\mu b}{}^{c} + \Omega_{a}{}^{d} B_{bd}{}^{c} \\ \\ -\Omega_{b}{}^{d} B_{ad}{}^{c} + \Omega_{d}{}^{c} B_{ab}{}^{d} - \Omega_{\mu}{}^{c} B_{\mu ab} \end{array} \right\} \tag{113}$$

For ${B_a}^{bc}$, we obtain

$$\delta {B_a}^{bc} = \left\{ \begin{array}{l} \Omega_{ad} B^{bcd} - \Omega_{\mu a} {B_\mu}^{bc} - \Omega^{cd} B^c_{ad} \\ \\ +\Omega^{bd} {B_{ad}}^c - {\Omega_a}^d {B_d}^{bc} - {\Omega_d}^b {B_a}^{cd} \\ \\ +{\Omega_d}^c {B_a}^{bd} - {\Omega_\mu}^b {B_{\mu a}}^c + {\Omega_\mu}^c {B_{\mu a}}^b \end{array} \right\} \tag{114}$$

Finally, for B^{abc}, we obtain

$$\delta B^{abc} = \left\{ \begin{array}{l} \Omega^{cd} {B_d}^{ab} - \Omega^{bd} {B_d}^{ac} + \Omega^{ad} {B_d}^{bc} \\ \\ +{\Omega_d}^a B^{bcd} - {\Omega_d}^b B^{acd} + {\Omega_d}^c B^{abd} \\ \\ -{\Omega_\mu}^a {B_\mu}^{bc} + {\Omega_\mu}^b {B_\mu}^{ac} - {\Omega_\mu}^c {B_\mu}^{ab} \end{array} \right\} \tag{115}$$

We can verify that, for any two tensor modules $\mathcal{A}$ and $\mathcal{B}$, the above infinitesimal transformations, acting in a like manner on the components of both, would leave invariant the following bilinear form:

$$\mathcal{A} \cdot \mathcal{B} = \left\{ \begin{array}{l} \frac{1}{3!} A_{\mu\nu\lambda} B_{\mu\nu\lambda} + \frac{1}{2} A_{\mu\nu a} {B_{\mu\nu}}^a + \frac{1}{2} {A_{\mu\nu}}^a B_{\mu\nu a} \\ \\ +\frac{1}{2} A_{\mu ab} {B_\mu}^{ab} - {A_{\mu a}}^b {B_{\mu b}}^a + \frac{1}{2} {A_\mu}^{ab} B_{\mu ab} \\ \\ +\frac{1}{3!} A_{abc} B^{abc} + \frac{1}{2} {A_{ab}}^c {B_c}^{ab} + \frac{1}{2} {A_a}^{bc} {B_{bc}}^a + \frac{1}{3!} A^{abc} B_{abc} \end{array} \right\} \tag{116}$$

6 The Dirac-Weyl Spinorial Representation of $\mathrm{O}_{1,9}$

A Dirac spinor in 10-spacetime has $2^5 = 32$ components. A Weyl (chiral) constraint would reduce this to 16 components. A Majorana constraint, relating the components to their Dirac conjugates, for fermions in 10-spacetime, would reduce the number of components further to 8. In order to construct a corresponding multiplet of Weyl spinors in 4-spacetime, with components that are described by SU_3 tensors, we must introduce the following set of operators:

$$\{L, R_a, L^a, R\} \tag{117}$$

The above objects are alternately left-handed and right-handed Weyl spinors of the 4-spacetime. As before, the symbols $(a, b, c, \ldots)$ do pertain to SU_3. We now write the commutators of the above operators with the generators of the $\mathrm{O}_{1,9}$ algebra.

For the commutators with $J_{\mu\nu}$, the generators of the 4-spacetime Lorentz algebra, all the foregoing operator elements would satisfy commutators like this:

$$[J_{\mu\nu}, L] = -\frac{1}{2}\gamma_{\mu\nu} L \tag{118}$$

In the above, $\gamma_{\mu\nu}$ is a member of the Dirac algebra, being equal to $\frac{1}{2}[\gamma_\mu, \gamma_\nu]$ in terms of the Dirac matrix operators γ_μ that satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.

