# Divergence-Free Versus Cutoff Quantum Field Theory

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**Abstract**

***We review the fundamental rules for constructing the regular and the gauge-invariant quantum field action both in the divergence-free approach and in the cutoff approach. Loop computations in quantum electrodynamics of fermionic spinor matter, and also in quantum gravity of fermionic spinor matter, are presented in both approaches. We explain how the results of the divergence-free method correspond to those of the cutoff method. We argue that in a fundamental theory that contains quantum gravity, the cutoff framework might be necessary, whereby the cutoff parameter and the gravitational coupling could be related to each other quite consistently.***

## 1 Introduction

The effective action framework[1]-[9] for computing the loop contributions of quantum field theory represents a very powerful and an extremely elegant scheme preserving underlying fundamental symmetries. In compact notation, for a theory with a classical action functional $W(\phi)$ describing a set of fields $\phi_i$, the effective quantum action $\Gamma(\phi)$ is defined by the functional integral

$$e^{\frac{i}{\hbar}\Gamma(\phi)} = \int (\,\mathrm{d}\varphi)\, e^{\frac{i}{\hbar}\{W(\phi+\varphi)-\varphi_i\Gamma_i\}} \tag{1}$$

Here $\Gamma_i$ denotes the functional derivative of $\Gamma(\phi)$ with respect to the effective fields $\phi_i$, while $\varphi_i$ represent the virtual (quantum) fields that are being integrated over. By first replacing $\varphi$ by $\sqrt{\hbar}\varphi$, the above functional integral computation of the effective action can be done iteratively[5], [6] in $\hbar$, such as $\Gamma = \Gamma_0 + \hbar\Gamma_1 + \hbar^2\Gamma_2 + \cdots$. Whereas $\Gamma_0$ is the classical action $W(\phi)$ itself, the first order (one-loop) contribution $\Gamma_1$ comes from evaluating the integral:

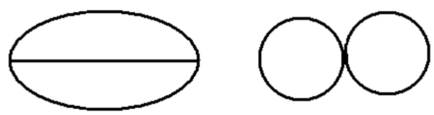
$$e^{\frac{i}{\hbar}\Gamma_1(\phi)} = \int (\,\mathrm{d}\varphi)\, e^{\frac{i}{2}W_{ij}\varphi_i\varphi_j} \tag{2}$$

Here $W_{ij}$ is the bilinear kernel, or the second functional derivative of the classical action $W$ with respect to its field arguments. Evaluating the above (Gaussian) integral, we obtain the one-loop contribution,

$$\Gamma_1(\phi) = \frac{i}{2}\mathbf{tr}(\ln W_{ij}) \tag{3}$$

While the 2-loop contribution can be computed analytically[5], [6], higher orders demand the recourse to graphical methods and rules. The latter are simply the following. Corresponding to any irreducible Feynman graph consisting of vertices and internal lines only, we must associate an effective propagator $W_{ij}^{-1}$, being the inverse of the bilinear kernel, with every internal line, and an effective vertex term $W_{ijk\cdots}$, being the $n$th derivative of the classical action, with each $n$-leg vertex. Each internal line and each vertex must have a factor of the imaginary unit $i$. The whole contribution must be multiplied by an overall factor of $-i$, and a combinatoric factor. The latter can be deduced from the symmetries of the graph[10]. Notice that the effective propagator $W_{ij}^{-1}$ and the effective vertices $W_{ijk\cdots}$ are all functions of the effective field $\phi$.

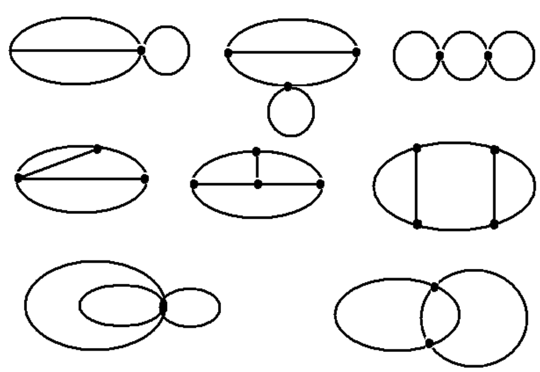
The 2-loop contributions can be described by the following graphs,



and the following analytic expression gives the corresponding terms:

$$\Gamma_2(\phi) = \frac{1}{12}W_{il}^{-1}W_{jm}^{-1}W_{kn}^{-1}W_{ijk}W_{lmn} - \frac{1}{8}W_{ij}^{-1}W_{kl}^{-1}W_{ijkl} \tag{4}$$

The following shows all possible 3-loop graphs:



Having shown how the basic formalism of the effective quantum action, for a generic field theory, can produce the successive loop contributions, the following subsections will introduce our two approaches of regularizing and computing these contributions, and how to control the divergences of quantum field theory, all in a manner preserving the fundamental gauge invariances of the theory.

### 1.1 The Divergence-Free Approach

In our divergence-free approach[6]-[8] for treating the divergences of the loop contributions, we begin by defining the logarithmic one-loop contribution as follows:

$$\frac{i}{2}\mathbf{tr}(\ln W_{ij}) \Rightarrow -\frac{i}{2}\varrho_\epsilon\, \mathbf{tr}\left(\frac{1}{\epsilon}W_{ij}^{-\epsilon}\right) \tag{5}$$

Here the symbol $\varrho_\epsilon$ stands for applying the operator $(\frac{\partial}{\partial\epsilon}\epsilon)$, and the subsequent process of taking the limit $\epsilon \to 0$. The necessary rule is that we should apply the latter operator and take the $\epsilon$ limit *after the integration over loop momentum is done*. Notice that the above procedure corresponds to the well-known definition of the logarithm as the limit of a power, such as $\ln A = -\lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}A^{-\epsilon}$.

Now correspondingly, higher-loop contributions will be regularized (actually becoming divergence-free) by replacing each effective propagator $W_{ij}^{-1}$ by a regular counterpart $W_{ij}^{-(1+\epsilon)}$, and associating with it a limiting operator $\varrho_\epsilon$ to be applied after all loop momenta are integrated over. It should be emphasized that *each* effective propagation entering a higher-loop expression should be associated with a *different* limiting parameter and associated operator, and all these operator prescriptions would have to be executed *after* all loop momenta are integrated over. This would preserve *gauge invariance*, ensure *freedom from divergences*, and guarantee *consistency*.[6]

In order to compute the various loop contributions as perturbative expansions with respect to the effective fields, we must split the effective bilinear kernel $W_{ij}(\phi)$ into a bare part $\Delta_{ij}$ and a field-dependent part $Y_{ij}(\phi)$. In matrix form, we write $W = \Delta + Y$. The one-loop contribution then takes the form:

$$-\frac{i}{2}\varrho_\epsilon\, \mathbf{tr}\left(\frac{1}{\epsilon}\frac{1}{(\Delta+Y)^\epsilon}\right) \tag{6}$$

This can be expanded with respect to $Y$, to give

$$-\frac{i}{2}\varrho_\epsilon\, \mathbf{tr}\left(\frac{1}{\Gamma(1+\epsilon)}\int_0^\infty \mathrm{d}\lambda\,\lambda^{\epsilon-1}\times\left\{\begin{array}{l} e^{-\lambda\Delta} - \lambda e^{-\lambda\Delta}Y \\ +\frac{\lambda^2}{2}\int_0^1 \mathrm{d}x\, e^{-(1-x)\lambda\Delta}Ye^{-x\lambda\Delta}Y + \cdots \end{array}\right\}\right) \tag{7}$$

The integration over $\lambda$ may be done once the operators are expressed in terms of matrix elements in momentum space. The resulting series takes the form:

$$-\frac{i}{2}\varrho_\epsilon\, \mathbf{tr}\left(\frac{1}{\epsilon}\frac{1}{\Delta^\epsilon} - \frac{1}{\Delta^{1+\epsilon}}Y + \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\left(\frac{1}{\Delta}Y\frac{1}{\Delta}\right)}_{(2+\epsilon)}Y - \cdots\right) \tag{8}$$

In the above series terms, it should be understood that two or several propagators that are separated by field insertions (the underbraced factor) are actually combined using Feynman parameters, with a power equal to the argument of the associated upper gamma function. For instance, the second-degree term may be represented in momentum space such as:

$$\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4p}{(2\pi)^4} \frac{Y(r)Y(-r)}{\{(1-x)\Delta(p)+x\Delta(p+r)\}^{2+\epsilon}} \tag{9}$$

where $p$ is the loop momentum, and $r$ is an external momentum carried by $Y$.

