Introducing Spin

A New Field of Mathematical Physics

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Abstract

This paper introduces spin - a new field of mathematics that gives us a deeper look into the quantum world. Euler did much of the foundational work in this field, without knowing that it applied to photons, electrons, sound and more.

Many elements of spin were already in our number system: the power series (for e^x, $\cos x$, and $\sin x$) are generated by spin equations, and the Schrodinger equation is a spin equation.

Spin would have been discovered earlier but there are some misconceptions in accepted math and quantum physics - primarily because assumptions were made about imaginary numbers that turned out not to be true.

Spin is based on newly discovered mathematical structures called unitary circles. They help us redefine the imaginary number system, and they give us a more fundamental way to look at the world.

1 A New Conception of Imaginary Numbers

1.1 Background

Imaginary numbers were first introduced into our number system with only a vague and ambiguous definition ($i = \sqrt{-1}$). But what are imaginary numbers? Imaginary numbers can not be found on the real number line, so where do they "live"? How do they interface with real numbers?

Those unanswered questions made Descartes uncomfortable with the constant i (sometimes called j). Real numbers, he said, represent real objects, and in his book *La Géométrie* in 1637, he disparagingly coined the term "imaginary numbers", suggesting they didn't represent anything at all.

Ironically, the diagrams used today to represent imaginary numbers are based on Descartes' Cartesian coordinate system. Argand diagrams use the x axis to represent real numbers, and the y axis to represent imaginary numbers. Since the coordinate points on these diagrams have both a real component and an imaginary component, these numbers are said to be complex. Similarly, the intersection of the real and imaginary axes are sometimes referred to as the complex plane.

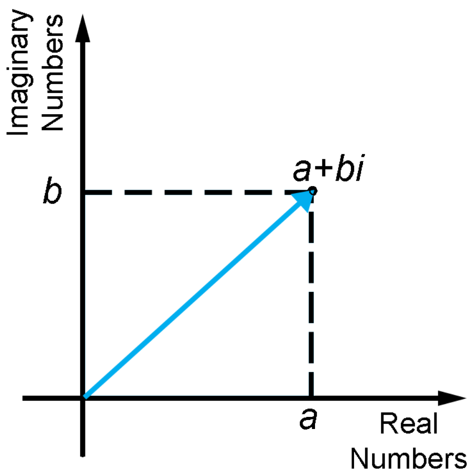


FIGURE 1: Argand Diagram

1.2 Competing Definitions of i

Early in the 20th century, it was discovered that i was a constant of nature - a real application was found for imaginary numbers! However today, there are still at least three different competing definitions/conceptions of i:

1. One view says i is defined as a shorthand symbol for working with negative square roots - and nothing more.
2. Another view (described above in the background section) is that i is a constant of nature, and that imaginary numbers form a linear dimension that intersects the real number line.
3. A different view is that i is a constant of nature, but it is limited to one of four points on a circle.

If we want the consistency and certainty of our real number system to extend to include imaginary numbers, then we need to know precisely and unambiguously how imaginary numbers relate to real numbers. Imaginary numbers as they are currently defined are invalid - they are vague and there is no mathematical bridge explicitly linking real and imaginary terms.

We need to know conceptually and mathematically how imaginary and real numbers relate. Yet the constant i is not decipherable and there are no mathematical operations or properties that allow us to relate imaginary numbers to real numbers. Without a way to relate real and imaginary numbers, terms such as $3i$ are mathematically meaningless - and not valid numbers.

All we know is that an imaginary number squared (whatever that means) yields -1.

1.3 Bridging Real and Imaginary Numbers

Conceptually i is a quarter of a turn on a circle. This concept is not new, but now it has evolved into the foundation for the property of spin.

Spin equations are complex and they generate answers that contain a mixture of real and imaginary terms. When a spin equation is "paired" with its complex conjugate, the complex terms in the answer always appear with their conjugate and cancel out. This means "pairing" a complex spin equation with its conjugate yields an all real answer. This is the mathematical foundation linking imaginary and real numbers.

Note that we still have no way to relate i with a real number. But we don't need to know how i directly relates to real numbers, because we know how to combine two imaginary numbers to get a real number!

