# Dirac Monopoles, Gauge Symmetry, Orientation-Entanglement, and the Fractional Quantum Hall Effect

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*Abstract: The purpose of this paper is to explain the pattern of fill factors observed in the Fractional Quantum Hall Effect (FQHE), which appears to be restricted to odd-integer denominators as well as the sole even-integer denominator of 2. The method is to use the mathematics of gauge theory to develop Dirac monopoles without strings as originally taught by Wu and Yang, while accounting for orientation / entanglement relationships between spinors and their environment in the physical space of spacetime. We find that the odd-integer denominators are included and the even-integer denominators are excluded if we regard two fermions as equivalent only if both their orientation and their entanglement are the same, i.e., only if they are separated by 4π not 2π. We also find that the even integer denominator of 2 is permitted because unit charges can pair into boson states which do not have the same entanglement considerations as fermions, and that all other even-integer denominators are excluded because only integer charges, and not fractional charges, can be so-paired. We conclude that the observed FQHE fill factor pattern can be fundamentally explained using nothing other than the mathematics of gauge theory in view of how orientation / entanglement applies to fermions but not to bosons, while restricting all but unfractionalized fermions from pairing into bosons.*

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## 1. Introduction: Wu and Yang and the Dirac Monopole without Strings

The Fractional Quantum Hall Effect (FQHE) observed in two-dimensional systems of electrons at low temperatures is characterized by observed filling factors $\nu = n/m$, where *n* and *m* are each integers, but where *m* is an *odd integer only*, with the exception that *m may also be the even integer 2*. In other words, the apparent pattern, widely reported and studied in the literature, for example, [1], [2], [3], [4], [5], is $n = 0, \pm1, \pm2, \pm3...$ and $m = 2, 3, 5, 7, 9, 11...$. Two questions arise from this effect: why are the denominators in the filling factor odd but not even, and why is the even denominator *m*=2 an apparent exception? We show that this pattern of filling factor denominators has a fundamental explanation based solely on using the mathematics of gauge theory to develop the Dirac Quantization Condition (DQC) for Dirac monopoles, in view of how orientation / entanglement applies to fermion spinors but not to bosons.

In 1931 Dirac discovered that the existence of magnetic monopoles implies that the electric charge must be quantized [6]. While charge quantization had been known for several decades based on the experimental work of Thompson [7] and Millikan [8], Dirac was apparently the first to lay out a possible theoretical imperative for this quantization. Using a hypothesized solenoid of singularly-thin width known as the Dirac string to shunt magnetic field lines out to mathematical infinity, Dirac established that a magnetic charge strength $\mu$ would be related to the electric charge strength *e* according to $e\mu = 2\pi n$, where *n* is an integer, which became known as the Dirac Quantization Condition (DQC). Subsequently, Wu and Yang used gauge potentials, which are locally- but not globally-exact, to obtain the exact same DQC without strings [9], [10]. Their approach is concisely summarized by Zee on pages 220-221 of [11] and will be briefly reviewed here, because it provides the methodological basis for understanding the pattern of filling factors observed for the FQHE. Throughout we use the natural units of $\hbar = c = 1$.

Using the differential one form $A = A_\mu dx^\mu$ for the electromagnetic gauge field a.k.a. potential and the differential two-form $F = \frac{1}{2!}F_{\mu\nu}dx^\mu \wedge dx^\nu = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$, a magnetic charge $\mu$ may be *defined* as the total net magnetic flux $\mu \equiv \oiint F$ passing through a closed two-dimensional surface $S^2$ which for convenience and symmetry we may take to be a sphere. Differential exterior calculus in spacetime geometry teaches that the exterior derivative of an exterior derivative is zero, *dd*=0, which means that the three-form equation $dF = ddA = 0$. Thus, via Gauss / Stokes, $\iiint 0 = \iiint dF = \oiint F = \mu$. In classical electrodynamics prior to Dirac, this was taken to mean that the magnetic charge $\mu$=0. But a close consideration of gauge symmetry, which is locally but not globally exact, tells a different story.