For the computation of higher-loop contributions, we must expand effective propagators of the form

$$\frac{1}{(\Delta+Y)^{1+\epsilon}} = \frac{1}{\Gamma(1+\epsilon)} \int_0^\infty \mathrm{d}\lambda\, \lambda^\epsilon\, e^{-\lambda(\Delta+Y)} \tag{10}$$

In momentum space, and after integrating over $\lambda$, the corresponding series in $Y$ takes the form

$$\frac{1}{\Delta^{1+\epsilon}} - \frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \underbrace{\left(\frac{1}{\Delta}Y\frac{1}{\Delta}\right)}_{(2+\epsilon)} + \frac{1}{2}\frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)} \underbrace{\left(\frac{1}{\Delta}Y\frac{1}{\Delta}Y\frac{1}{\Delta}\right)}_{(3+\epsilon)} - \cdots \tag{11}$$

The underbraced momentum-space propagators are understood to be combined using Feynman parameters with a total power equal to their number plus $\epsilon$ (argument of the associated upper gamma function).

### 1.2 The Cutoff Approach

In our cutoff approach[5] for regularizing the divergences of the loop contributions, we replace the logarithmic one-loop contribution as follows:

$$\frac{i}{2}\mathbf{tr}(\ln W) \Rightarrow -\frac{i}{2}\mathbf{tr}\left\{\int_{a^2}^\infty \frac{d\lambda}{\lambda} e^{-\lambda W}\right\} = -\frac{i}{2}\mathbf{tr}\left\{\int_{a^2}^\infty \frac{d\lambda}{\lambda} e^{-\lambda(\Delta+Y)}\right\} \tag{12}$$

Here $a$ is a cutoff length scale (since $W$ is usually like squared momentum). Again, in order to compute the various loop contributions as perturbative expansions with respect to the effective fields, we split the effective bilinear kernel $W$ into a bare part $\Delta$ and a field-dependent part $Y$. The above can be expanded with respect to $Y$,

$$-\frac{i}{2}\int_{a^2}^\infty \frac{d\lambda}{\lambda}\mathbf{tr}\left\{e^{-\lambda\Delta} - \lambda e^{-\lambda\Delta}Y + \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x\, e^{-(1-x)\lambda\Delta}Y e^{-x\lambda\Delta}Y + \cdots\right\} \tag{13}$$

For the computation of higher-loop contributions, we must expand effective propagators of the form

$$\frac{1}{\Delta+Y} = \int_{a^2}^\infty \mathrm{d}\lambda\, e^{-\lambda(\Delta+Y)} \tag{14}$$

We shall write for $e^{-\lambda(\Delta+Y)}$,

$$e^{-\lambda\Delta} - \lambda \int_0^1 \mathrm{d}x\, e^{-(1-x)\lambda\Delta} Y e^{-x\lambda\Delta} + \lambda^2 \int_0^1 \mathrm{d}x \int_0^x \mathrm{d}y\, e^{-\lambda(1-x)\Delta} Y e^{-\lambda(x-y)\Delta} Y e^{-\lambda\Delta} + \cdots \tag{15}$$

### 1.3 A Simple Comparison

Consider the Euclidean momentum-space integral

$$-\frac{1}{2} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \ln(p^2 + m^2) \tag{16}$$

In the divergence-free approach, we write

$$\varrho_\epsilon \frac{1}{2}\frac{1}{\epsilon} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{(p^2+m^2)^\epsilon} = \varrho_\epsilon \frac{1}{32\pi^2} \frac{\Gamma(\epsilon-2)}{\Gamma(\epsilon+1)} \left(m^2\right)^{2-\epsilon} \tag{17}$$

The effect of $\varrho_\epsilon$ is to pick the terms that are independent of $\epsilon$. Hence, expanding with respect to $\epsilon$, and picking the pertinent terms, we obtain

$$\frac{1}{64\pi^2} m^4 \left(\frac{3}{2} - \ln(m^2)\right) \tag{18}$$

In the cutoff approach, we write

$$\frac{1}{2} \int_{a^2}^{\infty} \frac{\mathrm{d}\lambda}{\lambda} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-\lambda(p^2+m^2)} = \frac{1}{32\pi^2} \int_{a^2}^{\infty} \frac{\mathrm{d}\lambda}{\lambda^3} e^{-\lambda m^2} \tag{19}$$

Expanding with respect to $a^2$, dropping the terms that vanish as $a \to 0$, we obtain

$$\frac{1}{64\pi^2} \left\{ \frac{1}{a^4} - \frac{2m^2}{a^2} + m^4 \left( \frac{3}{2} - \gamma - \ln(a^2 m^2) \right) \right\} \tag{20}$$

Comparing this result with the earlier one, we see that the divergence-free approach gives the same result obtained in the cutoff approach, provided that we drop the singular terms (like $1/a^2$ and $1/a^4$) and scale the logarithmic terms appropriately.

One-loop examples from quantum electrodynamics and from quantum gravity will be given in the following sections. Computations pertaining to higher loops will be the subject of other articles.

## 2 Quantum Electrodynamics

Let us take the action density

$$\bar{\psi}(i\gamma \cdot \nabla)\psi - m\bar{\psi}\psi \tag{21}$$

with $i\gamma\cdot\nabla = (i\gamma\cdot\partial + \gamma\cdot A)$. The above describes the coupling of the photon $A_\mu$ to a Dirac field $\psi$, having Dirac conjugate $\bar{\psi}$, and mass parameter $m$. From the above, we define the gauge-covariant bilinear kernel $W$, and its conjugate $\tilde{W}$,

$$W = (i\gamma\cdot\nabla - m) \qquad \tilde{W} = -(i\gamma\cdot\nabla + m) \tag{22}$$

These give the quadratic operator

$$\tilde{W}W = (\gamma\cdot\nabla)^2 + m^2 = \Delta - Y \qquad \begin{cases} \Delta = \partial^2 + m^2 \\ Y = i(\gamma\cdot\partial\gamma\cdot A + \gamma\cdot A\gamma\cdot\partial) + A^2 \end{cases} \tag{23}$$

The gauge-covariant inverse of the kernel may be given by

$$W^{-1} = \frac{1}{\tilde{W}W}\tilde{W} = -\frac{1}{\Delta - Y}(i\gamma\cdot\nabla + m) \tag{24}$$

The fermionic one-loop contribution is given by $-i\mathbf{tr}(\ln W)$, or equivalently $-i\mathbf{tr}(\ln\tilde{W})$. Taking the average of the two expressions, we can write for the one-loop contribution

$$-\frac{i}{2}\mathbf{tr}\ln(\tilde{W}W) = -\frac{i}{2}\mathbf{tr}\ln\left\{(\gamma\cdot\nabla)^2 + m^2\right\} = -\frac{i}{2}\mathbf{tr}\ln(\Delta - Y) \tag{25}$$

In the following subsections we shall compute the above fermionic loop up to second order in the photon field, first using the divergence-free approach, then using the cutoff approach, and shall compare the results.

## 2.1 Fermionic Loop in the Divergence-Free Approach

Here we regularize the gauge-invariant one-loop contributions[8] as follows:

$$-\frac{i}{2}\mathbf{tr}\ln(\Delta - Y) \Rightarrow \frac{i}{2}\varrho_\epsilon\,\mathbf{tr}\left(\frac{1}{\epsilon}\frac{1}{(\Delta - Y)^\epsilon}\right) \tag{26}$$

We shall expand the above to 2nd order with respect to $Y$,

$$\frac{i}{2}\varrho_\epsilon\mathbf{tr}\left\{\frac{1}{\epsilon}\frac{1}{\Delta^\epsilon} + \frac{1}{\Delta^{1+\epsilon}}Y + \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}Y + \cdots\right\} \tag{27}$$

We must recall that

$$\Delta = (\partial^2 + m^2) \qquad Y = i(\gamma\cdot\partial\gamma\cdot A + \gamma\cdot A\gamma\cdot\partial) + A^2 \tag{28}$$

### 2.1.1 The Vacuum Contribution

The vacuum contribution resulting from the foregoing series is

$$\frac{i}{2}\varrho_\epsilon\,\mathbf{tr}\left(\frac{1}{\epsilon}\frac{1}{(\partial^2 + m^2)^\epsilon}\right) \tag{29}$$