For the commutators with ${J_a}^b$, the generators of U_3, we have

$$\left[{J_a}^b, L\right] = -\frac{1}{2}{\delta_a}^b L \tag{119}$$

$$\left[{J_a}^b, R_c\right] = {\delta_c}^b R_a - \frac{1}{2}{\delta_a}^b R_c \tag{120}$$

$$\left[{J_a}^b, L^c\right] = -{\delta_a}^c L^b + \frac{1}{2}{\delta_a}^b L^c \tag{121}$$

$$\left[{J_a}^b, R\right] = \frac{1}{2}{\delta_a}^b R \tag{122}$$

For the commutators with Q_{ab}, we have

$$[Q_{ab}, L] = \epsilon_{abc} L^c \tag{123}$$

$$[Q_{ab}, R_c] = \epsilon_{abc} R \tag{124}$$

$$[Q_{ab}, L^c] = 0 \tag{125}$$

$$[Q_{ab}, R] = 0 \tag{126}$$

For the commutators with Q^{ab}, we have

$$\left[Q^{ab}, L\right] = 0 \tag{127}$$

$$\left[Q^{ab}, R_c\right] = 0 \tag{128}$$

$$\left[Q^{ab}, L^c\right] = -\epsilon^{abc} L \tag{129}$$

$$\left[Q^{ab}, R\right] = -\epsilon^{abc} R_c \tag{130}$$

For the commutators with $H_{\mu a}$, we have

$$[H_{\mu a}, L] = -\frac{1}{\sqrt{2}}\gamma_\mu R_a \tag{131}$$

$$[H_{\mu a}, R_b] = \frac{1}{\sqrt{2}}\epsilon_{abc}\gamma_\mu L^c \tag{132}$$

$$\left[H_{\mu a}, L^b\right] = -\frac{1}{\sqrt{2}}{\delta_a}^b \gamma_\mu R \tag{133}$$

$$[H_{\mu a}, R] = 0 \tag{134}$$

For the commutators with ${H_\mu}^a$, we have

$$[{H_\mu}^a, L] = 0 \tag{135}$$

$$[{H_\mu}^a, R_b] = \frac{1}{\sqrt{2}}{\delta_b}^a \gamma_\mu L \tag{136}$$

$$\left[H_\mu{}^a, L^b\right] = \frac{1}{\sqrt{2}}\epsilon^{abc}\gamma_\mu R_c \tag{137}$$

$$\left[H_\mu{}^a, R\right] = \frac{1}{\sqrt{2}}\gamma_\mu L^a \tag{138}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,9}$ generators $J_{\mu\nu}$, $J_a{}^b$, Q_{ab}, Q^{ab}, $H_{\mu a}$, or $H_\mu{}^a$, with any of the operators L, R_a, L^a, or R, are satisfied.

7 The Dirac Conjugate Spinorial Representation of $O_{1,9}$

In order to be able to write Lagrangian terms for fermionic fields we must introduce the conjugate spinorial representation. This can be done with the operator set

$$\left\{\overline{L}, \overline{R}^a, \overline{L}_a, \overline{R}\right\} \tag{139}$$

All these are Dirac conjugate spinors. We now write the commutators of all the $O_{1,9}$ generators with the elements of the above set.

First, all the above set of operators, being all Dirac conjugate spinors, would have commutators with $J_{\mu\nu}$, the 4-spacetime Lorentz generators, that are like this:

$$\left[J_{\mu\nu}, \overline{L}\right] = \frac{1}{2}\overline{L}\gamma_{\mu\nu} \tag{140}$$

For the commutators with $J_a{}^b$, the U_3 generators, we have

$$\left[J_a{}^b, \overline{L}\right] = \frac{1}{2}\delta_a{}^b\overline{L} \tag{141}$$

$$\left[J_a{}^b, \overline{R}^c\right] = -\delta_a{}^c\overline{R}^b + \frac{1}{2}\delta_a{}^b\overline{R}^c \tag{142}$$

$$\left[J_a{}^b, \overline{L}_c\right] = \delta_c{}^b\overline{L}_a - \frac{1}{2}\delta_a{}^b\overline{L}_c \tag{143}$$

$$\left[J_a{}^b, \overline{R}\right] = -\frac{1}{2}\delta_a{}^b\overline{R} \tag{144}$$