2 Imaginary Number System

2.1 Unitary Circles

Unitary circles model the property of spin. They start at the origin with a value of 1, and spin around (a full revolution) in one, two, or four steps. What makes them all unitary circles is the characteristic that the step unit, raised to the number of steps, equals one.

Types of Unitary Circles

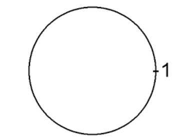
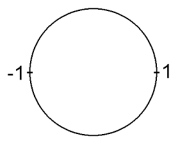
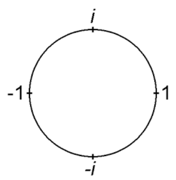
Number of Steps	Unitary Circle	Step Unit	Unitary Formula
1		1	$1^1 = 1$
2		-1	$-1^2 = 1$
4		i	$i^4 = 1$

FIGURE 2: Unitary circles

The last circle, with a step unit of i, is called the imaginary number circle.

2.2 Imaginary Number Circle

The imaginary number circle is a unitary circle. It has the property of spin, and i is the step unit. It can be graphically represented with the expressions i^n and i^{-n}.

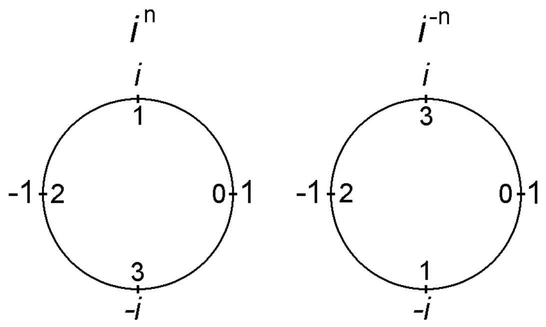


FIGURE 3: The imaginary number circle

The expression i^n starts at the origin (with a step number of 0), and steps around the circle in a counterclockwise direction. This endless looping process is spin. Similarly, the expression i^{-n} starts at the origin, but steps around the circle in a clockwise direction.

2.3 Complex Conjugates

The expressions i^n and i^{-n} generate both real and complex terms. It is therefore problematic to use either one of these expressions in formulas, because real world problems can not be solved with imaginary solutions (there is no way to interpret the result).

However i^n and i^{-n} are complex conjugates. The real terms add together and the complex terms cancel out. We can see in the table below that no matter what value n has, $i^n + i^{-n}$ always yields a real value (either 2, 0, or -2).

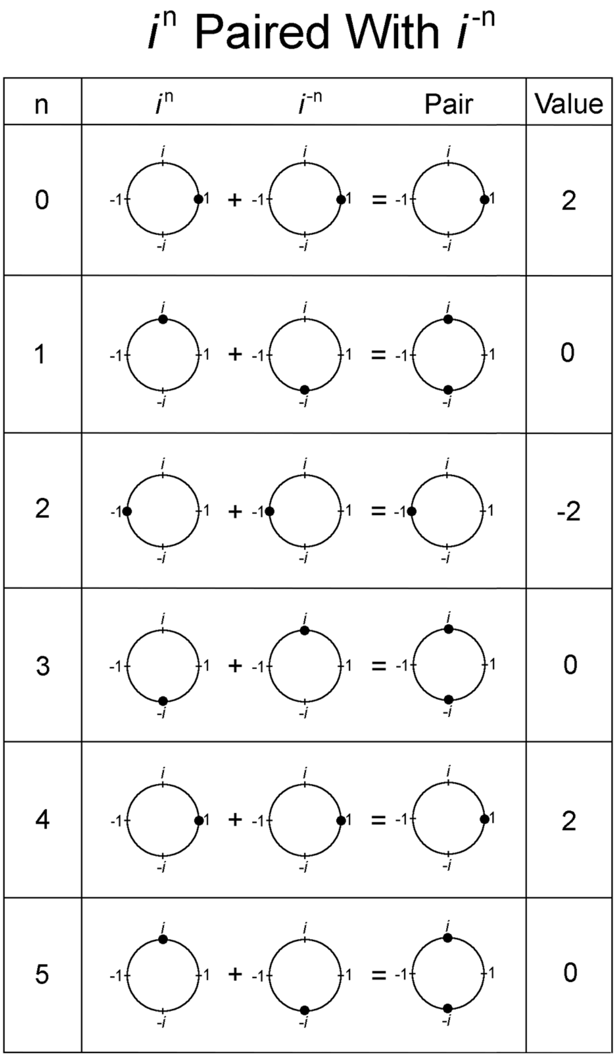


Figure 4: Complex conjugates paired together

2.4 Fundamental Spin Equations

The exponential functions e^i, e^x, and e^{ix} represent the three basic types of spin: imaginary, real, and complex.