In momentum space, this gives

$$2i\varrho_\epsilon \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{1}{\epsilon}\frac{1}{(-p^2+m^2)^\epsilon}\right) \tag{30}$$

where a factor of 4 has resulted from the trace over spinor matrices. Transforming the above to Euclidean momentum space,[1] then integrating over momentum, and executing the operator $\varrho_\epsilon$, we obtain

$$\frac{1}{16\pi^2}m^4\left(-\frac{3}{2}+\ln(m^2)\right) \tag{31}$$

#### 2.1.2 The Photon Bilinear

Substituting for $Y$, the bilinear contribution in the photon field is

$$\frac{i}{2}\mathbf{tr}\left\{\begin{array}{l}\frac{1}{(\partial^2+m^2)^{1+\epsilon}}A^2 \\ +\frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\partial^2+m^2}i(\gamma\cdot\partial\gamma\cdot A+\gamma\cdot A\gamma\cdot\partial)\frac{1}{\partial^2+m^2}}_{2+\epsilon}i(\gamma\cdot\partial\gamma\cdot A+\gamma\cdot A\gamma\cdot\partial)\end{array}\right\} \tag{32}$$

where the meaning of the underbrace notation is as explained in the introduction. Translating to momentum space, we obtain the bilinear $\frac{1}{2}A_\mu(r)A_\nu(-r)X_{\mu\nu}(r)$, where $r$ is the external photon momentum, and the kernel $X_{\mu\nu}(r)$ is given by

$$i\varrho_\epsilon \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\begin{array}{l}\frac{1}{(-p^2+m^2)^{1+\epsilon}}\eta_{\mu\nu} \\ +\frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1 \mathrm{d}x\,\frac{1}{\{(1-x)(-p^2+m^2)+x(-(p+r)^2+m^2)\}^{2+\epsilon}}\times(\cdots)\end{array}\right\} \tag{33}$$

$$(\cdots)=\{\gamma\cdot p\gamma_\mu+\gamma_\mu\gamma\cdot(p+r)\}\{\gamma\cdot(p+r)\gamma_\nu+\gamma_\nu\gamma\cdot p\} \tag{34}$$

Now the numerator involving the gamma matrices gives

$$(\cdots)=(2p_\mu+\gamma_\mu\gamma\cdot r)(2p_\nu+\gamma\cdot r\gamma_\nu) \tag{35}$$

Taking the trace over gamma matrices, this gives

$$4\left(4p_\mu p_\nu+2p_\mu r_\nu+2p_\nu r_\mu+r^2\eta_{\mu\nu}\right) \tag{36}$$

Also simplifying the denominator which involves the Feynman parameter, we obtain

$$4i\varrho_\epsilon \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\begin{array}{l}\frac{1}{(-p^2+m^2)^{1+\epsilon}}\eta_{\mu\nu}+ \\ \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1 \mathrm{d}x\,\frac{1}{\{-(p+xr)^2-x(1-x)r^2+m^2\}^{2+\epsilon}}\times \\ (4p_\mu p_\nu+2p_\mu r_\nu+2p_\nu r_\mu+r^2\eta_{\mu\nu})\end{array}\right\} \tag{37}$$

[1] **The Euclidean continuation of loop momentum is equivalent to Feynman's $i\epsilon$ prescription for propagators and is equivalent to $\mathrm{d}^4p\to i\,\mathrm{d}^4p$ and $p^2\to -p^2$.**

Now, we have to make a shift in the loop momentum of the second term with $p \to p - xr$. The numerator becomes under this shift and a subsequent symmetrization:

$$p^2\eta_{\mu\nu} - 4x(1-x)r_\mu r_\nu + r^2\eta_{\mu\nu} \tag{38}$$

Hence, we obtain

$$4i\varrho_\epsilon \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left\{ \begin{array}{l} \frac{1}{(-p^2+m^2)^{1+\epsilon}}\eta_{\mu\nu} + \\ \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)} \int_0^1 \mathrm{d}x \, \frac{1}{\{-p^2-x(1-x)r^2+m^2\}^{2+\epsilon}} \times \\ (p^2\eta_{\mu\nu} - 4x(1-x)r_\mu r_\nu + r^2\eta_{\mu\nu}) \end{array} \right\} \tag{39}$$

Converting the above to Euclidean loop momentum, then integrating over the latter, expanding with respect to external momentum to order $r^4$, integrating over the Feynman parameter $x$, and executing the operator $\varrho_\epsilon$, we obtain the gauge-invariant result:

$$\frac{1}{2\pi^2}\left(r^2\eta_{\mu\nu} - r_\mu r_\nu\right)\left(\ln(m^2) - \frac{1}{5}\frac{r^2}{m^2} + \cdots\right) \tag{40}$$

### 2.2 Fermionic Loop in the Cutoff Approach

The gauge-invariant one-loop contribution, in the cutoff approach, is expressed like

$$\frac{i}{2}\int_{a^2}^{\infty} \frac{\mathrm{d}\lambda}{\lambda}\,\mathbf{tr}\left(e^{-\lambda(\Delta - Y)}\right) \tag{41}$$

and this gives to 2nd order in $Y$,

$$\frac{i}{2}\int_{a^2}^{\infty} \frac{\mathrm{d}\lambda}{\lambda}\,\mathbf{tr}\left( \begin{array}{l} e^{-\lambda\Delta} + \lambda e^{-\lambda\Delta}Y + \\ \frac{\lambda^2}{2}\int_0^1 \mathrm{d}x\, e^{-(1-x)\lambda\Delta} Y e^{-x\lambda\Delta} Y + \cdots \end{array} \right) \tag{42}$$

We must recall that

$$\Delta = (\partial^2 + m^2) \qquad Y = i(\gamma\cdot\partial\gamma\cdot A + \gamma\cdot A\gamma\cdot\partial) + A^2 \tag{43}$$

#### 2.2.1 The Vacuum Contribution

Here we have

$$\frac{i}{2}\int_{a^2}^{\infty} \frac{\mathrm{d}\lambda}{\lambda}\,\mathbf{tr}\left(e^{-\lambda(\partial^2 + m^2)}\right) \tag{44}$$

Going to momentum space, and taking the trace over spinor indices, we get

$$2i\int_{a^2}^{\infty} \frac{\mathrm{d}\lambda}{\lambda}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\left(e^{-\lambda(-p^2+m^2)}\right) \tag{45}$$

Converting to Euclidean momentum, and integrating over $p$, we have

$$-\frac{1}{8\pi^2}\int_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda^3}\,e^{-\lambda m^2} \tag{46}$$

Expanding with respect to $a^2$, dropping the terms that vanish as $a \to 0$, we obtain

$$-\frac{1}{16\pi^2}\left\{\frac{1}{a^4}-\frac{2m^2}{a^2}+m^4\left(\frac{3}{2}-\gamma-\ln(a^2m^2)\right)\right\} \tag{47}$$

#### 2.2.2 The Photon Bilinear

Substituting for $Y$, we obtain the following bilinear contribution

$$\frac{i}{2}\int_{a^2}^{\infty}\mathrm{d}\lambda\,\mathbf{tr}\left\{\begin{array}{l} e^{-\lambda(\partial^2+m^2)}A^2 \\ -\frac{\lambda}{2}\int_0^1 \mathrm{d}x\,e^{-(1-x)\lambda(\partial^2+m^2)}(\gamma\cdot\partial\gamma\cdot A+\gamma\cdot A\gamma\cdot\partial)\times \\ e^{-x\lambda(\partial^2+m^2)}(\gamma\cdot\partial\gamma\cdot A+\gamma\cdot A\gamma\cdot\partial) \end{array}\right\} \tag{48}$$

Converting to momentum space, we obtain the bilinear $\frac{1}{2}A_\mu(r)A_\nu(-r)X_{\mu\nu}(r)$, with the kernel $X_{\mu\nu}(r)$ given by

$$i\int_{a^2}^{\infty}\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathbf{tr}\left\{\begin{array}{l} e^{-\lambda(-p^2+m^2)}\eta_{\mu\nu}+ \\ \frac{\lambda}{2}\int_0^1 \mathrm{d}x\,e^{-\lambda\left\{(1-x)(-p^2+m^2)+x[-(p+r)^2+m^2]\right\}}\times \\ \{\gamma\cdot p\gamma_\mu+\gamma_\mu\gamma\cdot(p+r)\}\{\gamma\cdot(p+r)\gamma_\nu+\gamma_\nu\gamma\cdot p\} \end{array}\right\} \tag{49}$$

Taking the spinorial trace, and simplifying the argument that involves the Feynman parameter $x$, we obtain

$$4i\int_{a^2}^{\infty}\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\begin{array}{l} e^{-\lambda(-p^2+m^2)}\eta_{\mu\nu}+ \\ \frac{\lambda}{2}\int_0^1 \mathrm{d}x\,e^{-\lambda\left\{-(p+xr)^2-x(1-x)r^2+m^2\right\}}\times \\ (4p_\mu p_\nu+2p_\mu r_\nu+2p_\nu r_\mu+r^2\eta_{\mu\nu}) \end{array}\right\} \tag{50}$$

