Multifractal Sets and the Dynamic Structure of the Standard Model

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Abstract

We show that the Standard Model (SM) represents a self-contained multifractal set on spacetime having arbitrarily small deviations from four-dimensionality ($D = 4 - \varepsilon,\ \ \varepsilon << 1$). All coupling charges residing on this background (gauge, Higgs and Yukawa) satisfy a closure relationship that a) tightly constrains the flavor and mass content of the SM and b) naturally solves the "hierarchy problem", without resorting to new concepts reaching beyond the physics of the SM.

Key words: Multifractals, Dimensional Reduction, Continuous Dimension, Fractional Field Theory, Standard Model, Hierarchy Problem.

1. Introduction and Motivation

The Standard Model for particle physics (SM) has been successfully tested at all accelerator facilities and is the best tool available for understanding the phenomena on the subatomic scale [1-3]. The prevailing view is that the SM represents only the low-energy limit of a more fundamental theory and that it can be consistently extrapolated to scales many orders of magnitude beyond the energy levels probed by the Large Hadron Collider. Despite its impressive performance, the SM leaves out a fairly large number of unsolved puzzles [2, 4]. We mention here three of these open questions that are relevant for the context of our work:

a) *Is the Higgs boson solely responsible for the electroweak symmetry breaking and the origin of mass?* The current view supports this assertion, although understanding of the Higgs sector remains widely open at this time [4]. There are two primary mass-generation mechanisms in the SM: the Higgs mechanism of electroweak symmetry breaking, accounting for the spectrum of massive gauge bosons and fermions, and dimensional transmutation, partially responsible for the mass of baryonic matter. While technical aspects of both mechanisms are well under control, neither one is able to uncover the origin of the electroweak scale or of the Higgs boson mass.

b) *Are fundamental parameters of the SM finely tuned?* The mass of the Higgs boson is sensitive to the physics at high energy scales. If there is no physics beyond the SM, the elementary Higgs mass parameter must be adjusted to an accuracy order of 1 part in 10^{32} in order to explain the large gap between the TeV scale and the Planck scale [2].

c) *What is the origin of quark, lepton and neutrino mass hierarchies and mixing angles?* These "flavor" parameters account for most of the basic parameters of the SM, and their pattern remains elusive. New particles at or above the TeV scale with flavor-dependent coupling charges are postulated in many scenarios, and observation of such particles would provide critical insights to these puzzles [2].

In contrast with the majority of mainstream proposals on "Beyond the SM Physics" (BSM) [5], the approach developed here exploits the idea that space-time dimensionality becomes scale-dependent near or above the low TeV scale. This conjecture has recently received considerable attention in theoretical physics and goes under several designations, from "*continuous dimension*" to "*dimensional reduction*", to *"non-integer metric"* and "*fractional field theory*" [6-9, 13-14]. The motivation for model building based on this conjecture can be also found in [6-9, 13-14].

Drawing from the idea of scale-dependent dimensionality, we show that the SM represents a *self-contained multifractal set* defined on ordinary spacetime having arbitrarily small deviations from four-dimensions $(\varepsilon << 1)$. In what follows, we refer to such spacetime as "*minimal fractal manifold*" (MFM). We find that all coupling charges residing on the MFM satisfy a closure relationship that a) fixes the flavor and mass content of the SM and b) naturally solves the hierarchy problem, without resorting to new concepts or degrees of freedom reaching beyond the physics of the SM.

The paper is organized in the following way: relevant definitions and assumptions are introduced in section 2; the modification of a generic action functional living on the MFM is detailed in section 3. The next section explores the consequences of placing classical electrodynamics of charged fermions on MFM. Expanding on these ideas, section 5 reveals how the mass and flavor content of the SM may be derived from the properties of the MFM. The ensuing multifractal structure of the SM and the proposed resolution of the hierarchy problem form the topic of sections 6 and 7. Two Appendix sections are included to make the paper self-contained.

We caution from the outset that ideas outlined here are entirely provisional. They require further consolidation and independent validation or rebuttal.