When a spin ½ fermion wavefunction (which we regard to be that of the electron) undergoes a local gauge (really, phase) transformation $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$, the gauge field one-form transforms as

$$A \to A' = A + e^{-i\Lambda} de^{i\Lambda} / ie \,. \tag{1.1}$$

If we represent $F$ in polar coordinates $(r,\varphi,\theta)$ as $F=(\mu/4\pi)d\cos\theta d\varphi$, then because $F=dA$ and $dd$=0, we can deduce that $A=(\mu/4\pi)\cos\theta d\varphi$. However, $d\varphi$ is not defined on the north and south poles. So we may define a north coordinate patch over which $A_N=(\mu/4\pi)(\cos\theta-1)d\varphi$ and a south patch over which $A_S=(\mu/4\pi)(\cos\theta+1)d\varphi$. But at places where these patches overlap, these gauge potentials are not the same, and specifically, the difference is $A_S-A_N=(\mu/2\pi)d\varphi$, or written slightly differently:

$$A_N \to A'_N \equiv A_S = A_N + (\mu/2\pi)d\varphi. \tag{1.2}$$

So comparing this with (1.1), we may regard $A_S$ as a gauge-transformed state $A'_N$ of $A_N$, for which the gauge transformation is simply:

$$\frac{1}{ie}e^{-i\Lambda}de^{i\Lambda}=\frac{\mu}{2\pi}d\varphi. \tag{1.3}$$

This equation is satisfied by:

$$\exp(i\Lambda)=\exp\left(ie\mu\frac{\varphi}{2\pi}\right), \tag{1.4}$$

as can be seen simply by plugging $e^{i\Lambda}$ from (1.4) into the left hand side of (1.3) and reducing. This relates the azimuth angle $\varphi$ which is one of the three spacetime coordinates, to the local gauge (phase) angle $\Lambda$, and thereby connects rotations through $\varphi$ in physical space to rotations through $\Lambda$ in the gauge space in a manner that we shall now explore in detail.

In polar coordinates, $\varphi=0$ and $\varphi=2\pi$ in (1.4) describe exactly the same *orientation* (but not entanglement) on the surface of $S^2$. So to make sense of (1.4) at like-orientations, we must have:

$$\exp(i\Lambda)=\exp(ie\mu\cdot 0)=1=\exp(ie\mu\cdot 1), \tag{1.5}$$

Specifically, this means that $\exp(ie\mu)=1$. Mathematically, the general solution for an equation of this form is $\exp(i2\pi n)=1$ for any integer $n=0,\pm1,\pm2,\pm3...$, which is infinitely degenerate but quantized. As a result, the solution to (1.5) is:

$$\Lambda=e\mu=2\pi n. \tag{1.6}$$

This, of course, is the Dirac Quantization Condition, which may also specified in relation to the gauge (phase) parameter $\Lambda$ which is seen to be an quantized integer multiple of $2\pi$. Specifically, (1.6) with simple rearrangement tells us that the electric charge is quantized according to:

$$e = n\frac{2\pi}{\mu} = ne_{\mathrm{u}} = \frac{\Lambda}{\mu}, \tag{1.7}$$

where the $n$=1 "unit" (u) of electric charge is $e_{\mathrm{u}} \equiv 2\pi/\mu$, defined as $2\pi$ times the inverse of the magnetic charge. If this monopole "exists," then the electric charge is quantized in units of $e_{\mathrm{u}}$.

We may then go back to the original definition $\mu \equiv \oiint F$ and use (1.6) to write:

$$\oiint F = \mu = \frac{2\pi}{e} n = n\mu_{\mathrm{u}} = \frac{\Lambda}{e}, \tag{1.8}$$

where we also define an $n$=1 unit of magnetic charge $\mu_{\mathrm{u}} \equiv 2\pi/e$. By appropriate local gauge transformation, and specifically by choosing $n$=0 which is the same as choosing the phase angle $\Lambda = 0$, the nonzero surface integral can be made to vanish, $\oiint F = 0$. But this does not invalidate (1.7) and (1.8) nor does it prevent us from seeking to draw physical conclusions from these. It simply means that $n$=0 with no monopoles and no electric charges is one of the permitted states. But the meaning of the whole range of charges $e = ne_{\mathrm{u}}$ for $n \neq 0$ has long been physically-interpreted as suggesting charge quantization. In the lowest positive non-zero $n$=1 state, from (1.6), we have $\Lambda = e\mu = 2\pi$. If we define a reduced $\not{\Lambda} \equiv \Lambda/2\pi$, then by (1.6), the reduced gauge parameter $\not{\Lambda} = n$ is simply the charge quantum number $n$. So every gauge transformation adding an angle of $2\pi$ adds one unit of electric and magnetic charge.