With a momentum shift $p \to (p-xr)$, in the loop momentum of the second term, and a subsequent momentum symmetrization, then converting to Euclidean momentum integration, we obtain

$$-4\int_{a^2}^{\infty}\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\begin{array}{l} e^{-\lambda(p^2+m^2)}\eta_{\mu\nu}+ \\ \frac{\lambda}{2}\int_0^1 \mathrm{d}x\,e^{-\lambda\left\{p^2-x(1-x)r^2+m^2\right\}}\times \\ (-p^2\eta_{\mu\nu}-4x(1-x)r_\mu r_\nu+r^2\eta_{\mu\nu}) \end{array}\right\} \tag{51}$$

Now integrating over loop momentum $p$, expanding to 4th order in the external momentum $r$, integrating over the the Feynman parameter $x$, and expanding with respect to $a^2$ (dropping the terms that vanish as $a \to 0$), we obtain the gauge-invariant result:

$$\frac{1}{12\pi^2}\left(r^2\eta_{\mu\nu} - r_\mu r_\nu\right)\left(\gamma + \ln(a^2m^2) - \frac{1}{5}\frac{r^2}{m^2} + \cdots\right) \tag{52}$$

### 2.3 Comparing Results

For the vacuum contribution of the fermionic loop in the divergence-free approach, we obtain

$$-\frac{m^4}{16\pi^2}\left(\frac{3}{2} - \ln(m^2)\right) \tag{53}$$

The corresponding contribution in the cutoff approach is

$$-\frac{1}{16\pi^2}\left\{\frac{1}{a^4} - \frac{2m^2}{a^2} + m^4\left(\frac{3}{2} - \gamma - \ln(a^2m^2)\right)\right\} \tag{54}$$

For the fermionic loop contribution to the photon kernel, in the divergence-free approach, we obtain

$$\frac{1}{2\pi^2}\left(r^2\eta_{\mu\nu} - r_\mu r_\nu\right)\left(\ln(m^2) - \frac{1}{5}\frac{r^2}{m^2} + \cdots\right) \tag{55}$$

The corresponding contribution in the cutoff approach is

$$\frac{1}{2\pi^2}\left(r^2\eta_{\mu\nu} - r_\mu r_\nu\right)\left(\gamma + \ln(a^2m^2) - \frac{1}{5}\frac{r^2}{m^2} + \cdots\right) \tag{56}$$

## 3 Quantum Gravity

We consider the following action density,

$$\frac{1}{2}V\bar{\psi}\left(iV^{\mu a}\gamma_a\nabla_\mu - m\right)\psi + h.c. \tag{57}$$

$$\nabla_\mu\psi = \left(\partial_\mu + \frac{1}{4}\gamma_{ab}\omega_\mu{}^{ab}\right)\psi \tag{58}$$

Here we have the coupling of the vierbein field $V_\mu{}^a$ (inverse $V^{\mu a}$, $V = \det V_\mu{}^a$) to the Dirac field $\psi$, having Dirac conjugate $\bar{\psi}$, and mass parameter $m$. The field $\omega_\mu{}^{ab}$ is the Lorentz gauge field. In fact, we shall not need the coupling of $\omega_\mu{}^{ab}$ in the following computations. We shall scale the spinor field like $\psi \to V^{-1/2}\psi$, and shall expand the vierbein field about flat spacetime,

$$\begin{cases} V_{\mu a} = \eta_{\mu a} + \frac{1}{2}\phi_{\mu a} \\ V^{\mu a} = \eta^{\mu a} - \frac{1}{2}\phi^{\mu a} + \frac{1}{4}(\phi^2)^{\mu a} + \cdots \end{cases} \tag{59}$$

Here $\phi_{\mu\nu}$ represents the (symmetric) graviton field. Notice that the metric tensor is related to the vierbein by $g_{\mu\nu} = V_\mu{}^a V_{\nu a}$, and $\sqrt{g} = V$, and with the above expansion,

$$g_{\mu\nu} = \eta_{\mu\nu} + \phi_{\mu\nu} + \frac{1}{4}(\phi^2)_{\mu\nu} \tag{60}$$

We obtain for the foregoing action density, to 2nd order in $\phi$,

$$\bar{\psi}(i\gamma\cdot\partial - m)\psi - h_{\mu\nu}\left\{\bar{\psi}\gamma_\mu(\partial_\nu\psi) - (\partial_\mu\bar{\psi})i\gamma_\nu\psi\right\} \tag{61}$$

$$h_{\mu\nu} = \left(\frac{1}{2}\phi_{\mu\nu} - \frac{1}{8}(\phi^2)_{\mu\nu}\right) \tag{62}$$

Notice that $(\phi^2)_{\mu\nu}$ would represent $\phi_{\mu\lambda}\phi_{\lambda\nu}$, and is symmetric.

From the above, we can define the bilinear spinorial kernel $W$, and its conjugate $\tilde{W}$,

$$\begin{cases} W = (i\gamma\cdot\partial - m) - i\gamma_\mu\left(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}\right) \\ \tilde{W} = -(i\gamma\cdot\partial + m) + i\gamma_\mu\left(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}\right) \end{cases} \tag{63}$$

Hence we have $W\tilde{W} = \Delta - Y$, with $\Delta = (\partial^2 + m^2)$, and

$$Y = \left\{ \begin{array}{l} \gamma\cdot\partial\gamma_\mu(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}) + \gamma_\mu(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu})\gamma\cdot\partial \\ -\gamma_\mu(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu})\gamma_\lambda(h_{\lambda\rho}\partial_\rho + \partial_\rho h_{\lambda\rho}) \end{array} \right\} \tag{64}$$

The fermionic one-loop contribution is given by

$$-\frac{i}{2}\mathbf{tr}\ln(W\tilde{W}) = -\frac{i}{2}\mathbf{tr}\ln(\Delta - Y) \tag{65}$$

In the following subsections we shall compute the above fermionic loop up to second order in the graviton field, first using the divergence-free approach, then using the cutoff approach, and shall compare the results.

### 3.1 Fermionic Loop in the Divergence-Free Approach

Here we regularize the gauge-invariant one-loop contributions as follows:

$$-\frac{i}{2}\mathbf{tr}\ln(\Delta - Y) \Rightarrow \frac{i}{2}\varrho_\epsilon\,\mathbf{tr}\left(\frac{1}{\epsilon}\frac{1}{(\Delta - Y)^\epsilon}\right) \tag{66}$$

We shall expand the above to 2nd order with respect to $Y$,

$$\frac{i}{2}\varrho_\epsilon\,\mathbf{tr}\left\{\frac{1}{\epsilon}\frac{1}{\Delta^\epsilon} + \frac{1}{\Delta^{1+\epsilon}}Y + \frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}Y + \cdots\right\} \tag{67}$$

We must recall that $\Delta = (\partial^2 + m^2)$, and

$$Y = \left\{ \begin{array}{l} \gamma \cdot \partial \gamma_\mu (h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}) + \gamma_\mu (h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}) \gamma \cdot \partial \\ -\gamma_\mu (h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}) \gamma_\lambda (h_{\lambda\rho}\partial_\rho + \partial_\rho h_{\lambda\rho}) \end{array} \right\} \tag{68}$$

$$h_{\mu\nu} = \left( \frac{1}{2}\phi_{\mu\nu} - \frac{1}{8}(\phi^2)_{\mu\nu} \right) \tag{69}$$

#### 3.1.1 Vacuum Contribution

The vacuum contribution from the fermionic one-loop is

$$\frac{i}{2}\varrho_\epsilon \,\mathbf{tr} \left( \frac{1}{\epsilon} \frac{1}{\Delta^\epsilon} \right) = \frac{i}{2}\varrho_\epsilon \,\mathbf{tr} \left( \frac{1}{\epsilon} \frac{1}{(\partial^2 + m^2)^\epsilon} \right) \tag{70}$$

Translating to momentum space, and taking the spinorial trace, we obtain

$$2i\varrho_\epsilon \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left( \frac{1}{\epsilon} \frac{1}{(-p^2 + m^2)^\epsilon} \right) \tag{71}$$

Converting to Euclidean momentum and integrating, we obtain

$$-\frac{1}{8\pi^2}\varrho_\epsilon \frac{\Gamma(-2+\epsilon)}{\Gamma(1+\epsilon)} \left(m^2\right)^{2-\epsilon} \tag{72}$$

Executing the $\varrho_\epsilon$ operator, we get

$$-\frac{m^4}{16\pi^2} \left( \frac{3}{2} - \ln(m^2) \right) \tag{73}$$

#### 3.1.2 Linear Contribution

The one-loop fermionic contribution that is linear in the graviton field comes from

$$\frac{i}{2}\varrho_\epsilon \,\mathbf{tr} \left( \frac{1}{\Delta^{1+\epsilon}} Y \right) = i\mathbf{tr} \left( \frac{1}{\Delta^{1+\epsilon}} \gamma_\mu \left( h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu} \right) \gamma \cdot \partial \right) \tag{74}$$

