A proof of nonexistence of Green's functions for the Maxwell equations

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Abstract

Arguments in favor of existence of Green's functions for all linear equations are analyzed. In case of equation for electromagnetic field, these arguments have been widely used through formal considerations according to which electromagnetic field equations are nothing but some non-covariant scalar equations. We criticize these considerations and show that justification of applying the method of Green's functions to equations of classical electrodynamics are invalid. Straightforward calculations are presented which show that in case of dipole radiation the method gives incorrect results.

1 Introduction

Existence of Green's functions for all linear differential equations was always believed to be something evident. In physical theories, Green's functions appear as fields of point-like sources that leaves no questions of their existence as well as of that of point-like sources themselves. Coulomb's law combined with the linear superposition principle is the most clear demonstration of the method of Green's functions. Thus, existence of Green's functions for physical fields is a straightforward consequence of linearity and existence of point-like sources. Another reason why they must exist is found in quantum field theory where they are identified with propagators of quantum particles, which are defined as amplitudes of probability that a particle emitted in one space-time point reaches another. Since there is no question of existence of amplitude of such a phenomenon, there is no question of existence of the corresponding propagators. However, identity of propagators and Green's has been proved only for scalar fields. Proofs of this identity for other fields are based on an assumption that Green'functions of all the rest fields is nothing but that for a scalar field multiplied by a constant factor, that needs another proof.

In this work we show that in fact, no Green's functions for the Maxwell equations exist. This fact poses a question of equivalence of Green's function and particle propagator for non-scalar fields which will not be considered here. Below we only consider Maxwell equations and show non-existence of Green's functions for them. The present work is organized as follows. We start with three-dimensional version of the equation for the vector potential known in magnetostatics and show that this equation has no Green's function referring to a more rigorous proof presented in our preprint [1]. Then we pass to the space-time version of this equation and repeat our considerations which show that Green's functions for electromagnetic field do not exist at all. This conclusion is then verified by straightforward calculation in case of oscillating point-like dipole. Point-like oscillating dipole is chosen as the simplest non-stationary source of electromagnetic field to which the method of Green's functions can easily be applied, besides, there exist numerous other sources for which the fact that the method gives erroneous results. In fact, the method gives erroneous results in all non-trivial cases, for example, for a point-like charge [2] and though it suffices to present one example to show that the method is wrong, we present another example. Since there is no reason to conduct calculations for all possible sources of non-stationary electromagnetic field to see that it is so, we find fit to claim that Maxwell equations have no Green's function.

2 Non-existence of Green's function for the main equation of magnetostatics

Recently, we have presented a proof of non-existence of Green's function for the well-known equation of magnetostatics [1]

$$\nabla \times \nabla \times \vec{A} = \vec{I}. \tag{1}$$

The proof is based on the fact that point-like sources of magnetic fields cannot be represented in the form of current density $\vec{I}$ required by the equation. Point-like magnetic dipoles can be can be represented this way, but, first, explicit form of their fields has nothing in common with Coulomb potential as expected and second, it is impossible to compose any continuous current density of point-like magnetic dipoles. Hence, point-like dipole should be kept apart from all considerations related to the question of existence of Green's functions for the equation (1). By a point-like source of magnetic field one usually means so-called "element of current" defined as an infinitesimally short segment carrying electric current. However, such a "source" of magnetic field cannot me used in this capacity for the following reason.

An immediate consequence of the equation (1) reads that current density $\vec{I}$ satisfies the charge conservation law in the form

$$\nabla \cdot \vec{I} = 0 \tag{2}$$

which serves as integrability condition for it. In other words, if the right-hand side of this equation has non-zero divergence, it has no solutions and thus is meaningless. Nevertheless, whatever the right-hand side of this equation will be, the method of Green's function can be applied to it that gives some plausible formal solution which, evidently, has no meaning. Since, on one hand, the equation is meaningful if and only if its right-hand side satisfies the equation (2) and on the other hand, formal solutions obtained by the method of Green's functions are meaningless, a special attention should be paid to selection of point-like sources to underlie the method. This touches, first of all, the notion of "element of current" used in this capacity.