2. Definitions and assumptions

A1) Our work deals exclusively with the behavior of field theory on MFM, defined as a continuous spacetime of dimension $D = 4 \pm \varepsilon$, where $\varepsilon << 1$. This cross-over regime between $\varepsilon \neq 0$ and $\varepsilon = 0$ is the *only sensible setting* where the dynamics of interacting fields is likely to asymptotically approach all consistency requirements imposed by Quantum Field Theory (QFT) and the SM [10-11]. Large deviations from four dimensions ($\varepsilon \sim O(1)$) may signal the breakdown of these requirements. Particular attention needs to be paid, for example, to the potential violation of Lorentz invariance in Quantum Gravity theories advocating the emergence of spacetime of lower dimensionality at high energy scales [12-14].

From the standpoint of interacting field theory, a non-vanishing and arbitrarily small deviation from four dimensions is equivalent to allowing the Renormalization Group (RG) equations to slide outside the isolated fixed points solutions (FP) [15]. Recalling that FP are synonymous with equilibria in the dynamical systems theory, it follows that, in general, the evolution of quantum fields is no longer required to settle down to equilibrium states. The end result is that the condition $\varepsilon << 1$ enables the isolated FP of the RG equations to morph into attractors with a more complex structure [15-16].

A2) u_0 is the reference charge distribution on MFM for a *fixed* $\varepsilon << 1$ (fixed number of dimensions),

A3) $\bar{u}$ is the effective charge distribution on MFM when $\varepsilon << 1$ is allowed to vary (i.e., the number of dimensions is allowed to evolve with the energy scale),

A4) λ_0, g_0, $y_{0,f}$ are the coupling charges for the scalar, gauge and Yukawa sectors of the Standard Model, measured at the electroweak scale defined by M_{EW} in ordinary four dimensional spacetime ($\varepsilon = 0$).

A5) Any theory exploring physics beyond the Standard Model (BSM) must fully recover the principles and the framework of perturbative QFT at energy scales approaching M_{EW}. In particular, it needs to preserve unitarity, renormalizability and local gauge invariance and be compatible with precision electroweak data [10, 17].

3. The minimal fractal manifold (MFM)

Field theory on fractional four-dimensional spacetime is described by the action

$$S=\int_{-\infty}^{+\infty} d\rho(x)L=\int_{-\infty}^{+\infty}(\mathrm{v}(x)d^4x)L \tag{1}$$

where the measure $d\rho(x)$ denotes the ordinary four-dimensional volume element multiplied by a weight function $\mathrm{v}(x)$ [13-14]. If the weight function is factorizable in coordinates and positive semidefinite, $\mathrm{v}(x)$ assumes the form

$$\mathrm{v}(x)=\prod_{\eta=0}^{3}\frac{\left|x^{\eta}\right|^{\alpha_\eta-1}}{\Gamma(\alpha_\eta)} \tag{2}$$

in which

$$0<\alpha_\eta\leq 1 \tag{3}$$

are four independent parameters. An isotropic spacetime of dimension $D=4\pm\varepsilon$ is characterized by

$$\alpha=1\pm\varepsilon=\frac{\sum_\eta \alpha_\eta}{4} \tag{4}$$

which turns (2) into

$$\mathrm{v}(x) \approx (|x|^4)^{\pm\varepsilon} \tag{5}$$

Dimensional analysis requires all coordinates entering (2) and (5) to be scalar quantities. They can be generically specified relative to a characteristic length and time scale, as in

$$x = \frac{x_0}{L} = \frac{\mu}{\mu_0} \tag{6}$$

in which $\mu,\ \mu_0$ are positive-definite energy scales. Relation (5) becomes

$$\mathrm{v}(x) = (\frac{\mu}{\mu_0})^{\pm 4\varepsilon} \tag{7}$$

such that

$$\lim_{|x|\to 0} \mathrm{v}(x) = \begin{cases} 0,\ if\ \pm\varepsilon > 0 \\ \infty,\ if\ \pm\varepsilon < 0 \end{cases} \tag{8}$$