This is how Wu and Yang obtain Dirac monopoles and the DQC without strings, which leads to the quantization of both electric and magnetic charges.

## 2. Orientation – Entanglement and Odd-Numbered Fractional Quantum Hall Denominators

If we closely study this derivation by Wu and Yang, we see that there are some additional quantum states that have not yet been considered. Referring to (1.5), not only do $\varphi = 0$ and $\varphi = 2\pi$ describe exactly the same *orientation* (sans entanglement), but so too do $\varphi = 4\pi$, $\varphi = 6\pi$, $\varphi = 8\pi$, etc. So we now extend (1.5) to:

$$\exp(i\Lambda) = 1 = \exp(ie\mu\cdot 1) = \exp(ie\mu\cdot 2) = \exp(ie\mu\cdot 3) = \exp(ie\mu\cdot 4) = \exp(ie\mu\cdot 5) = \exp(ie\mu\cdot 6)\ldots \tag{2.1}$$

Each of the above is a separate relationship of the general form $\exp(ie\mu\cdot m) = 1$, where $m = 1,2,3,4,5,6...$ is an integer not the same as the $n$ already in use. At the same time, as noted after (1.5), the general solution for an equation of this form is $\exp(i2\pi n) = 1$ with this integer

$n = 0, \pm 1, \pm 2, \pm 3 \ldots$. Comparing $\exp(ie\mu \cdot m) = 1$ with $\exp(i2\pi n) = 1$ means that more generally, $e\mu \cdot m = 2\pi n$, or, restated:

$$e = \frac{n}{m}\frac{2\pi}{\mu} = \frac{n}{m}e_{\mathrm{u}} = \nu e_{\mathrm{u}}, \tag{2.2}$$

where we define a "filling factor"

$$\nu \equiv \frac{n}{m}; \quad n = 0, \pm 1, \pm 2, \pm 3 \ldots; \quad m = 1, 2, 3, 4, 5, 6 \ldots. \tag{2.3}$$

So this tells us that the unit of electric charge $e_{\mathrm{u}}$ can be fractionalized into any $\nu = n/m$ fraction of itself.

Now, while fractional charges are observed in the Fractional Quantum Hall Effect, (2.2) and (2.3) give us *too many fractional states*, because with the exception of *m*=2, the only states which appear to be found experimentally are states in which *m* is an *odd* integer. So the question now becomes, whether (2.3) with further development be used to explain the FQHE solely on the basis of gauge symmetry. But to do so, we have to find out why the denominator is restricted to only odd integers, with the exception of the even integer 2. That is, (2.3) allows $m = 1, 2, 3, 4, 5, 6 \ldots$, but based on what is observed, we should only have $m = 1, 2, 3, 5, 7, 9 \ldots$, so we now need to account for this discrepancy. This is where we start to consider not only orientation, but also entanglement.

So far, we have regarded two different fermion states to be physically equivalent if their azimuth angles $\varphi$ differ by $2\pi$. So (2.1) and thus (2.2) and (2.3) were developed by regarding $\varphi = 2\pi$, $\varphi = 4\pi$, $\varphi = 6\pi$, $\varphi = 8\pi$, etc. to be equivalent states because they are all states of like-orientation. But let us now return to $\psi(x) \to \psi'(x) = e^{i\Lambda(x)}\psi(x)$, which is the gauge transformation with which we originally started at (1.1). This transformation acts on a Dirac fermion wavefunction taken to be that of an electron. Because Dirac fermions behave as spinors, they have not only orientation, but also entanglement in relation to their environment in the physical space of spacetime. One of the most widely-regarded discussions of this, is that of Misner, Thorne and Wheeler in [12] at section 41.5.

So let's start with a Dirac fermion, e.g., electron with a unit of charge $e_{\mathrm{u}} \equiv 2\pi/\mu$ in the *n*=1 state as obtained in the DQC (1.6), which is the state for which the azimuth angle $\varphi = 2\pi$. Now, let us rotate this electron into an equivalent state which maintains not only orientation, *but also entanglement*. If we start at $\varphi = 2\pi$ and rotate the fermion wavefunction $\psi(x)$ to $\varphi = 4\pi$, the fermion will have been restored to its same orientation, *but not to its same entanglement*.