$$e^i = \sum_{n=0}^{\infty} i^n \tag{1}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} \tag{3}$$

These elements of spin are used in the cosine and sine of x functions.

$$\cos(x) = \sum_{n=0}^{\infty} \left(\frac{i^n x^n}{n!} + \frac{i^{-n} x^n}{n!} \right) /2 \tag{4}$$

$$\sin(x) = \sum_{n=0}^{\infty} \left(\frac{i^n x^n}{n!} + \frac{i^{-n} x^n}{n!} \right) / -2i \tag{5}$$

2.5 Euler's Spin Equations

Leonard Euler discovered the infinite series that represented $\cos(x)$, $\sin(x)$, and e^x functions.

$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \tag{6}$$

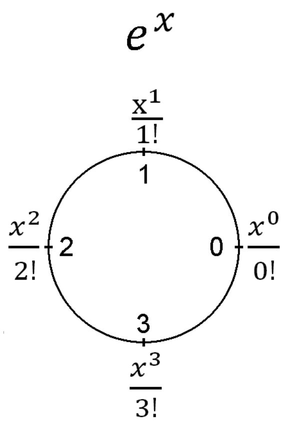
$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{1}1}{11!} + \dots \tag{7}$$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \tag{8}$$

He also came up with the formula for e^x.

$$e^x = \sum_{i=o}^{\infty} \frac{x^n}{n!} \tag{9}$$

e^x can be graphically represented by starting at the origin ($n = 0$) and stepping counter clockwise around the circle (spinning). With each step, a new term in the series is created.



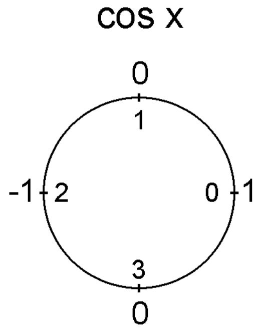
n	$x^n/n!$
0	$x^0/0!$
1	$x^1/1!$
2	$x^2/2!$
3	$x^3/3!$
4	$x^4/4!$
5	$x^5/5!$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \tag{10}$$

Cosine function

The cosine and sine functions are very similar to the function e^x. They both (cosine and sine) start at the origin and step counterclockwise around the circle. What is different is that they use a mask for determining the value of the term. The mask times the term is the value that appears in the series.

$$TermValue_a = (i^n + i^{-n})/2 \times \frac{x^a}{a!} \tag{11}$$



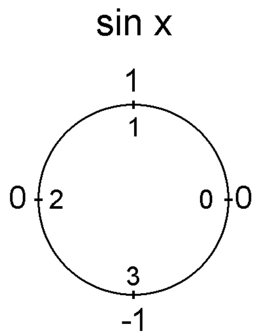
n	x	Cosine Mask $(i^n + i^{-n})/2$	Term $x^n/n!$
0	0	1	$x^0/0!$
1	$1\tau/4$	0	$x^1/1!$
2	$2\tau/4$	-1	$x^2/2!$
3	$3\tau/4$	0	$x^3/3!$
4	$4\tau/4$	1	$x^4/4!$
5	$5\tau/4$	0	$x^5/5!$
6	$6\tau/4$	-1	$x^6/6!$

$$\cos x = \sum_{n=0}^{\infty} \left(\frac{i^n x^n}{n!} + \frac{i^{-n} x^n}{n!} \right) /2 = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots \tag{12}$$

Sine function

Similarly, determining each value of the $\sin x$ series also involves using a mask. The value for any term a in the $\sin x$ power series, is equal to the sine mask times the $x^a/a!$ term.

$$TermValue_a = (i^n + i^{-n})/(-2i) \times \frac{x^a}{a!} \tag{13}$$



n	x	Sine Mask $(i^n + i^{-n})/(-2i)$	Term $x^n/n!$
0	0	0	$x^0/0!$
1	$1\tau/4$	1	$x^1/1!$
2	$2\tau/4$	0	$x^2/2!$
3	$3\tau/4$	-1	$x^3/3!$
4	$4\tau/4$	0	$x^4/4!$
5	$5\tau/4$	1	$x^5/5!$
6	$6\tau/4$	0	$x^6/6!$

$$\sin x = \sum_{n=0}^{\infty} \left(\frac{i^n x^n}{n!} + \frac{i^{-n} x^n}{n!} \right) /(-2i) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots \tag{14}$$

2.6 Derivatives

Spin functions (such as e^x and $\cos x$) evolve with steps that are derivatives. The derivative of the e^x function is calculated by taking the derivative of each of the terms in the series. Note that the derivative of e^x is simply e^x.