Element of current is defined as the limit of length tending to zero of a straight segment carrying direct current. Evidently, passage to the limit does not affect the main properties of this “source” of magnetic field, therefore it suffices to consider hypothetic physical properties of this object. A segment of straight line currying direct current is possible, but it must be taken into account that its endpoints accumulate electric charges of opposite sign. Electric field of these growing charges in the endpoints of the segment should also be taken into account. Since this field is non-stationary, it produces some magnetic field that also should be taken unto account. Thus, commonplace considerations read that first, a segment carrying direct current produces magnetic field, second, linearly growing point-like charges at its endpoints produce non-stationary electric field and third, this electric field also produces some magnetic field in accord with Maxwell equations. The endpoints disappear when attaching segments to each other and so do all effects related to their endpoints, but the notion of Green function is based on considerations of the field of a segment alone as if it existed independently. Therefore, first one should find it completely, then cancel endpoints along with their fields when joining segments together. In other words, entire field of an element of current should be obtained first, then it should be shown that this field can be divided into stationary and non-stationary parts and finally, if the field is divided, show that one of its parts can be associated with the direct current in the segment. Nothing of this sort has ever been made and most probably, cannot be done. What was done is an assumption that such a part exists and all the rest disappears when joining endpoints of two segments.

The only way to construct the field of two opposite growing point-like charges and direct current in a straight conductor between them known to majority of physicists, is the same method of Green's functions, but in the space-time. Note that the problem in question is stationary by its nature, hence expected to be solved in the frame of magnetostatics with (1) as the master equation. In this frame one encounters only direct current from one end of a segment to another that contradicts the charge conservation law and thereby makes the master equation meaningless. In other words, this “source” of magnetic field cannot be used as an underlying idea of the method of Green's functions. As such, usage of the method in magnetostatics is without foundation and it remains to check out whether it gives wrong results. This verification have been completed in our work [1] where it was shown that in one of the simplest cases of V-shaped wire the integral of the Green function can be calculated and the expression obtained gives an erroneous representation of the field in question. Consequently, the equation (1) cannot be solved by the method of Green's functions, or, in other words, Green's function for this equation does not exist.

This conclusion sounds somewhat strange after widely adopted considerations [3] which formally justify usage of the method. According to these considerations, it suffices to choose a special gauge $\nabla \cdot \vec{A} = 0$ to turn the equation (1) into the equation

$$\triangle \vec{A} = \vec{I}. \tag{3}$$

It leaps the eyes that this equation is not covariant because the operator $\triangle$ has covariant definition only for scalars. Moreover, components of vectors as functions of coordinates depend not only on the shape of the vector field but also on choice of the field of local frames to which the vector is referred. In other words, this equation can only be right in Cartesian coordinates and breaks in all the rest coordinate systems whereas the only covariant equation for the vector potential is the original equation (1). Thus, the only justification for employing the method of Green's functions in magnetostatics, is substituting a non-covariant equation for the original one that cannot be justified itself. And finally, straightforward verification shows that application of the method gives erroneous results [1]. Thus, we claim that Green's function for the master equation of magnetostatics does not exist.

3 Non-existence of Green's function for electromagnetic field

Electromagnetic field is known to obey Maxwell equations in inertial frames, but they are not covariant because passage to accelerated frames changes their form [2]. Even if they are written in terms of vector analysis with operator ∇, they cannot be made manifestly covariant because this operator is attached to an inertial frame of reference. A manifestly covariant form of equations for electromagnetic field needs another mathematical toolkit which is exterior calculus. In terms of this calculus the equations have the form of single equation

$$\mathrm{d}^*\mathrm{d}\alpha = I, \tag{4}$$

where 1-form α stands for the vector potential. In this section we consider the question of existence of Green's function for this equation.

An immediate consequence of the field equation is the equation

$$\mathrm{d}I = 0 \tag{5}$$

which has the same meaning as the equation (2) and expresses electric charge conservation law. Again, this equation serves also as integrability condition for the master equations (4) so that the latter is meaningless if its right-hand side does not satisfy it. All the rest considerations are similar to that conducted above when discussing this problem which appeared first in magnetostatics. The generally accepted approach to it is the same, first to justify somehow that the equation (4) is identic to a scalar equation, then to use Green function for it. For this end, so-called Lorentz gauge $\mathrm{d}^*\alpha = 0$ was introduced which justifies some obfuscating way that the field equation (4) is the same as the D'Alembert equation for Cartesian components of the vector potential

$$\Box A_i = J_i.$$

Again, the way lies through turning electrodynamics into a non-covariant theory and all the rest problems remain.

If Green's function for the equation exists, it provides formal solution for any right-hand side disregard of its properties, thus, in general, solutions which do not exist when the current density does not satisfy the equation (5). However, the commonplace belief that it exists, needed to be verified first. As before, the basic idea which underlies this belief, is that of existence of point-like sources of electromagnetic field, which in the space-time are events which have certain place and time for example, certain points or infinitesimal segments of a particle world line. The idea of "element of current" used in magnetostatics for this end, was spread to the space-time and used the same way and naturally, brought the same error to electrodynamics.