To keep the entanglement the same, i.e., to maintain the same *version* of the fermion, we must rotate the fermion through an azimuth of $4\pi$, not merely $2\pi$. This is because of the double covering whereby a $2\pi$ rotation (a single "winding") for SU(2) corresponds to a $4\pi$

rotation (a double winding) in the O(3) physical space of the spacetime rotation group in which the azimuth $\varphi$ coordinate subsists. Consequently, starting from the $\varphi = 2\pi$ state of an $n$=1 unit charge $e_{\mathrm{u}} \equiv 2\pi / \mu$, the angles with like-entanglement will be $\varphi = 6\pi$, $\varphi = 10\pi$, $\varphi = 14\pi$ etc. The angles with opposite-entanglement will be those with $\varphi = 4\pi$, $\varphi = 8\pi$, $\varphi = 12\pi$, etc. So to restore the same version of the fermion we must keep the former but discard the latter set of azimuth angles.

With this in mind we go back to (2.1), start with the $\varphi = 2\pi$ state for the $n$=1 unit charge $e_{\mathrm{u}} = 2\pi / \mu$, and rotate this through a succession of $4\pi$ windings, eliminating the intermediate $2\pi$ rotations which give the same orientation but produce opposite entanglement. When we discard these oppositely entangled states, (2.1) now becomes:

$$\exp(i\Lambda) = 1 = \exp(ie\mu \cdot 1) = \exp(ie\mu \cdot 3) = \exp(ie\mu \cdot 5) = \exp(ie\mu \cdot 7) = \exp(ie\mu \cdot 9) \dots . \tag{2.4}$$

Each of the above is now a separate relationship $\exp(ie\mu(2m+1)) = 1$ where $m$ is an integer with the values $m = 0,1,2,3\dots$, so that $2m+1 = 1,3,5,7\dots$ is an *odd integer*. Comparing $\exp(ie\mu(2m+1)) = 1$ with $\exp(i2\pi n) = 1$ means that $e\mu(2m+1) = 2\pi n$, or, restated:

$$e = \frac{n}{1+2m}\frac{2\pi}{\mu} = \frac{n}{1+2m} e_{\mathrm{u}} = \nu e_{\mathrm{u}} , \tag{2.5}$$

with a redefined filling factor:

$$\nu = \frac{n}{1+2m}; \quad n = 0, \pm 1, \pm 2, \pm 3 \dots; \quad m = 0,1,2,3 \dots . \tag{2.6}$$

In contrast to (2.3), this filling factor *will always have an odd denominator*. So the appearance of odd but not even denominators (other than the even denominator 2) in the FQHE appears to be explained by (2.6), and is the result of requiring that physically-equivalent gauge transformations be those which, when applied to a fermion, produce the same orientation *and the same entanglement*.

So now we have a the gauge theory explanation for why the FQHE filling factor denominators are restricted to only odd integers but not even ones: It is opposite entanglement which excludes the even denominators. Now we come to the next question: Why does nature also appear to permit the even denominator 2, in addition to the odd denominators $1+2m$ in (2.6), but not permit any other even denominators?

# 3. Even-Numbered Fractional Quantum Hall Denominators Restricted to be Equal to 2

Equation (1.6) for the Dirac Quantization Condition (1.6) specifies charges $e = n(2\pi/\mu) = ne_{\mathrm{u}}$ which are integer multiples of the unit charge $e_{\mathrm{u}} = 2\pi/\mu$. For this set of integer charges, (1.6) tells us that the gauge parameter $\Lambda = 2\pi n$. This entire set of integer-quantized charges corresponds to the single azimuth $\varphi = 2\pi$, which means that there is a one-to-infinite quantized mapping of $\varphi$ to $\Lambda$. That is, an infinite set of gauge states $\Lambda = 2\pi n$ can all be used to equivalently describe the same azimuth state $\varphi = 2\pi$, yielding quantized multiples of the unit charge.