Choosing $\mu < \mu_0$ we can expand (7) as [18]:

$$a^{\varepsilon} = e^{\varepsilon \ln a} \approx 1 + \varepsilon \ln a \tag{9}$$

which yields

$$\mathrm{v}(x) = 1 \pm 4\varepsilon \ln(x) = 1 \pm 4\varepsilon \ln(\frac{\mu}{\mu_0}) \tag{10}$$

4. Emergence of effective field charges on the MFM

A remarkable property of fractal spacetime is the emergence of “effective” coupling charges induced by polarization in non-integer dimensions [13, 19]. To fix ideas, consider the case of classical electrodynamics coupled to spinor fields in a MFM with evolving dimensionality [13]. From (10) we obtain

$$\bar{e}^2 = \mathrm{v}(x) e_0^2 \approx \frac{e_0^2}{1 \mp 4\varepsilon \ln(\frac{\mu}{\mu_0})} \tag{11}$$

where, following definitions A2) and A3),

$$\bar{e} = \bar{u}, \quad e_0 = u_0$$

In light of assumption A5), (11) has to match the expression of the running charge in perturbative Quantum Electrodynamics (QED). At one loop, this expression reads [20]

$$e^2 = \frac{e_0^2}{1 - \frac{e_0^2}{6\pi^2} \ln(\frac{\mu}{\mu_0})} \tag{12}$$

Comparing (11) with (12) leads to:

$$\boxed{e_0^2 = O(\varepsilon)} \tag{13}$$

This finding reveals that the dimensional parameter ε represents the physical source of the field charge in ordinary four-dimensional spacetime. As previously alluded to, this "dynamic generation" of effective field charges can be traced back to the intrinsic polarization induced by fractal spacetime. The process is strikingly similar to the emergence of non-trivial fixed points in the Landau-Ginzburg-Wilson model of critical behavior in $D = 4 - \varepsilon$ dimensions [15, 18]. The discussion may be extrapolated from electrodynamics to classical gauge theory and, as we show next, it sets the stage for a novel interpretation of mass and flavor hierarchies present in the SM.

5. The mass and flavor hierarchies of the Standard Model

Analysis of the RG equations in slightly less than four-dimensions reveals that, near the electroweak scale, the normalized masses of fermions (m_f), weak bosons (M) and electroweak gauge charges (g_o) scale as [9, 15, 21]

$$m_f \sim \varepsilon \tag{14}$$

$$g_0^2 \sim \varepsilon \tag{15}$$

$$g_0^2 M^2 = const \to M^2 \sim \varepsilon^{-1} \tag{16}$$

It can be also shown that the system of RG equations lead in general to a transition to chaos via period-doubling bifurcations as $\varepsilon \to 0$ [15, 21]. The sequence of critical values ε_n, $n = 1, 2, ...$ driving this transition to chaos satisfies the geometric progression

$$\varepsilon_n - \varepsilon_\infty = \varepsilon_n - 0 \sim k_n \, \overline{\delta}^{\,-n} \tag{17}$$

Here, $n >> 1$ is the index counting the number of cycles created through the period-doubling cascade, $\overline{\delta}$ is the rate of convergence and k_n is a coefficient that becomes asymptotically independent of n as $n \to \infty$. Period-doubling cycles are characterized by $n = 2^i$, for i >> 1. Substituting (17) in (14) and (15) yields the following ladder-like progression of critical couplings

$$\boxed{m_{f,i} \sim g_{0,i}^2 \sim \overline{\delta}^{\,-2^i}} \tag{18}$$

Scaling (18) recovers the full mass and flavor content of the SM, including neutrinos, together with the coupling strengths of gauge interactions. Specifically,

- The trivial FP of the RG equations consists of the massless photon (γ) and the massless UV gluon (g).
- The non-trivial FP of the RG equations is degenerate and consists of massive quarks (q), massive charged leptons and their neutrinos (l,ν) and massive weak bosons (W, Z).
- Gauge interactions develop near the non-trivial FP and include electrodynamics, the weak interaction and the strong interaction.