For the commutators with Q_{ab}, we have

$$\left[Q_{ab}, \overline{L}\right] = 0 \tag{145}$$

$$\left[Q_{ab}, \overline{R}^c\right] = 0 \tag{146}$$

$$\left[Q_{ab}, \overline{L}_c\right] = \epsilon_{abc}\overline{L} \tag{147}$$

$$\left[Q_{ab}, \overline{R}\right] = \epsilon_{abc}\overline{R}^c \tag{148}$$

For the commutators with Q^{ab}, we have

$$\left[Q^{ab}, \overline{L}\right] = -\epsilon^{abc}\overline{L}_c \tag{149}$$

$$\left[Q^{ab}, \overline{R}^{c}\right] = -\epsilon^{abc}\overline{R} \tag{150}$$

$$\left[Q^{ab}, \overline{L}_{c}\right] = 0 \tag{151}$$

$$\left[Q^{ab}, \overline{R}\right] = 0 \tag{152}$$

For the commutators with $H_{\mu a}$, we have

$$\left[H_{\mu a}, \overline{L}\right] = 0 \tag{153}$$

$$\left[H_{\mu a}, \overline{R}^{b}\right] = \frac{1}{\sqrt{2}}\delta_{a}{}^{b}\overline{L}\gamma_{\mu} \tag{154}$$

$$\left[H_{\mu a}, \overline{L}_{b}\right] = -\frac{1}{\sqrt{2}}\epsilon_{abc}\overline{R}^{c}\gamma_{\mu} \tag{155}$$

$$\left[H_{\mu a}, \overline{R}\right] = \frac{1}{\sqrt{2}}\overline{L}_{a}\gamma_{\mu} \tag{156}$$

For the commutators with $H_{\mu}{}^{a}$, we have

$$\left[H_{\mu}{}^{a}, \overline{L}\right] = -\frac{1}{\sqrt{2}}\overline{R}^{a}\gamma_{\mu} \tag{157}$$

$$\left[H_{\mu}{}^{a}, \overline{R}^{b}\right] = -\frac{1}{\sqrt{2}}\epsilon^{abc}\overline{L}_{c}\gamma_{\mu} \tag{158}$$

$$\left[H_{\mu}{}^{a}, \overline{L}_{b}\right] = -\frac{1}{\sqrt{2}}\delta_{b}{}^{a}\overline{R}\gamma_{\mu} \tag{159}$$

$$\left[H_{\mu}{}^{a}, \overline{R}\right] = 0 \tag{160}$$

We can verify that all the Jacobi identities involving any two of the $O_{1,9}$ generators $J_{\mu\nu}$, $J_{a}{}^{b}$, Q_{ab}, Q^{ab}, $H_{\mu a}$, or $H_{\mu}{}^{a}$, with anyone of the operators $\overline{L}$, $\overline{R}^{a}$, $\overline{L}_{a}$, or $\overline{R}$, are satisfied.

Having constructed the manifestly 4-spacetime Lorentz covariant, as well as the SU_3 covariant, algebraic representations for a fundamental $O_{1,9}$ spinor, and for its Dirac conjugate, we can verify that all generators $J_{\mu\nu}$, $J_{a}{}^{b}$, Q_{ab}, Q^{ab}, $H_{\mu a}$, or $H_{\mu}{}^{a}$, of $O_{1,9}$ do commute with the following quadratic operator:

$$\overline{L}L + \overline{R}^{a}\overline{R}_{a} - \overline{L}_{a}L^{a} - \overline{R}R \tag{161}$$

We now proceed to construct the spinorial multiplet modules, giving the infinitesimal transformations of the components, and the invariant bilinear. Notice, that if the L's and the R's are truly left-handed and right-handed Weyl spinors, the above quandratic form would be zero. However, the above is written for the purpose of normalizing the algebraic operators corresponding to unconstrained spinors, before the chirality constraints are applied.

8 Fundamental Spinorial Multiplet

We introduce an SU_3 covariant multiplet of Weyl spinors in 4-spacetime, represented by the following module:

$$\Psi = \overline{R}\psi + \overline{L}_a\xi^a + \overline{R}^a\psi_a + \overline{L}\xi \tag{162}$$

Notice that the ψ's are left-handed Weyl spinors, while the ξ's are right-handed. We also introduce the Dirac conjugate module:

$$\overline{\Psi} = \overline{\psi}R + \overline{\xi}_a L^a + \overline{\psi}^a R_a + \overline{\xi}L \tag{163}$$

Now, with the $\mathrm{O}_{1,9}$ parameter module,

$$\mathcal{W} = \frac{1}{2}\Omega_{\mu\nu}J_{\mu\nu} + \Omega_a{}^b J_b{}^a + \frac{1}{2}\Omega_{ab}Q^{ab} + \frac{1}{2}\Omega^{ab}Q_{ab} + \Omega_{\mu a}H_\mu{}^a + \Omega_\mu{}^a H_{\mu a} \tag{164}$$

we can compute the commutators $[\mathcal{W}, \Psi]$ and $\left[\mathcal{W}, \overline{\Psi}\right]$. These give the corresponding spinorial modules whose components define the $\mathrm{O}_{1,9}$ infinitesimal transformations.