Taking the spinorial trace, substituting for $\Delta$, and $h_{\mu\nu} \to \frac{1}{4}\phi_{\mu\nu}$,

$$i\varrho_\epsilon \,\mathbf{tr} \left( \frac{1}{(\partial^2 + m^2)^{1+\epsilon}} \left( \phi_{\mu\nu}\partial_\nu + \partial_\nu \phi_{\mu\nu} \right) \partial_\mu \right) \tag{75}$$

Manipulating objects under the trace, this gives

$$2i\phi_{\mu\nu}\varrho_\epsilon \,\mathbf{tr} \left( \frac{1}{(\partial^2 + m^2)^{1+\epsilon}} \partial_\mu \partial_\nu \right) \tag{76}$$

Translating to momentum space, then symmetrizing with respect to loop momentum,

$$-2i\phi_{\mu\nu}\varrho_\epsilon \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{p_\mu p_\nu}{(-p^2+m^2)^{1+\epsilon}}\right) \Rightarrow -\frac{i}{2}\phi\,\varrho_\epsilon \int \frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{p^2}{(-p^2+m^2)^{1+\epsilon}}\right) \tag{77}$$

Here $\phi$ denote $\phi_{\mu\mu}$. Converting to Euclidean loop momentum, and integrating,

$$-\frac{1}{16\pi^2}\phi\,\varrho_\epsilon \frac{\Gamma(-2+\epsilon)}{\Gamma(1+\epsilon)} \tag{78}$$

Executing the $\varrho_\epsilon$ operator, we get

$$-\frac{m^4}{32\pi^2}\left(\frac{3}{2}-\ln(m^2)\right)\phi \tag{79}$$

#### 3.1.3 The Graviton Bilinear

The one-loop fermionic contribution that is bilinear in the graviton field comes from:

$$\frac{i}{2}\varrho_\epsilon\,\mathbf{tr}\left\{\frac{1}{\Delta^{1+\epsilon}}Y+\frac{1}{2}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\underbrace{\frac{1}{\Delta}Y\frac{1}{\Delta}}_{2+\epsilon}Y+\cdots\right\} \tag{80}$$

Substituting for $Y$, and subsequently for $h_{\mu\nu}$ in terms of $\phi_{\mu\nu}$, we obtain to second order in the latter,

$$\frac{i}{8}\varrho_\epsilon\,\mathbf{tr}\left\{\begin{array}{l} -\frac{1}{\Delta^{1+\epsilon}}\gamma_\mu\left((\phi^2)_{\mu\nu}\partial_\nu+\partial_\nu(\phi^2)_{\mu\nu}\right)\gamma\cdot\partial \\ -\frac{1}{4}\frac{1}{\Delta^{1+\epsilon}}\gamma_\mu\left(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu}\right)\gamma_\lambda\left(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho}\right)+ \\ \frac{1}{8}\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\left(\begin{array}{l}\underbrace{\frac{1}{\Delta}\left(\gamma\cdot\partial\gamma_\mu(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu})+\gamma_\mu(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu})\gamma\cdot\partial\right)\frac{1}{\Delta}}_{2+\epsilon}\times \\ \left(\gamma\cdot\partial\gamma_\lambda(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho})+\gamma_\lambda(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho})\gamma\cdot\partial\right)\end{array}\right)\end{array}\right\} \tag{81}$$

The above three terms will be computed successively.

#### 3.1.4 Computing the First Bilinear Term

We shall compute the first bilinear term

$$-\frac{i}{8}\varrho_\epsilon\,\mathbf{tr}\left\{\frac{1}{\Delta^{1+\epsilon}}\gamma_\mu\left((\phi^2)_{\mu\nu}\partial_\nu+\partial_\nu(\phi^2)_{\mu\nu}\right)\gamma\cdot\partial\right\} \tag{82}$$

Manipulating operators under the trace, translating to momentum space, doing the spinorial trace, and symmetrizing loop momenta, we obtain

$$\frac{i}{4}(\phi^2)_{\mu\mu}\,\varrho_\epsilon \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{p^2}{(-p^2+m^2)^{1+\epsilon}} \tag{83}$$

Converting to Euclidean loop momentum, integrating, and applying $\varrho_\epsilon$, we obtain

$$\frac{m^4}{64\pi^2}\left(\frac{3}{2}-\ln(m^2)\right)(\phi^2)_{\mu\mu} \tag{84}$$

#### 3.1.5 Computing the Second Bilinear Term

We shall compute the second bilinear term

$$-\frac{i}{32}\varrho_\epsilon\,\mathbf{tr}\left\{\frac{1}{\Delta^{1+\epsilon}}\gamma_\mu\left(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu}\right)\gamma_\lambda\left(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho}\right)\right\} \tag{85}$$

Doing the spinorial trace, and going to momentum space, we obtain

$$\frac{i}{8}\varrho_\epsilon\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\frac{(2p_\nu+r_\nu)(2p_\rho+r_\rho)}{(-p^2+m^2)^{1+\epsilon}}\right\}\phi_{\mu\nu}(-r)\phi_{\mu\rho}(r) \tag{86}$$

Here $r$ is the momentum carried by the external graviton. Symmetrizing with respect to the loop momentum $p$, converting to Euclidean loop momentum, integrating, and applying $\varrho_\epsilon$, we obtain

$$\frac{1}{128\pi^2}\left(m^4\left\{\frac{3}{2}-\ln(m^2)\right)\eta_{\nu\rho}+m^2(1-\ln(m^2))r_\nu r_\rho\right\}\phi_{\mu\nu}(-r)\phi_{\mu\rho}(r) \tag{87}$$

#### 3.1.6 Computing the Third Bilinear Term

We shall compute the third bilinear term

$$\frac{i}{64}\varrho_\epsilon\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\mathbf{tr}\left\{\begin{array}{l}\underbrace{\frac{1}{\Delta}\left(\gamma\cdot\partial\gamma_\mu(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu})+\gamma_\mu(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu})\gamma\cdot\partial\right)\frac{1}{\Delta}}_{2+\epsilon}\times\\[2ex] \left(\gamma\cdot\partial\gamma_\lambda(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho})+\gamma_\lambda(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho})\gamma\cdot\partial\right)\end{array}\right\} \tag{88}$$

Translating to momentum space, combining the underbraced propagators with a Feynman parameter $x$, and taking the spinorial trace, we obtain for the coefficient of $\phi_{\mu\nu}(-r)\phi_{\lambda\rho}(r)$,

$$\frac{i}{16}\varrho_\epsilon\frac{\Gamma(2+\epsilon)}{\Gamma(1+\epsilon)}\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\begin{array}{l}\frac{1}{(-(p+xr)^2-x(1-x)r^2+m^2)^{2+\epsilon}}\times\\[2ex] (4p_\mu p_\lambda+2p_\mu r_\lambda+2p_\lambda r_\mu+r^2\eta_{\mu\lambda})\times\\[2ex] (2p_\nu+r_\nu)(2p_\rho+r_\rho)\end{array}\right\} \tag{89}$$

Making the shift $p\to(p-xr)$ in the loop momentum, symmetrizing with respect to the latter, converting to Euclidean loop momentum, and integrating, then expanding to 4th order with respect to the external graviton momentum $r$, and subsequently integrating with respect to the Feynman parameter, finally applying the operator $\varrho_\epsilon$, we obtain our result for the third bilinear term. The latter consists of three parts. For the part that is independent of graviton momentum, we have

$$-\frac{m^4}{128\pi^2}\left(\frac{3}{2}-\ln(m^2)\right)(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}+\eta_{\mu\nu}\eta_{\lambda\rho}) \tag{90}$$

For the part that is quadratic in the graviton momentum, we have

$$-\frac{m^2}{384\pi^2}(1-\ln m^2)\begin{pmatrix}\eta_{\mu\lambda}r_\nu r_\rho+\eta_{\nu\lambda}r_\mu r_\rho+\eta_{\lambda\rho}r_\mu r_\nu+\eta_{\mu\nu}r_\lambda r_\rho+\eta_{\mu\rho}r_\lambda r_\nu\\ -2\eta_{\nu\rho}r_\mu r_\lambda+r^2(2\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\nu}\eta_{\lambda\rho})\end{pmatrix} \tag{91}$$

For the part that is quartic in the graviton momentum, we have

$$\frac{\ln m^2}{3840\pi^2}\left\{\begin{array}{l}r^2\left(4\eta_{\mu\lambda}r_\nu r_\rho-\eta_{\nu\lambda}r_\mu r_\rho-\eta_{\lambda\rho}r_\mu r_\nu-\eta_{\mu\nu}r_\lambda r_\rho-\eta_{\mu\rho}r_\nu r_\lambda+4\eta_{\nu\rho}r_\mu r_\lambda\right)\\ -2r_\mu r_\nu r_\lambda r_\rho-r^4\left(4\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\nu}\eta_{\lambda\rho}\right)\end{array}\right\} \tag{92}$$