6. Multifractal structure of the Standard Model

A key parameter of the RG analysis is the dimensionless ratio (μ/Λ_{UV}), in which μ is the sliding scale and $\Lambda_{UV} >> \mu$ the high-energy cutoff of the underlying theory. With reference to a field theory embedded in four dimensions $(D=4)$, the connection between the parameter $\varepsilon = 4 - D$ and Λ_{UV} is given by [15, 21-22]

$$\varepsilon \sim \frac{1}{\log(\frac{\Lambda_{UV}^2}{\mu^2})} \tag{19}$$

The large numerical disparity between μ and Λ_{UV} enables one to approximate ε as in

$$\varepsilon \sim (\frac{\mu}{\Lambda_{UV}})^2 \tag{20}$$

Let m_i denote the full spectrum of particle masses present in the SM. Relation (20) can be written as

$$\varepsilon_i = (\frac{m_i}{\Lambda_{UV}})^2 = \frac{m_i^2}{M_{EW}^2}\frac{M_{EW}^2}{\Lambda_{UV}^2} = r_i^2\varepsilon_0 \tag{21}$$

in which

$$r_i = \frac{m_i}{M_{EW}}, \quad \varepsilon_0 = \frac{M_{EW}^2}{\Lambda_{UV}^2} \tag{22}$$

and

$$r_i^2 = \frac{\varepsilon_i}{\varepsilon_0} \tag{23}$$

With reference to (b.3) of Appendix B, we find that (23) obeys a closure relationship typically associated with multifractal sets, namely [22]:

$$\boxed{\sum_i r_i^2 = \sum_i (\frac{m_i}{M_{EW}})^2 = 1} \tag{24}$$

in which the sum in the left-hand side extends over all SM fermions (leptons and quarks).

The sum-rule (24) may be alternatively cast in terms of SM field charges. We obtain

$$2\lambda_0 + \frac{g_0^2}{4} + \frac{g_0^2 + (g_0\,')^2}{4} + \sum_{l,q}\frac{y_{0,f}^2}{2} = 1 \tag{25}$$

where

$$\lambda_0 = \frac{(u_0)_{scalar}}{\varepsilon_0}$$

$$g_0^2 = \frac{(u_0)_{gauge}}{\varepsilon_0}$$

$$g'^2_0 = \frac{(u'_0)_{gauge}}{\varepsilon_0}$$

From either (24) or (25) one derives

$$\boxed{M_{EW} \sim V = 246.2 \ GeV} \tag{26}$$

in close agreement with the vacuum expectation value of the SM Higgs boson (V). In closing, we mention that the existence of (25) was first brought up in [24], with no attempt of formulating a theoretical interpretation. It is instructive to note that the power exponent $D_H = 2$ entering (24) is identical with the Hausdorff dimension of both random walks and quantum-mechanical paths [18, 25-26].

7. Discussion: solving the flavor and hierarchy problems on the MFM

Relations (18), (24) and (25) tightly constrain the particle content of the SM. They naturally fix its number of independent field flavors near the electroweak scale. Also, since all scaling ratios in (24) must have a magnitude of less than one unit, (24) and (25) necessarily imply that the mass of the Higgs boson cannot grow beyond M_{EW}, at least near the electroweak scale. This conclusion brings closure to the *hierarchy problem*, whose formulation is briefly outlined in Appendix A.

Appendix A: the Hierarchy Problem

Electroweak (EW) symmetry in the SM is broken by a scalar field having the following doublet structure [23]:

$$\Phi = \begin{pmatrix} G^{+} \\ 1/\sqrt{2}[(H+\mathrm{V})+iG^{0}] \end{pmatrix} \tag{a.1}$$

Here, G^{+} and G^{0} represent the charged and neutral Goldstone bosons arisen from spontaneous symmetry breaking, H is the SM Higgs boson, $\mathrm{V} \approx M_{EW} = 246\, GeV$ is the Higgs vacuum expectation value. Symmetry breaking is caused by the Higgs potential, whose form satisfies the requirements of renormalizability and gauge-invariance:

$$V(\Phi,\Phi^{\div}) = \mu_H^2 \Phi^{+}\Phi + \lambda_0 (\Phi^{+}\Phi)^2 \tag{a.2}$$