For the infinitesimal transformations of the Ψ components, we obtain

$$\delta\psi = \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\psi - \frac{1}{2}\Omega_a{}^a\psi - \frac{1}{2}\epsilon^{abc}\Omega_{ab}\psi_c - \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\xi^a \tag{165}$$

$$\delta\xi^a = \begin{pmatrix} \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi^a + \Omega_b{}^a\xi^b - \frac{1}{2}\Omega_b{}^b\xi^a \\ \\ -\frac{1}{2}\Omega^{abc}\Omega_{bc}\xi - \frac{1}{\sqrt{2}}\epsilon^{abc}\Omega_{\mu b}\gamma_\mu\psi_c + \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\psi \end{pmatrix} \tag{166}$$

$$\delta\psi_a = \begin{pmatrix} \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\psi_a - \Omega_a{}^b\psi_b + \frac{1}{2}\Omega_b{}^b\psi_a \\ \\ +\frac{1}{2}\epsilon_{abc}\Omega^{bc}\psi - \frac{1}{\sqrt{2}}\Omega_{\mu a}\gamma_\mu\xi - \frac{1}{\sqrt{2}}\epsilon_{abc}\Omega_\mu{}^b\gamma_\mu\xi^c \end{pmatrix} \tag{167}$$

$$\delta\xi = \frac{1}{4}\Omega_{\mu\nu}\gamma_{\mu\nu}\xi + \frac{1}{2}\Omega_a{}^a\xi + \frac{1}{2}\epsilon_{abc}\Omega^{ab}\xi^c + \frac{1}{\sqrt{2}}\Omega_\mu{}^a\gamma_\mu\psi_a \tag{168}$$

For the infinitesimal transformations of the $\overline{\Psi}$ components, we obtain

$$\delta\overline{\psi} = -\frac{1}{4}\Omega_{\mu\nu}\overline{\psi}\gamma_{\mu\nu} + \frac{1}{2}\Omega_a{}^a\overline{\psi} + \frac{1}{2}\epsilon_{abc}\Omega^{ab}\overline{\psi}^c - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\overline{\xi}_a\gamma_\mu \tag{169}$$

$$\delta\overline{\xi}_a = \begin{pmatrix} -\frac{1}{4}\Omega_{\mu\nu}\overline{\xi}_a\gamma_{\mu\nu} - \Omega_a{}^b\overline{\xi}_b + \frac{1}{2}\Omega_b{}^b\overline{\xi}_a \\ \\ +\frac{1}{2}\epsilon_{abc}\Omega^{bc}\overline{\xi} + \frac{1}{\sqrt{2}}\Omega_{\mu a}\overline{\psi}\gamma_\mu + \frac{1}{\sqrt{2}}\epsilon_{abc}\Omega_\mu{}^b\overline{\psi}^c\gamma_\mu \end{pmatrix} \tag{170}$$

$$\delta\overline{\psi}^a = \begin{pmatrix} -\frac{1}{4}\Omega_{\mu\nu}\overline{\psi}^a\gamma_{\mu\nu} + \Omega_b{}^a\overline{\psi}^b - \frac{1}{2}\Omega_b{}^b\overline{\psi}^a \\ \\ -\frac{1}{2}\epsilon^{abc}\Omega_{bc}\overline{\psi} + \frac{1}{\sqrt{2}}\epsilon^{abc}\Omega_{\mu b}\overline{\xi}_c\gamma_\mu - \frac{1}{\sqrt{2}}\Omega_\mu{}^a\overline{\xi}\gamma_\mu \end{pmatrix} \tag{171}$$

$$\delta\overline{\xi} = -\frac{1}{4}\Omega_{\mu\nu}\overline{\xi}\gamma_{\mu\nu} - \frac{1}{2}\Omega_a{}^a\overline{\xi} - \frac{1}{2}\epsilon^{abc}\Omega_{ab}\overline{\xi}_c + \frac{1}{\sqrt{2}}\Omega_{\mu a}\overline{\psi}^a\gamma_\mu \tag{172}$$

9 The Composition of a Vector Mutiplet from Fermionic Bilinears, and the Fermionic Kinetic Terms

Here we give the composition of an $\mathrm{O}_{1,9}$ vector multiplet with components $\{V_\mu, V_a, V^a\}$ from the ψ and ξ components of a fundamental spinorial multiplet:

$$V_\mu = \frac{1}{\sqrt{2}}\left(\overline{\psi}\gamma_\mu\psi - \overline{\xi}_a\gamma_\mu\xi^a - \overline{\psi}^a\gamma_\mu\psi_a + \overline{\xi}\gamma_\mu\xi\right) \tag{173}$$

$$V_a = \overline{\xi}\psi_a - \overline{\xi}_a\psi - \epsilon_{abc}\overline{\psi}^b\xi^c \tag{174}$$

$$V^a = \overline{\psi}^a\xi - \overline{\psi}\xi^a - \epsilon^{abc}\overline{\xi}_b\psi_c \tag{175}$$