#### 3.1.7 Collected Cosmological Term

The cosmological term corresponds to $\sqrt{g}=V$ in the effective action. Expanding to 2nd order in the graviton field we have

$$e^{\mathbf{tr}\ln(\eta+\frac{1}{2}\phi)}=e^{(\frac{1}{2}\phi-\frac{1}{8}\phi\cdot\phi)}\approx 1+\frac{1}{2}\phi+\frac{1}{8}\left(\phi^2-\phi_{\mu\nu}\phi_{\mu\nu}\right)+\cdots \tag{93}$$

In our foregoing fermionic loop computations, we obtained the vacuum term:

$$-\frac{m^4}{16\pi^2}\left(\frac{3}{2}-\ln(m^2)\right) \tag{94}$$

We obtained the linear term:

$$-\frac{m^4}{32\pi^2}\left(\frac{3}{2}-\ln(m^2)\right)\phi \tag{95}$$

And obtained the (momentum-independent) bilinear:

$$-\frac{m^4}{128\pi^2}\left(\frac{3}{2}-\ln(m^2)\right)\left(\phi^2-\phi_{\mu\nu}\phi_{\mu\nu}\right) \tag{96}$$

Hence combining, we obtain the contribution to the cosmological term $\sqrt{g}$, with coefficient:

$$-\frac{m^4}{16\pi^2}\left(\frac{3}{2}-\ln(m^2)\right) \tag{97}$$

#### 3.1.8 Collected Einstein Term

The bilinear terms of the graviton that come from the Einstein action density $\sqrt{g}g^{\mu\nu}R_{\mu\nu}$, with $R_{\mu\nu}$ the Ricci tensor, take the usual form for a massless spin-2 field:

$$\frac{1}{4}(\partial_\lambda\phi_{\mu\nu})^2 - \frac{1}{4}(\partial_\mu\phi_{\nu\nu})^2 + \frac{1}{2}\partial_\mu\phi_{\mu\nu}\partial_\nu\phi_{\lambda\lambda} - \frac{1}{2}\partial_\mu\phi_{\mu\lambda}\partial_\nu\phi_{\nu\lambda} \tag{98}$$

In momentum space this gives

$$\phi_{\mu\nu}(r)\phi_{\lambda\rho}(-r)\left(\frac{1}{4}r^2\eta_{\mu\lambda}\eta_{\nu\rho} - \frac{1}{4}r^2\eta_{\mu\nu}\eta_{\lambda\rho} + \frac{1}{2}r_\mu r_\nu\eta_{\lambda\rho} - \frac{1}{2}r_\mu r_\lambda\eta_{\nu\rho}\right) \tag{99}$$

The collected bilinears with quadratic momentum that we obtained in the preceding sections correspond to the above, with the coefficient:

$$-\frac{m^2}{96\pi^2}(1-\ln m^2) \tag{100}$$

#### 3.1.9 Collected Curvature Squared Terms

For the graviton bilinear terms with quartic momentum, we obtained

$$\frac{\ln m^2}{3840\pi^2}\left\{\begin{array}{l} r^2\left(4\eta_{\mu\lambda}r_\nu r_\rho - \eta_{\nu\lambda}r_\mu r_\rho - \eta_{\lambda\rho}r_\mu r_\nu - \eta_{\mu\nu}r_\lambda r_\rho - \eta_{\mu\rho}r_\nu r_\lambda + 4\eta_{\nu\rho}r_\mu r_\lambda\right) \\ -2r_\mu r_\nu r_\lambda r_\rho - r^4\left(4\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\rho}\right) \end{array}\right\} \tag{101}$$

multiplied by $\phi_{\mu\nu}(-r)\phi_{\lambda\rho}(r)$. In fact we can show that this corresponds to the bilinears of the following Einstein invariant action density $\sqrt{g}\left(3{R_{\mu\nu}}^2 - R^2\right)$, with coefficient

$$\frac{\ln m^2}{960\pi^2} \tag{102}$$

### 3.2 Fermionic Loop in the Cutoff Approach

Like the preceding section that was concerned with the divergence-free approach, we shall use the following fermionic kernel operator $W$ and its conjugate $\tilde{W}$,

$$\left\{\begin{array}{l} W = (i\gamma\cdot\partial - m) - i\gamma_\mu\left(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}\right) \\ \tilde{W} = -(i\gamma\cdot\partial + m) + i\gamma_\mu\left(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}\right) \end{array}\right. \tag{103}$$

with

$$h_{\mu\nu} = \left(\frac{1}{4}\phi_{\mu\nu} - \frac{1}{8}(\phi^2)_{\mu\nu}\right) \tag{104}$$

We shall write $W\tilde{W} = (\Delta - Y)$, with $\Delta = (\partial^2 + m^2)$ and

$$Y = \left\{\begin{array}{l} \gamma\cdot\partial\gamma_\mu\left(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}\right) + \gamma_\mu\left(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}\right)\gamma\cdot\partial \\ -\gamma_\mu\left(h_{\mu\nu}\partial_\nu + \partial_\nu h_{\mu\nu}\right)\gamma_\lambda\left(h_{\lambda\rho}\partial_\rho + \partial_\rho h_{\lambda\rho}\right) \end{array}\right\} \tag{105}$$

The cutoff-regularized fermionic one-loop contribution will be given by

$$-\frac{i}{2}\mathbf{tr}\ln(W\tilde{W}) \Rightarrow \frac{i}{2}\int_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda}\mathbf{tr}\left\{e^{-\lambda(\Delta+Y)}\right\} \tag{106}$$

And we have the following expansion to 2nd order in $Y$,

$$\frac{i}{2}\int_{a^2}^{\infty}\mathrm{d}\lambda\,\mathbf{tr}\left\{\begin{array}{l}\frac{1}{\lambda}e^{-\lambda\Delta}+e^{-\lambda\Delta}Y+\\ \frac{\lambda}{2}\int_0^1\mathrm{d}x\,e^{-(1-x)\lambda\Delta}Y\,e^{-x\lambda\Delta}Y+\cdots\end{array}\right\} \tag{107}$$

#### 3.2.1 Vacuum Contribution

The fermionic one-loop contribution to the vacuum term is

$$\frac{i}{2}\int_{a^2}^{\infty}\mathrm{d}\lambda\,\mathbf{tr}\left(\frac{1}{\lambda}e^{-\lambda(\partial^2+m^2)}\right) \tag{108}$$

Translating to momentum space, and taking the spinorial trace, we obtain

$$2i\int_{a^2}^{\infty}\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left(\frac{1}{\lambda}e^{-\lambda(-p^2+m^2)}\right) \tag{109}$$

Converting to Euclidean loop momentum, and integrating over the latter,

$$-\frac{1}{8\pi^2}\int_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda^3}\,e^{-\lambda m^2} \tag{110}$$

Expanding the above with respect to $a^2$, dropping the terms that vanish as $a\to 0$, we obtain

$$-\frac{1}{16\pi^2}\left\{\frac{1}{a^2}-\frac{2m^2}{a^2}+m^2\left(\frac{3}{2}-\gamma-\ln(a^2m^2)\right)\right\} \tag{111}$$

#### 3.2.2 Linear Contribution

The fermionic one-loop contribution that is linear in the graviton field is given by

$$\frac{i}{2}\int_{a^2}^{\infty}\mathrm{d}\lambda\,\mathbf{tr}\left(e^{-\lambda\Delta}Y\right) \tag{112}$$

Substituting for $\Delta$ and $Y$, manipulating the operators under the trace, and taking the spinorial trace, we obtain

$$8i\int_{a^2}^{\infty}\mathrm{d}\lambda\,\mathbf{tr}\left(e^{-\lambda(\partial^2+m^2)}h_{\mu\nu}\partial_\mu\partial_\nu\right) \tag{113}$$

Translating to momentum space, and with $h_{\mu\nu} \to \frac{1}{4}\phi_{\mu\nu}$, then symmetrizing with respect to loop momentum, we obtain

$$-\frac{i}{2}\int\limits_{a^2}^{\infty} \mathrm{d}\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left(e^{-\lambda(-p^2+m^2)} p^2\right) \phi_{\mu\mu} \tag{114}$$

Converting to Euclidean loop momentum, and integrating over the latter, we obtain

$$-\frac{1}{16\pi^2}\int\limits_{a^2}^{\infty} \frac{\mathrm{d}\lambda}{\lambda^3}\, e^{-\lambda m^2} \tag{115}$$