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \tag{15}$$

$$\frac{d}{dx}e^x = 0 + \frac{1x^0}{1!} + \frac{2x^1}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \dots \tag{16}$$

$$\frac{d}{dx}e^x = 0 + \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{4!} + \frac{x^4}{4!} + \dots = e^x \tag{17}$$

$$\frac{d}{dx}e^x = e^x \tag{18}$$

Similarly, the derivative of the cos(x) function is calculated by taking the derivative of each of the individual terms. One derivative of the cosine function (sin) is a quarter of a turn. Four derivatives (or four quarter turns) returns you to the origin (to the cos function).

$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \tag{19}$$

$$\frac{d}{dx}\cos x = 0 - \frac{2x^1}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} - \frac{10x^9}{10!} + \dots \tag{20}$$

$$= 0 - \frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \dots = -\sin x \tag{21}$$

$$\frac{d^2}{dx^2}\cos x = 0 - \frac{1x^0}{1!} + \frac{3x^2}{3!} - \frac{5x^4}{5!} + \frac{7x^6}{7!} - \frac{9x^8}{9!} \dots \tag{22}$$

$$= 0 - \frac{x^0}{0!} + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \dots = -\cos x \tag{23}$$

$$\frac{d^3}{dx^3}\cos x = 0 - 0 + \frac{2x^1}{2!} - \frac{4x^3}{4!} + \frac{6x^5}{6!} - \frac{8x^7}{8!} \dots \tag{24}$$

$$= 0 - 0 + \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sin x \tag{25}$$

$$\frac{d^4}{dx^4}\cos x = 0 - 0 + \frac{1x^0}{1!} - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} \dots \tag{26}$$

$$= 0 - 0 + \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \dots = \cos x \tag{27}$$

This illustrates that the cos(x) derivatives loop endlessly every four steps - a characteristic of spin exhibited also by i^n.

3 Complex Spin Functions

Euler knew much about how the e^x, $\cos(x)$ and $\sin(x)$ functions integrated with imaginary numbers. Below is his elegant proof showing how they are related.

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \tag{28}$$

$$e^{ix} = \frac{i^0x^0}{0!} + \frac{i^1x^1}{1!} + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \dots \tag{29}$$

$$e^{ix} = \frac{x^0}{0!} + \frac{ix^1}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \dots \tag{30}$$

$$e^{ix} = \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + \left(\frac{ix^1}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \dots\right) \tag{31}$$

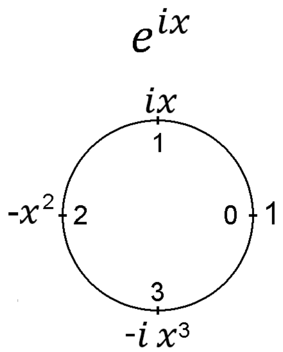
$$e^{ix} = \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \tag{32}$$

$$e^{ix} = \cos x + i \sin x \tag{33}$$

Formula 29 shows x replaced with ix. Formula 30 is a simplification of the imaginary terms. Formula 31 groups the terms with even and odd exponents. Formula 32 factors out an i from the the odd exponents group.

Remarkably, the terms grouped within parentheses in formula 32, are the power series for the $\cos x$ and $\sin x$ respectively. Formula 33 simply substitutes the terms "$\cos x$" and "$\sin x$" for the respective series.