Using the $\mathrm{O}_{1,9}$ infinitesimal transformations of the components on both sides, we can verify that the above expressions are identities, in the sense that they are constructed properly to be covariant with respect to the full $\mathrm{O}_{1,9}$ algebra.

The importance of being able to compose an $\mathrm{O}_{1,9}$ vector from the fermionic spinorial components is connected with the need to translate the 10-spacetime kinetic term to its 4-spacetime counterpart. From the foregoing composition of the vector V_μ, we can deduce for the kinetic terms:

$$\bar{\Psi}(i\gamma\cdot\partial)\Psi \Rightarrow \frac{1}{2}\left\{\bar{\psi}(i\gamma\cdot\partial)\psi - \bar{\psi}^a(i\gamma\cdot\partial)\psi_a + \bar{\xi}(i\gamma\cdot\partial)\xi - \bar{\xi}_a(i\gamma\cdot\partial)\xi^a\right\} \tag{176}$$

Now we proceed to the application of the *Majorana constraint* that takes the form

$$\begin{cases} \xi \to C\tilde{\bar{\psi}} & \bar{\xi} \to -\tilde{\psi}C^{-1} \\ \\ \xi^a \to C\tilde{\bar{\psi}}^a & \bar{\xi}_a \to -\tilde{\psi}_a C^{-1} \end{cases} \tag{177}$$

where again, the tilde symbol and the matrix C are as described in the introduction. Hence, the fermionic kinetic terms become

$$\left\{\bar{\psi}(i\gamma\cdot\partial)\psi - \bar{\psi}^a(i\gamma\cdot\partial)\psi_a\right\} \tag{178}$$

We can replace ψ by the antineutrino ν^*, and ψ_a by the SU_3 triplet (ν, e, e^*), as we shall do later in connection with the writing out of the coupling terms.

10 The Composition of a 3rd Rank Tensor Multiplet from Fermionic Bilinears

Here we give the composition of an $\mathrm{O}_{1,9}$ tensor multiplet of 3rd rank with components $V_{\mu\nu\lambda}$, $V_{\mu\nu a}$, $V^a_{\mu\nu}$, etc., from the ψ and the ξ components of a fundamental spinorial multiplet:

$$V_{\mu\nu\lambda} = \frac{1}{2\sqrt{2}}\left(\overline{\xi}\gamma_{\mu\nu\lambda}\xi - \overline{\psi}^a\gamma_{\mu\nu\lambda}\psi_a - \overline{\xi}_a\gamma_{\mu\nu\lambda}\xi^a + \overline{\psi}\gamma_{\mu\nu\lambda}\psi\right) \tag{179}$$

$$V_{\mu\nu a} = \frac{1}{2}\left(\bar{\xi}\gamma_{\mu\nu}\psi_a - \bar{\xi}_a\gamma_{\mu\nu}\psi - \epsilon_{abc}\overline{\psi}^b\gamma_{\mu\nu}\xi^c\right) \tag{180}$$

$$V_{\mu\nu}{}^a = \frac{1}{2}\left(\overline{\psi}^a\gamma_{\mu\nu}\xi - \overline{\psi}\gamma_{\mu\nu}\xi^a - \epsilon^{abc}\bar{\xi}_b\gamma_{\mu\nu}\psi_c\right) \tag{181}$$

$$V_{\mu a}{}^b = \frac{1}{\sqrt{2}}\left\{\begin{array}{l}\left(\bar{\xi}_a\gamma_\mu\xi^b - \overline{\psi}^b\gamma_\mu\psi_a\right) \\ \\ -\frac{1}{2}\delta_a{}^b\left(\bar{\xi}\gamma_\mu\xi - \overline{\psi}^c\gamma_\mu\psi_c + \bar{\xi}_c\gamma_\mu\xi^c - \overline{\psi}\gamma_\mu\psi\right)\end{array}\right\} \tag{182}$$

$$V_\mu{}^{ab} = \frac{1}{\sqrt{2}}\epsilon^{abc}\left(\bar{\xi}_c\gamma_\mu\xi - \overline{\psi}\gamma_\mu\psi_c\right) \tag{183}$$