#### 3.2.3 The Graviton Bilinear

The one-loop fermionic contribution that is bilinear in the graviton field comes from:

$$\frac{i}{2}\int\limits_{a^2}^{\infty} \mathrm{d}\lambda\, \mathbf{tr} \left( e^{-\lambda\Delta} Y + \int\limits_0^1 \mathrm{d}x\, e^{-(1-x)\lambda\Delta} Y e^{-x\lambda\Delta} Y \right) \tag{116}$$

Substituting for $Y$, and subsequently for $h_{\mu\nu}$ in terms of $\phi_{\mu\nu}$, we obtain to second order in the latter,

$$-\frac{i}{8}\int\limits_{a^2}^{\infty} \mathrm{d}\lambda\, \mathbf{tr} \left\{ \begin{array}{l} e^{-\lambda(\partial^2+m^2)}\gamma_\mu \left((\phi^2)_{\mu\nu}\partial_\nu + \partial_\nu(\phi^2)_{\mu\nu}\right)\gamma\cdot\partial \\ +\frac{1}{4}e^{-\lambda(\partial^2+m^2)}\gamma_\mu \left(\phi_{\mu\nu}\partial_\nu + \partial_\nu\phi_{\mu\nu}\right)\gamma_\lambda \left(\phi_{\lambda\rho}\partial_\rho + \partial_\rho\phi_{\lambda\rho}\right) \\ -\frac{\lambda}{8}\int_0^1 \mathrm{d}x\, e^{-(1-x)\lambda(\partial^2+m^2)} \times \\ (\gamma\cdot\partial\gamma_\mu(\phi_{\mu\nu}\partial_\nu + \partial_\nu\phi_{\mu\nu}) + \gamma_\mu(\phi_{\mu\nu}\partial_\nu + \partial_\nu\phi_{\mu\nu})\gamma\cdot\partial) \times \\ e^{-x\lambda(\partial^2+m^2)} \left(\gamma\cdot\partial\gamma_\lambda(\phi_{\lambda\rho}\partial_\rho + \partial_\rho\phi_{\lambda\rho}) + \gamma_\lambda(\phi_{\lambda\rho}\partial_\rho + \partial_\rho\phi_{\lambda\rho})\gamma\cdot\partial\right) \end{array} \right\} \tag{117}$$

The above three terms will be computed successively.

#### 3.2.4 Computing the First Bilinear Term

We shall compute the first bilinear term

$$-\frac{i}{8}\int\limits_{a^2}^{\infty} \mathrm{d}\lambda\, \mathbf{tr} \left\{ e^{-\lambda(\partial^2+m^2)}\gamma_\mu \left((\phi^2)_{\mu\nu}\partial_\nu + \partial_\nu(\phi^2)_{\mu\nu}\right)\gamma\cdot\partial \right\} \tag{118}$$

Taking the spinorial trace, translating to momentum space, and symmetrizing over loop momentum,

$$\frac{i}{4}\int\limits_{a^2}^{\infty} \mathrm{d}\lambda \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \left\{ e^{-\lambda(-p^2+m^2)} p^2 \right\} (\phi^2)_{\mu\mu} \tag{119}$$

Converting to Euclidean loop momentum, and integrating over the latter, we obtain

$$\frac{1}{32\pi^2}\int_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda^3}e^{-\lambda m^2}\left(\phi^2\right)_{\mu\mu} \tag{120}$$

#### 3.2.5 Computing the Second Bilinear Term

We shall compute the second bilinear term

$$-\frac{i}{32}\int_{a^2}^{\infty}\mathrm{d}\lambda\,\mathbf{tr}\left\{e^{-\lambda(\partial^2+m^2)}\gamma_\mu\left(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu}\right)\gamma_\lambda\left(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho}\right)\right\} \tag{121}$$

Taking the spinorial trace, translating to momentum space, and symmetrizing with respect to loop momentum, we obtain

$$\frac{i}{8}\int_{a^2}^{\infty}\mathrm{d}\lambda\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{e^{-\lambda(-p^2+m^2)}\left(p^2\eta_{\nu\rho}+r_\nu r_\rho\right)\right\}\phi_{\mu\nu}(r)\phi_{\mu\rho}(-r) \tag{122}$$

Converting to Euclidean loop momentum, and integrating over the latter, we obtain

$$-\frac{1}{128\pi^2}\int_{a^2}^{\infty}\mathrm{d}\lambda\,e^{-\lambda m^2}\left(\frac{1}{\lambda^2}r_\nu r_\rho-\frac{2}{\lambda^3}\eta_{\nu\rho}\right)\phi_{\mu\nu}(r)\phi_{\mu\rho}(-r) \tag{123}$$

#### 3.2.6 Computing the Third Bilinear Term

We compute the third bilinear term

$$\frac{i}{64}\int_{a^2}^{\infty}\mathrm{d}\lambda\,\lambda\int_0^1\mathrm{d}x\,\mathbf{tr}\left\{\begin{array}{l} e^{-(1-x)\lambda(\partial^2+m^2)}\times \\ (\gamma\cdot\partial\gamma_\mu(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu})+\gamma_\mu(\phi_{\mu\nu}\partial_\nu+\partial_\nu\phi_{\mu\nu})\gamma\cdot\partial)\times \\ e^{-x\lambda(\partial^2+m^2)}\times \\ (\gamma\cdot\partial\gamma_\lambda(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho})+\gamma_\lambda(\phi_{\lambda\rho}\partial_\rho+\partial_\rho\phi_{\lambda\rho})\gamma\cdot\partial) \end{array}\right\} \tag{124}$$

Taking the spinorial trace, and translating to momentum space, we obtain for the coefficient of $\phi_{\mu\nu}(r)\phi_{\lambda\rho}(-r)$

$$\frac{i}{64}\int_{a^2}^{\infty}\mathrm{d}\lambda\,\lambda\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^4p}{(2\pi)^4}\left\{\begin{array}{l} e^{-\lambda\left\{-(p+xr)^2-x(1-x)r^2+m^2\right\}}\times \\ (4p_\mu p_\lambda+2p_\mu r_\lambda+2p_\lambda r_\mu+\eta_{\mu\lambda}r^2)\times \\ (2p+r)_\nu(2p+r)_\rho \end{array}\right\} \tag{125}$$

Making the shift $p \to (p - xr)$ in the loop momentum, symmetrizing with respect to the latter, then converting to Euclidean loop momentum, and integrating over the latter, then expanding to 4th order in the external graviton momentum $r$, and integrating over the Feynman parameter $x$, we obtain the result for our third bilinear term. The latter consists of three parts. The part that is independent of graviton momentum is

$$-\frac{1}{64\pi^2}\int_{a^2}^{\infty}\frac{d\lambda}{\lambda^3}e^{-\lambda m^2}\left(\eta_{\mu\lambda}\eta_{\nu\rho}+\eta_{\mu\rho}\eta_{\nu\lambda}+\eta_{\mu\nu}\eta_{\lambda\rho}\right) \tag{126}$$

The part that is quadratic in the graviton momentum is

$$\frac{1}{384\pi^2}\int_{a^2}^{\infty}\frac{d\lambda}{\lambda^2}e^{-\lambda m^2}\left\{\begin{array}{l}\eta_{\mu\lambda}r_\nu r_\rho+\eta_{\nu\lambda}r_\mu r_\rho+\eta_{\lambda\rho}r_\mu r_\nu+\eta_{\mu\nu}r_\lambda r_\rho+\eta_{\mu\rho}r_\nu r_\lambda \\ -2\eta_{\nu\rho}r_\mu r_\lambda+r^2(2\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\nu}\eta_{\lambda\rho})\end{array}\right\} \tag{127}$$

The part that is quartic in the graviton momentum is

$$-\frac{1}{3840\pi^2}\int_{a^2}^{\infty}\frac{d\lambda}{\lambda}e^{-\lambda m^2}\left\{\begin{array}{l}r^2\left(4\eta_{\mu\lambda}r_\nu r_\rho-\eta_{\nu\lambda}r_\mu r_\rho-\eta_{\lambda\rho}r_\mu r_\nu\right. \\ \left.-\eta_{\mu\nu}r_\lambda r_\rho-\eta_{\mu\rho}r_\nu r_\lambda+4\eta_{\nu\rho}r_\mu r_\lambda\right) \\ -2r_\mu r_\nu r_\lambda r_\rho-r^4\left(4\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\nu}\eta_{\lambda\rho}\right)\end{array}\right\} \tag{128}$$

#### 3.2.7 Collected Cosmological Term

The cosmological term corresponds to $\sqrt{g} = V$ in the effective action. Expanding to 2nd order in the graviton field we have

$$e^{\mathbf{tr}\ln(\eta+\frac{1}{2}\phi)} = e^{\left(\frac{1}{2}\phi-\frac{1}{8}\phi\cdot\phi\right)} \approx 1+\frac{1}{2}\phi+\frac{1}{8}\left(\phi^2-\phi_{\mu\nu}\phi_{\mu\nu}\right)+\cdots \tag{129}$$