Now, if we take a single fermion e.g. electron in the $\varphi = 2\pi$ state and do a $4\pi$ rotation to a $\varphi = 6\pi$ state which restores the electron to its original orientation and entanglement version, then $\exp(i\Lambda) = 1 = \exp(ie\mu \cdot 3)$ is the portion of (2.4) which describes this new state. Referring again to the general relationship $\exp(i2\pi n) = 1$, the solution is $\Lambda = 2\pi n = 3e\mu$, restated as $e = \Lambda/3\mu = (n/3)(2\pi/\mu) = ne_{\mathrm{u}}/3$. This specifies integer $n$ multiples *of 1/3 of the unit charge* $e_{\mathrm{u}}$. As before, the gauge parameter $\Lambda = 2\pi n$, from which we earlier defined a reduced gauge $\not{\Lambda} = n$ after (1.8). Similarly, if we define a reduced azimuth $\not{\varphi} \equiv \varphi/2\pi = 3$, we see that the fractional denominator *m* is equal to the reduced azimuth, $m = \not{\varphi} = 3$. It is readily seen that for the $\varphi = 10\pi$, $\varphi = 14\pi$ etc. states which also maintain the $\varphi = 2\pi$ version of the fermion, that the odd number denominators in (2.5) and (2.6) may be written as $\not{\varphi} = 1 + 2m$. Using this information and notation, we rewrite the filling factor of (2.6) as:

$$\nu = \frac{\not{\Lambda}}{\not{\varphi}}; \quad \not{\Lambda} = 0, \pm1, \pm2, \pm3...; \quad \not{\varphi} = 1 + 2m = 1,3,5,7,9...; \quad m = 0,1,2,3.... \tag{3.1}$$

So the *fractionalization* of charge is determined directly by the number of "windings" $\not{\varphi} = 1 + 2m$ *in the physical space of spacetime*, and the odd numbers in the denominators occur because one must use two windings, not one, to restore a fermion to its original version. On the other hand, the *quantization* of charge into integer multiples of a fractional charge is related to the quantized degeneracy of the gauge parameter $\not{\Lambda} = n$ which describes an infinite number of equivalent gauge states, and mathematically to the fact that phase angles which differ from one another by $2\pi$ are degenerately equivalent. Now let's turn to the only even denominator, 2, which is phenomenologically-observed in the FQHE.

A denominator of 2 corresponds to a winding number $\not{\varphi} = 2$. The set of quantized states for the unit charge – not a fractional charge – which we now write as $e = \not{\Lambda}(2\pi/\mu) = \not{\Lambda} e_{\mathrm{u}}$, corresponds to the winding number $\not{\varphi} = 1$. So to get from an electron with $\not{\varphi} = 1$ to some state with $\not{\varphi} = 2$ we are only making one turn of the azimuth. Thus while we are restoring orientation, we are not restoring entanglement. Nonetheless, $\not{\varphi} = 2$ is the only even winding number which nature permits, so we have to figure out why we observe a state that is only one turn above $\cancel{\varphi}=1$.

An electron will not be restored to its original version at $\cancel{\varphi}=2$, because fermions need to do two windings to regain their original version. Only bosons can maintain equivalent version with one winding, because for bosons, entanglement is not an issue because they are not spinors. So for an electron to go from $\cancel{\varphi}=1$ to $\cancel{\varphi}=2$, that electron must "disguise" itself as a boson. How might the electron do that? By finding a second electron to "conspire" with the first electron and "pair up" into a single boson system. Then, that pair of electrons can be wound from $\cancel{\varphi}=1$ to $\cancel{\varphi}=2$ without changing its entanglement. And based on (3.1), the filling factor will now be $\nu=\cancel{\Lambda}/\cancel{\varphi}=n/2$, which yields the denominator of 2.

What does this mean?: it means that while all of the permitted windings of individual electrons yield fractional charges with the 3, 5, 7, 9, etc. odd denominators, the permitted winding for a boson *pair of electrons* yields the one permitted even denominator, namely 2. While the "Cooper pairs" model of electron pairing [13] may well come to mind, for the moment let us not be that specific. Let us simply talk in terms of the requirement that a first electron needs to find some way to pair together with a second electron if a $\cancel{\varphi}=2$ winding state and thus a ½ unit of charge is to be empirically displayed under the right set of conditions – as it is at extremely low temperatures in suitable materials under experimentally-replicable circumstances – while leaving open the mechanism by which that pairing take place. So, the pairing of electrons into boson states would appear to explain why 2 is *permitted* as an even denominator, and we know that there is some grounding in established theory for such pairing to occur. Now, we have left to explain why 4, 6, 8 and even denominators *other than 2* are *not permitted*.