Since e^{ix} is a spin equation, we can graph it and look at the table values to better understand its properties.



n	x	$\sin x$	$i\sin x$	$\cos x$	e^{ix}
0	0	0	0	1	1
1	$1/4\tau$	1	i	0	i
2	$2/4\tau$	0	0	-1	-1
3	$3/4\tau$	-1	$-i$	0	$-i$
4	$4/4\tau$	0	0	1	1
5	$5/4\tau$	1	i	0	i

$$e^{ix} = \cos x + i\sin x \tag{34}$$

The table for e^{ix} is similar to previous tables except there is a new "$i\sin x$" column which is i times the "$\sin x$" column. The last column (e^{ix}), is simply "$i\sin x$+$\cos x$".

The equation e^{ix} starts at the origin ($n = 0$) and steps counterclockwise around the circle. Note that every other term generated by this equation is imaginary (this can be seen by looking under the e^{ix} column in the table above). So before using this equation it must be paired with its complex conjugate (e^{-ix}) - to cancel out the imaginary terms.

3.1 Euler's' Identity

Starting with the complex Euler equation (formula 35), if you substitute π for x and simplify, you get an equation referred to as Euler's identity (shown in formula 38).

$$\begin{aligned} e^{ix} &= \cos x + i \sin x & (35) \\ e^{i\pi} &= \cos \pi + i \sin \pi & (36) \\ e^{i\pi} &= -1 + 0 & (37) \\ e^{i\pi} + 1 &= 0 & (38) \end{aligned}$$

It is a famous identity because it integrates three mathematical constants: e, i, and π. The physicist Richard Feynman called Euler's identity "the most remarkable formula in mathematics". In 1988, a survey by the Mathematical Intelligencer reported that its readers voted this equation the "Most beautiful mathematical formula ever".

Mathematicians appreciated the elegance of this identity, even though its meaning was unclear. Now we know that this identity is really a spin equation, and it is easy to understand. If we look back at the e^{ix} table (used with formula 34) we can see that e^{ix} is equal to -1 when $x = \pi$.

Or we can look at this equation using just spin concepts. We know this equation spins π radians (or one half of a revolution), and we know the value at that point is -1. Though technically this identity is not valid - it must first be paired with its complex conjugate.

3.2 Euler's Formula Completed

To eliminate the imaginary terms generated by the spin equation e^{ix}, we need to pair it with its complex conjugate e^{-ix}. Below is a way to derive an equation that pairs e^{ix} (formula 39) with its complex conjugate. The first step is to determine the complex conjugate of e^{ix} which is e^{-ix}. We can get this by starting with the formula for e^{ix} and then substituting $-x$ for x (formula 40).

$$e^{ix} = \cos(x) + i\sin(x) \tag{39}$$

$$e^{i(-x)} = \cos(-x) + i\sin(-x) \tag{40}$$

Then with the help of the following two identities (formulas 41 and 42), $e^{i(-x)}$ can be simplified to formula 43.

$$\cos(-x) = \cos(x) \tag{41}$$

$$\sin(-x) = -\sin(x) \tag{42}$$

$$e^{-ix} = \cos(x) - i\sin(x) \tag{43}$$

At this point we have a formula for both e^{ix} (formula 39) and and its complex conjugate e^{-ix} (formula 43). All we have to do is add those two equations together and we get our fundamental spin equation:

$$e^{ix} + e^{-ix} = \cos(x) + i\sin(x) + \cos(x) - i\sin(x) \tag{44}$$

$$e^{ix} + e^{-ix} = 2\cos(x) \tag{45}$$

For mathematical consistency, there also must be an imaginary form of brackets placed around $e^{ix} + e^{-ix}$ to indicate that the two paired terms must be treated as a single mathematical entity.

$$[e^{ix} + e^{-ix}] = 2\cos(x) \tag{46}$$

4 Using spin concepts to compute energy

4.1 Basic energy equations

Energy formula

The energy (E) of a particle of light is equal to Planck's constant (h) times the rate-of-spin (f).

$$f = \text{Rate-of-spin (frequency in revolutions per second)} \quad (47)$$
$$h = \text{Planck's constant} \quad (48)$$
$$E = hf \quad (49)$$

The rate-of-spin (f) is also the rate-of-energy. If you double the rate-of-spin you double the energy.