In our foregoing fermionic loop computations, we obtained the vacuum term:

$$-\frac{1}{8\pi^2}\int_{a^2}^{\infty}\frac{d\lambda}{\lambda^3}e^{-\lambda m^2} \tag{130}$$

We obtained the linear term:

$$-\frac{1}{16\pi^2}\int_{a^2}^{\infty}\frac{d\lambda}{\lambda^3}e^{-\lambda m^2} \tag{131}$$

And obtained the (momentum-independent) bilinear:

$$-\frac{1}{64\pi^2}\int_{a^2}^{\infty}\frac{d\lambda}{\lambda^3}e^{-\lambda m^2} \tag{132}$$

Hence combining, we obtain the contribution to the cosmological term $\sqrt{g}$, with coefficient

$$-\frac{1}{8\pi^2}\int_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda^3}e^{-\lambda m^2} \tag{133}$$

Expanding the above integral with respect to $a^2$, dropping the terms that vanish as $a \to 0$, we obtain

$$-\frac{1}{16\pi^2}\left\{\frac{1}{a^4}-\frac{2m^2}{a^2}+m^4\left(\frac{3}{2}-\gamma-\ln(a^2m^2)\right)\right\} \tag{134}$$

#### 3.2.8 Collected Einstein Term

The bilinear terms of the graviton that come from the Einstein action density $\sqrt{g}g^{\mu\nu}R_{\mu\nu}$, with $R_{\mu\nu}$ the Ricci tensor, take the usual form for a massless spin-2 field:

$$\frac{1}{4}(\partial_\lambda\phi_{\mu\nu})^2-\frac{1}{4}(\partial_\mu\phi_{\nu\nu})^2+\frac{1}{2}\partial_\mu\phi_{\mu\nu}\partial_\nu\phi_{\lambda\lambda}-\frac{1}{2}\partial_\mu\phi_{\mu\lambda}\partial_\nu\phi_{\nu\lambda} \tag{135}$$

In momentum space this gives

$$\phi_{\mu\nu}(r)\phi_{\lambda\rho}(-r)\left(\frac{1}{4}r^2\eta_{\mu\lambda}\eta_{\nu\rho}-\frac{1}{4}r^2\eta_{\mu\nu}\eta_{\lambda\rho}+\frac{1}{2}r_\mu r_\nu\eta_{\lambda\rho}-\frac{1}{2}r_\mu r_\lambda\eta_{\nu\rho}\right) \tag{136}$$

The collected bilinears with quadratic momentum that we obtained in the preceding sections correspond to the above, with the coefficient:

$$\frac{1}{96\pi^2}\int_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda^2}e^{-\lambda m^2} \tag{137}$$

Expanding the above integral with respect to $a^2$, dropping the terms that vanish as $a \to 0$, we obtain

$$\frac{1}{96\pi^2}\left\{\frac{1}{a^2}-m^2\left(1-\gamma-\ln(a^2m^2)\right)\right\} \tag{138}$$

#### 3.2.9 Collected Curvature Squared Terms

For the graviton bilinear terms with quartic momentum, we obtained

$$-\frac{1}{3840\pi^2}\int_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda}e^{-\lambda m^2}\left\{\begin{array}{l} r^2\left(4\eta_{\mu\lambda}r_\nu r_\rho-\eta_{\nu\lambda}r_\mu r_\rho-\eta_{\lambda\rho}r_\mu r_\nu\right.\\ \left.-\eta_{\mu\nu}r_\lambda r_\rho-\eta_{\mu\rho}r_\nu r_\lambda+4\eta_{\nu\rho}r_\mu r_\lambda\right)\\ -2r_\mu r_\nu r_\lambda r_\rho-r^4\left(4\eta_{\mu\lambda}\eta_{\nu\rho}-\eta_{\mu\rho}\eta_{\nu\lambda}-\eta_{\mu\nu}\eta_{\lambda\rho}\right)\end{array}\right\} \tag{139}$$

multiplied by $\phi_{\mu\nu}(-r)\phi_{\lambda\rho}(r)$. In fact we can show that this corresponds to the bilinears of the following Einstein invariant action density $\sqrt{g}\left(3R_{\mu\nu}{}^2 - R^2\right)$ with coefficient

$$-\frac{1}{960\pi^2}\int\limits_{a^2}^{\infty}\frac{\mathrm{d}\lambda}{\lambda}e^{-\lambda m^2} \tag{140}$$

Expanding the above integral with respect to $a^2$, dropping the terms that vanish as $a \to 0$, we obtain

$$\frac{1}{960\pi^2}\left(\gamma + \ln(a^2m^2)\right) \tag{141}$$

### 3.3 Comparing Results

For the fermionic one-loop contribution to the cosmological term $\sqrt{g}$, we obtained in the divergence-free approach the coefficient

$$-\frac{m^4}{16\pi^2}\left(\frac{3}{2} - \ln(m^2)\right) \tag{142}$$

while in the cutoff approach, we obtained

$$-\frac{1}{16\pi^2}\left\{\frac{1}{a^4} - \frac{2m^2}{a^2} + m^4\left(\frac{3}{2} - \gamma - \ln(a^2m^2)\right)\right\} \tag{143}$$

For the Einstein term $\sqrt{g}R$, in the divergence-free approach, we obtained the coefficient

$$-\frac{m^2}{96\pi^2}\left(1 - \ln(m^2)\right) \tag{144}$$

while in the cutoff approach,

$$\frac{1}{96\pi^2}\left\{\frac{1}{a^2} - m^2\left(1 - \gamma - \ln(a^2m^2)\right)\right\} \tag{145}$$

For the curvature squared contribution $\sqrt{g}\left(3R_{\mu\nu}{}^2 - R^2\right)$, in the divergence-free approach, we obtained the coefficient

$$\frac{1}{960\pi^2}\ln(m^2) \tag{146}$$

while in the cutoff approach,

$$\frac{1}{960\pi^2}\left\{\gamma + \ln(a^2m^2)\right\} \tag{147}$$

## 4 Discussion

We have presented detailed computations of the one-loop contributions of a Dirac field in quantum electrodynamics and in quantum gravity both, in the divergence-free, and in the cutoff approaches for constructing regular, and gauge-invariant, effective action. Comparing our results shows that the divergence-free approach eliminates the quadratic and the quartic divergences (terms that diverge like $1/a^2$ and like $1/a^4$, respectively, where $a$ is the cutoff length parameter). However, the omnipresent logarithmic terms of the divergence-free approach, such as $\ln(m^2)$, always correspond to the combination $(\gamma+\ln(a^2m^2))$, where $\gamma$ is the Euler constant. This means that these logarithmic terms (characterized by arbitrary scale) of the divergence-free approach do in fact correspond to the logarithmic divergences of the cutoff approach (having a definite scale determined by the cutoff). Hence, it seems quite acceptable to do regular computations in either approach. If computations are done in the cutoff approach, then discarding the power-like divergences would seem to give us the equivalent of the divergence-free approach. Whereas this correspondence is perfect as far as one-loop computations can show, we still *need to show whether this correspondence is also perfect in higher-loop computations*.

On the other hand, we tend to believe that *a cutoff parameter must exist in a truly fundamental field theory*. This cutoff is likely to be related to the gravitational coupling. Notice, for instance, that the one-loop contributions to the Einstein term would generate a coefficient like $1/a^2$, showing that perhaps the origin of the gravitational coupling is the cutoff parameter itself. However, a better comprehension of the relationship between the gravitational coupling and the cutoff parameter can be obtained from the realization that while the effective quantum action is computed in flat spacetime, *consistency requires that any generated cosmological constant must be equated to zero.* We can easily see that computing the cosmological constant to several loop orders in quantum gravity would give us a series of the form

$$\sim \frac{1}{a^4}\left\{c_0 + c_1\frac{\kappa^2}{a^2} + c_2\frac{\kappa^4}{a^4} + \dots\right\} \tag{148}$$

Here, $\kappa$ is the gravitational coupling constant, and $a$ is the cutoff parameter, while the coefficients $c_i$ are numerical constants that *depend on the bosonic and the fermionic particle content* of the fundamental theory. Whether the above series can be put equal to zero, leading to an inverted series relating $\kappa$ and $a$, remains *to be seen in a definite fundamental unification theory.* If this idea can be successful then the problem of quantum gravity and the inherent problem of quantum field theory divergences can both be solved simultaneously and consistently.

What is needed now is to settle on a definite fundamental unification theory containing the gravitational interaction, and proceed to compute the contributions to several loop orders, all within a gauge-invariant effective action framework, and in the cutoff approach.

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