The next even denominator of course is *m*=4 which we now know corresponds to the winding azimuth $\cancel{\varphi}=4$. And we know that this fractionalization $\nu=n/4$ is *not observed*. So let us start with an electron in the $\cancel{\varphi}=3$ fractional state for which $\nu=n/3$. These are fractional fermion charges, so to get them to an $\cancel{\varphi}=4$ winding which would correspond adding one azimuth turn to $\cancel{\varphi}=3$. This would result in an oppositely-entangled state, which is inequivalent to $\cancel{\varphi}=3$. So, as we did to get from $\cancel{\varphi}=1$ to $\cancel{\varphi}=2$, we would have to "pair up" two of these $\cancel{\varphi}=3$ fractional $\nu=n/3$ fermions into a boson state to get to $\cancel{\varphi}=4$ with a quarter-integer fraction $\nu=n/4$. The fact that we do not observe $\cancel{\varphi}=4$, nor do we observe any other even windings $\cancel{\varphi}=6,8,10...$, is nature's way of telling us that *fractional charges with cannot be paired up into boson states*. *All boson pairs must be constructed from unfractioned charge units* $\mu_{\mathrm{u}}=2\pi/e$. Given that fractional charges are commonly regarded as quasiparticles while unit charges are not, this simply means that only "real" particles, not quasiparticles, can form pairs.

Pulling together all of these results, we now supplement (3.1) with $\cancel{\varphi}=2$. Thus, the final result for the overall observed pattern is:

$$\nu=\frac{\cancel{\Lambda}}{\cancel{\varphi}};\quad \cancel{\Lambda}=0,\pm1,\pm2,\pm3...;\quad \cancel{\varphi}=2 \quad \text{--or--} \quad \cancel{\varphi}=1+2m;\quad m=0,1,2,3.... \tag{3.2}$$

The $\not\varphi = 1+2m$ odd-denominator states represent fermions exhibiting fractional charges; the $\not\varphi = 2$ state represents a boson pair of unit charges that are not fractional; and the absence of $\not\varphi = 4,6,8...$ states tells us that fractional charges are not capable of forming into boson pairs. In Figure 1 below, which is reproduced from [14] and [15], we have annotated the unit electron charge, the $\nu = \not\Lambda / \not\varphi = 1/3$ fractional charge, and the ground state for a pair of unit electrons forming a boson with $\not\Lambda / \not\varphi = 1/2$. Also added as annotations are apparent $\nu = 5/11$, $\nu = 6/11$ and $\nu = 7/9$ fractional states.