$$V_{abc} = \epsilon_{abc}\bar{\xi}\psi \tag{184}$$

$$V_{ab}{}^c = \left\{\frac{1}{2}\left(\bar{\xi}\psi_a + \bar{\xi}_a\psi\right)\delta_b{}^c - (a \leftrightarrow b)\right\} + \frac{1}{2}\epsilon_{abd}\left(\overline{\psi}^d\xi^c + \overline{\psi}^c\xi^d\right) \tag{185}$$

$$V_a{}^{bc} = -\left\{\frac{1}{2}\left(\overline{\psi}^b\xi + \overline{\psi}\xi^b\right)\delta_a^c - (b \leftrightarrow c)\right\} + \frac{1}{2}\epsilon^{bcd}\left(\bar{\xi}_d\psi_a + \bar{\xi}_a\psi_d\right) \tag{186}$$

$$V^{abc} = \epsilon^{abc}\overline{\psi}\xi \tag{187}$$

Using the infinitesimal $O_{1,9}$ transformations of the components on both sides, we can verify that the above expressions are identities, in the sense that they are constructed properly to be covariant with respect to the full $O_{1,9}$ algebra.

11 The $O_{1,9}$ Couplings of a 3rd Rank Tensor to the Majorana-Weyl Fermion

Using the foregoing composition of a 3rd rank $O_{1,9}$ tensor in terms of the components of a Weyl fermion, we can now construct the couplings. Starting with the bilinear invariant $\mathcal{V} \cdot \mathcal{W}$, for two tensor modules $\mathcal{V}$ and $\mathcal{W}$, we would replace the components of $\mathcal{W}$ by their compositions in terms of the ψ and ξ fermionic components. We then trade the ξ components for the ψ counterparts according to the Majorana constraint described earlier. We obtain the manifestly 4-spacetime Lorentz invariant, and SU_3 invariant, coupling terms. These will be given according to the respective bosonic field component.

11.1 Couplings to the Tensor $V_{\mu\nu\lambda}$

The couplings of the fermions to the tensor $V_{\mu\nu\lambda}$ are given by

$$\frac{1}{12}V_{\lambda\mu\nu}\left(\overline{\psi}\gamma_{\lambda\mu\nu}\psi - \overline{\psi}^a\gamma_{\lambda\mu\nu}\psi_a\right) \tag{188}$$

Here we have an SU_3 singlet Weyl fermion ψ, and a triplet ψ_a. Trading the tensor $V_{\mu\nu\lambda}$ to an axial vector using $V_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda\rho} A_\rho$, and reducing the Dirac element $\gamma_{\lambda\mu\nu}$ likewise to $\gamma_\mu \gamma_5$, noting that we deal with Weyl fermions, we obtain the couplings in the form:

$$\frac{1}{2} A_\mu \left(\overline{\psi} \gamma_\mu \psi - \overline{\psi}^a \gamma_\mu \psi_a \right) \tag{189}$$

Assigning the singlet ψ to the antineutrino ν^*, and as in the SU_3 theory, the triplet of fermions to (ν, e, e^*), we can expand the above couplings to

$$-\frac{1}{2} A_\mu \left(\overline{\nu} \gamma_\mu \nu + \overline{e} \gamma_\mu e + \overline{e^*} \gamma_\mu e^* - \overline{\nu^*} \gamma_\mu \nu^* \right) \tag{190}$$

11.2 Couplings to the Conjugate Field Triplets $V_{\mu\nu a}$ and $V_{\mu\nu}{}^a$

Here we give the couplings to $V_{\mu\nu a}$. The couplings to $V_{\mu\nu}{}^a$ can be obtained by simple conjugation. We have

$$\frac{1}{2\sqrt{2}} V_{\mu\nu a} \left(\overline{\psi}^a \gamma_{\mu\nu} C \tilde{\overline{\psi}} + \frac{1}{2} \epsilon^{abc} \tilde{\psi}_b C^{-1} \gamma_{\mu\nu} \psi_c \right) \tag{191}$$