with $\lambda_0 \approx O(1)$ and $\mu_H^2 \approx O(M_{EW}^2)$. A vanishing quartic coupling ($\lambda_0 = 0$) represents the critical value that separates the ordinary EW phase from an unphysical phase where the Higgs field assumes unbounded values. Likewise, the coefficient μ_H^2 plays the role of an order parameter whose sign describes the transition between a symmetric phase and a broken phase. Minimizing the Higgs potential yields an expectation value given by:

$$\mathrm{V}^2 = -(\mu_H^2/\lambda_0) \tag{a.3}$$

where the physical mass of the Higgs is:

$$M_H^2 = 2\lambda_0 \mathrm{V}^2 = -2\mu_H^2 \tag{a.4}$$

The renormalized mass squared of the Higgs scalar contains two contributions:

$$\mu_H^2 = \mu_{0,H}^2 + \Delta\mu^2 \tag{a.5}$$

in which $\mu_{0,H}^2$ represents the ultraviolet (bare) value. This mass parameter picks up quantum corrections $\Delta\mu^2$ that depend quadratically on the ultraviolet cutoff Λ_{UV} of the theory. Consider for example the contribution of radiative corrections to μ_H^2 from top quarks. The complete one-loop calculation of this contribution reads:

$$\Delta\mu^2 = \frac{N_c\lambda_t^2}{16\pi^2}[-2\Lambda_{UV}^2 + 6M_t^2\ln(\frac{\Lambda_{UV}}{M_t}) + ...] \tag{a.6}$$

in which λ_t and M_t are the Yukawa coupling and mass of the top quark. If the bare Higgs mass is set near the cutoff $\mu_{0,H}^2 = O(\Lambda^2) = O(M_{Pl}^2)$, then $\Delta\mu^2 \approx -10^{35}\,\text{GeV}^2$. This large correction must precisely cancel against $\mu_{0,H}^2$ to protect the EW scale. This is the root cause of the hierarchy problem, which boils down to the implausible requirement that $\mu_{0,H}^2$ and $\Delta\mu^2$ should offset each other to about 32 decimal places.

Appendix B: A primer of fractals and multifractals

We highlight here few basic concepts and terminology pertaining to fractals and multi-fractals. Fractals are geometrical objects with non-integer dimensions that display self-similarity on all scales of observation [18]. The concept of *dimension* plays a key role in the geometry of fractal sets. It is customary to characterize fractals by an ensemble of three dimensions, namely:

1) The Euclidean dimension "$D = 1, 2, 3...$" represents the dimension of the space where the object resides and is always an integer.

2) The topological dimension "$d_T \leq D$" describes the dimensionality of continuous primitive objects such as points, curves, surfaces or volumes ($d_T = 0, 1, 2, 3$ in ordinary four-dimensional spacetime).

3) The definition of the fractal (or *Hausdorff*) dimension is as follows: Cover the fractal object by d – dimensional balls of radius "Δ" and let "$N(\Delta)$" be the minimum number of balls needed for this operation. The fractal dimension "D_H" satisfies the inequality $d_T \leq D_H \leq D$ and is given by

$$\lim_{\Delta \to 0} N(\Delta) = \Delta^{-D_H} \tag{b.1}$$

leading to

$$D_H = \lim_{\Delta \to 0} [\frac{\log N(\Delta)}{\log \Delta^{-1}}] \tag{b.2}$$

Many of the self-similar structures in fractal geometry are built recursively, a typical example being the *Cantor set*. To construct a Cantor set in one dimension ($D = 1$), take a line segment called the *generator*, split it into thirds and remove the middle third. Iterate this process arbitrarily many times. One is left with a countable set of isolated points having a non-integer fractal dimension D_H, with $d_T = 0 \leq D_H \leq D = 1$. A simple Cantor set generated from segments of equal length is defined by a single scaling factor $r = 1/3 < 1$. By contrast, more general fractals (such as *multifractals*) can be created using generator segments of different scaling factors $r_i < 1$, $i = 1, 2, ..., N$ satisfying the closure relation

$$\sum_{i=1}^{N} r_i^{D_H} = 1 \tag{b.3}$$

Many strange attractors of nonlinear dynamical systems represent multifractals and are typically characterized by a continuous spectrum of Hausdorff dimensions [18].

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