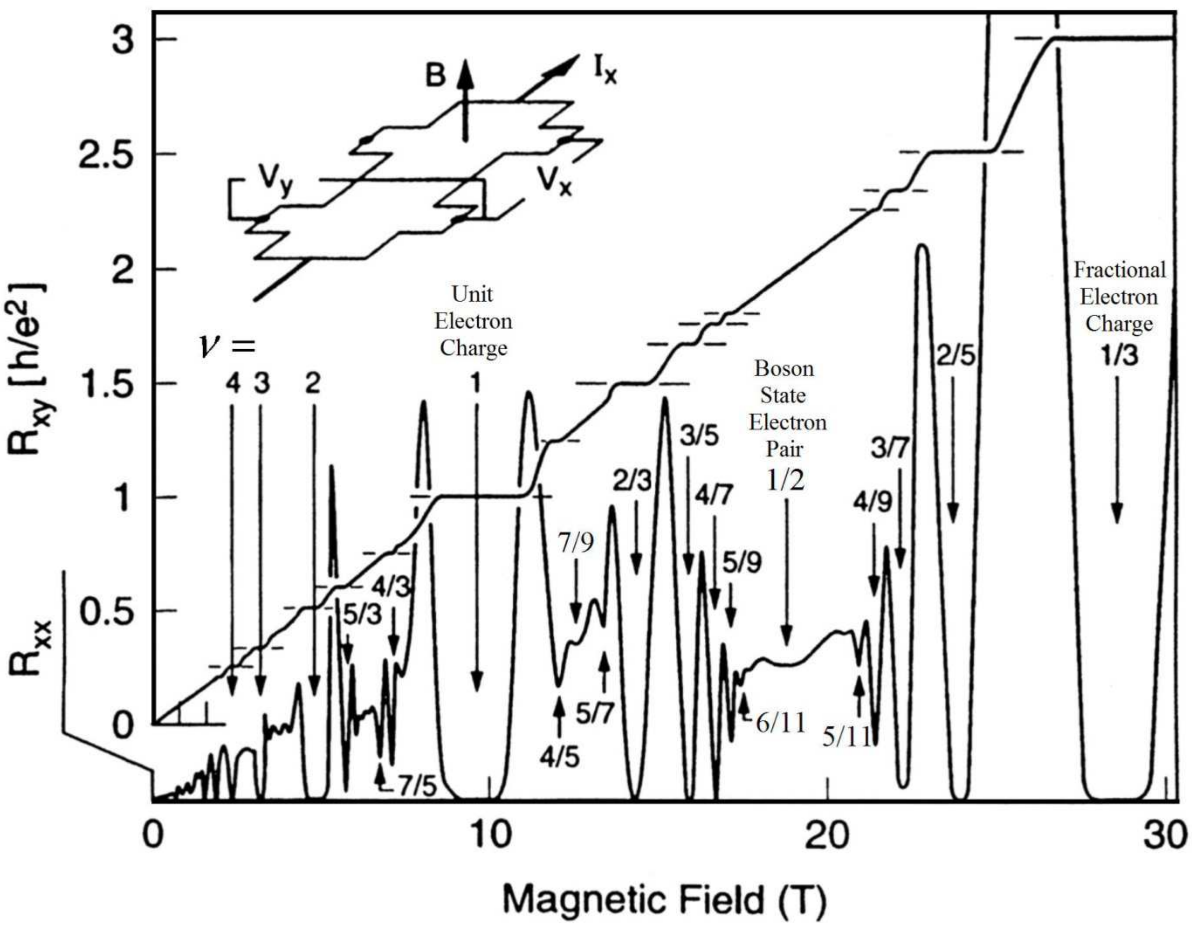


Figure 1: Fractional Quantum Hall Effect, reproduced from [14], [15], with annotation

It is worthwhile comparing the $\not\Lambda = n = 1$ ground state of the $\not\varphi = 1$, $\not\varphi = 2$ and $\not\varphi = 3$ windings, which are the three states annotated above. For $\nu = \not\Lambda / \not\varphi = 1$ we of course have a unit electron charge. For $\nu = \not\Lambda / \not\varphi = 1/3$ we have 1/3 fractional charge. But for the boson pair with $\nu = \not\Lambda / \not\varphi = 1/2$ with a ½ unit of fractional charge, there are *two electrons not one* contributing to the half unit of charge. Therefore, each electron actually contributes a ¼ unit of charge. Given that electrons naturally repel one another so that any pair formation mechanism must overcome this repulsion, it will be easier for two electrons to assume charges of ¼ unit apiece and then pair into a boson, than to stay in the unit charge state or in the 1/3 charge state and then pair up. It is the ¼ charge-per-electron paired state which minimizes the repulsion and therefore provides the most energetically-favored configuration.

Finally, we return to the original definition $\mu \equiv \oiint F$ of the Dirac monopoles. If rewrite $e = \nu e_{\mathrm{u}} = \nu\left(2\pi / \mu\right)$ with the complete filling factor (3.2) in terms of $\mu$, then using the "unit" of magnetic charge $\mu_{\mathrm{u}} = 2\pi / e$, what we learn about the permitted monopole fluxes is that:

$$\oiint F = \nu\mu_{\mathrm{u}} = \left(\cancel{A} / \cancel{\varphi}\right)\mu_{\mathrm{u}}; \quad \cancel{A} = 0, \pm 1, \pm 2, \pm 3 \ldots; \quad \cancel{\varphi} = 2 \quad \text{--or--} \quad \cancel{\varphi} = 1 + 2m; \quad m = 0,1,2,3\ldots \tag{3.3}$$

It is often said that the Dirac Quantization Condition demonstrates that if magnetic charges exist, then electric charge is quantized. The existence of quantized electric charge is then used to infer the possible existence of Dirac monopoles, even though there have apparently been no such monopoles observed. But when we use Wu and Yang's gauge theory without strings [9], [10] to develop the DQC to its logical conclusion, we see that fractional charges of the FQHE emerge right alongside of quantized charges. So the DQC nomenclature somewhat misrepresents this result, because the complete result is really a Dirac Quantization *and Fractionalization* Condition, DQFC.