Making the fermionic assignments together with the following bosonic one:

$$V_{\mu\nu 1} \to \mathcal{N}_{\mu\nu} \qquad V_{\mu\nu 2} \to \mathcal{E}_{\mu\nu} \qquad V_{\mu\nu 3} \to \mathcal{F}_{\mu\nu} \tag{192}$$

we obtain

$$\frac{1}{4\sqrt{2}} \left\{ \begin{array}{l} \mathcal{N}_{\mu\nu} \left(2\overline{\nu} \gamma_{\mu\nu} C \tilde{\overline{\nu^*}} + \tilde{e} C^{-1} \gamma_{\mu\nu} e^* - \tilde{e^*} C^{-1} \gamma_{\mu\nu} e \right) \\ \\ +\mathcal{E}_{\mu\nu} \left(2\overline{e} \gamma_{\mu\nu} C \tilde{\overline{\nu^*}} - \tilde{\nu} C^{-1} \gamma_{\mu\nu} e^* + \tilde{e^*} C^{-1} \gamma_{\mu\nu} \nu \right) \\ \\ +\mathcal{F}_{\mu\nu} \left(2\overline{e^*} \gamma_{\mu\nu} C \tilde{\overline{\nu^*}} - \tilde{e} C^{-1} \gamma_{\mu\nu} \nu + \tilde{\nu} C^{-1} \gamma_{\mu\nu} e \right) \end{array} \right\} \tag{193}$$

11.3 Couplings to the Vector Bosons $V_{\mu a}{}^b$

The couplings to $V_{\mu a}{}^b$ are

$$V_{\mu a}{}^b \overline{\psi}^a \gamma_\mu \psi_b - \frac{1}{2} V_{\mu a}{}^a \left(\overline{\psi} \gamma_\mu \psi + \overline{\psi}^b \gamma_\mu \psi_b \right) \tag{194}$$

Splitting the U_3 tensor $V_{\mu a}{}^b$ into a traceless part, that of the SU_3 octet called again $V_{\mu a}{}^b$, and a trace V_μ, we obtain

$$V_{\mu a}{}^b \overline{\psi}^a \gamma_\mu \psi_b + \frac{1}{2} V_\mu \left(\overline{\psi} \gamma_\mu \psi + \frac{1}{3} \overline{\psi}^b \gamma_\mu \psi_b \right) \tag{195}$$

Making the fermionic assignments, and the SU_3 assignments for ${V_{\mu a}}^b$, with $V_\mu \to \mathcal{B}_\mu$, we obtain

$$\left\{\begin{array}{l} \mathcal{A}_\mu\left(-\overline{e}\gamma_\mu e + \overline{e^*}\gamma_\mu e^*\right) \\ +\frac{1}{2}\mathcal{Z}_\mu\left(-\overline{e}\gamma_\mu e + 2\overline{\nu}\gamma_\mu \nu - \overline{e^*}\gamma_\mu e^*\right) \\ -\frac{1}{6}\mathcal{B}_\mu\left(\overline{e}\gamma_\mu e + \overline{\nu}\gamma_\mu \nu + \overline{e^*}\gamma_\mu e^* + 3\overline{\nu^*}\gamma_\mu \nu^*\right) \\ +\mathcal{W}_\mu \overline{e}\gamma_\mu \nu + \mathcal{W}^*_\mu \overline{\nu}\gamma_\mu e \\ +\mathcal{X}_\mu \overline{e}\gamma_\mu e^* + \mathcal{X}^*_\mu \overline{e^*}\gamma_\mu e \\ +\mathcal{Y}_\mu \overline{\nu}\gamma_\mu e^* + \mathcal{Y}^*_\mu \overline{e^*}\gamma_\mu \nu \end{array}\right\} \tag{196}$$

11.4 Couplings to the Vector Bosons $V_{\mu ab}$ and ${V_\mu}^{ab}$

We give here the couplings to $V_{\mu ab}$, while those pertaining to ${V_\mu}^{ab}$ can be obtained by simple conjugation. We have

$$-\frac{1}{2}\epsilon^{abc}V_{\mu ab}\overline{\psi}\gamma_\mu\psi_c \tag{197}$$

Now we make the replacement $V_{\mu ab} \to \epsilon_{abc}{V_\mu}^c$, to obtain

$$-{V_\mu}^a\overline{\psi}\gamma_\mu\psi_a \tag{198}$$

Making the fermionic assignments, together with

$${V_\mu}^1 \to \mathcal{N}^*_\mu \qquad {V_\mu}^2 \to \mathcal{E}^*_\mu \qquad {V_\mu}^3 \to \mathcal{F}_\mu \tag{199}$$

we obtain

$$-\mathcal{N}^*_\mu\overline{\nu^*}\gamma_\mu\nu - \mathcal{E}^*_\mu\overline{\nu^*}\gamma_\mu e - \mathcal{F}_\mu\overline{\nu^*}\gamma_\mu e^* \tag{200}$$