Indeed, "element of current" as it was defined in the space-time, does not satisfy the equation (5), hence, does not exist physically and is meaningless mathematically. To show this, we consider "element of current" of a more general nature. Let I be a current density which satisfies the condition, B – a domain in the space-time and ∂B – its boundary. The part of the source confined inside of B turns into a point-like source under reducing B to a point. Therefore, it suffices to establish whether the confined part of the source satisfies the condition. For this end, we introduce a step function θ which is equal to unity inside and zero outside of B. Its gradient $\delta \equiv \mathrm{d}\theta$ possesses a δ-function singularity on ∂B and is orthogonal to this 3-surface everywhere. Evidently, the confined part of the source has the form θI and does not satisfy the condition (5):

$$\mathrm{d}(\theta I) = \delta \wedge I.$$

This 4-form has δ-function singularity on the surface ∂B that signifies that if the surface cuts the current lines, it must serve as a source or drain of charge. In any case, since a part of the source extracted this way, breaks the condition (5), so does any point-like source, consequently, foundations of the method of Green's functions are incompatible with the integrability condition of the field equation and application of the method of Green's functions to the equation (4) is without foundation. Now, to show this, it remains to present an example of an apparently wrong formal solution of this form.

Such an example is given by the well-studied case of oscillating electric dipole. Intensities of the field obtained by the method of Green's functions are presented by the equation (9.18) on the page 271 of the book [3]. Magnetic intensity $\vec{H}$ is purely azimuthal, thus, its only non-zero component is H^φ. Electric intensity of the field is given as $\vec{E} = \vec{H} \times \vec{n}$, where $\vec{n}$ is unit radial vector, $\vec{n} = \partial_r$ in spherical coordinates $\{t, r, \theta, \varphi\}$. Consequently, both of intensities are tangent to the spheres $r = \text{const}$ and orthogonal to each other, hence, $\vec{E}$ has single non-zero component E^θ. This fact alone signifies that $\nabla\vec{E} \neq 0$. In other words, this field cannot be produced by an oscillating dipole, but corresponds to a non-zero charge density. Let us find the charge density in the space needed to produce the field presented in the book [3] as if it was source-free.

The 1-form of electric strength is $E = f(t, r, \theta)\mathrm{d}\theta$ with some non-zero θ-component $f(t, r, \theta)$. Then, the corresponding induction is $\Delta = {}^*E = f \sin\theta\, \mathrm{d}\varphi \wedge \mathrm{d}r$ and its exterior derivative $\mathrm{d}\Delta = \rho$ is equal to the charge density which represents the source of the electric field:

$$\rho = \mathrm{d}\Delta = \frac{\partial f \sin\theta}{\partial\theta}\, \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}\varphi \neq 0.$$

This 3-form is non-zero for any continuous electric strength E which has the only non-zero component E^φ. The result obtained shows that solutions of the field equation (4) cannot be obtained by the method of Green's functions or, in other words, this equation has no Green's functions.

4 Conclusion

Linearity of a differential equation was always believed to be sufficient condition of existence of Green's function for it. In physics, one encounters a broad variety of partial differential equations, mainly linear or linearized, so the method of Green's function could well be widely used. Early physical theories which describe scalar potentials of gravitational and electrostatic fields or propagation of heat and elastic waves, led to scalar equations. However, this method was actually of no use.The point is that scalar covariant equations can be solved by the method of variables separation that provides complete solutions in form of Hilbert spaces, whereas first, Green's give only particular solutions and second, as a rule, to find them, one has to take complicated integrals that often is impossible.

Scalar equations have scalar right-hand sides so there is no question of existence of point-like sources and their fields, correspondingly, a question of existence of Green's functions for them never arose. Appearance of vector fields, particularly, as vector potentials for magnetic and electromagnetic fields, would make essential changes in views upon mathematical tools used in physics. Instead, theoretical physicists tried to reduce Maxwell equations to scalar equations studied before and employ methods which are valid only for scalar equations. For this end, they sacrificed covariance and confined theory of electromagnetic field within only Cartesian coordinates. This restriction allowed them actually to replace a vector potential with three or four scalars called its components and employ commonplace scalar mathematics when describing electromagnetic fields. All the rest was just a technical task. They successively introduced point-like source of electromagnetic field called "element of current" and as a result, we have electrodynamics with retarded potentials, photon propagator in sense of R. P. Feynman and its generalizations created in gauge theories. Now it turn out that Green's functions for the equation for electromagnetic field does not exist because there is difference between scalar and non-scalar fields as well as between the corresponding equations.

References

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