This extended understanding of Dirac monopoles should put into a somewhat different perspective how one thinks about these monopoles, at least based on Dirac's quantization absent further developments such as t'Hooft / Polyakov monopoles [16], [17] which rely on Yang-Mills gauge theory which is not needed for Dirac monopoles alone. Although the Dirac monopoles when fully developed using Wu and Yang's gauge approach are *fractionalized as well as quantized*, these fractional charges are not observed except under very limited conditions at extremely low temperatures in suitable superconducting materials. Thus, to the degree that the filling factors (3.2) do describe a feature of the natural world but only under these specialized conditions, and because (3.3) is integrally related to (3.2), it would appear that the non-zero magnetic fluxes $\oiint F = \nu\mu_{\mathrm{u}}$ of Dirac monopoles (as distinguished from other types of monopole) would only evidence themselves in nature under equally-restricted conditions.

## 4. Conclusion: Dirac Quantization and Fractionalization

We conclude that a complete analysis of the gauge symmetries of Dirac Monopoles following the approach pioneered by Wu and Yang [9], [10] results in electric and magnetic charges which are quantized *and fractionalized* in the manner observed in the Fractional Quantum Hall Effect. Because fermions rotated through an azimuth over $2\pi$ regain their orientation but not their entanglement, the $4\pi$ rotation needed to restore both orientation and entanglement is responsible for the observation of odd-integer denominators and the skipping of most even-integer denominators. The only observed even-integer denominator of 2 appears to be the result of pairing two integer-charged fermions into a boson, and the absence of any larger even denominators appears to indicate that only integer charges, and not fractional charges, can be so-paired.

## References

[1] Laughlin, R. B. Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations, *Physical Review Letters* 50 18 (1983)
[2] Goldman, V. J. and Su, B. *Resonant Tunneling in the Quantum Hall Regime: Measurement of Fractional Charge*, *Science* 267 5200 (1995)
[3] http://physicsworld.com/cws/article/news/1997/oct/24/fractional-charge-carriers-discovered
[4] de-Picciotto, R., Reznikov, M., Heiblum, M., Umansky, V., Bunin, G., Mahalu, D., *Direct observation of a fractional charge*, Nature 389 (6647): 162 (1997)
[5] J. Martin, J., S. Ilani, S., B. Verdene, B., J. Smet, J., V. Umansky, V., D. Mahalu, D., D. Schuh, D., G. Abstreiter, G., A. Yacoby, A., *Localization of Fractionally Charged Quasi Particles*, Science ***305*** (5686): 980–3 (2004)
[6] Dirac, P.A.M., *Quantized Singularities in the Electromagnetic Field*, Proceedings of the Royal Society A 133 (821): 60–72, (September 1931)
[7] Thomson, J. J., *Cathode Rays*, The Electrician 39, 104 (1897)
[8] Millikan, R. A., *On the Elementary Electric charge and the Avogadro Constant*, Physical Review, series II, **2**, pp. 109-143 (1913)
[9] Wu, T. T. and Yang, C.N., *Concept of non-integrable phase factors and global formulation of gauge fields*, Phys. Rev. D 12 (1975) 3845
[10] Wu, T. T. and Yang, C. N., *Dirac Monopole without Strings: Classical Lagrangian theory*, Phys. Rev. D 14 (1976) 437
[11] Zee, A., *Quantum Field Theory in a Nutshell*, Princeton (2003)
[12] Misner, C. W., Thorne, K. S., and Wheeler, J. A., *Gravitation*, Freeman (1973)
[13] Cooper, L. N., Bound electron pairs in a degenerate Fermi gas, Physical Review 104 (4): 1189–1190 (1956)
[14] http://www.nap.edu/openbook.php?record_id=10001&page=6
[15] Stormer, H.L., D.C. Tsui, and A.C.Gossard, *The fractional quantum Hall effect*, Reviews of Modern Physics, vol.71 (no.2) (Feb. 1999)
[16] t'Hooft, G., *Magnetic Monopoles in Unified Gauge Theories*, Nuclear Physics B79, 276-284, (1974)
[17] Polyakov, A. M., *Particle Spectrum in the Quantum Field Theory*, JETP Lett. 20, 194-195 (1974)