11.5 Coupling to the Scalar Bosons V_{abc} and V^{abc}

Here we give the couplings to the scalars V_{abc}, while the couplings to V^{abc} can be obtained by simple conjugation. We have

$$\frac{1}{6\sqrt{2}}\epsilon^{abc}V_{abc}\overline{\psi}C\tilde{\overline{\psi}} \tag{201}$$

Making the replacement $V_{abc} \to \epsilon_{abc}\phi^*$, we obtain

$$\frac{1}{\sqrt{2}}\phi^*\overline{\psi}C\tilde{\overline{\psi}} \tag{202}$$

Now with the fermionic replacement $\psi \to \nu^*$, we have

$$\frac{1}{\sqrt{2}}\phi^*\overline{\nu^*}C\tilde{\overline{\nu^*}} \tag{203}$$

Notice that this is the term which gives the antineutrino its *Majorana mass*, when the Higgs scalar ϕ gets a real vacuum value.

11.6 Couplings to the Scalar Bosons $V_{a,b}{}^c$ and $V_a{}^{bc}$

Here we give the couplings to the scalars $V_{a,b}{}^c$, while the couplings to $V_a{}^{bc}$ can be obtained by simple conjugation. We have

$$\frac{1}{\sqrt{2}} V_{ab}{}^a \overline{\psi}^b C \tilde{\overline{\psi}} - \frac{1}{2\sqrt{2}} \epsilon^{abc} V_{ab}{}^d \overline{\psi}_c C^{-1} \psi_d \tag{204}$$

We make the replacement $V_{a,b}{}^c \to \epsilon_{abd}\phi^{dc}$, where ϕ^{ab} is 9-component scalar, to obtain

$$-\frac{1}{\sqrt{2}} \phi^{ab} \overline{\psi}_a C^{-1} \psi_b + \frac{1}{\sqrt{2}} \epsilon_{abc} \phi^{ab} \overline{\psi}^c C \psi \tag{205}$$

We can make the fermionic assignments in the above after splitting the SU_3 indices, however, we shall give here the terms that contribute to fermionic masses. The term that gives a *Dirac mass* to the electron comes from the *symmetric* part of ϕ^{ab}, with indices 2 and 3,

$$-\sqrt{2}\phi^{23} \tilde{e} C^{-1} e^* + \text{h.c.} \tag{206}$$

The term that gives a *Dirac mass* to the neutrino comes from the *antisymmetric* part of ϕ^{ab}, again with indices 2 and 3,

$$\sqrt{2}\phi^{23} \overline{\nu} C \tilde{\overline{\nu^*}} + \text{h.c.} \tag{207}$$

A *Majorana mass* to the neutrino comes from

$$-\frac{1}{\sqrt{2}} \phi^{11} \tilde{\nu} C^{-1} \nu + \text{h.c.} \tag{208}$$

12 Discussion

We have constructed the couplings of bosons to fermions in a 10-spacetime gravidynamic model where the fundamental fermions (the electron, the neutrino, and antiparticles) are described unitedly by a Majorana-Weyl fermionic spinor of $O_{1,9}$. Likewise, the vector bosons which gauge the electroweak interactions of the four Weyl fermions, as well as the Higgs bosons that provide them with masses, are all consolidated in an antisymmetric 3rd rank tensor of $O_{1,9}$. Whereas the dynamics and the self-interactions of the bosonic particles are not described in this article, we intend to make provisions in other articles.

The foregoing 10-spacetime theory is an economic analogue to the 18-spacetime theory. Whereas the latter theory[3] would be able to describe the grand unified electroweak and strong interactions of four generations of leptons and quarks in the framework of a 256-component Majorana-Weyl fermion via an underlying SU_7 internal symmetry, the present theory can describe the electroweak interactions of only four fundamental fermions (the electron, the neutrino, and their antiparticles) in the framework of a 16-component Majorana-Weyl fermion via an underlying SU_3 internal symmetry. Whereas we have proposed[4] in connection with the 14-spacetime unification theory for leptons that the latter, and *for the sake of theoretical economy*, could be able to describe hadronic structure rather than introducing quarks, the simplicity and the elegance of the present 10-spacetime theory would implore us to think of the four fundamental fermions (the electron, the neutrino, and antiparticles) as possible *constituents of all leptonic as well as hadronic matter*. How this can be done is a matter yet to be formulated. However, we must realize that the implications underlying the gauge and the geometrical structure of the present 10-spacetime theory is *a realm yet to be explored*. We must explore the *solitonic* and the *magnetic-like* interactions of the four fundamental particles and their bosonic allies, and their possibilities for generating both the leptonic and the hadronic worlds.

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