Convert to radians

The frequency is often expressed in terms of radians instead of revolutions. Tau (τ) is a constant equal to 6.28.... There are τ radians for each revolution - so the rate-of-spin expressed in radians (ω) will be about 6.28 times greater than the frequency of revolutions (f).

$$\tau = 6.28... \quad (50)$$
$$\omega = \mathrm{f} \times \tau \quad (51)$$

Since the frequency in radians is about 6 times greater than revolutions, we need a reduced version of Planck's constant ($\hbar$) in an energy formula based on radians.

$$\hbar = h/\tau \quad (52)$$
$$E = \hbar\omega \quad (53)$$

Relating energy and momentum

The energy (E) is equal to the speed of light (c) times the momentum (p).

$$E = cp \tag{54}$$

The rate of momentum (k) and the rate of energy (ω) are related by the speed of light (c).

Constant	Description	Measured in
k	Rate of momentum	radians/300,000km
ω	Rate of energy	radians/sec
c	Speed of light	300,000km/sec

$$\omega = ck \tag{55}$$

4.2 The Schrodinger Equation

The Greek letter psi (Ψ) is often used to describe the "motion" of a particle of light. It is often expressed as a complex exponential using the rate-of-energy (ω) and the rate-of-momentum (k).

$$\Psi = e^{i(\mathrm{k}x-\omega t)} \tag{56}$$

$$\mathrm{k} = p/\hbar \tag{57}$$

$$\omega = E/\hbar \tag{58}$$

Using the first derivative of $\Psi(t)$ to compute energy

The first derivative of $\Psi(t)$ is how fast Ψ is changing in time - which is how fast it is spinning. The rate of spin is also the rate-of-energy, so we can use it with Planck's reduced constant to determine $E\Psi$ (the energy of Ψ).

So we take the derivative of Ψ with respect to time, and then multiply both sides by $i\hbar$ to get an energy equation for $E\Psi$:

$$\frac{\partial \Psi}{\partial t} = -i\omega\Psi = -i\frac{E}{\hbar}\Psi \tag{59}$$

$$i\hbar\frac{\partial \Psi}{\partial t} = E\Psi \tag{60}$$

The i in the last equation basically means the energy will oscillate back and forth from positive to negative. The rest of the equation says Planck's reduced constant times the rate of spin gives us the energy.

Using the first derivative of $\Psi(x)$ to compute momentum

The derivative of $\Psi(x)$ similarly gives us a formula for momentum. The change in Ψ with respect to x is the rate of momentum. If we multiply this rate of momentum times Planck's reduced constant, we get $p\Psi$ (the momentum of Ψ).

$$\frac{\partial \Psi}{\partial x} = i\frac{p}{\hbar}\Psi \tag{61}$$

$$-i\hbar\frac{\partial \Psi}{\partial x} = p\Psi \tag{62}$$

This formula tells us that the momentum of Ψ is equal to the rate-of-momentum ($\partial\Psi/\partial x$) times Planck's reduced constant. The $-i$ makes the sign oscillate back and forth from negative to positive.

Using the second derivative $\Psi(x)$ to compute the kinetic energy

The second derivative of $\Psi(x)$ is closely related to the kinetic energy - they only differ by a factor of $-\frac{\hbar^2}{2m}$. So to compute the kinetic energy, we just take the second derivative of Ψ with respect to x, and multiply both sides by that factor.

$$\frac{\partial \Psi}{\partial x} = i\frac{p}{\hbar}\Psi \tag{63}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\Psi = \frac{m^2v^2}{\hbar^2}\Psi \tag{64}$$

$$\frac{-\hbar^2}{2m} \times \frac{\partial^2 \Psi}{\partial x^2}\Psi = \frac{-\hbar^2}{2m} \times \frac{m^2v^2}{\hbar^2}\Psi = \frac{1}{2}mv^2\Psi \tag{65}$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}\Psi = \frac{1}{2}mv^2\Psi = \text{Kinetic Energy of } \Psi \tag{66}$$

Relating the energy equations

The Schrodinger equation is based on the fundamental energy equation that says the total energy ($E\Psi$) is equal to the kinetic energy ($KE\Psi$) plus the potential energy ($PE\Psi$).

$$E\Psi = KE\Psi + PE\Psi \tag{67}$$

So we can substitute formula 60 for the total energy (EΨ), formula 66 for the kinetic energy, and we can let $\upsilon\Psi$ equal the potential energy. This resulting energy equation is the time independent Schrodinger equation.

$$i\hbar\frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}\Psi + \upsilon\Psi \tag{68}$$